Northern Technical University

Technical Engineering College / Mosul

Department of Power Mechanics Engineering Technology

Strength of Materials

For

Second Year Students

Strength of Materials:

Strength of Materials, or Mechanics of Materials, is the study of how solid objects withstand stresses and strains. It covers concepts like stress, strain, elasticity, plasticity, and failure mechanisms. Key properties include Young's modulus (stiffness), yield strength (onset of permanent deformation), and ultimate strength (maximum stress before failure). The field also examines fracture mechanics, fatigue (failure from repeated loading), and creep (slow deformation under stress). Applications span civil, mechanical, aerospace engineering, and materials science, ensuring structures and components are designed to safely endure various forces and conditions without failing.

The Aim of This Course:

- To know different types of the stresses which may subjected to the mechanical elements and their expected effects such as strain.
- To study the shear forces and bending moment diagrams with essential stresses

Assessments:

Classwork	**10 Marks**
Homework	**10 Marks**
Reports	**10 Marks**
Quiz	**10 Marks**
Mid Term	**10 Marks**
Overall	**50 Marks**
Final Exam	**50 Marks**
Total	**100 Marks**

Course Contents

Week no.	Materials to be covered
1	Introduction to Strength of Materials
2	Simple stress and strain
3	Stress-strain diagram
4	Shearing stress
5	Bearing stress Thermal stress
6	Thin wall cylinders
7	Stress in compound section
8	Shear force and bending moment in beam
9	
10	Bending stress
11	Slope and Deflection of Beams
12	
13	Torsion
14	Combined stresses
15	

Main Reference:

- **Strength of Materials,** By Andrew Pytel and Ferdinand L. Singer 4th edition.

Useful References:

- **A Textbook of Strength of Materials,** by R. K. Rajput, 7th edition
- **Mechanics of Materials**, by Ferdinand P. Beer, 8th edition.
- **Mechanics of Materials I**, by E. J. Hearn, 3rd edition.

Units

The units and symbols of the four fundamental quantities of mechanics for the International System of metric units SI And U.S. customary system are summarized in the following table:

Quantity	Dimensional Symbol	SI Units		U.S. Customary Units (FPS)	
		Unit	Symbol	Unit	Symbol
Mass	M	kilogram	kg	slug	-
Length	L	meter	m	foot	ft
Time	T	second	s	second	sec
Force	F	newton	N	pound	lb

SI Prefixes

Multiple	*Exponential Form*	*Prefix*	*SI Symbol*
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k
Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

Conversion Factors (FPS) to (SI)

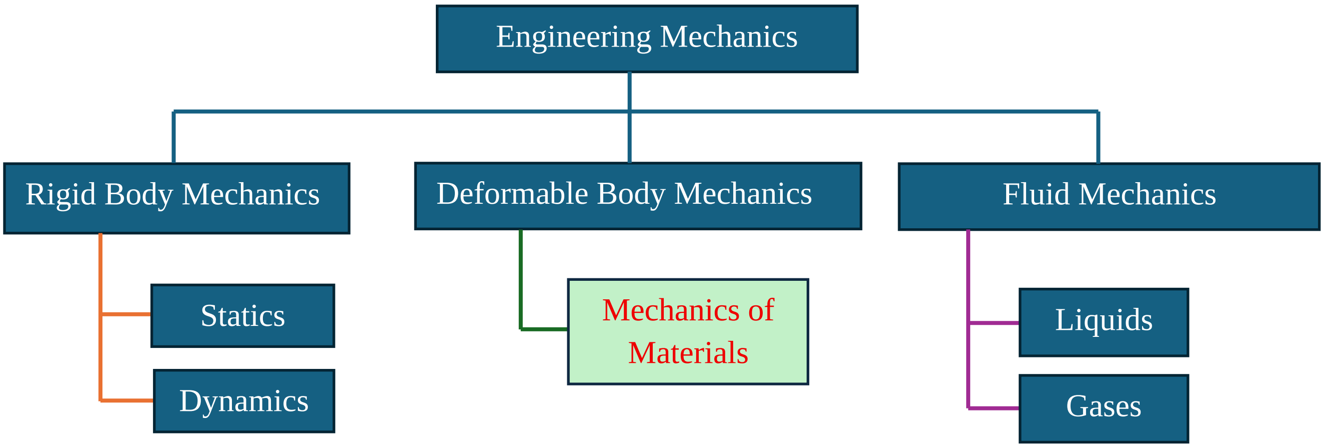
Quantity	Unit of Measurement (FPS)	Equals	Unit of Measurement (SI)
Force	1 lbf	=	4.448 N
Mass	1 slug	=	14.59 kg
Length	1 ft	=	0.3048 m

Conversion Factors (SI) to (FPS)

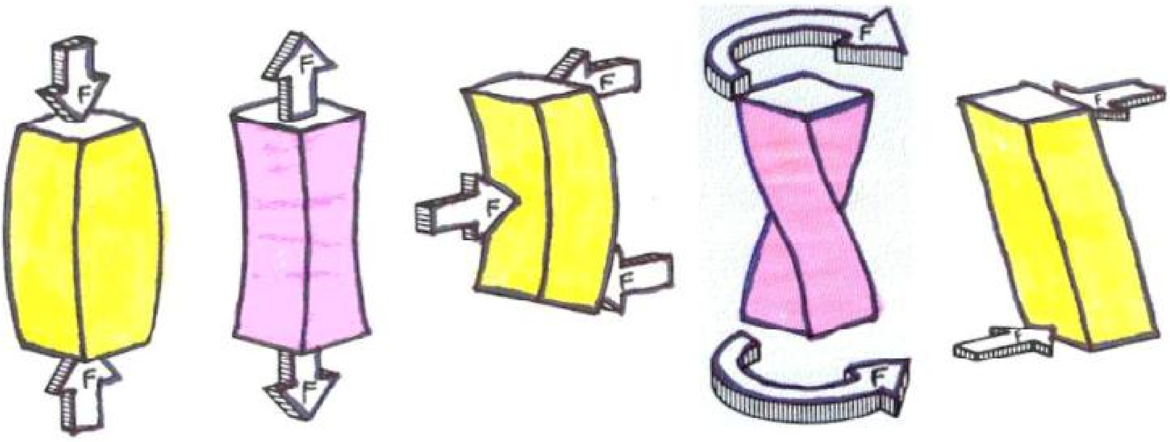
Quantity	Unit of Measurement (SI)	Equals	Unit of Measurement (FPS)
Force	1N	=	0.22481 lbf
Mass	1 kg	=	0.0685218 Slug
Length	1 m	=	3.2808ft

Introduction to Mechanics of materials

Definition: Mechanics of materials is a branch of applied mechanics that deals with the behaviour of solid bodies subjected to various types of loading.



Types of Stress



Compression **Tension (stretched)** **Bending** **Torsion (twisted)** **Shearing**

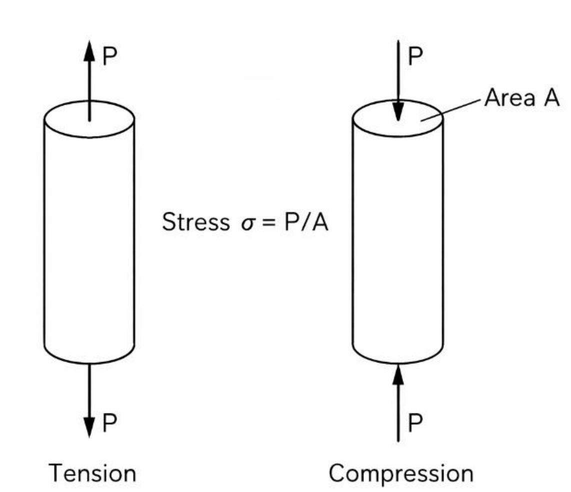
Why do we study Mechanics of Materials?

Anyone concerned with the strength and physical performance of natural/man-made structures should study Mechanics of Materials

Simple Stress and Strain

If a cylindrical bar is subjected to a direct pull or push along its axis as shown, then it is said to be subjected to tension or compression. Typical examples of tension are the forces present in towing ropes or lifting hoists, whilst compression occurs in the legs of your chair as you sit on it or in the support pillars of buildings.

Force can be denoted by several letters such as: F, P, N, etc.



In the SI system of units load is measured in Newton's, although a single Newton, in engineering terms, is a very small load. In most engineering applications, therefore, loads appear in SI multiples, i.e. Kilo-Newton (kN) or Mega-Newton (MN).

There are a number of different ways in which load can be applied to a member. Typical loading types are:

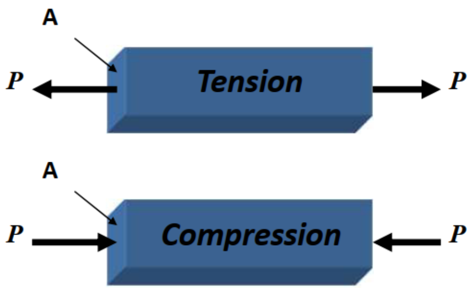
A- **Static or dead loads**, i.e. non-fluctuating loads, generally caused by gravity effects.
B- **Live loads**, as produced by, for example, lorries crossing a bridge.
C- **Impact** or **shock loads** caused by sudden blows.
D- **Fatigue**, **fluctuating** or **alternating loads**, the magnitude and sign of the load changing with time.

Direct or normal stress (σ)

When a bar is subjected to a uniform tension or compression, i.e. a direct force (F or P), which is uniformly or equally applied across the cross-section, then the internal forces set up are also distributed uniformly and the bar is said to be subjected to a uniform direct or normal stress, the stress being defined as:

$$Stress = \frac{load}{area}$$

$$\boldsymbol{\sigma} = \frac{\boldsymbol{P}}{\boldsymbol{A}} \quad \dots\dots\dots\dots (1)$$

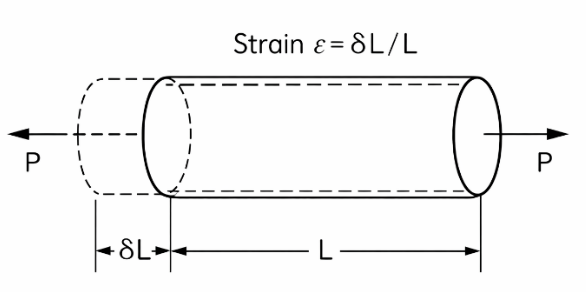


Stress (σ) may thus be compressive or tensile depending on the nature of the load and will be measured in units of Newton per square meter (N/m^2) or multiples of this.

Direct strain (ε)

If a bar is subjected to a direct load, and hence a stress, the bar will change in dimension (such as length). If the bar has an original length L and changes in length by an amount L, the strain produced is defined as follows:

$$Strain = \frac{change\ in\ length}{original\ length} \quad \rightarrow \quad \boldsymbol{\varepsilon = \frac{\delta L}{L}} \quad ....(2)$$



Since, in practice, the extensions of materials under load are very small, it is often convenient to measure the strains in the form of strain x 10^{-6} i.e. Micro strain, when the symbol used becomes $\mu\varepsilon$. Alternatively, strain can be expressed as a percentage strain

$$\varepsilon = \frac{\delta L}{L} \times 100\%$$

Sign convention for direct stress and strain

Tensile stresses and strains are considered POSITIVE in sense producing an increase in length. Compressive stresses and strains are considered NEGATIVE in sense producing a decrease in length.

Example 1:

A metal wire is 2.5 mm diameter and 2 m long. A force of 12 N is applied to it and it stretches 0.3 mm. Assume the material is elastic. Determine the following.

A- The stress in the wire σ.
B- The strain in the wire ε.

Solution:

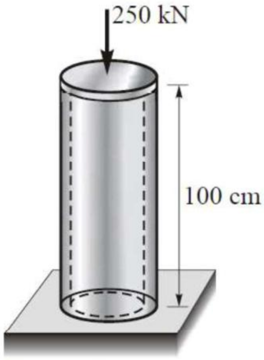
$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (2.5 \times 10^{-3})^2 = 0.0000049\ m^2$$

$$\sigma = \frac{P}{A} = \frac{12}{0.0000049} = 2448979.59\ Pa = 2.44\ MPa$$

$$\varepsilon = \frac{\delta L}{L} = \frac{0.3}{2 \times 1000} = 0.00015\ or\ 150\ \mu\varepsilon$$

Example 2:

A short post constructed from a hollow circular tube of aluminium supports a compressive load of 250 kN (see figure). The inner and outer diameters of the tube are 9 cm and 13 cm, respectively, and its length is 100 cm. The shortening of the post due to the load is measured as 0.5 mm. Determine the compressive stress and strain in the post.



Hollow aluminum post in compression

Solution:

$$d_o = \frac{13}{100} = 0.13\,m \quad , \quad d_i = \frac{9}{100} = 0.09\,m$$

$$L = \frac{100}{100} = 1\,m \quad , \delta L = \frac{0.5}{1000} = 0.0005\,m$$

$$A = \frac{\pi}{4}(d_o^2 - d_i^2) = \frac{\pi}{4}(0.13^2 - 0.09^2) = 0.0069\,m^2$$

$$\sigma = \frac{P}{A} = \frac{250 \times 10^3}{0.0069} = -36231884.05\,N/m^2 = -36.2\,MPa$$

$$\varepsilon = \frac{\delta L}{L} = \frac{0.0005}{1} = -0.0005$$

Elastic Materials - Hooks's Law

A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed. Since loads are proportional to the stresses they produce and deformations are proportional to the strains, this also implies that, whilst materials are elastic, stress is proportional to strain. Hook's law, in its simplest form, therefore states that:


Robert Hooke (1635-1703)

$$Stress(\sigma) \propto Strain(\varepsilon)$$

$$\frac{Stress}{Strain} = constant$$

this law is obeyed within certain limits by most ferrous alloys and it can even be assumed to apply to other engineering materials such as concrete, timber and non-ferrous alloys with reasonable accuracy. Whilst a material is elastic the deformation produced by any load will be completely recovered when the load is removed; there is no permanent deformation.

Modulus of Elasticity – Young's Modulus

Within the elastic limits of materials, i.e. within the limits in which law applies, it has been shown that


Thomas Young (1773-1829)

$$\frac{Stress}{Strain} = constant$$

This constant is given the symbol E and termed the modulus of elasticity or modulus.

$$E = \frac{Stress}{Strain} = \frac{\sigma}{\varepsilon} = \frac{\frac{P}{A}}{\frac{\delta L}{L}} \rightarrow \quad \boldsymbol{E = \frac{PL}{A\delta L}} \quad \ldots\ldots\ldots(3)$$

Young s modulus E is generally assumed to be the same in tension or compression and for most engineering materials has a high numerical value. Typically, E = 200×10^9 N/m^2 for steel or (E=200 GN/m^2), so that it will be observed that strains are normally very small.

Example 3:

A 25 mm diameter bar is subjected to an axial tensile load of 100 kN. Under the action of this load a 200mm gauge length is found to extend 0.19 mm. Determine the modulus of elasticity for the bar material.

Solution

$$E = \frac{\sigma}{\varepsilon} = \frac{PL}{A\delta L} = \frac{(100\times10^3)\times(200\times10^{-3})}{\frac{\pi}{4}(25\times10^{-3})^2\times(0.19\times10^{-3})} = 214440344376.45\,Pa = 214.4\,GPa$$

Example 4:

A Steel column is 3 m long and 0.4 m diameter. It carries a load of 50 MN. Given that the modulus of elasticity is 200 GPa, calculate the compressive stress and strain and determine how much the column is compressed.

Solution:

$$A = \frac{\pi d^2}{4} = \frac{\pi(0.4)^2}{4} = 0.1257\ \mathrm{m}^2$$

$$\sigma = \frac{P}{A} = \frac{50\times10^6}{0.1257} = 397772474.14\ \mathrm{Pa} = 397.7\,MPa$$

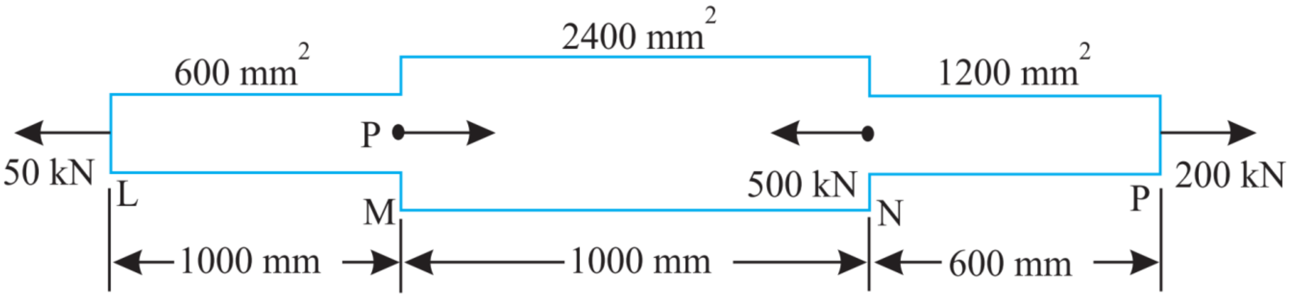
$$\varepsilon = \frac{\sigma}{E} = \frac{397.7\times10^6}{200\times10^9} = 0.00198$$

$$\delta L = \varepsilon L = 0.00198\times3 = 0.00597\ \mathrm{m} = 5.97\ \mathrm{mm} \approx 6\ \mathrm{mm}$$

Homework No 1

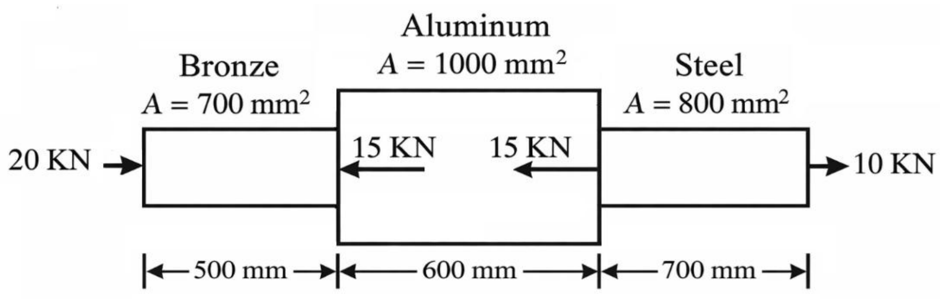
A member LMNP is subjected to point loads as shown in below figure, if E = 210 GN/m^2, Calculate:

A- Force P necessary for equilibrium.
B- Total elongation of the bar.



Example 5:

An aluminum rod is rigidly fastened between a bronze and a steel rod as shown in the below figure. Axial loads are applied at the positions indicated. Determine the stress in each rod and the total elongation of the rods, take $E_{\text{aluminum}} = 70\ \text{GN/m}^2$, $E_{\text{bronze}} = 110\ \text{GN/m}^2$, $E_{\text{steel}} = 200\ \text{GN/m}^2$



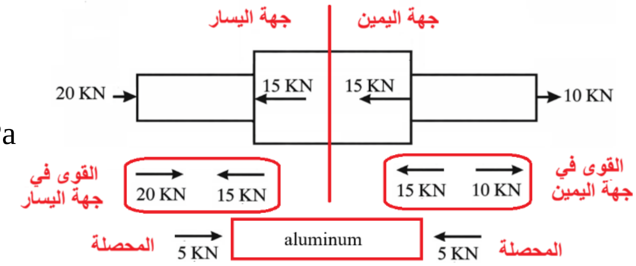
Solution:

Stress for **bronze** rod

$$\sigma_b = \frac{P_b}{A_b} = \frac{20 \times 10^3}{700 \times 10^{-6}} = -28571428.57\ N/m^2 = -28.57\ \text{MPa}$$

Stress for **aluminum** rod

$$\sigma_a = \frac{P_a}{A_a} = \frac{5 \times 10^3}{1000 \times 10^{-6}} = -5000000\ N/m^2 = -5\ \text{MPa}$$



Stress for **steel** rod

$$\sigma_s = \frac{P_s}{A_s} = \frac{10 \times 10^3}{800 \times 10^{-6}} = 12500000\ N/m^2 = 12.5\ \text{MPa}$$

The elongation of the **bronze** rod

$$\delta L_b = \frac{P_b L_b}{A_b E_b} \rightarrow \delta L_b = \frac{(20 \times 10^3) \times (500 \times 10^{-3})}{(700 \times 10^{-6}) \times (110 \times 10^9)} = -0.000129\ m = -0.129\ mm$$

The elongation of the **aluminum** rod

$$\delta L_a = \frac{P_a L_a}{A_a E_a} \rightarrow \delta L_a = \frac{(5 \times 10^3) \times (600 \times 10^{-3})}{(1000 \times 10^{-6}) \times (70 \times 10^9)} = -0.0000428\ m = -0.0428\ mm$$

The elongation of the **steel** rod

$$\delta L_s = \frac{P_s L_s}{A_s E_s} \rightarrow \delta L_s = \frac{(10 \times 10^3) \times (700 \times 10^{-3})}{(800 \times 10^{-6}) \times (200 \times 10^9)} = 0.0000437\ m = 0.0437\ mm$$

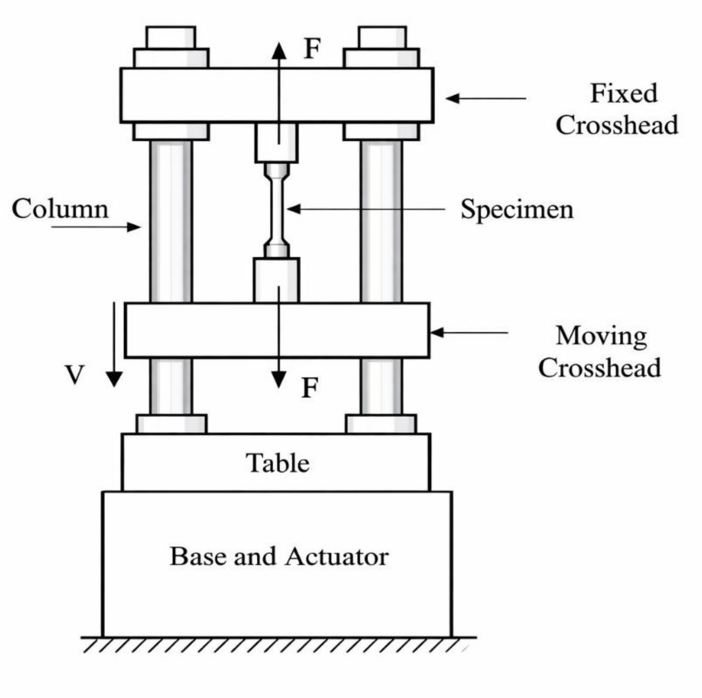
The elongation of the rods

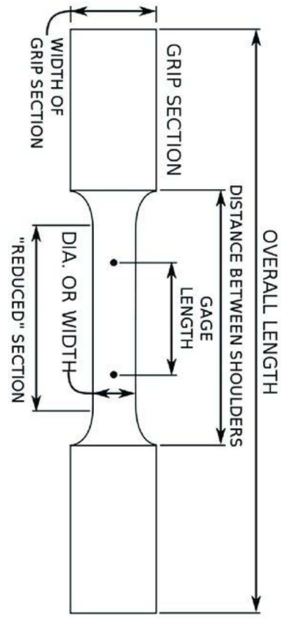
$$\delta L_t = \delta L_b + \delta L_a + \delta L_s$$

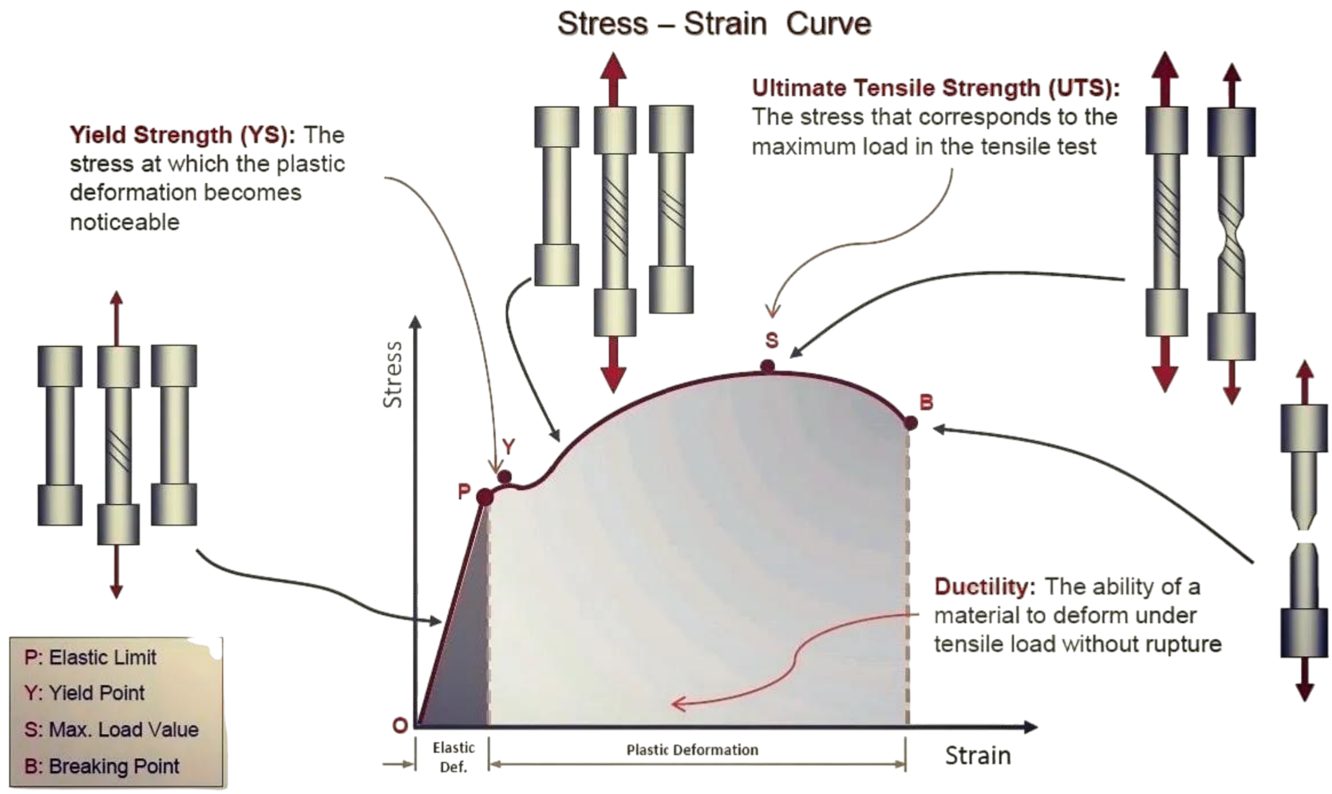
$$\delta L_t = (-0.129) + (-0.0428) + (0.0437) = -0.1281\ mm$$

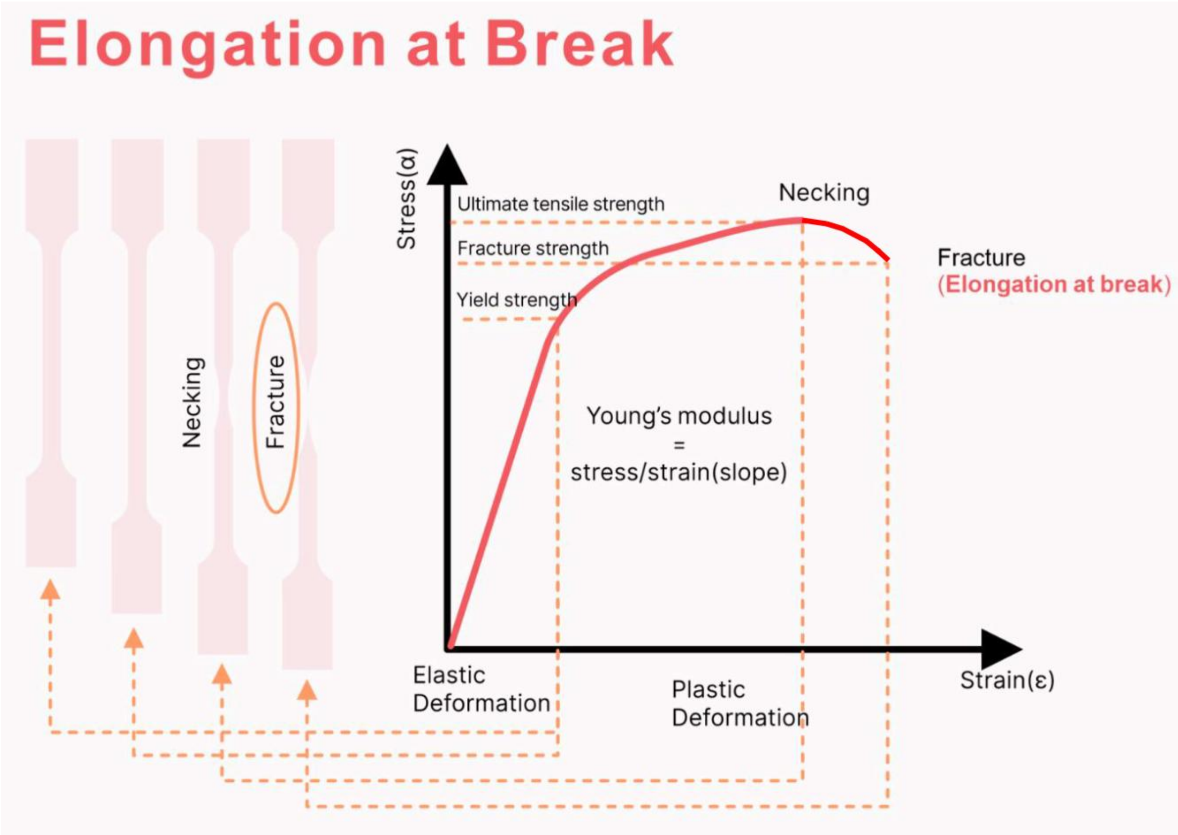
Stress-strain diagram

In order to compare the strengths of various materials it is necessary to carry out some standard form of test to establish their relative properties. One such test is the standard tensile test. Measurements of the change in length of a selected gauge length of the bar are recorded throughout the loading operation by means of extensometers and a graph of load against extension or stress against strain is produced as follows.









Ductile Materials

The capacity of a material to allow large extensions, referred to the ability to be drawn out plastically, is termed its ductility. Materials with high ductility are termed ductile materials, members with low ductility are termed brittle materials. A quantitative value of the ductility is obtained by measurements of the percentage elongation or percentage reduction in area; both being defined below.

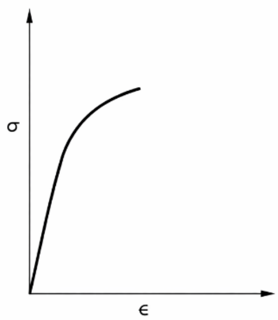
$$\text{Percentage elongation} = \frac{\text{Increase in gauge length at fracture}}{\text{Original gauge length}} \times 100$$

$$\text{Percentage reduction in area} = \frac{\text{Reduction in cross-sectional area of necked portion}}{\text{Original area}} \times 100$$

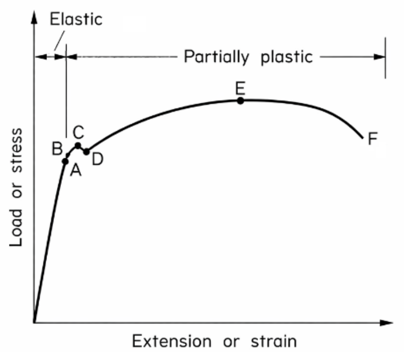
A property closely related to ductility is malleability, which defines a material's ability to be hammered out into thin sheets. A typical example of a malleable material is lead. This is used extensively in the plumbing trade where it is hammered or beaten into corners or joints to provide a weatherproof seal.

Brittle Materials

A brittle material is one which exhibits relatively small extensions to fracture so that the partially plastic region of the tensile test graph is much reduced. There is little or no necking at fracture for brittle materials.



Brittle material diagram

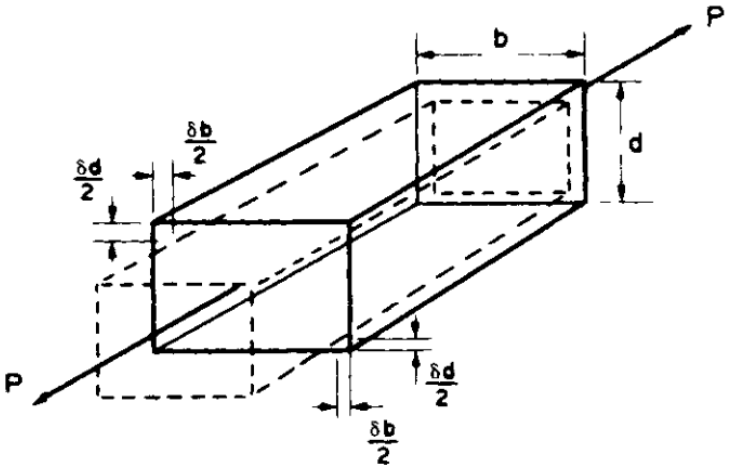


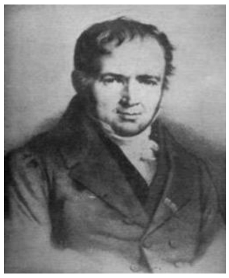
Ductile material diagram

Poisson's Ratio

Consider the rectangular bar as shown below, subjected to a tensile load. Under the action of this load the bar will increase in length by an amount of δL giving a longitudinal strain in the bar of

$$\varepsilon_L = \frac{\delta L}{L}$$





Simeon Denis Poisson
(1781-1840)

The bar will also exhibit, however, a reduction in dimensions laterally, i.e. its breadth and depth will both reduce. The associated lateral strains will both be equal, will be of opposite sense to the longitudinal strain, and will be given by

$$\boldsymbol{\varepsilon_{lat} = -\frac{\delta b}{b} = -\frac{\delta d}{d}} \quad \ldots\ldots\ldots (4)$$

Provided the load on the material is retained within the elastic range the ratio of the lateral and longitudinal strains will always be constant. This ratio is termed ***Poisson's ratio***.

$$\textbf{Poisson's ratio } (\nu) = \frac{\textbf{lateral strain}}{\textbf{longitudinal strain}} = \frac{-\,\delta d/d}{\delta L/L} \quad \ldots\ldots\ldots (5)$$

The negative sign of the lateral strain is normally ignored to leave Poisson's ratio simply as a ratio of strain magnitudes. It must be remembered, however, that the longitudinal strain induces a lateral strain of opposite sign, e.g. tensile longitudinal strain induces compressive lateral strain.

For most engineering materials the value of v lies between 0.25 and 0.33.

Since

$$\text{Longitudinal strain} = \frac{\text{Longitudinal stress}}{\text{Young's modulus}} = \frac{\sigma}{E} \quad \ldots\ldots\ldots (6)$$

Hence

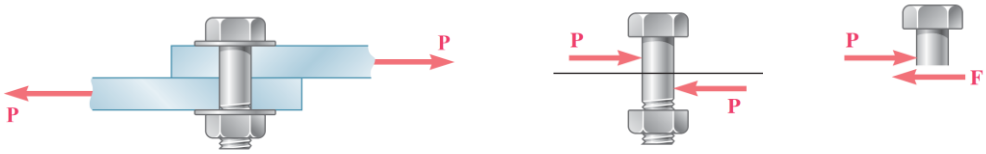
$$\textbf{Lateral strain} = \nu \frac{\sigma}{E} \quad \ldots\ldots\ldots \ (7)$$

Shear Stress

Consider a block or portion of material as shown below (a), subjected to a set of equal and opposite forces *P*. There is a tendency for one layer of the material to slide over another to produce the form of failure shown in (b). If this failure is restricted, then a shear stress (τ) is set up, defined as follows:

$$\text{Shear Stress} = \frac{\text{Shear Load}}{\text{Area Resisting Shear}} \rightarrow \tau = \frac{P}{A} = \frac{F}{A} \quad \ldots\ldots\ldots(8)$$

This shear stress will always be tangential to the area on which it acts; direct stresses, however, are always normal to the area on which they act.



Shear Strain

If one again considers the block to be a bicycle brake block it is clear that the rectangular shape of the block will not be retained as the brake is applied and the shear forces introduced. The block will in fact change shape or into the form shown below. The angle of deformation (ϕ) is then termed the shear strain.

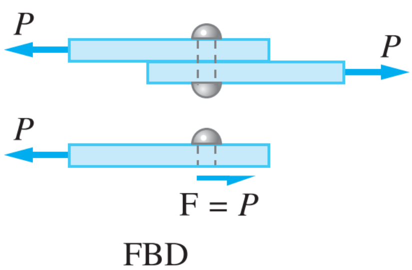
Shear strain is measured in radians and hence is non-dimensional, i.e. it has no units. For materials within the elastic range the shear strain is proportional to the shear stress producing it, i.e.

$$\frac{\text{Shear stress}}{\text{Shear strain}} = \frac{\tau}{\gamma} = \text{constant} = G \quad \ldots\ldots\ldots\ldots(9)$$

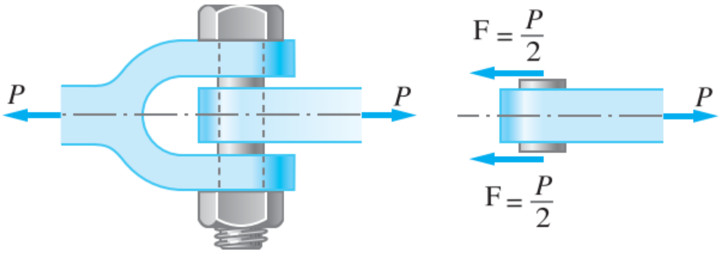
The constant G is termed the modulus of rigidity or shear modulus and is directly comparable to the modulus of elasticity used in the direct stress application. The term modulus thus implies a ratio of stress to strain in each case.

Double Shear

Consider the simple riveted lap joint as shown below. When load is applied to the plates the rivet is subjected to shear forces tending to shear it on one plane as indicated.



In the butt joint with two cover plates, however, each rivet is subjected to possible shearing on two faces, i.e. double shear.

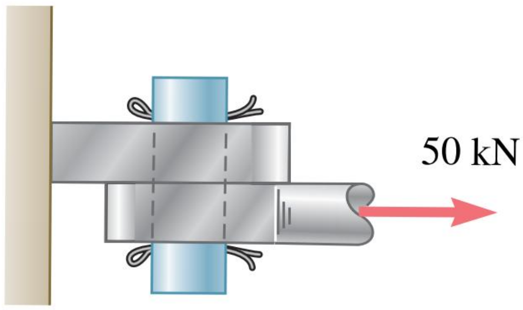


In such cases twice the area of metal is resisting the applied forces so that the shear stress set up is given by

$$\text{Shear Stress } (in\ double\ shear) \quad \tau = \frac{F}{A} = \frac{P}{2A} \quad \ldots\ldots\ldots\ldots (10)$$

Example 6:

A pin connection is subjected to a tensile force of 50 kN. Given that the pin has a diameter of 25 mm, calculate the average shear stress developed within the pin.



Solution:

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(25 \times 10^{-3})^2 = 0.00049\ m^2$$

$$\tau_{\text{avg}} = \frac{F}{A} = \frac{50 \times 10^3}{0.00049} = 102040816.32 \text{ Pa} = 102.04 \text{ MPa}$$

Example 7:

For the connection shown in below figure, determine the average shear stress produced in the 20 mm. diameter bolts if the applied load is 140 kN.

Solution

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(20 \times 10^{-3})^2 = 0.000314\ m^2$$

Total area (A_t) for 3 bolts:

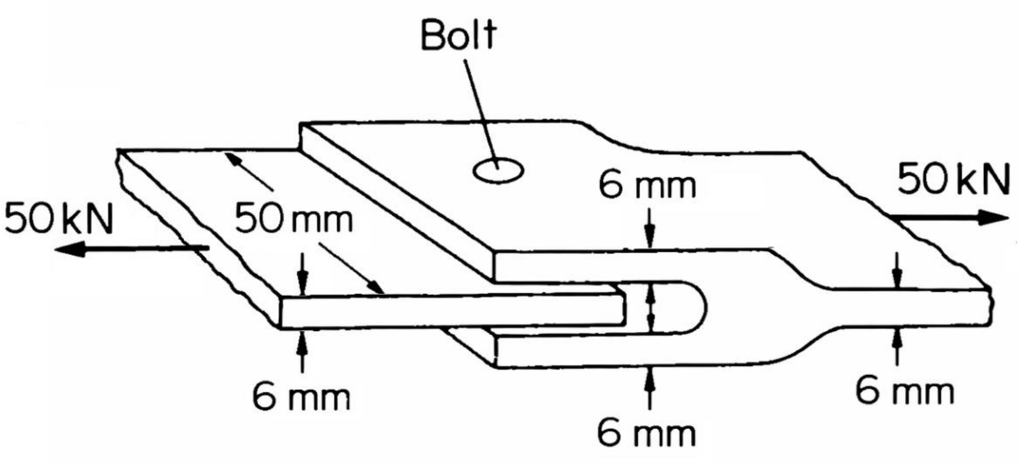
$$A_t = n \times A \quad \rightarrow \quad A_t = 3 \times A \quad \rightarrow \quad A_t = 3 \times 0.000314 = 0.000942\ m^2$$

$$\tau_{\text{avg}} = \frac{F}{A_t} = \frac{140 \times 10^3}{0.000942} = 148619957.5 \text{ Pa} = 148.6 \text{ MPa}$$

Example 8:

The coupling shown in below figure is constructed from steel of rectangular cross-section and is designed to transmit a tensile force of 50 kN. If the bolt is of 15 mm diameter calculate:

A- the shear stress in the bolt;
B- the direct stress in the plate;
C- the direct stress in the forked end of the coupling.



Solution:

(a) The bolt is subjected to double shear

$$\text{shear stress in bolt } \tau = \frac{P}{2A} = \frac{50 \times 10^3}{2 \times \frac{\pi}{4}(15 \times 10^{-3})^2} = 141471060.5\ Pa = 141.47 \text{ MN/m}^2$$

(b) The plate will be subjected to a direct tensile stress given by

$$\sigma = \frac{P}{A} = \frac{50 \times 10^3}{(50 \times 10^{-3}) \times (6 \times 10^{-3})} = 166666666.6\ Pa = 166.6 \text{ MN/m}^2$$

(c) The force in the coupling is shared by the forked end pieces, each being subjected to a direct stress

$$\sigma = \frac{P}{A} = \frac{25 \times 10^3}{50 \times 6 \times 10^{-6}} = 83333333.3\ Pa = 83.3 \text{ MN/m}^2$$

Allowable working stress - factor of safety

The most suitable strength or stiffness criterion for any structural element or component is normally some maximum stress or deformation which must not be exceeded. In the case of stresses, the value is generally known as the maximum allowable working stress.

Because of uncertainties of loading conditions, design procedures, production methods, etc., designers generally introduce a factor of safety into their designs, defined as follows:

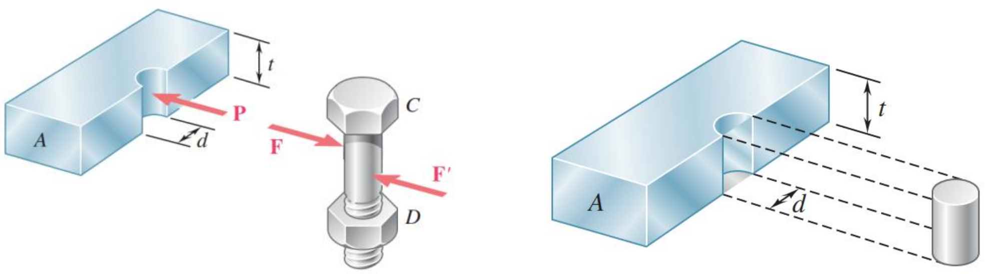
$$\text{factor of safety} = \frac{\text{maximum stress}}{\text{allowable working stress}} \quad \ldots\ldots\ldots\ldots (11)$$

However, in view of the fact that plastic deformations are seldom accepted this definition is sometimes modified to

$$\text{factor of safety} = \frac{\text{yield stress (or proof stress)}}{\text{allowable working stress}} \quad \ldots\ldots\ldots\ldots (12)$$

Bearing Stress in Connections

Bearing stress is a normal stress that is produced by the compression of one surface against another. The contact pressure between the separate bodies. It differs from compressive stress, as it is an internal stress caused by compressive forces. Bolts, pins, and rivets create stresses in the members they connect, along the bearing surface, or surface of contact.

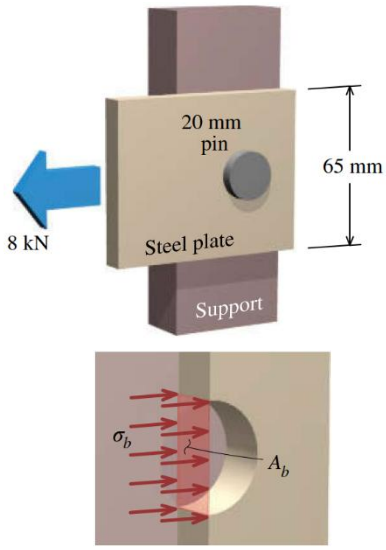


When one body presses against another, bearing stress occurs. Bearing stress is given by:

$$\sigma_b = \frac{P}{A} = \frac{P}{t \times d} \quad \ldots\ldots\ldots\ldots (13)$$

Example 9:

A 65 mm wide by 13 mm thick steel plate is connected to a support with a 20 mm diameter pin. The steel plate carries an axial load of 8 kN. Determine the bearing stress in the steel plate.



Solution:

Projected bearing area:

$$A_b = d \times t = (20 \times 10^{-3}) \times (13 \times 10^{-3}) = 0.00026 \ \text{m}^2$$

Bearing stress:

$$\sigma_b = \frac{P}{A_b} = \frac{8 \times 10^3}{0.00026} = 30769230.7 \ Pa \ = \ 30.7 \ MPa$$

Temperature Stresses

When the temperature of a component is increased or decreased the material respectively expands or contracts. If this expansion or contraction is not resisted in any way then the processes take place free of stress. If, however, the changes in dimensions are restricted then stresses termed temperature stresses will be set up within the material.

Consider a bar of material with a linear coefficient of expansion. Let the original length of the bar be L and let the temperature increase be Δt. If the bar is free to expand the change in length would be given by

$$\Delta L = L\alpha\Delta t \quad \ldots\ldots\ldots(14)$$

and the new length

$$L' = L + L\alpha\Delta t = L(1 + \alpha\Delta t) \quad \ldots\ldots\ldots(15)$$

If this extension were totally prevented, then a compressive stress would be set up equal to that produced when a bar of length $\boldsymbol{L(1+\alpha\Delta t)}$ is compressed through a distance of $\boldsymbol{L\alpha\Delta t}$. In this case the bar experiences a compressive strain

$$\varepsilon = \frac{\Delta L}{L} = \frac{L\alpha\Delta t}{L(1 + \alpha\Delta t)} \quad \ldots\ldots\ldots\ldots(16)$$

In most cases at is very small compared with unity so that

$$\varepsilon = \frac{L\alpha\Delta t}{L} = \alpha\Delta t \quad \ldots\ldots\ldots\ldots(17)$$

$$\varepsilon = \alpha\Delta t \quad \ldots\ldots\ldots\ldots(18)$$

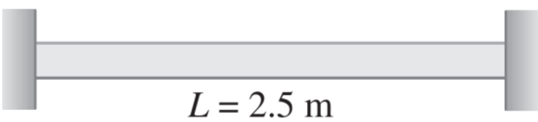
$$E = \frac{\sigma_{th}}{\varepsilon} \quad \ldots\ldots\ldots\ldots(19)$$

$$\sigma_{th} = E\varepsilon \quad \rightarrow \quad \sigma_{th} = E\alpha\Delta t \quad \ldots\ldots\ldots\ldots(20)$$

This is the stress set up owing to total restraint on expansions or contractions caused by a temperature rise, or fall, Δt. In the former case the stress is compressive, in the latter case the stress is tensile.

Example 10:

The horizontal steel rod of 2.5 m long is secured between two walls as shown in below figure. If the rod is stress-free at 30 °C, compute the change in length and thermal stress when the temperature has increased to 90 °C. The coefficient of thermal expansion of steel is $\alpha = 11.7\times10^{-6}$ /°C and E = 200 GPa.



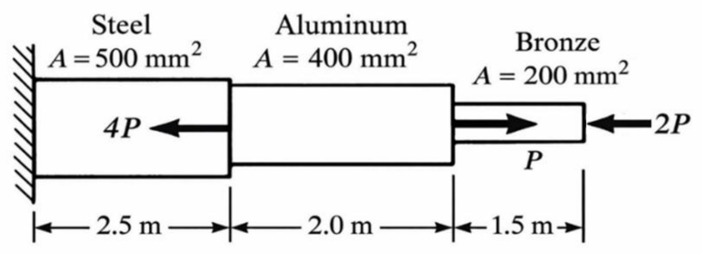
Solution

$$\Delta L = L\alpha\Delta t \quad \rightarrow \quad \Delta L = 2.5 \times 11.7 \times 10^{-6} \times (90 - 30) \;=\; 0.001755\, m$$

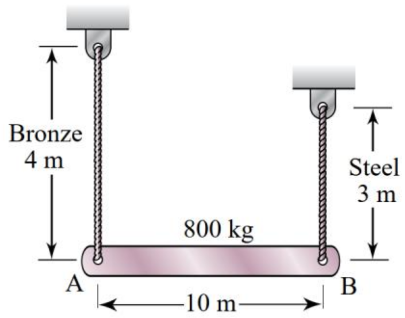
$$\sigma_{th} = E\alpha\Delta t \quad \rightarrow \quad \sigma_{th} = (200 \times 10^{9}) \times (11.7 \times 10^{-6}) \times (90 - 30) \;=\; 140400000\, Pa = 140.4\, MPa$$

Homework No. 2

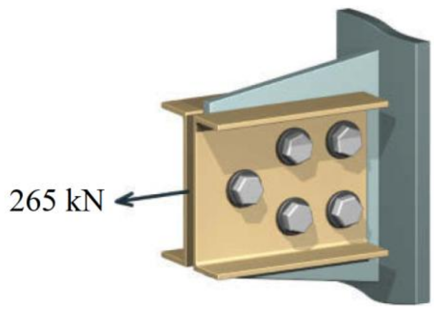
1. An aluminum rod is rigidly attached between a steel rod and a bronze rod as shown in the below figure. Axial loads are applied at the positions indicated. Find the maximum value of P that will not exceed a stress in steel of 140 MPa, in aluminum of 90 MPa, or in bronze of 100 MPa.
(ANS: P= 10 kN)



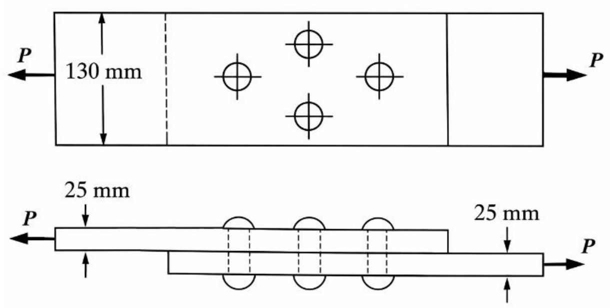
2. A homogeneous 800 kg bar AB is supported at either end by a cable as shown in the below figure. Calculate the smallest area of each cable if the stress is not to exceed 90 MPa in bronze and 120 MPa in steel. **(ANS: A_b= 43.6 mm^2, A_s =32.7 mm^2)**



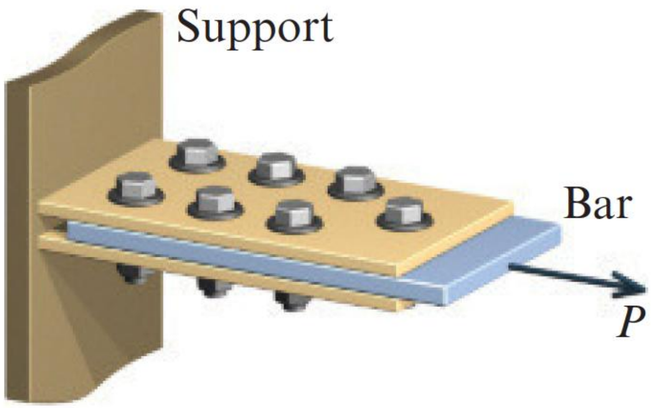
3. The five-bolt connection shown in the below figure must support an applied load of P = 265 kN. If the average shear stress in the bolts must be limited to 120 MPa, what is the minimum bolt diameter that may be used for this connection? **(ANS: 16.77 mm ≈ 18 mm)**



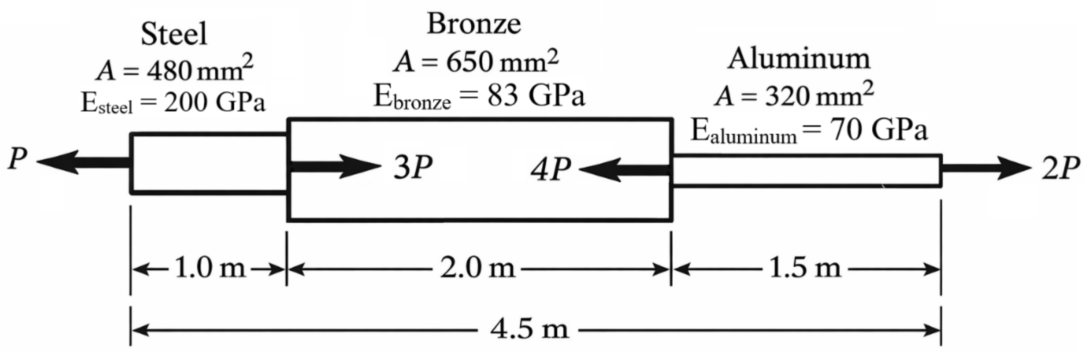
4. The lap joint shown in the below figure is fastened by four 20 mm diameter rivets. Calculate the maximum safe load P that can be applied if the shearing stress in the rivets is limited to 100 MPa and the bearing stress in the plates is limited to 150 MPa. Assume the applied load is uniformly distributed among the four rivets. **(ANS: 125.7 kN)**



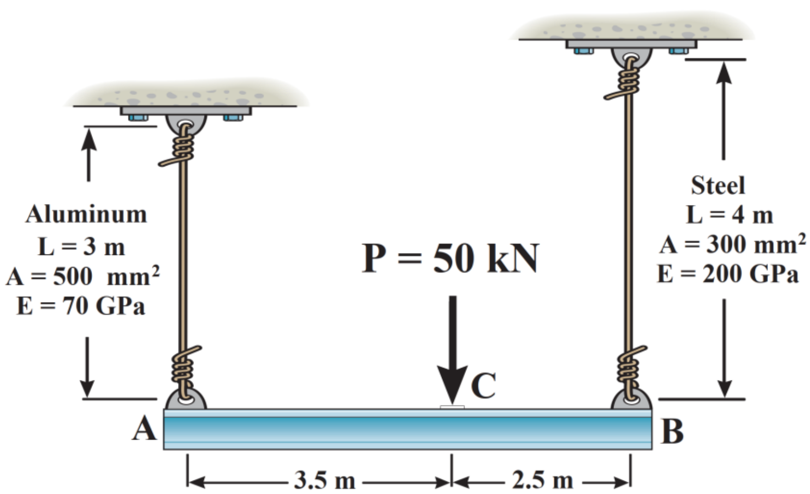
5. Seven bolts are used in the connection between the bar and the support shown in below Figure. The **ultimate shear strength** of the bolts is **320 MPa**, and a **factor of safety** of **2.5** is required with respect to fracture. Determine the **minimum** allowable **bolt diameter** required to support an applied load of **P = 1225 kN**.



6. A bronze bar is fastened between a steel bar and an aluminum bar as shown in below Fig. Axial loads are applied at the positions indicated. Find the largest safe value of P that will not exceed an **overall deformation of 3.0 mm**, or the following stresses: **140 MPa** in the steel, **120 MPa** in the bronze, and **80 MPa** in the aluminum. Assume that the assembly is suitably braced to prevent buckling. Use E_{steel} = **200 GPa**, $E_{aluminum}$ = **70 GPa**, and E_{bronze} = **83 GPa**.



7. The rigid bar AB of negligible weight, attached to two vertical cables as shown in below Figure, is horizontal before the load **P** is applied. Determine the stress in each cable and vertical movement of **P** if its magnitude is 50 kN.



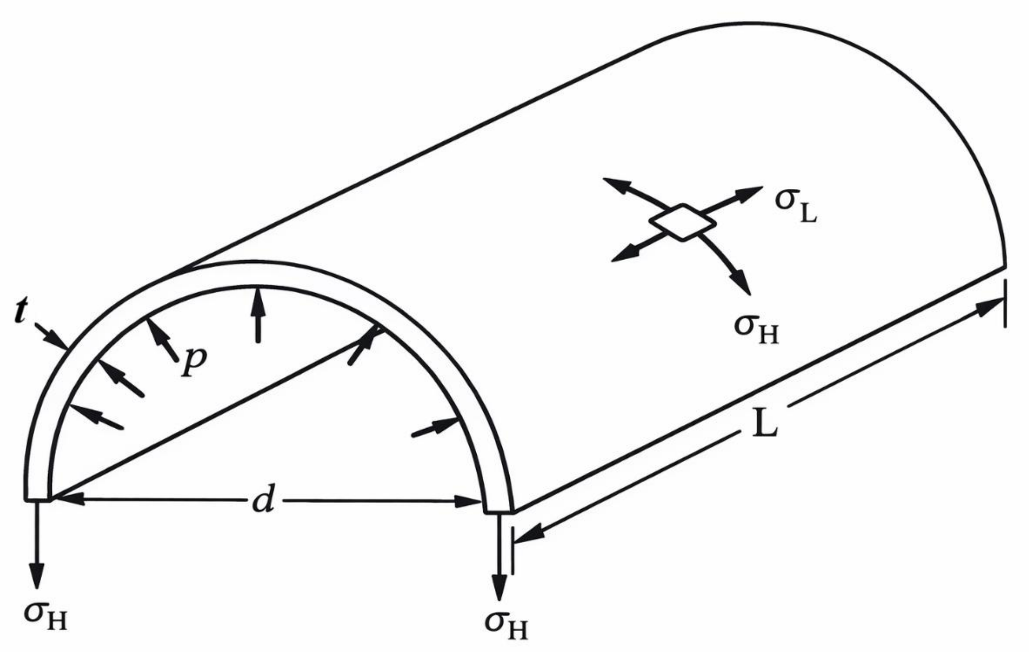
Thin Wall Cylinders

When a thin-walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder material, namely the circumferential or hoop stress, the radial stress and the longitudinal stress. Provided that the ratio of thickness to inside diameter of the cylinder is less than **1/10**, it is reasonably accurate to assume that the hoop and longitudinal stresses are constant across the wall thickness and that the magnitude of the radial stress set up is so small in comparison with the hoop and longitudinal stresses that it can be neglected. This is obviously an approximation since, in practice, it will vary from zero at the outside surface to a value equal to the internal pressure at the inside surface. For the purpose of the initial derivation of stress formulae, it is also assumed that the ends of the cylinder and any riveted joints present have no effect on the stresses produced.

$$\frac{t}{d} < \frac{1}{10} \rightarrow \frac{t}{d} < 0.1$$

Hoop or Circumferential Stress

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of half of the cylinder as shown in below figure.



Half of a thin cylinder subjected to internal pressure showing the hoop and longitudinal stresses acting on any element in the cylinder surface.

Total force on half-cylinder owing to internal pressure = $p \times$ ***projected area*** $= p \times dL$

Total resisting force owing to hoop stress on set up in the cylinder walls $= 2\sigma_h \times L \times t$

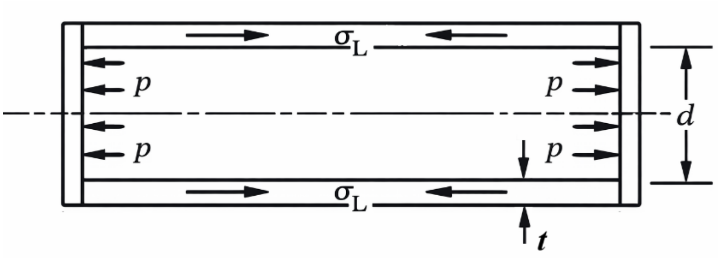
$$\therefore \quad 2\sigma_h \times L \times t = p \times d \times L$$

circumferential or hoop stress

$$\sigma_H = \frac{pd}{2t} \quad \dots\dots\dots\dots (21)$$

Longitudinal Stress

Consider now the cylinder shown in below figure.



Cross-section of a thin cylinder

Total force on the end of the cylinder owing to internal pressure

$$= pressure \times area = p \times \frac{\pi d^2}{4}$$

Area of metal resisting this force = π×d×t (approximately)

$$\therefore \; stress\ set\ up \;=\; \frac{force}{area} = \frac{p \times \frac{\pi}{4} d^2}{\pi \times d \times t} = \frac{pd}{4t}$$

i.e. **longitudinal stress**

$$\boldsymbol{\sigma_L = \frac{pd}{4t}} \quad \ldots\ldots\ldots\ldots (22)$$

Example 11:

A cylindrical steel pressure vessel 400 mm in diameter with a wall thickness of 20 mm is subjected to an internal pressure of 4.5 MN/m^2

1. Calculate the hoop and longitudinal stresses in the steel
2. To what value may the internal pressure be increased if the stresses in the steel is limited to 120 MN/m^2

Solution:

a)

$$\sigma_L = \frac{pd}{4t} = \frac{(4.5 \times 10^6) \times (400 \times 10^{-3})}{4 \times (20 \times 10^{-3})} = 22500000\, Pa \;=\; 22.5\, MPa$$

$$\sigma_H = \frac{pd}{2t} = \frac{(4.5 \times 10^6) \times (400 \times 10^{-3})}{2 \times (20 \times 10^{-3})} = 45000000\, Pa \;=\; 45\, MPa$$

b)

$$\sigma_H = \frac{pd}{2t}$$

$$120 \times 10^6 = \frac{p \times (400 \times 10^{-3})}{2 \times (20 \times 10^{-3})}$$

$$p = 12000000\, Pa \;=\; 12\, MPa$$

Example 12:

Calculate the minimum required wall thickness of a cylindrical pressure vessel designed to contain gas at an internal pressure of 10 MPa. The vessel has an internal diameter of 600 mm, and the allowable stress in the material is limited to 85 MPa. Additionally, verify whether the thin-walled cylinder assumption is applicable for the calculated thickness.

Solution:

The critical stress is the circumferential (hoop) stress

$$\boldsymbol{\sigma_H = \frac{pd}{2t} \rightarrow t = \frac{pd}{2\sigma_H} \rightarrow t = \frac{(10 \times 10^6) \times (600 \times 10^{-3})}{2 \times (85 \times 10^6)} = 0.035\, m = 35\, mm}$$

Check for thin-wall assumption

$$\boldsymbol{\frac{t}{d} = \frac{35}{600} = 0.058 < 0.1}$$

So, the thin-cylinder assumption is valid.

Example 13:

A cylindrical pressure vessel is fabricated from steel plating that has a thickness of 20 mm. The diameter of the pressure vessel is 450 mm and its length is 2 m. Determine the maximum internal pressure that can be applied if the longitudinal stress is limited to 120 MPa, and the circumferential stress is limited to 60 MPa.

Solution

Check for thin-wall assumption

$$\boldsymbol{\frac{t}{d} = \frac{20}{450} = 0.044 < 0.1}$$

So, the thin-cylinder assumption is valid.

Pressure based on longitudinal stress

$$\sigma_L = \frac{pd}{4t} \rightarrow p = \frac{4t\sigma_L}{d}$$

$$p = \frac{4 \times (20 \times 10^{-3}) \times (120 \times 10^6)}{(450 \times 10^{-3})} = 21333333.3\, Pa = 21.3\, MPa$$

Pressure based on circumferential (hoop) stress

$$\sigma_H = \frac{pd}{2t} \rightarrow p = \frac{2t\sigma_H}{d}$$

$$p = \frac{2 \times (20 \times 10^{-3}) \times (60 \times 10^6)}{(450 \times 10^{-3})} = 5333333.3\, Pa = 5.3\, MPa$$

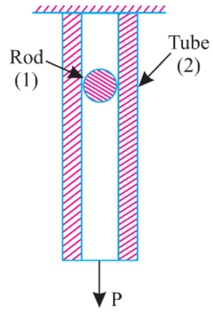
Maximum allowable internal pressure

The vessel must satisfy **both** stress limits, so the allowable pressure is the smaller value:

$$p_{\text{max}} = 5.3 \text{ MPa}$$

Stress in Compound Section

Frequently ties consist of two materials, fastened together to prevent uneven straining of the two materials. In these cases, it is interesting to calculate the distribution of the load between the materials. It will be assumed that the two materials also are symmetrically distributed about the axis of the bar, as with a cylindrical rod encased in a tube.



If an axial load P is applied to the bar,

$$P = P_1 + P_2 = \sigma_1 \times A_1 + \sigma_2 \times A_2 \quad ....(23)$$

where, σ_1 and σ_2 are the stresses induced and A_1 and A_2 are the cross-sectional areas of the materials.

The strains produced, ε_1 and ε_2 are equal.

$$\therefore \ \varepsilon_1 = \varepsilon_2 \quad .... \ (24)$$

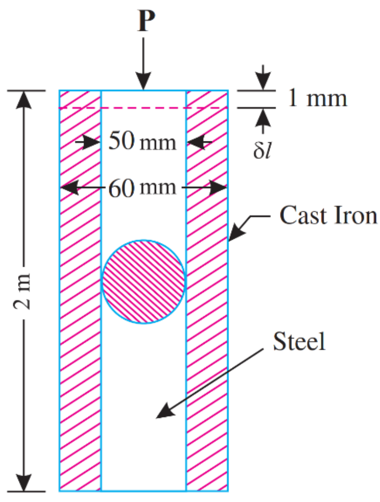
$$\therefore \ \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \ \rightarrow \ \frac{\sigma_1}{\sigma_2} = \frac{E_1}{E_2} \quad ....(25)$$

$$E_1 = \frac{\sigma_1}{\varepsilon} \ \rightarrow \ \sigma_1 = \varepsilon \times E_1 \quad ....(26)$$

$$E_2 = \frac{\sigma_2}{\varepsilon} \ \rightarrow \ \sigma_2 = \varepsilon \times E_2 \quad ....(27)$$

Example 14:

A compound bar of length 2 m is subjected to axial load P. The bar is formed cast iron and steel. The steel diameter is 50 mm and thickness for the cast iron is 5mm. The deflection in both materials is 1 mm. Determine the axial load P if the modules of elasticity for steel 200 GN/m^2 and for cast iron is 100 GN/m^2.



Solution:

$$P = P_S + P_{CI}$$

$$A_s = \frac{\pi}{4} d_i^2 = \frac{\pi}{4}(50 \times 10^{-3})^2 = 0.00196 \, m^2$$

$$A_{CI} = \frac{\pi}{4}(d_i^2 - d_0^2) = \frac{\pi}{4}[(60 \times 10^{-3})^2 - (50 \times 10^{-3})^2] = 0.00086 \, m^2$$

$$P_s = \frac{\delta L \times A_s \times E_s}{L} = \frac{(1 \times 10^{-3}) \times (0.00196) \times (200 \times 10^9)}{2} = 196000 \, N = 196 \, kN$$

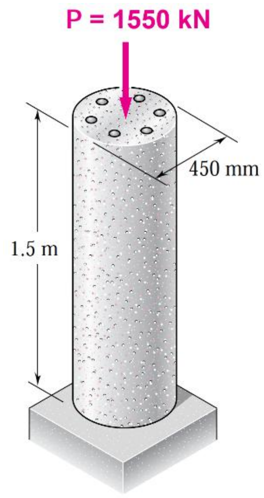
$$P_{CI} = \frac{\delta L \times A_{CI} \times E_{CI}}{L} = \frac{(1 \times 10^{-3}) \times (0.00086) \times (100 \times 10^9)}{2} = 43000 \, N = 43 \, kN$$

$$P = P_S + P_{CI} = 196 + 43 = 239 \, kN$$

Example 15:

The 1.5 m concrete post with 450 mm diameter is reinforced with six steel bars, each with a 28 mm diameter. Knowing that the modules of elasticity for steel 200 GN/m^2 and for concrete is 25 GN/m^2, determine the normal stresses in the steel and in the concrete when a 1550 kN axial centric force P is applied to the post.

Solution:



$$A_s = 6 \times \frac{\pi}{4} d_s^2 = 6 \times \frac{\pi}{4} (28 \times 10^{-3})^2 = 0.00369\, m^2$$

$$A_C = \frac{\pi}{4} d_o^2 - A_s = \frac{\pi}{4} (450 \times 10^{-3})^2 - 0.0369 = 0.155\, m^2$$

$$P = \frac{\delta L \times A_c \times E_c}{L} + \frac{\delta L \times A_s \times E_s}{L}$$

$$P = \frac{\delta L}{L} (A_c \times E_c + A_s \times E_s)$$

$$P = \varepsilon \times (A_c \times E_c + A_s \times E_s)$$

$$\varepsilon = \frac{P}{(A_c \times E_c + A_s \times E_s)}$$

$$\varepsilon = \frac{1550 \times 10^3}{0.155 \times (25 \times 10^9) + 0.00369 \times (200 \times 10^9)}$$

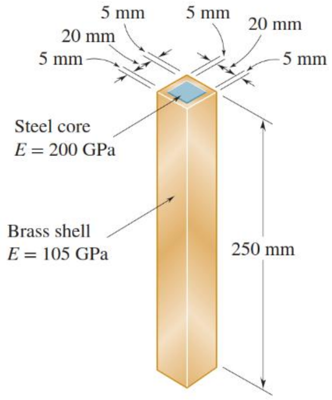
$$\varepsilon = 0.000336$$

$$E_c = \frac{\sigma_c}{\varepsilon} \rightarrow \sigma_c = \varepsilon \times E_c \rightarrow \sigma_c = 0.000336 \times 25 \times 10^9 = 8400000\, Pa = 8.4 \text{ MPa}$$

$$E_s = \frac{\sigma_s}{\varepsilon} \rightarrow \sigma_s = \varepsilon \times E_s \rightarrow \sigma_s = 0.000336 \times 200 \times 10^9 = 67200000\, Pa = 67.2 \text{ MPa}$$

Homework No. 3

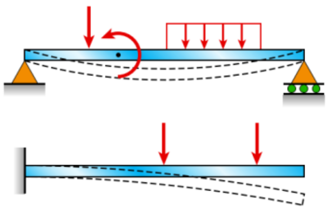
The length of the assembly decreases by 0.15 mm when an axial force is applied by means of rigid end plates. Determine (a) the magnitude of the applied force, (b) the corresponding stress in the steel core.



Shear Force and Bending Moment in Beam

What are beams?

A structural member which is long when compared with its lateral dimensions, subjected to transverse forces so applied as to induce bending of the member in an axial plane, is called a beam.

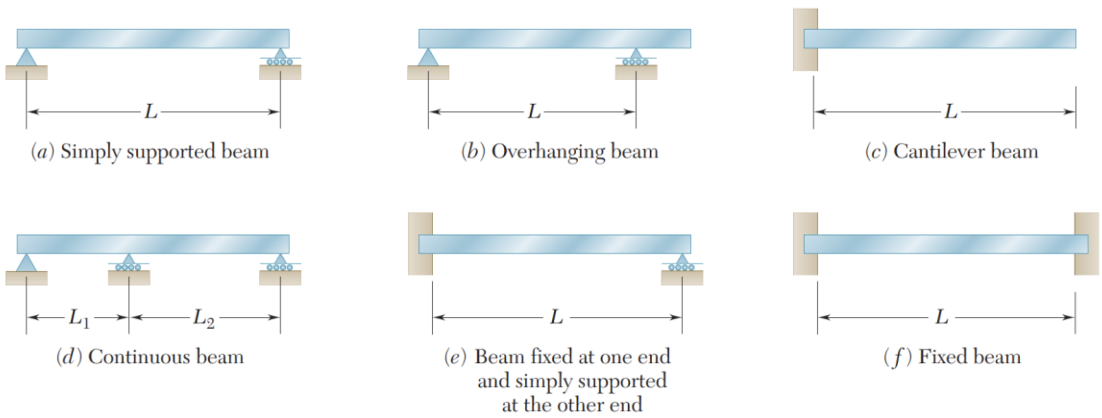


When a beam is loaded by forces or couples, stresses and strains are created throughout the interior of the beam. To determine these stresses and strains, the internal forces and internal couples that act on the cross sections of the beam must be found.

The resultant of the stresses acting on the cross section can be reduced to a shear force (S.F.) and a bending moment (B.M.).

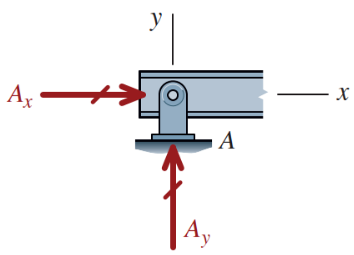
Beam Types

Types of beams-depending on how they are supported.

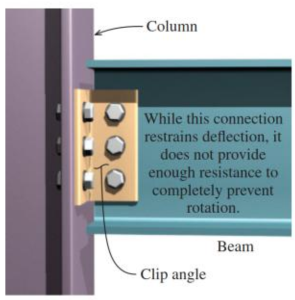


Types of Supports

Pin support: A pin support prevents translation in two orthogonal directions. For beams, this condition means that displacements parallel to the longitudinal axis of the beam (i.e., the x direction in below figure) and displacements perpendicular to the longitudinal axis (i.e., the y direction in below figure) are restrained at the supported joint.

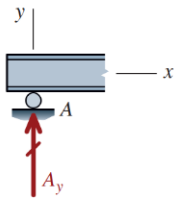


(a) Pin support

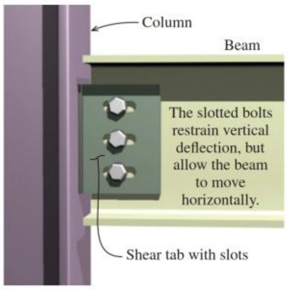


(a) Pin support

Roller support: A roller support prevents translation perpendicular to the longitudinal axis of the beam (i.e., the y direction in below figure); however, the joint is free to translate in the x direction and to rotate about the z axis. Unless specifically stated otherwise, a roller support should be assumed to prevent joint displacement both in the +y and −y directions. The roller support in in below figure provides a reaction force to the beam in the y direction only.

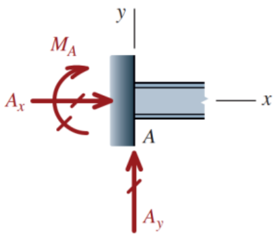


(*b*) Roller support

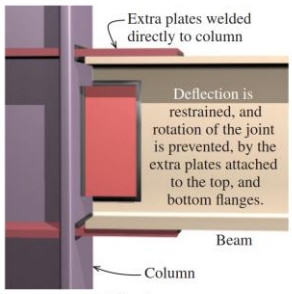


(*b*) Roller support

Fixed support. A fixed support prevents both translation and rotation at the supported joint. The fixed support shown in figure below provides reaction forces to the beam in the x and y directions, as well as a reaction moment in the z direction. This type of support is sometimes called a moment connection.



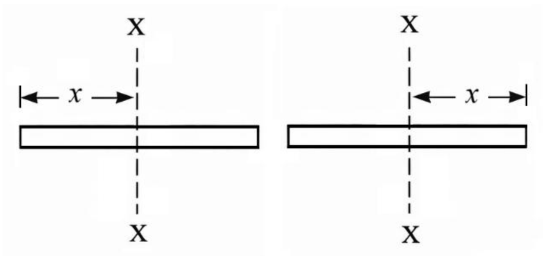
(*c*) Fixed support



(*c*) Fixed support

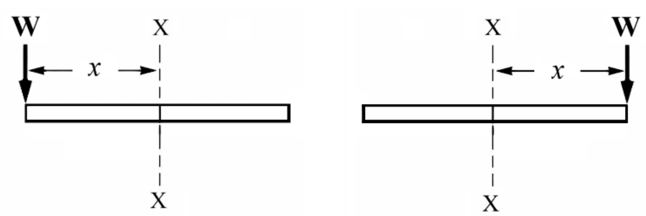
Shear Force (S.F.)

The shearing force at the section is defined as the algebraic sum of the vertical forces acting (taken on one side of the section) to the left or right of the section.



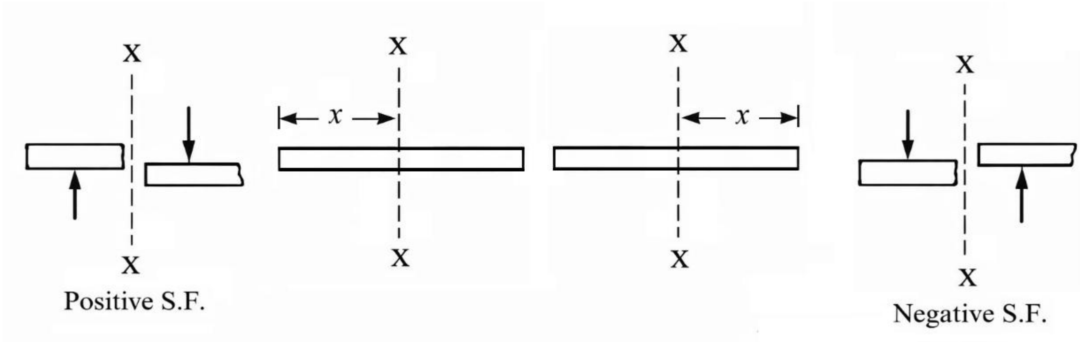
Bending Moment (B.M.)

The bending moment at the section is defined as the algebraic sum of the moment of the forces (taken on either side of the section) to the left or to the right of the section.



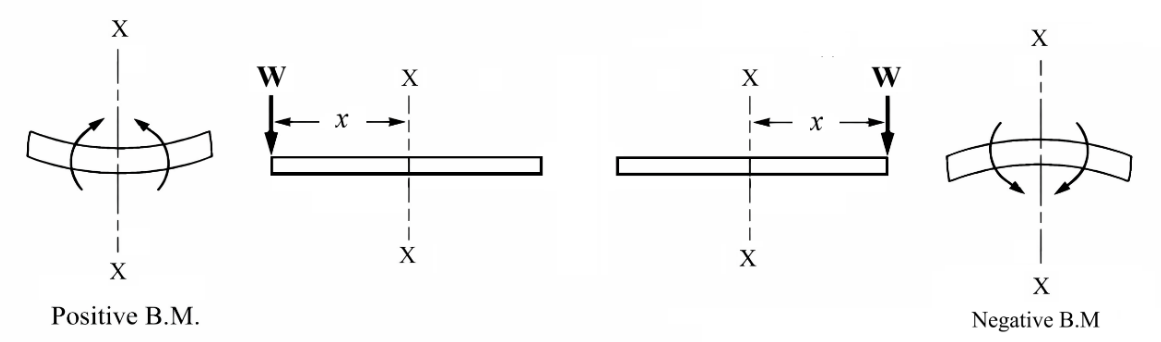
Shearing force (S.F.) sign convention

Forces upwards to the left of a section or downwards to the right of the section are positive.



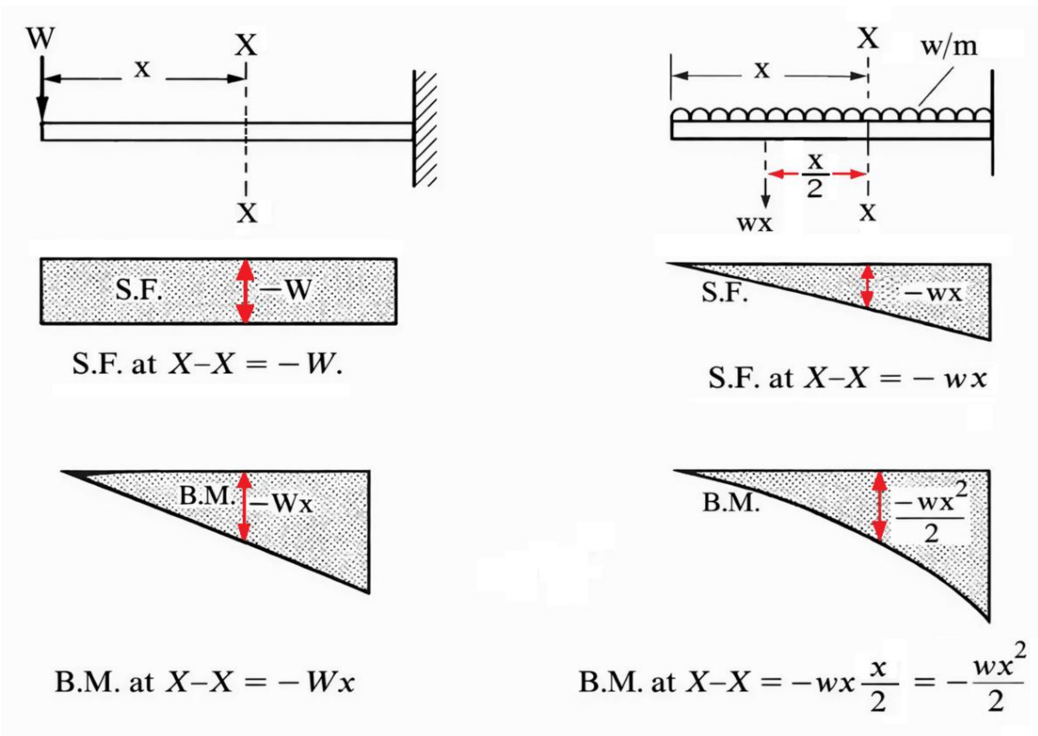
Bending Moment (B.M.) sign convention

Clockwise moments to the left and counterclockwise to the right are positive.

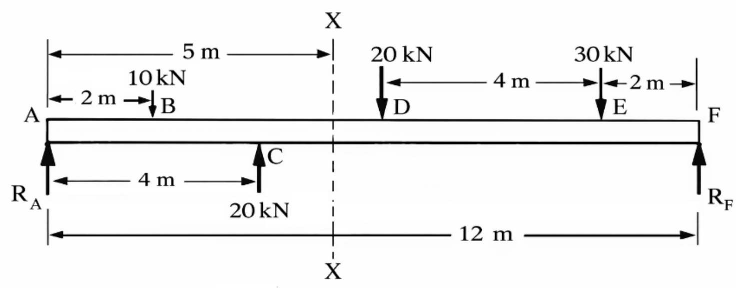


Types of loading beams:

There are two general forms of loading to which structures may be subjected, namely: Concentrated and Distributed loads



S.F. and B.M. diagrams for beams carrying concentrated loads only



The values of the reactions at the ends of the beam may be calculated by applying both normal and vertical equilibrium conditions (Taking a moment and summation of forces).

$$\Sigma M_f = 0$$

$$-R_A \times 10 + 10 \times (12 - 2) - 20 \times (12 - 4) + 20 \times 6 + 30 \times 2 = 0$$

$$R_A = 10\ kN$$

For Vertical equilibrium

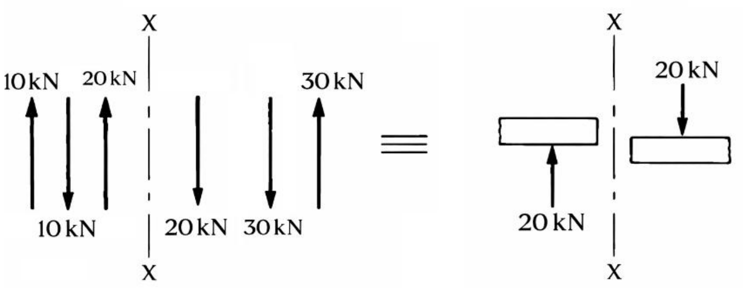
$$\Sigma F_y = 0$$

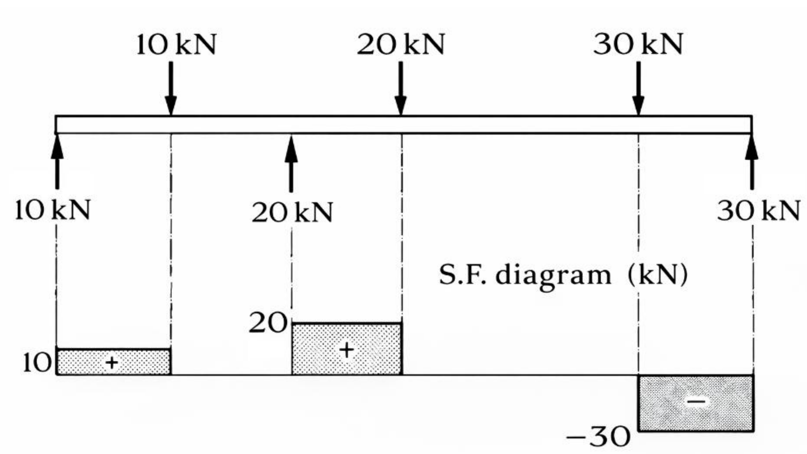
$$10 + R_F + 20 - 10 - 20 - 30 = 0$$

$$R_F = 30\ kN$$

S.F. and B.M. diagrams for beams carrying concentrated loads only

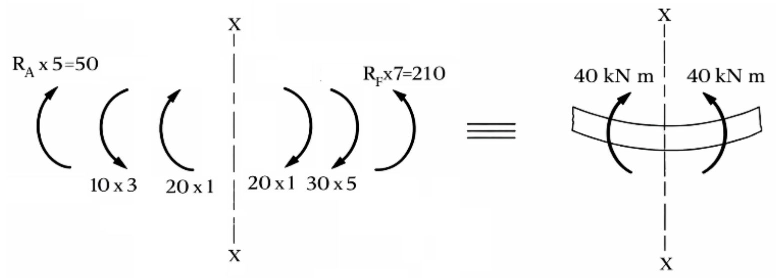
Summing up the forces on either side of X-X, using the sign convention listed above, the shear force at X-X is therefore +20 kN, i.e. The resultant force at X-X tending to shear the beam is 20 kN





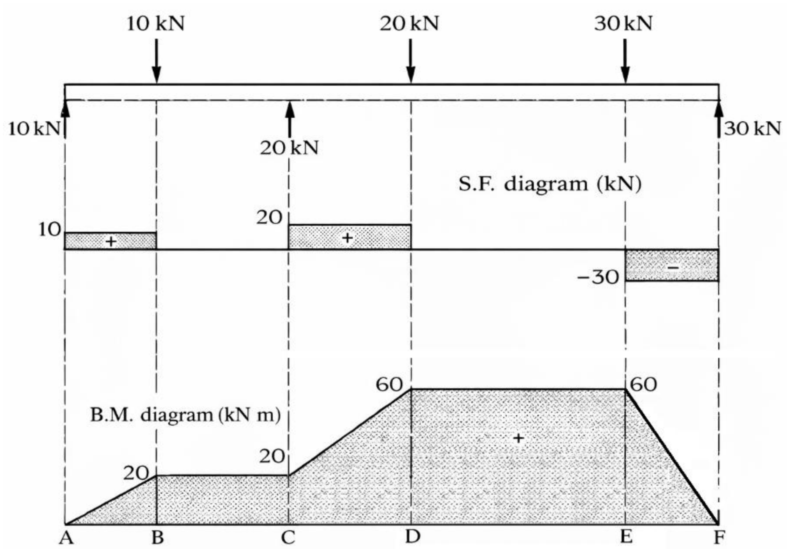
S.F. and B.M. diagrams for beams carrying concentrated loads only

The summation of the moments of the forces at X-X, the resultant, B.M. being 40 kN.m.



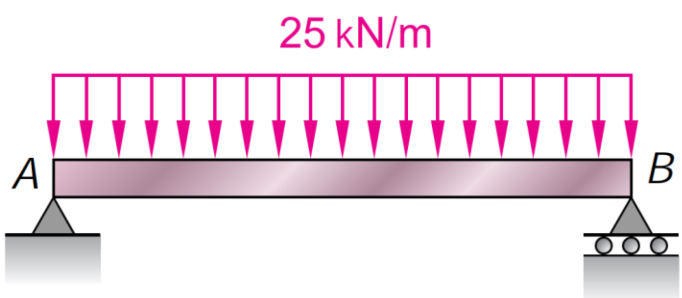
B.M. at A	= 0
B.M. at B = + (10 × 2)	= + 20 kN·m
B.M. at C = + (10 × 4) − (10 × 2)	= + 20 kN·m
B.M. at D = + (10 × 6) + (20 × 2) − (10 × 4)	= + 60 kN·m
B.M. at E = + (30 × 2)	= + 60 kN·m
B.M. at F	= 0

All the above values have been calculated from the moments of the forces to the left of each section considered except for E where forces to the right of the section are taken.



S.F. and B.M. diagrams for uniformly distributed loads (U.D.I.)

Consider the simply supported beam carrying a uniformly distributed load (U.D.L.), w = 25 kN/m across the complete span.



The values of the reactions at the ends of the beam may be calculated by applying both normal and vertical equilibrium conditions (Taking a moment and summation of forces).

$$\Sigma M_B = 0$$

$$-R_A \times 12 + 25 \times 12 \times \frac{12}{2} = 0 \quad \rightarrow \quad R_A = 150\ kN$$

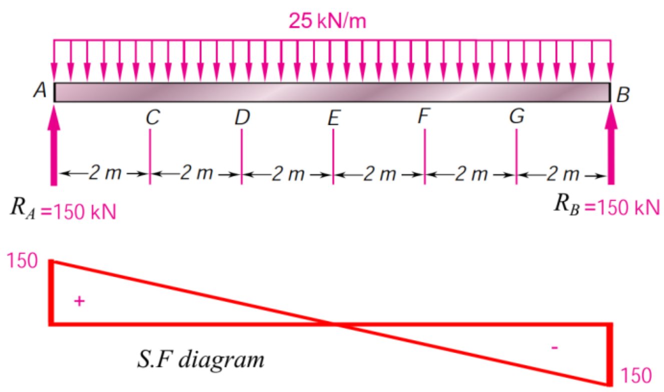
For Vertical equilibrium

$$\Sigma F_y = 0$$

$$150 + R_B - 25 \times 12 = 0 \quad \rightarrow \quad R_B = 150\ kN$$

The S.F. at A, using the usual sign convention, is therefore + 150kN. Consider now the beam divided into six equal parts 2 m long. The S.F. at any other point C is, therefore,

150 - load downwards between A and C = 150 - (25× 2) = + 100 kN



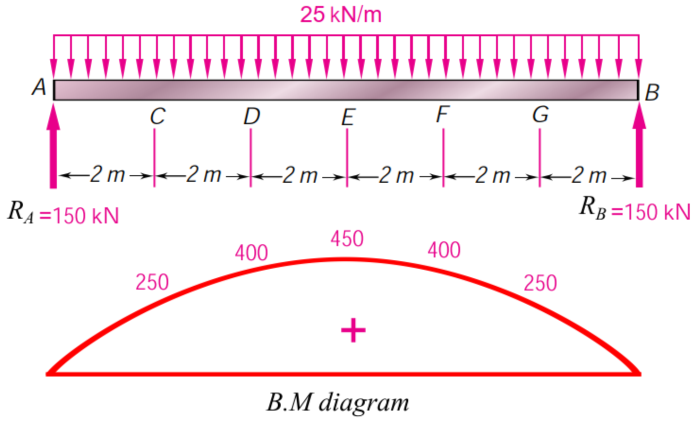
When evaluating B.M. it is assumed that a (U.D.L.) can be replaced by a concentrated load of equal value acting at the middle of its spread. When taking moments about C, therefore, the portion of the (U.D.L.) between A and C has an effect equivalent to that of a concentrated load of 25× 2 = 50 kN acting the center of AC, i.e. 1 m from C.

B.M. at *C* = + (R_A × 2) − (50 × 1) = + 250 kN·m

Similarly, for moments at D the U.D.L. on AD can be replaced by a concentrated load of 25 × 4 = 100 kN at the center of AD, i.e. at C.

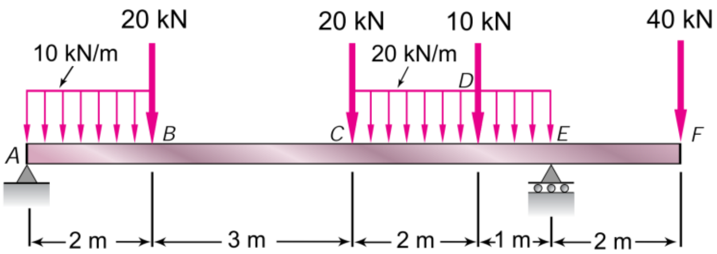
B.M. at *D* = + (R_A × 4) − (25 × 4) × 2 = 600 – 200 = + 400 kN·m

B.M. at *E* = + (R_A × 6) − (25 × 6) × 3 = 900 – 450 = + 450 kN·m

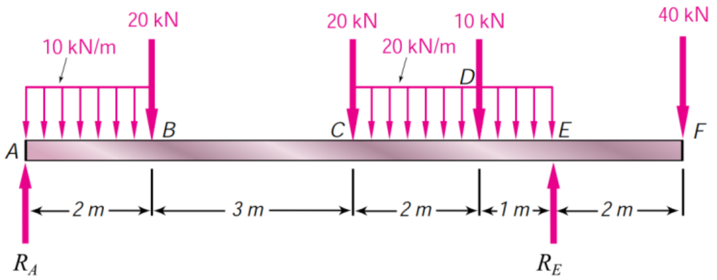


S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads (U.D.L.)

Consider the beam shown in below figure loaded with a combination of concentrated loads and U.D.L.s.



Considering the entire beam as a free body, to find the reactions at points A and E



Taking moments about E, $\Sigma M_A = 0$

$$-R_A \times (8) + 10 \times 2 \times (7) + 20 \times (6) + 20 \times (3) + 20 \times 3 \times (1.5) - 40 \times (2) = 0 \;\rightarrow\; R_A = 42.5\, kN$$

For Vertical equilibrium, $\Sigma F_y = 0$

$$42.5 + R_B - 10 \times 2 - 20 - 20 - 20 \times 3 - 40 = 0 \;\rightarrow\; R_E = 127.5\, kN$$

Working from the left-hand support it is now possible to construct the S.F. diagram, as follow:

For section 1: $SF = R_A - W_1 \times x \quad [0 \le x \le 2]$

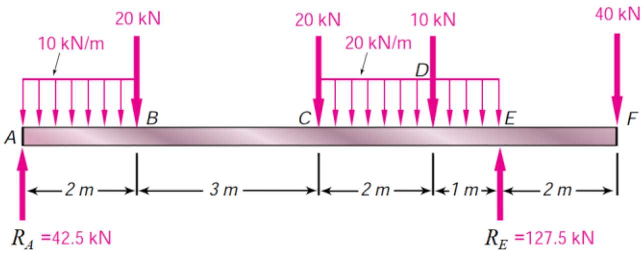
For section 2: $SF = R_A - W_1 \times 2 - 20 \quad [2 \le x \le 5]$

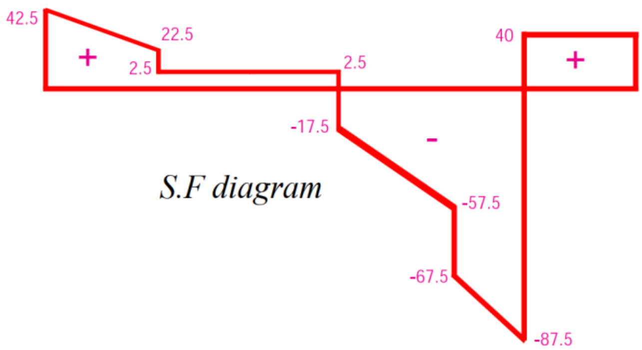
For section 3: $SF = R_A - W_1 \times 2 - 20 - 20 - W_2(x - 5) \quad [5 \le x \le 7]$

For section 4: $SF = R_A - W_1 \times 2 - 20 - 20 - 10 - W_2(x - 7) \quad [7 \le x \le 8]$

For section 5: $SF = R_A - W_1 \times 2 - 20 - 20 - 10 - W_2 \times 3 + 127.5 \quad [8 \le x \le 10]$

It is possible as indicated previously, by following the direction arrows of the loads. In the case of the U.D.L.'s the S.F. diagram will decrease gradually by the amount of the total load until the end of the U.D.L. or the next concentrated load is reached. Where there is no U.D.L. the S.F. diagram remains horizontal between load points.





Working from the left-hand support it is now possible to construct the B.M. diagram from the following equations

For section 1:

$$SF = R_A \times x - W_1 \times x \times \frac{x}{2} \qquad [\mathbf{0 \leq x \leq 2}]$$

For section 2:

$$SF = R_A \times x - W_1 \times 2 \times (x-1) - 20 \times (x-2) \qquad [\mathbf{2 \leq x \leq 5}]$$

For section 3:

$$SF = R_A \times x - W_1 \times 2 \times (x-1) - 20 \times (x-2) - 20(x-5) - W_2(x-5)\frac{(x-5)}{2} \quad [\mathbf{5 \leq x \leq 7}]$$

For section 4:

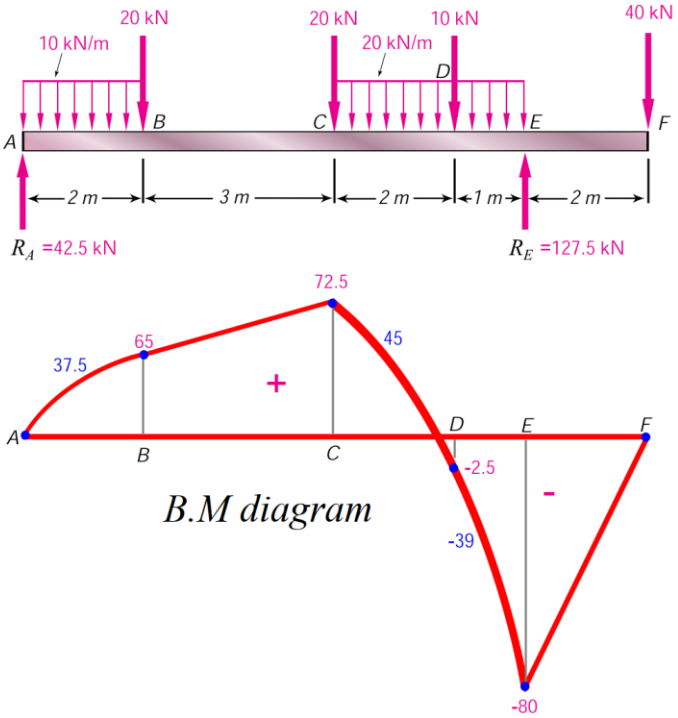
$$SF = R_A \times x - W_1 \times 2 \times (x-1) - 20(x-2) - 20(x-5) - W_2(x-7)\frac{(x-7)}{2} - 10(x-7) \quad [\mathbf{7 \leq x \leq 8}]$$

For section 5:

$$SF = R_A \times x - W_1 \times 2 \times (x-1) - 20(x-2) - 20(x-5) - W_2 \times 3 \times (x-6.5) - 10(x-7) + 127.5$$
$$[\mathbf{8 \leq x \leq 10}]$$

In order to plot the B.M. diagram the following values must be determined:

B.M. at *A* = **0**
B.M. at *B* = (42.5 × 2) − (10 × 2 × 1) = **65 kN·m**
B.M. at *C* = (42.5 × 5) − (10 × 2 × 4) − (20 × 3) = **72.5 kN·m**
B.M. at *D* = (42.5 × 7) − (10 × 2 × 6) − (20 × 5) − (20 × 2) − (20 × 2 × 1) = **−2.5 kN·m**
B.M. at *E* = (42.5 × 8) − (10 × 2 × 7) − (20 × 6) − (20 × 3) − (20 × 3 × 1.5) − (10 × 1) = **−80 kN·m**
B.M. at *F* = **0**



For complete accuracy one or two intermediate values should be obtained along each U.D.L. portion of the beam,

e.g. B.M. midway between A and B = (42.5 × 1) − (10 × 1 × 0.5) = 42.5 – 5 = 37.5 kN·m

Similarly, B.M. midway between C and D = 45 kN·m

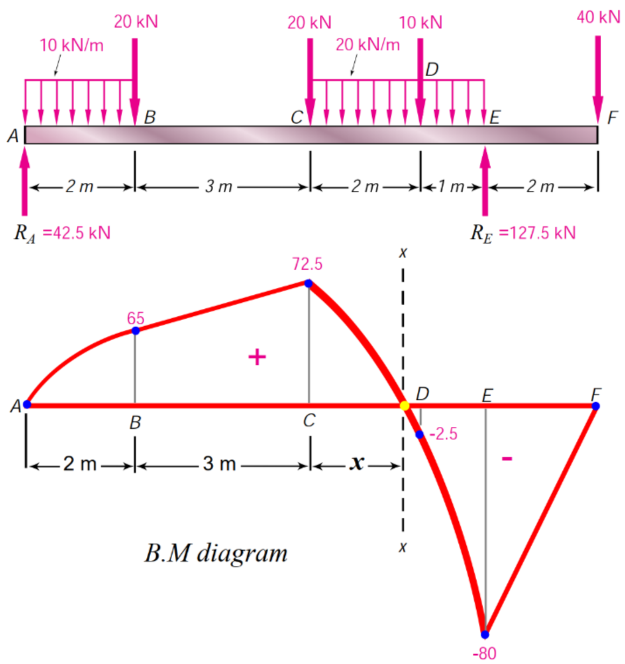
B.M. midway between D and E = −39 kN·m

The B.M. and S.F. diagrams are then as shown in above figure.

Point of Contraflexure

A point of contraflexure is a point where the curvature of the beam changes sign. It is sometimes referred to as a point of inflexion and will be shown later to occur at the point, or points, on the beam where the B.M. is zero.

For the beam, the B.M. diagram that this point lies somewhere between C and D (B.M. at C is positive, B.M. at D is negative). If the required point is a distance x from C then at that point.



Taking the bending moments about the section x–x, we obtain the following equation:

$$BM_{xx} = (42.5)(5 + x) - (10 \times 2)(4 + x) - 20(3 + x) - 20x - (20x\frac{x}{2})$$

$$BM_{xx} = 212.5 + 42.5x - 80 - 20x - 60 - 20x - 20x - 10x^2$$

$$BM_{xx} = 72.5 - 17.5x - 10x^2$$

$$Thus, the\ BM_{xx}\ is\ zero\ where$$

$$0 = 72.5 - 17.5x - 10x^2$$

Rearrange the equation:

$$i.e \qquad 10x^2 + 17.5x - 72.5 = 0$$

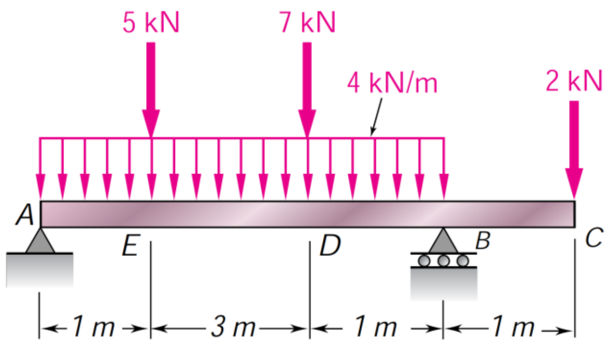
$$Then \qquad x = \frac{-(17.5) \mp \sqrt{(17.5)^2 - 4 \times 10 \times (-72.5)}}{2 \times 10}$$

$$x = 1.956 \quad or \ -3.7$$

The point of contraflexure must be situated at 1.956 m to the right of C.

Example 16:

Draw the S.F. and B.M. diagrams for the beam loaded as shown in below figure, and determine (a) the position and magnitude of the maximum B.M., and (b) the position of any point of contraflexure.



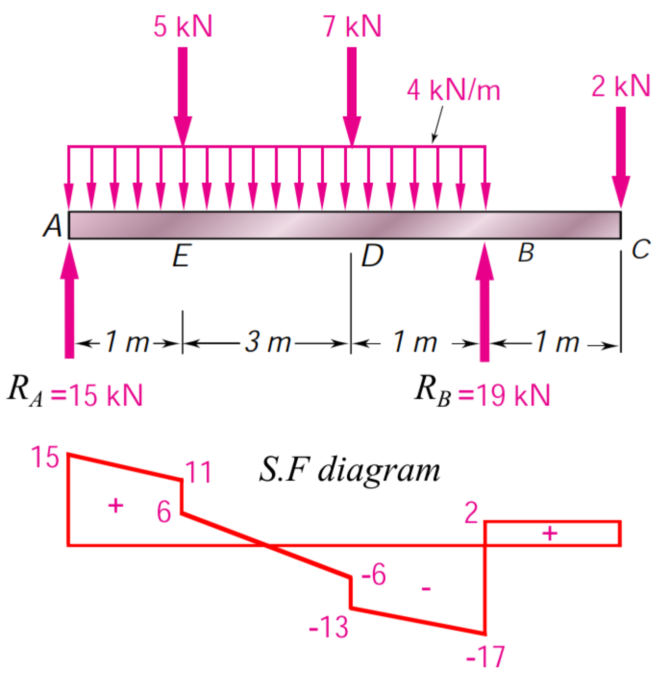
Solution:

Taking the moments about A,

$$\Sigma M_A = 0 \quad \rightarrow \quad R_B \times 5 - 5 \times 1 - 7 \times 4 - 2 \times 6 - (4 \times 5) \times 2.5 = 0 \quad \rightarrow \quad R_B = 19\ kN$$

For Vertical equilibrium

$$\Sigma F_y = 0 \quad \rightarrow \quad 19 + R_A - 5 - 7 - 2 - 4 \times 5 = 0 \quad \rightarrow \quad R_A = 15\ kN$$



The S.F. diagram may now be constructed as described in previous section and is shown in below figure.

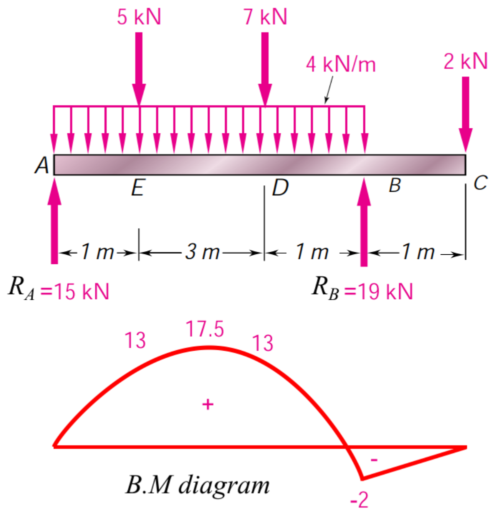
Calculation of bending moments

B.M. at A and C = 0

B.M. at B = −2 × 1 = −2 kN·m

B.M. at D = −(2 × 2) + (19 × 1) − (4 × 1 × 0.5) = +13 kN·m

B.M. at E = +(15 × 1) − (4 × 1 × 0.5) = +13 kN·m



The maximum B.M. will be given by the point (or points) at which *dBM/dx* (Le. The shear force) is zero. By inspection of the S.F. diagram this occurs midway between D and E, i.e. at 1.5 m from E.

$$B.M.\,at\,this\,point\; = (2.5 \times 15) - (5 \times 1.5) - (4\; \times 2.5 \times \frac{2.5}{2}) \;=\; +17.5\,kN \cdot m$$

The B.M. at any point between D and E at a distance of *x* from A will be given by

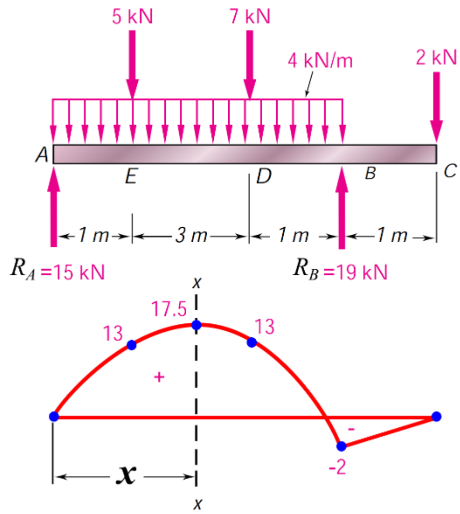
$$M_{xx} = 15x - 5(x-1) - 4\frac{x^2}{2} \;=\; 10x + 5 - 2x^2$$

$$M_{xx} = 15x - 5(x-1) - 4\frac{x^2}{2} \;= 10x + 5\; - 2x^2$$

$$\frac{dM}{dx} \;=\; 10 \;-\; 4x \;=\; 0$$

$$\therefore\; x \;=\; 2.5\,m$$

1.5 m from E, as found previously



(b) Since the B.M. diagram only crosses the zero axis once there is only one point of contraflexure, i.e. between B and D. Then, B.M. at distance y from A will be given by

$$M_{yy} = 15y - 5(y-1) - 7(y-4) - 4y\frac{y}{2}$$

$$M_{yy} = 15y - 5y + 5 - 7y + 28 - 2y^2$$

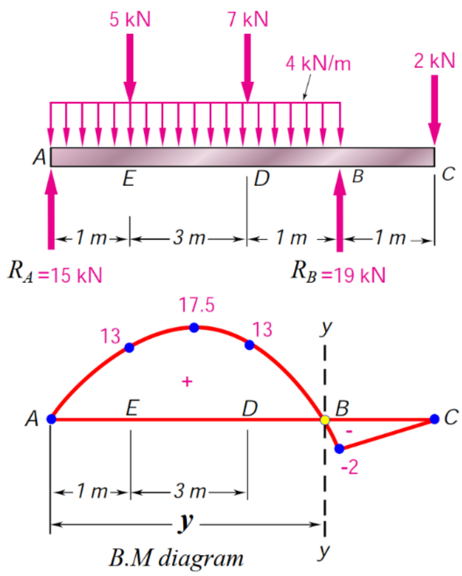
The point of contraflexure occurs where B.M. = 0, i.e. where M_{yy}, = 0,

$$0 = -2y^2 \;+ 3y + 33$$

$$i.e \quad 2y^2 \;- 3y - 33 = 0$$

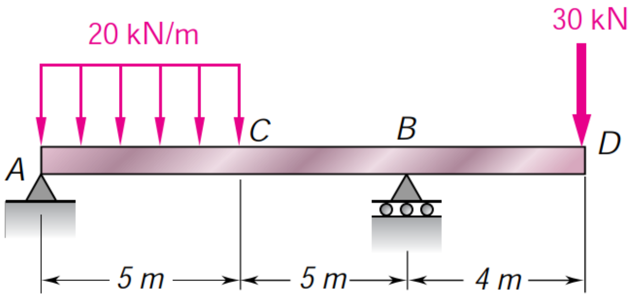
$$Then \qquad y = \frac{-(-3) \mp \sqrt{(-3)^2 - 4 \times 2 \times (-33)}}{2 \times 2} = 4.88\,m$$

i.e. point of contraflexure occurs 0.12 m to the left of B.

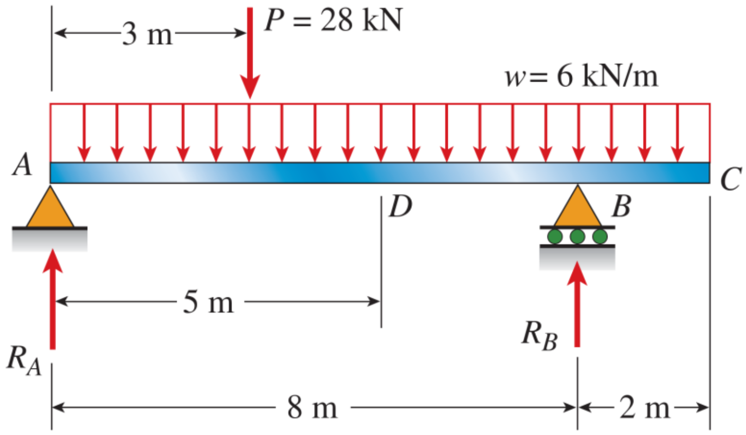


Homework No. 4

1. Write shear and moment equations for the beam loaded as shown in below figure and sketch the shear and moment diagrams.



2. A simple beam with an overhang is supported at points A and B. A uniform load of intensity w=6 kN/m acts throughout the length of the beam and a concentrated load acts at a point 3 m from the left-hand support. The span length is 8 m and the length of the overhang is 2 m. Calculate the shear force and bending moment at cross section D located 5 m from the left-hand support then plot the shear and moment diagrams.



Bending Stress

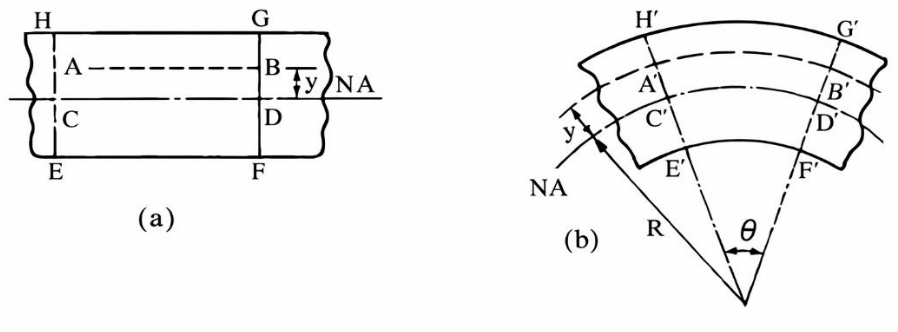
When a beam is loaded it is bent and subjected to bending moments. Consequently, longitudinal or bending stresses are induced in cross-section. In order to determine the practical utility of any beam, it is very necessary to establish a relationship between the radius of curvature to which the beam bends, the bending moment, the bending stress and its cross-sectional dimensions. The equation which connects these quantities is known as the "bending equation".

Assumptions in 'Theory of bending':

1. The beam is initially straight and unstressed.
2. The material of the beam is perfectly homogeneous and isotropic, i.e. of the same density and elastic properties throughout.
3. The elastic limit is nowhere exceeded.
4. Young's modulus for the material is the same in tension and compression.
5. Plane cross-sections remain plane before and after bending.
6. Every cross-section of the beam is symmetrical about the plane of bending, i.e. about an axis perpendicular to the N.A.
7. There is no resultant force perpendicular to any cross-section.

Simple bending theory

If we now consider a beam initially unstressed and subjected to a constant B.M. a long its length, i.e. pure bending, as would be obtained by applying equal couples at each end, it will bend to a radius R as shown in figure. b below. As a result of this bending the top fibres of the beam will be subjected to tension and the bottom to compression. It is reasonable to suppose, therefore, that somewhere between the two there are points at which the stress is zero. The locus of all such points is termed the neutral axis. The radius of curvature R is then measured to this axis. For symmetrical sections the N.A. is the axis of symmetry, but whatever the section the N.A. will always pass through the center of area or centroid.



Beam subjected to pure bending (a) before, and (b) after, the moment M has been applied.

Consider now two cross-sections of a beam, HE and GF, originally parallel (Fig. a). When the beam is bent (Fig. b) it is assumed that these sections remain plane; i.e. H'E' and G'F', the final positions of the sections, are still straight lines. They will then subtend some angle θ.

Consider now some fibre AB in the material, distance y from the N.A. When the beam is bent, this will stretch to A'B'.

$$\text{Strain in fibre } AB = \frac{\text{extension}}{\text{original length}} = \frac{A'B' - AB}{AB} \quad \dots\dots\dots \quad (28)$$

But A B = CD, and, since the N.A. is unstressed, CD = C'D'.

$$\therefore \quad \text{strain} = \frac{A'B' - C'D'}{C'D'} = \frac{(R + y)\theta - R\theta}{R\theta} = \frac{y}{R} \quad \ldots\ldots\ldots\ldots (29)$$

But

$$\frac{\text{stress}}{\text{strain}} = \text{Young's modulus } E \quad \rightarrow \quad \text{strain} = \frac{\sigma}{E} \quad \ldots\ldots\ldots\ldots (30)$$

Equating the two equations for strain,

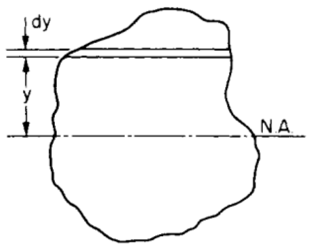
$$\frac{\sigma}{E} = \frac{y}{R}$$

or

$$\frac{\sigma}{y} = \frac{E}{R} \quad \ldots\ldots\ldots\ldots (32)$$

Consider now a cross-section of the beam (the below figure). From the above eqn. the stress on a fibre at distance y from the N.A. is:

$$\sigma = \frac{E}{R} y \quad \ldots\ldots\ldots\ldots (33)$$



If the strip is of area δA, the force on the strip is

$$F = \sigma\,\delta A = \frac{E}{R}\, y\,\delta A$$

This has a moment about the N.A. of

$$Fy = \frac{E}{R}\, y^2\,\delta A$$

The total moment for the whole cross-section is therefore

$$M = \sum \frac{E}{R}\, y^2\,\delta A$$
$$= \frac{E}{R} \sum y^2\,\delta A$$

since E and R are assumed constant.

The term $\sum y^2 \delta A$ is called the **second moment of area** of the cross-section and is given the symbol I.

$$\therefore M = \frac{E}{R} I \ \ and \ \ \frac{M}{I} = \frac{E}{R} \quad \ldots\ldots\ldots\ldots (34)$$

Combining equations (32) and (34), we have the bending theory equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \quad \ldots\ldots\ldots\ldots (36)$$

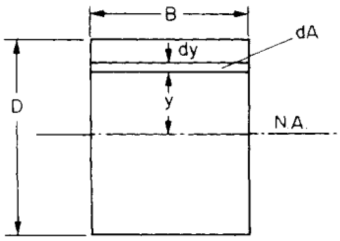
From eqn. (34) it will be seen that if the beam is of uniform section, the material of the beam is homogeneous and the applied moment is constant, the values of I, E and M remain constant and hence the radius of curvature of the bent beam will also be constant. Thus, for pure bending of uniform sections, beams will deflect into circular arcs and for this reason the term circular bending is often used. From eqn. (34) the radius of curvature to which any beam is bent by an applied moment M is given by:

$$R = \frac{EI}{M} \quad \ldots\ldots\ldots\ldots (37)$$

and is thus directly related to the value of the quantity E l . Since the radius of curvature is a direct indication of the degree of flexibility of the beam (the larger the value of R, the smaller the deflection and the greater the rigidity) the quantity *EI* is often termed the *flexural rigidity* or *flexural stiffness* of the beam. The relative stiffnesses of beam sections can then easily be compared by their *EI* values. It should be observed here that the above proof has involved the assumption of pure bending without any shear being present.

Second moment of area

Consider the rectangular beam cross-section shown in below Figure and an element of area dA, thickness dy, breadth B and distance y from the N.A. which by symmetry passes through the center of the section.



The second moment of area I has been defined earlier as

$$I = \int y^2 \, dA$$

Thus for the rectangular section the second moment of area about the N.A., i.e. an axis through the center, is given by

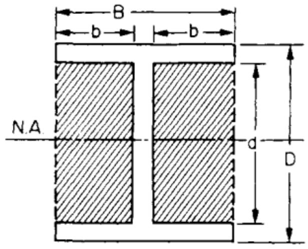
$$I_{N.A.} = \int_{-D/2}^{D/2} y^2 \, B \, dy = B \int_{-D/2}^{D/2} y^2 \, dy$$

$$= B \left[\frac{y^3}{3}\right]_{-D/2}^{D/2} = \frac{BD^3}{12} \quad \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of 0 to D,

$$I = B \left[\frac{y^3}{3}\right]_{0}^{D} = \frac{BD^3}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

These standard forms prove very convenient in the determination of $I_{N.A.}$ values for built-up sections which can be conveniently divided into rectangles. For symmetrical sections as, for instance, the I-section shown in below Figure.



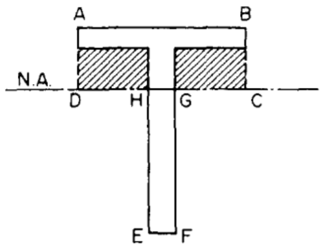
$$I_{N.A.} = I \text{ of dotted rectangle } - I \text{ of shaded portions}$$
$$= \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

It will be found that any symmetrical section can be divided into convenient rectangles with the N.A. running through each of their centroids and the above procedure can then be employed to effect a rapid solution.

For unsymmetrical sections such as the T-section of the below figure. it is more convenient to divide the section into rectangles with their edges in the N.A., when the second type of standard form may be applied.

$$I_{N.A.} = I_{ABCD} \text{ (about DC)} - I_{\text{shaded areas}} \text{ (about DC)} + I_{EFGH} \text{ (about HG)}$$

(each of these quantities may be written in the form $BD^3/3$).



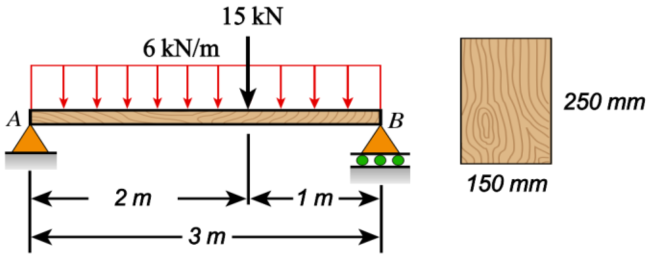
As an alternative procedure it is possible to determine the second moment of area of each rectangle about an axis through its own centroid ($I_G = 8D^3/12$) and to "shift" this value to the equivalent value about the N.A. by means of the parallel *axis theorem*.

$$I_{N.A.} = I_G + Ah^2 \quad \ldots\ldots\ldots\ldots (42)$$

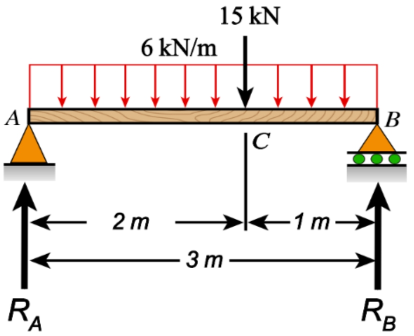
where A is the area of the rectangle and *h* the distance of its centroid *G* from the N.A. Whilst this is perhaps not so quick or convenient for sections built-up from rectangles, it is often the only procedure available for sections of other shapes, e.g. rectangles containing circular holes.

Example 17:

A beam 150 mm wide by 250 mm deep supports the loads shown in below Figure. Determine the maximum bending (flexural) stress.



Solution



Taking the moments about A,

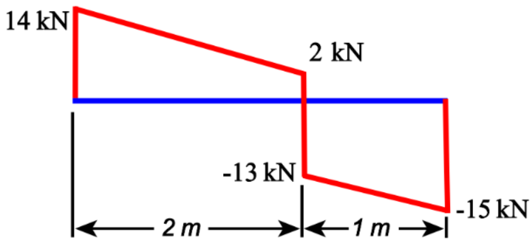
$$\Sigma M_A = 0 \quad \rightarrow \quad R_B \times 3 - 6 \times 3 \times 1.5 - 15 \times 2 = 0$$

$$R_B = 19\ kN$$

For Vertical equilibrium

$$\Sigma F_y = 0 \quad \rightarrow \quad 19 + R_A - 6 \times 3 - 15 = 0 \quad \rightarrow \quad R_A = 14\ kN$$

We begin by computing the maximum bending moment. We draw the shear force diagram to look for the locations of zero shear which occurs at *x* = 2 m.



Shear Force Diagram

Using the moment equation or the area of this diagram to compute the bending moment, we have at *x* = 2 m

$$M_{max} = 14 \times x - 6x\frac{x}{2} = 14x - 3x^2 = 14 \times 2 - 3 \times (2)^2 = 16\ kN.m$$

The bending moment could be evaluated by calculating area of this diagram up to *x* = 2 m.

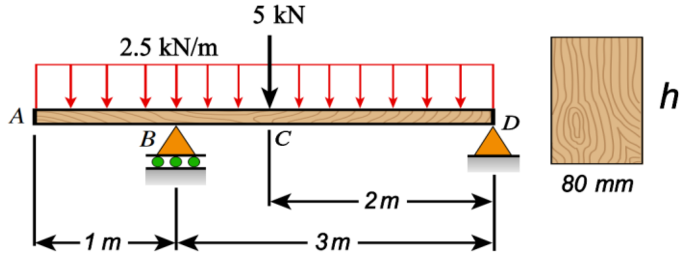
$$M_{max} = \frac{1}{2} \times 2 \times (14 - 2) + 2 \times 2 = 16\ kN.m$$

We now apply the flexure formula, being careful to use consistent units for the various quantities. the second moment of area is $I = bh^3/12$, and $y=h/2$ so

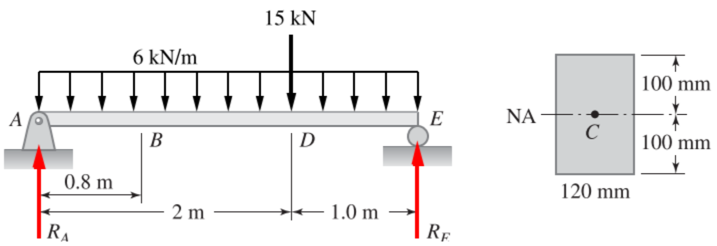
$$\sigma = \frac{M_{max} y}{I} = \frac{M_{max} \times \frac{h}{2}}{\frac{bh^3}{12}} = \frac{6M_{max}}{bh^2} = \frac{6 \times 16 \times 10^3}{0.15 \times (0.25)^2} = 10240000 \text{ Pa} = 10.24 \text{ MPa}$$

Homework No. 5

1. **Determine the minimum height h of the beam shown in below figure if the bending (flexural) stress is not to exceed 20 MPa.**



2. **The simply supported beam in the below Figure has a rectangular cross section 120 mm wide and 200 mm high. Compute:**
   - **A- The maximum bending stress in the beam.**
   - **B- The bending stress at a point on section B that is 25 mm below the top of the beam.**



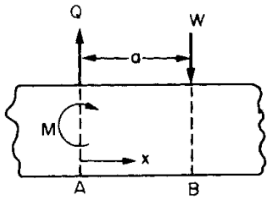
Slope and Deflection of Beams

In practically all engineering applications limitations are placed upon the performance and behavior of components and normally they are expected to operate within certain set limits of, for example, stress or deflection. The stress limits are normally set so that the component does not yield or fail under the most severe load conditions which it is likely to meet in service. In certain structural or machine linkage designs, however, maximum stress levels may not be the most severe condition for the component in question. In such cases it is the limitation in the maximum deflection which places the most severe restriction on the operation or design of the component. It is evident, therefore, that methods are required to accurately predict the deflection of members under lateral loads since it is this form of loading which will generally produce the greatest deflections of beams, struts and other structural types of members.

Relationship between loading, S.F., B.M., slope and deflection

Macaulay Method

Consider, a small portion of a beam in which, at a particular section A, the shearing force is Q and the B.M. is M , as shown in the below Figure At another section B, distance a along the beam, a concentrated load W is applied which will change the B.M. for points beyond B.



Between A and B,

$$M = EI\frac{d^2y}{dx^2} = M + Qx \quad ....(43)$$

$$\therefore EI\frac{dy}{dx} = Mx + \frac{Qx^2}{2} + C_1 \quad ....(44)$$

$$EIy = \frac{Mx^2}{2} + \frac{Qx^3}{6} + C_1x + C_2 \quad ....(45)$$

Beyond *B*

$$M = EI\frac{d^2y}{dx^2} = M + Qx - W(x - a) \quad ....(46)$$

$$\therefore EI\frac{dy}{dx} = Mx + \frac{Qx^2}{2} - \frac{Wx^2}{2} + Wax + C_3 \quad ....(47)$$

$$EIy = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{Wx^3}{6} + \frac{Wax^2}{2} + C_3x + C_4 \quad ....(48)$$

Same slope at *B*

Equating (44) and (45):

$$Mx + \frac{Qx^2}{2} + C_1 = Mx + \frac{Qx^2}{2} - \frac{Wx^2}{2} + Wax + C_3$$

At B, where $x = a$

$$C_1 = -\frac{Wa^2}{2} + Wa^2 + C_3$$
$$C_3 = C_1 - \frac{Wa^2}{2}$$

Substituting in (47)

$$EI\frac{dy}{dx} = Mx + \frac{Qx^2}{2} - \frac{Wx^2}{2} + Wax + C_1 - \frac{Wa^2}{2}$$
$$EI\frac{dy}{dx} = Mx + \frac{Qx^2}{2} - \frac{W}{2}(x-a)^2 + C_1 \quad \ldots\ldots\ldots \quad (49)$$

Also, for the Same deflection at B

Equating (45) and (48), with $x = a$:

$$\frac{Ma^2}{2} + \frac{Qa^3}{6} + C_1 a + C_2 = \frac{Ma^2}{2} + \frac{Qa^3}{6} - \frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3 a + C_4$$
$$C_1 a + C_2 = -\frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3 a + C_4$$

Substitute $C_3 = C_1 - \frac{Wa^2}{2}$:

$$C_1 a + C_2 = -\frac{Wa^3}{6} + \frac{Wa^3}{2} + \left(C_1 - \frac{Wa^2}{2}\right)a + C_4$$
$$C_4 = C_2 + \frac{Wa^3}{6}$$

Substituting in (48)

$$EIy = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{Wx^3}{6} + \frac{Wax^2}{2} + \left(C_1 - \frac{Wa^2}{2}\right)x + \frac{Wa^3}{6} + C_2$$
$$EIy = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{W}{6}(x-a)^3 + C_1 x + C_2 \quad \ldots\ldots\ldots \quad (50)$$

Thus, inspecting (46), (49) and (50), we can see that the general method of obtaining slopes and deflections (i.e. integrating the equation for M) will still apply provided that the term $W(x - a)$ is integrated with respect to $(x - a)$ and not x. Thus, when integrated, the term becomes

$$\frac{W(x-a)^2}{2} \quad and \quad \frac{W(x-a)^3}{6}$$

successively.

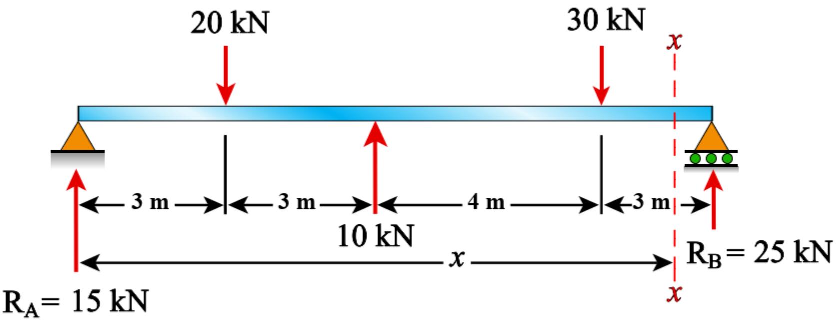
In addition, since the term $W(x - a)$ applies only after the discontinuity, i.e. when $x > a$, it should be considered only when $x > a$ or when $(x - a)$ is positive. For these reasons such terms are conventionally put into square or curly brackets and called *Macaulay terms*. Thus, Macaulay terms must be (a) *integrated with respect to themselves* and (b) *neglected when negative.*

For the whole beam, therefore,

$$EI\frac{dy}{dx} = M + Qx - W(x-a)$$

Example 18:

Determine the slope and deflection under the 30 kN load for the beam loading system shown in below Figure. Find also the position and magnitude of the maximum deflection. Take E = 208 GN/m^2 and I = 82×10^{-6} m^4.



Solution:

Using the Macaulay method the equation for the B.M. at any general section *X-X* is then given by

$$B.M_{xx} = 15x - 20(x-3) + 10(x-6) - 30(x-10)$$

Here it must be emphasized that all loads in the right-hand side of the equation are in units of kN (i.e. newtons × 10^3). *In subsequent working, therefore, it is convenient to carry through this factor as a denominator on the left-hand side in order that the expressions are dimensionally correct.*

$$\frac{EI}{10^3}\frac{d^2y}{dx^2} = 15x - 20(x-3) + 10(x-6) - 30(x-10)$$

Integrating,

$$\frac{EI}{10^3}\frac{dy}{dx} = 15\frac{x^2}{2} - 20\left[\frac{(x-3)^2}{2}\right] + 10\left[\frac{(x-6)^2}{2}\right] - 30\left[\frac{(x-10)^2}{2}\right] + A$$

$$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] + Ax + B$$

where A and B are two constants of integration.

Now when x =0, y = 0 , ∴ B = 0, and when x = 12, y = 0

$$0 = 15\frac{12^3}{6} - 20\left[\frac{(12-3)^3}{6}\right] + 10\left[\frac{(12-6)^3}{6}\right] - 30\left[\frac{(12-10)^3}{6}\right] + 12A$$

$$0 = 4320 - 2430 + 360 - 40 + 12A \quad \rightarrow \quad A = -184.2$$

The deflection at any point is given by

$$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] - 184.2x$$

The deflection at mid-span is thus found by substituting x = 6 in the above equation, bearing in mind that the dimensions of the equation are kNm^3.

$$y = \frac{10^3}{EI}\left[15 \times \frac{6^3}{6} - 20\left[\frac{(6-3)^3}{6}\right] - 184.2 \times 6\right]$$

N.B.-Two of the Macaulay terms then vanish since one becomes zero and the other negative and therefore neglected.

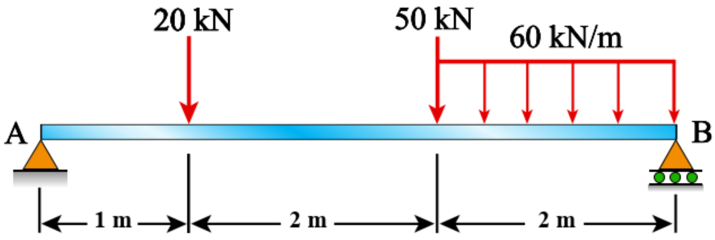
$$y = \frac{10^3}{208 \times 10^9 \times 82 \times 10^{-6}}\left[15 \times \frac{6^3}{6} - 20\frac{(3)^3}{6} - 184.2 \times 6\right]$$

$$y = \frac{10^3}{208 \times 10^9 \times 82 \times 10^{-6}}[-655.2 \times 10^3]$$

$$y = 0.384\, m = 38.4\, mm$$

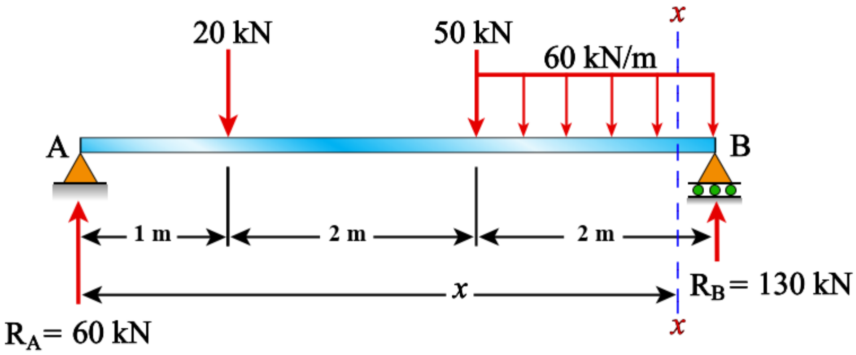
Example 19:

Determine the deflection under the 50 kN load for the beam loading system shown in below figure. Find also the position and magnitude of the maximum deflection. Take E = 200 GN/m^2; I = 83 × 10^{-6} m^4.



Solution

Taking moments about either end of the beam gives R_A = 60 kN and R_B = 130 kN



Applying Macaulay's method,

$$B.M_{xx} = \frac{EI}{10^3}\frac{d^2y}{dx^2} = 60x - 20(x-1) - 50(x-3) - 60\left[\frac{(x-3)^2}{2}\right]$$

The load unit of kilonewton is accounted for by dividing the left-hand side of (1) by lo3and the U.D.L. term is obtained by treating the U.D.L. to the left of X-X as a concentrated load of 60(x - 3) acting at its mid-point of (x - 3)/2 from X-X.

Integrating,

$$\frac{EI}{10^3}\frac{dy}{dx} = 60\frac{x^2}{2} - 20\left[\frac{(x-1)^2}{2}\right] - 50\left[\frac{(x-3)^2}{2}\right] - 60\left[\frac{(x-3)^3}{6}\right] + A$$

$$\frac{EI}{10^3}y = 60\frac{x^3}{6} - 20\left[\frac{(x-1)^3}{6}\right] - 50\left[\frac{(x-3)^3}{6}\right] - 60\left[\frac{(x-3)^4}{24}\right] + Ax + B$$

where A and B are two constants of integration.

Now when x =0, y = 0 , ∴ B = 0

when x = 5, y = 0, ∴ substituting in in above equation to find A

$$0 = 60\frac{5^3}{6} - 20\left[\frac{(5-1)^3}{6}\right] - 50\left[\frac{(5-3)^3}{6}\right] - 60\left[\frac{(5-3)^4}{24}\right] + 5A$$

$$0 = 1250 - 213.3 - 66.7 - 40 + 12A \quad \rightarrow \quad A = -186$$

The deflection at any point is given by

$$\frac{EI}{10^3}y = 60\frac{x^3}{6} - 20\left[\frac{(x-1)^3}{6}\right] - 50\left[\frac{(x-3)^3}{6}\right] - 60\left[\frac{(x-3)^4}{24}\right] - 186x$$

The deflection under the 50 kN load is thus found by substituting x = 3 in the above equation, bearing in mind that the dimensions of the equation are kNm^3.

$$y = \frac{10^3}{EI}\left[60\frac{x^3}{6} - 20\left[\frac{(x-1)^3}{6}\right] - 50\left[\frac{(x-3)^3}{6}\right] - 60\left[\frac{(x-3)^4}{24}\right] - 186x\right]$$

$$y = \frac{10^3}{EI}\left[60\frac{3^3}{6} - 20\left[\frac{(3-1)^3}{6}\right] - 50\left[\frac{(3-3)^3}{6}\right] - 60\left[\frac{(3-3)^4}{24}\right] - 186 \times 3\right]$$

$$y = \frac{10^3}{200 \times 10^9 \times 83 \times 10^{-6}}\left[60\frac{3^3}{6} - 20\left[\frac{(3-1)^3}{6}\right] - 186 \times 3\right]$$

$$y = -0.01896\, m = 19\, mm$$

In order to determine the maximum deflection, its position must first be estimated. The position occurs once the slop equals zero (i.e dy/dx=0), it is reasonable to assume that the maximum deflection point will occur somewhere between the 20 kN and 50 kN loads (x between 1 and 3 m)

$$\frac{EI}{10^3}\frac{dy}{dx} = 60\frac{x^2}{2} - 20\left[\frac{(x-1)^2}{2}\right] - 50\left[\frac{(x-3)^2}{2}\right] - 60\left[\frac{(x-3)^3}{6}\right] - 186$$

$$0 = 30\frac{x^2}{2} - 10\left[\frac{(x-1)^2}{2}\right] - 186$$

$$0 = 30x^2 - 10x^2 - 20x - 10 - 186$$

$$0 = 20x^2 - 20x - 196$$

$$x = 2.67\, m$$

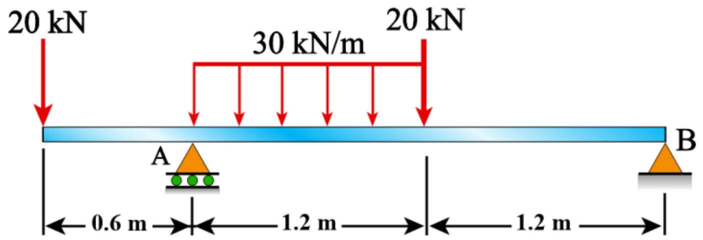
Then, the maximum deflection is given by

$$y_{max} = \frac{10^3}{EI}\left[60\frac{2.67^3}{6} - 20\left[\frac{(2.67-1)^3}{6}\right] - 186 \times 2.67\right]$$

$$y_{max} = -0.0194\ m = -19.4\ mm$$

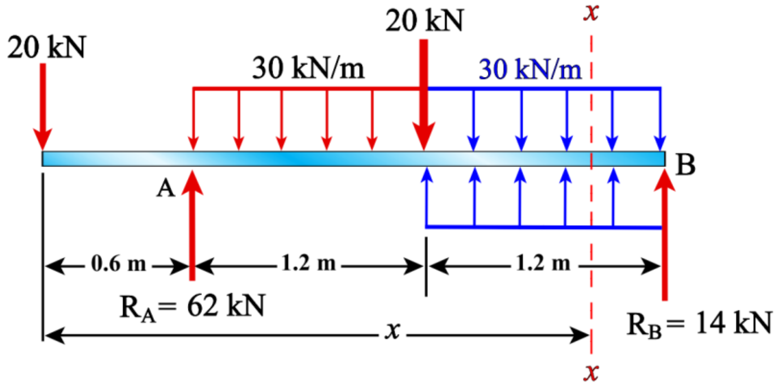
Example 20:

Determine the deflection at a point 1 m from the left-hand end of the beam loaded as shown in below figure using Macaulay's method. EI = 0.65 MN m^2.



Solution

Here the distributed load extends only over the middle segment. We can create continuity, however, by assuming that the distributed load extends beyond the middle segment and adding an equal upward distributed load to cancel its effect beyond the middle segment, as shown



$$B.M_{xx} = \frac{EI}{10^3}\frac{d^2y}{dx^2} = -20x + 62(x-0.6) - 30\left[\frac{(x-0.6)^2}{2}\right] - 20(x-1.8) + 30\left[\frac{(x-1.8)^2}{2}\right]$$

$$\frac{EI}{10^3}\frac{dy}{dx} = -20\frac{x^2}{2} + 62\left[\frac{(x-0.6)^2}{2}\right] - 30\left[\frac{(x-0.6)^3}{6}\right] - 20\left[\frac{(x-1.8)^2}{2}\right] + 30\left[\frac{(x-1.8)^3}{6}\right] + A$$

$$\frac{EI}{10^3}y = -20\frac{x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] - 20\left[\frac{(x-1.8)^3}{6}\right] + 30\left[\frac{(x-1.8)^4}{24}\right] + Ax + B$$

Now when $x = 0.6$, $y = 0$,

$$0 = -20\frac{0.6^3}{6} + 0.6A + B \quad \rightarrow \quad 0.72 = 0.6A + B \quad \dots\dots\dots (1)$$

y = 0, and when $x = 3$,

$$0 = -20\frac{3^3}{6} + 62\left[\frac{(3-0.6)^3}{6}\right] - 30\left[\frac{(3-0.6)^4}{24}\right] - 20\left[\frac{(3-1.8)^3}{6}\right] + 30\left[\frac{(3-1.8)^4}{24}\right] + 3A + B$$

$$-8.208 = 3A + B \quad \dots\dots\dots (2)$$

Solve eqs. (1) and (2) we get: A = -3.72 and B = 2.952

Substituting into the Macaulay deflection equation,

$$\frac{EI}{10^3}y = -20\frac{x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] - 20\left[\frac{(x-1.8)^3}{6}\right] + 30\left[\frac{(x-1.8)^4}{24}\right] - 3.72x + 2.952$$

At $x = 1$

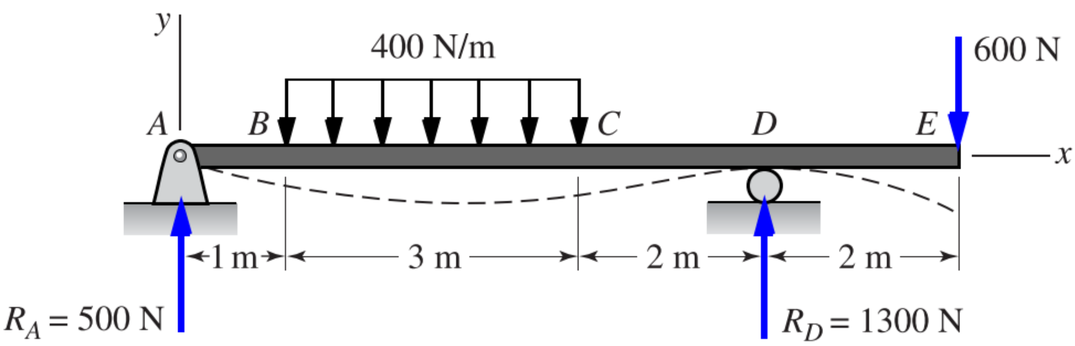
$$y = \frac{10^3}{EI}\left[-20\frac{x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] - 20\left[\frac{(x-1.8)^3}{6}\right] + 30\left[\frac{(x-1.8)^4}{24}\right] - 3.72x + 2.952\right]$$

$$y = \frac{10^3}{0.65 \times 10^6}\left[-20\frac{1^3}{6} + 62\left[\frac{(1-0.6)^3}{6}\right] - 30\left[\frac{(1-0.6)^4}{24}\right] - 3.72 \times 1 + 2.952\right]$$

$$y = -0.00534\, m = -5.34\, mm$$

Homework No. 6

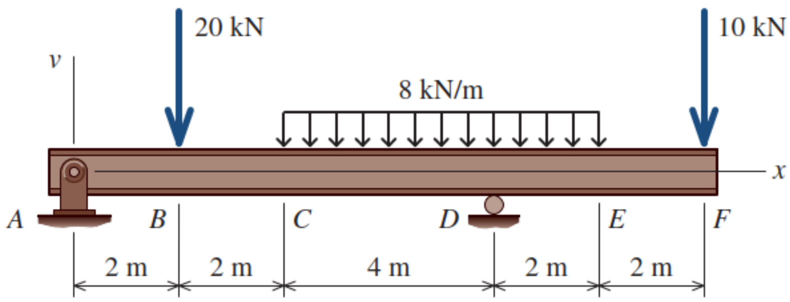
1- Determine the deflection at a point 4 m from the left-hand end of the beam loaded as shown in below figure using Macaulay's method. Take EI = 0.65 MN m^2.



2- The simply supported beam shown in below figure structural steel wide-flange shape has E = 200 GPa and I = 60.8 × 10^{-6} m^4, using Macaulay's method compute the following:

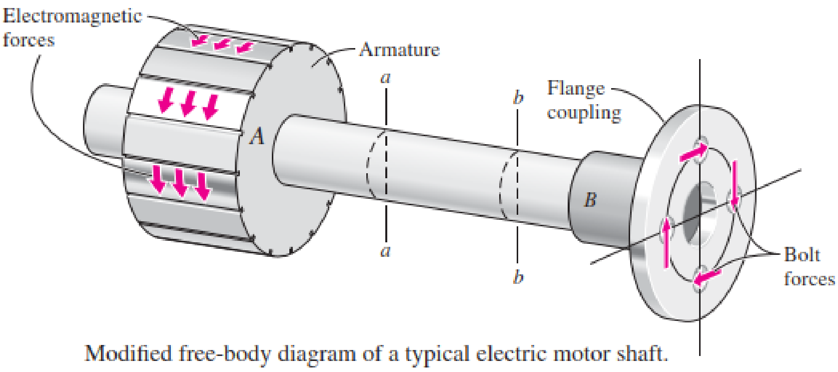
A- The deflection of the beam at C.

B- The deflection of the beam at F.



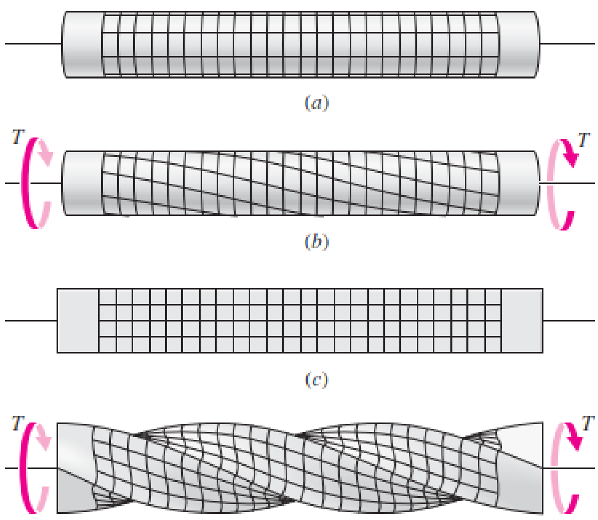
Torsion

Torque is a moment that tends to twist a member about its longitudinal axis. In the design of machinery (and some structures), the problem of transmitting a torque from one plane to a parallel plane is frequently encountered. The simplest device for accomplishing this function is called a shaft. Shafts are commonly used to connect an engine or a motor to a pump, compressor, axle, or similar device. Shafts connecting gears and pulleys are a common application involving torsion members. Most shafts have circular cross sections, either solid or tubular. A modified free-body diagram of a typical device is shown in below Figure



Modified free-body diagram of a typical electric motor shaft.

In 1784, C. A. Coulomb, a French engineer, experimentally developed the relationship between the applied torque and the angle of twist for circular bars. A. Duleau, another French engineer, in a paper published in 1820, analytically derived the same relationship by making the assumptions that a plane section before twisting remains plane after twisting and that a radial line on the cross section remains plane after twisting. Visual examination of twisted models indicates that these assumptions are apparently correct for either solid or hollow circular sections (provided that the hollow section is circular and symmetrical with respect to the axis of shaft), but incorrect for any other shape. For example, compare the distortions evident in the two prismatic rubber shaft models shown in below Figure. Figures a and b show a circular rubber shaft before and after an external torque T is applied to its ends. When torque T is applied to the end of the round shaft, the circular cross sections and longitudinal grid lines marked on the shaft deform into the pattern shown in Figure b. Each longitudinal grid line is twisted into a helix that intersects the circular cross sections at equal angles. The length of the shaft and its radius remain unchanged. Each cross section remains plane and undistorted as it rotates with respect to an adjacent cross section. Figures c and d show a square rubber shaft before and after an external torque T is applied to its ends. Plane cross sections in Figure c before the torque is applied do not remain plane after T is applied (Figure d). The behavior exhibited by the square shaft is characteristic of all but circular sections; therefore, the analysis that follows is valid only for solid or hollow circular shafts.



Torsion is our introduction to problems in which the stress is not uniform, or assumed to be uniform, over the cross section of the member. Another problem in this category, which we will treat later, is the bending of beams. Derivation of the equations used in the analysis of both torsion and bending follows these steps:

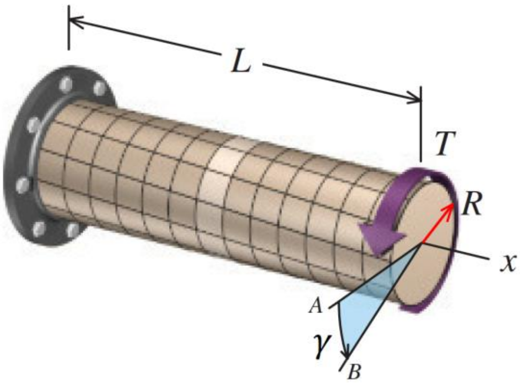
- Make simplifying assumptions about the deformation based on experimental evidence.
- Determine the strains that are geometrically compatible with the assumed deformations.
- Use Hooke's law to express the equations of compatibility in terms of stresses.
- Derive the equations of equilibrium. (These equations provide the relationships between the stresses and the applied loads.)

Torsion of Circular Shafts

If a twisting load is transmitted through a member (like shaft), then it is subjected to torsion

Angle of twist

Consider now the solid circular shaft of radius $\boldsymbol{R}$ subjected to a torque $\boldsymbol{T}$ at one end, the other end being fixed (below figure). Under the action of this torque a radial line at the free end of the shaft twists through an angle $\boldsymbol{\theta}$, point $\boldsymbol{A}$ moves to $\boldsymbol{B}$, and $\boldsymbol{AB}$ subtends an angle γ at the fixed end. This is then the angle of distortion of the shaft, i.e. the shear strain.



Since angle in radians = arc / radius

$$\text{arc } \boldsymbol{AB} = \boldsymbol{R\theta} = \boldsymbol{L\gamma}$$

$$\gamma = \frac{R\theta}{l} \quad \ldots\ldots\ldots\ldots\ldots.. (51)$$

From the definition of rigidity modulus

$$G = \frac{shear\ stress}{shear\ strain} = \frac{\tau}{\gamma}$$

$$G = \frac{\tau}{\gamma}$$

$$\gamma = \frac{\tau}{G} \quad \ldots\ldots\ldots\ldots\ldots (52)$$

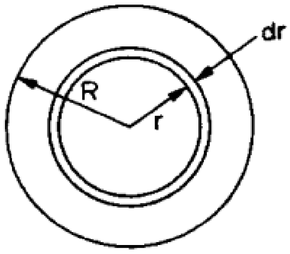
where $\boldsymbol{\tau}$ is the shear stress set up at radius $\boldsymbol{R}$.

Therefore equating eqns. (51) and (52),

$$\frac{\tau}{R} = \frac{G\theta}{l} \quad \ldots\ldots\ldots (53)$$

Stresses

Let the cross-section of the shaft be considered as divided into elements of radius ***r*** and thickness ***dr*** as shown in below Figure each subjected to a shear stress τ'.



The force set up on each element

= stress × area

= τ' × 2πr dr (approximately)

This force will produce a moment about the center axis of the shaft, providing a contribution to the torque

= (**τ' × 2πr dr**) × **r**

= **$2\pi\,\tau'\,r^2$ dr**

The total torque on the section ***T*** will then be the sum of all such contributions across the section,

$$T = \int_0^R 2\pi\,\tau' r^2\,dr$$

Now the shear stress $\boldsymbol{\tau'}$ will vary with the radius rand must therefore be replaced in terms of **r** before the integral is evaluated. From eqn. (53)

$$\tau' = \frac{G\theta}{L}\,r$$

$$T = \int_0^R 2\pi \frac{G\theta}{L} r^3\,dr$$

$$T = \frac{G\theta}{L}\int_0^R 2\pi\,r^3\,dr$$

The integral $\int_0^R 2\pi\,r^3\,dr$ is called the polar second moment of area ***J*** , and may be evaluated as a standard form for solid and hollow shafts as shown below.

$$T = \frac{G\theta}{L}\,J$$

Or

$$\frac{T}{J} = \frac{G\theta}{L} \quad \ldots\ldots\ldots (54)$$

Combining eqns. (53) and (54) produces the so-called simple theory of torsion:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \quad \ldots\ldots\ldots (55)$$

Polar second moment of area

As stated above the polar second moment of area ***J*** is defined as

$$J = \int_0^R 2\pi\, r^3\, dr$$

For a solid shaft,

$$J = 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$J = \frac{2\pi R^4}{4} \quad Or \quad J = \frac{\pi D^4}{32} \quad \ldots\ldots\ldots (56)$$

For a hollow shaft of internal radius ***r***,

$$J = 2\pi \int_r^R r^3\ dr\ = 2\pi \left[\frac{r^4}{4}\right]_r^R$$

$$J = \frac{\pi}{2}(R^4 - r^4) \quad Or \quad J = \frac{\pi}{32}(D^4 - d^4) \quad \ldots\ldots\ldots (57)$$

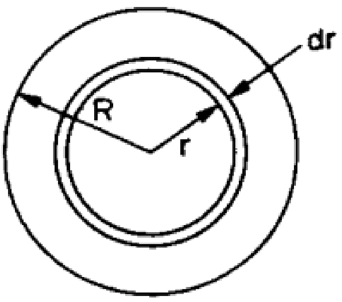
For thin-walled hollow shafts the values of ***D*** and d may be nearly equal, and in such cases, there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area.

Now

$$J = \int_0^R 2\pi\, r^3\, dr = \sum (2\pi r\, dr) r^2$$

$$J = \sum A r^2$$

where ***A*** (= 2πr dr) is the area of each small element of below Fig.,



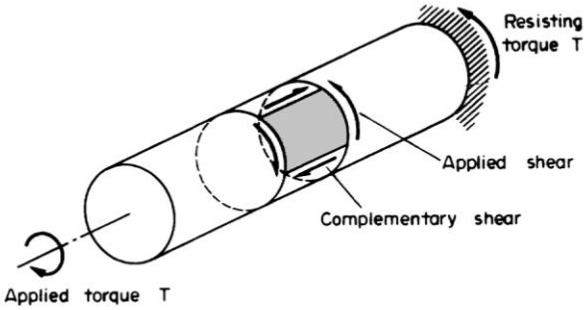
i.e. ***J*** is the sum of the ***Ar²*** terms for all elements. If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness ***t = dr***, then

$$J = Ar^2 = (2\pi r t) r^2$$

$$J \approx 2\pi r^3 t \quad \ldots\ldots\ldots (58)$$

Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in below Figure,



The shear stresses values being given by the simple torsion theory as:

$$\tau = \frac{G\theta}{L}R$$

Now from the definition of the shear or rigidity modulus G,

$$\tau = G\gamma$$

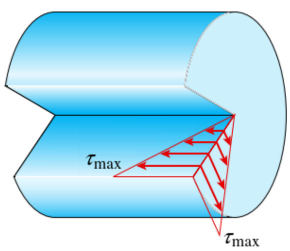
It therefore follows that the two equations may be combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

$$\tau = \frac{G\theta}{L}R = G\gamma \quad \dots\dots\dots (59)$$

or, in terms of some internal radius ***r***,

$$\tau' = \frac{G\theta}{L}r = G\gamma \quad \dots\dots\dots (60)$$

These equations indicate that the shear stress and shear strain vary linearly with radius and have their maximum value at the outside radius (below Figure).



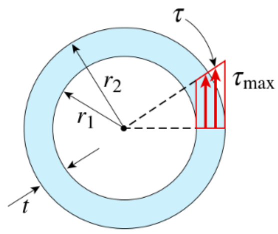
Section modulus

It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\tau = \frac{TR}{J}$$

With ***R*** the outside radius of the shaft, the above equation yields the greatest possible value of τ, i.e.,



$$\tau_{\max} = \frac{TR}{J}$$

$$\tau_{\max} = \frac{T}{Z} \quad \ldots\ldots\ldots (61)$$

where $Z = \frac{J}{R}$ is termed the polar section modulus. It will be seen from the preceding section that:

for solid shafts,

$$Z = \frac{\pi D^3}{16} \quad \ldots\ldots\ldots (62)$$

For hollow shafts,

$$Z = \frac{\pi(D^4 - d^4)}{16D} \quad \ldots\ldots\ldots (63)$$

Torsion of hollow shafts

It has been shown above that the maximum shear stress in a solid shaft is developed in the outer surface, values at other radii decreasing linearly to zero at the center. It is clear, therefore, that if there is to be some limit set on the maximum allowable working stress in the shaft material then only the outer surface of the shaft will reach this limit. The material within the shaft will work at a lower stress and, particularly near the center, will not contribute as much to the torque-carrying capacity of the shaft. In applications where weight reduction is of prime importance as in the aerospace industry, for instance, it is often found advisable to use hollow shafts.

The relevant formulae for hollow shafts have been introduced in previous sections and will not be repeated here. As an example of the increased torque-to-weight ratio possible with hollow shafts, however, it should be noted for a hollow shaft with an inside diameter half the outside diameter that the maximum stress increases by 6 % over that for a solid shaft of the same outside diameter whilst the weight reduction achieved is approximately 25 %.

Power transmitted by shafts

If a shaft carries a torque T (N.m) and rotates at ω (rad/s) it will do work at the rate of

$$\mathbf{P = T \times \omega}$$

The angular velocity for a given frequency of rotation ***N*** (rev/min) can be written as

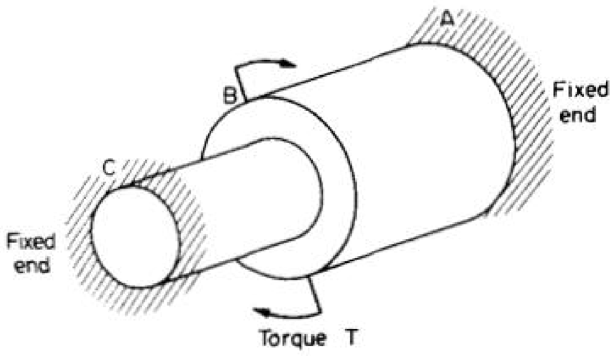
$$\omega = \frac{2\pi N}{60} \quad \text{(rad/s)}$$

Thus, the power transmitted by the shaft given by

$$P = \frac{T \times 2\pi N}{60} \quad \ldots\ldots\ldots\ldots\ldots (64)$$

Composite shafts – parallel connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel.



For parallel connection, total torque given by:

$$T = T_1 + T_2 \quad \dots\dots\dots \quad (65)$$

In this case, the angles of twist of each portion are equal:

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \quad \dots\dots\dots \quad (66)$$

i.e. for equal lengths (as is normally the case for parallel shafts)

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2} \quad \dots\dots\dots\dots \quad (67)$$

Thus, two equations are obtained in terms of the torques in each part of the composite shaft, and these torques can therefore be determined.

The maximum stresses in each part can then be found from

$$\tau_1 = \frac{T_1 R_1}{J_1} \qquad and \quad \tau_2 = \frac{T_2 R_2}{J_2}$$

Example 21:

A solid shaft, 100 mm diameter, transmits 75 kW at 150 rpm. Determine the value of the maximum shear stress set up in the shaft and the angle of twist in a length of 1 m, if G = 80 GN/m^2.

Solution

The polar moment of inertia for the shaft is

$$J = \frac{\pi D^4}{32} = \frac{\pi (0.1)^4}{32} = 9.82 \times 10^{-6} \quad m^4$$

The torque in given by

$$T = \frac{60P}{2\pi N} = \frac{60 \times 75 \times 10^2}{2\pi \times 150} = 4774.6 \quad N.m$$

The maximum shear stress is computed from elastic torsion formula:

$$\tau_{\max} = \frac{TR}{J} = \frac{4774.6 \times 0.05}{9.82 \times 10^{-6}} = 24310290.6\ Pa = 24.3\ MPa$$

Also, from the torsion theory

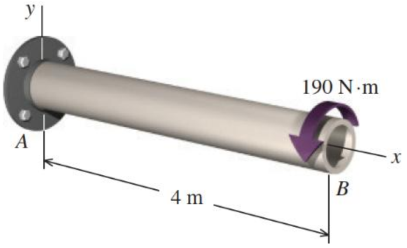
$$\frac{T}{J} = \frac{G\theta}{L}$$

$$\theta = \frac{TL}{GJ} = \frac{4774.6 \times 1}{(80 \times 10^{9}) \times (9.82 \times 10^{-6})} = 0.00607\ rad$$

$$\theta = 0.00607 \times \frac{180}{\pi} = 0.348\,°$$

Example 22:

A hollow circular steel shaft with an outside diameter of 38 mm and a wall thickness of 3 mm is subjected to a pure torque of 190 N·m. The shaft is 2.3 m long. The shear modulus of the steel is G = 83 GPa. Determine



A- The maximum shear stress in the shaft.
B- The magnitude of the angle of twist in the shaft.

Solution

Inside diameter:

$$D_i = D_o - 2t = 38 - 2 \times 3 = 32 \text{ mm}$$

The polar moment of inertia for the shaft is

$$J = \frac{\pi}{32}\left(D_o{}^4 - D_i{}^4\right) = \frac{\pi}{32}(0.038^4 - 0.032^4) = 1.017 \times 10^{-7}\ \ m^4$$

(a) The maximum shear stress is computed from the elastic torsion formula:

$$\tau_{\max} = \frac{TR}{J} = \frac{190 \times 0.038}{1.017 \times 10^{-7}} = 70993117.01\ Pa = 70.993\ MPa$$

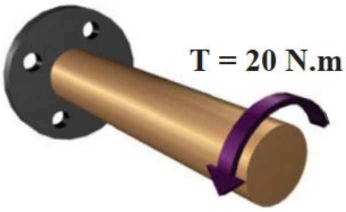
(b) Also, from the torsion theory

$$\theta = \frac{TL}{GJ} = \frac{190 \times 2.3}{(83 \times 10^{9}) \times (1.017 \times 10^{-7})} = 0.0517\ \ rad$$

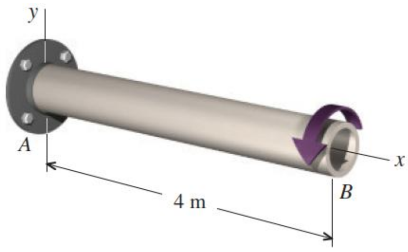
$$\theta = 0.0517 \times \frac{180}{\pi} = 2.96\,°$$

Homework No 8

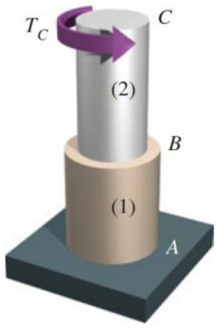
1. A 500 mm long solid steel [G = 80 GPa] shaft is being designed to transmit a torque T = 20 N · m. The maximum shear stress in the shaft must not exceed 70 MPa, and the angle of twist must not exceed 3° in the 500 mm length. Determine the minimum diameter d required for the shaft.



2. Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rpm. The maximum shear stress in the shaft is limited to 70 MN/m^2 and the angle of twist to 3.8° in a length of 4 m. For the shaft material G = 80 GN/m^2.



3. A compound shaft consists of brass segment (1) and aluminum segment (2). Segment (1) is a solid brass shaft with an allowable shear stress of 60 MPa. Segment (2) is a solid aluminum shaft with an allowable shear stress of 90 MPa. If a torque of T_C = 23 kN · m is applied at C, determine the minimum required diameter of (a) the brass shaft and (b) the aluminum shaft.



Combined stresses

Shaft subjected to bending moment and torque

1) To find stress due to bending effect

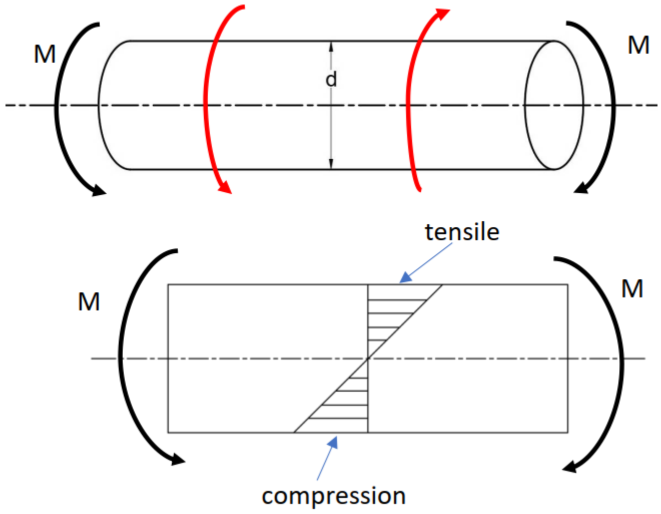
$$\frac{M}{I} = \frac{\sigma}{y}$$

$$I = \frac{\pi d^4}{64} \quad and \quad y = \frac{d}{2}$$

$$\sigma = \frac{My}{I}$$

$$\sigma = \frac{M\left(\frac{d}{2}\right)}{\frac{\pi d^4}{64}}$$

$$\sigma = \frac{32M}{\pi d^3} \quad (Direct\ stress\ due\ to\ bending)$$



2) To find stress due to torsion

$$\frac{T}{J} = \frac{\tau}{R}$$

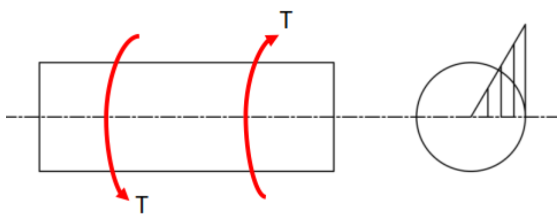
J = Polar moment of inertia

$$J = \frac{\pi d^4}{32}, R = \frac{d}{2}$$

$$\tau = \frac{TR}{J}$$

$$\tau = \frac{T\left(\frac{d}{2}\right)}{\frac{\pi d^4}{32}}$$

$$\tau = \frac{16T}{\pi d^3} \quad (shear\ stress\ due\ to\ torsion)$$



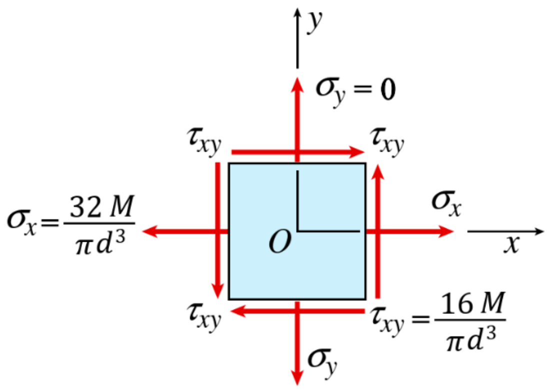
∴ An element at the top is loaded as shown: -

Maximum bending stress due to combined loading given as:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Since $\sigma_y = 0$:

$$\sigma_1 = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$



Substitute:

$$\sigma_1 = \frac{32M}{\pi d^3}\left[\frac{1}{2} + \frac{1}{2}\sqrt{M^2 + T^2}\right]$$

$$\sigma_{max} = \frac{32}{\pi d^3}\left[\frac{M}{2} + \frac{1}{2}\sqrt{M^2 + T^2}\right]$$

Maximum shear stress due to combined bending and torsion given as:

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3}\right)^2}$$

$$\tau_{max} = \frac{1}{2} \times \frac{32}{\pi d^3}\sqrt{M^2 + T^2}$$

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$$

Equivalent bending moment (Me)

Let (Me) be the equivalent bending moment (that bending moment if acting alone will cause the same direct stress which is caused by the combined effect of (M & T)

equivalent bending moment producing same direct stress:

$$\frac{M_e}{I} = \frac{\sigma_{max}}{y}$$

$$I = \frac{\pi d^4}{64}, y = \frac{d}{2}$$

$$\frac{M_e}{\frac{\pi d^4}{64}} = \frac{\sigma_{max}}{\frac{d}{2}}$$

$$\sigma_{max} = \frac{32M_e}{\pi d^3}$$

$$M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$$

Equivalent torque (Te)

Let (Te) be the equivalent torque (That torque if acting alone will cause the same maximum shear stress which caused by the combined effect of (M & T)

$$\frac{T_e}{J} = \frac{\tau_{max}}{R}$$

$$J = \frac{\pi d^4}{32}, R = \frac{d}{2}$$

$$\frac{16T_e}{\pi d^3} = \tau_{max}$$

Since:

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$$

Therefore:

$$T_e = \sqrt{M^2 + T^2}$$

Example 23:

A hollow shaft of 200 mm outside diameter and 125 mm bore is subjected simultaneously to a bending moment of 43 kN.m and a torque of 65 kN.m. calculate the bending stress and the torsional shear stress. Hence find the maximum shear stress in the shaft due to the combined torque and bending moment.

Solution

bending stress in the shaft:

$$\frac{M}{I} = \frac{\sigma_b}{y} \quad \rightarrow \quad \sigma_b = \frac{M \times y}{I}$$

$$I = \frac{\pi}{64}\left({D_o}^4 - {D_i}^4\right) = \frac{\pi}{64}(0.2^4 - 0.125^4) = 66.5 \times 10^{-6} \quad m^4$$

$$y = \frac{D_o}{2} = \frac{0.2}{2} = 0.1\, m$$

$$\therefore \quad \sigma_b = \frac{M \times y}{I} = \frac{43 \times 10^3 \times 0.1}{66.5 \times 10^{-6}} = 64.6\, MPa$$

The torsional shear stress

$$\frac{T}{J} = \frac{\tau}{R} \quad \rightarrow \quad \tau = \frac{TR}{J}$$

$$J = \frac{\pi}{32}\left({D_o}^4 - {D_i}^4\right) = \frac{\pi}{32}(0.2^4 - 0.125^4) = 133.1 \times 10^{-6} \quad m^4$$

$$R = \frac{d}{2} = \frac{0.2}{2} = 0.1 \;\, m$$

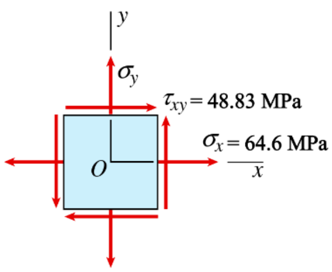
$$\therefore \quad \tau = \frac{TR}{J} = \frac{65 \times 10^3 \times 0.1}{133.1 \times 10^{-6}} = 48.83\, MPa$$

The maximum shear stress in the shaft due to the combined torque and bending moment

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

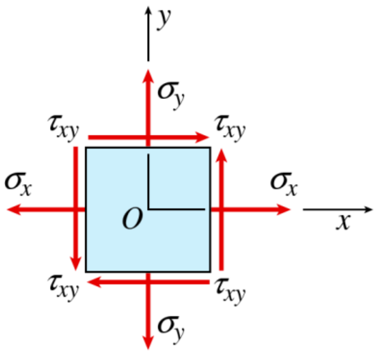
$$\tau_{max} = \frac{1}{2}\sqrt{(64.6 - 0)^2 + 4(48.83)^2}$$

$$\tau_{max} = 58.53\, MPa$$



Material Subjected to Direct and Shear Stress

If the material is subjected to the two types of stresses at the same time as shown in fig.



The maximum and minimum stress will be found as follows:

$$\sigma_{max,min} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

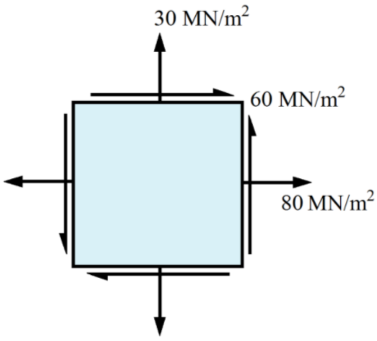
Maximum shear stress:

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

These are called principal stresses, acting on mutually perpendicular planes called principal planes.

Example 24:

Under certain loading conditions, the stresses in the walls of a cylinder are as follows: (a) 80 MN/m^2 tensile. (b) 30 MN/m^2 tensile at right angles to (a). (c) shear stresses of 60 MN/m^2 on the planes on which the stresses (a) and (b) act; the shear couple acting on planes carrying the 30 MN/m^2 stress is clockwise in effect. Calculate the principal stresses and the maximum shear stress.



Solution:

the max. and min. stress will be found as follows:

$$\sigma_{max,min} = \frac{1}{2}(80 + 30) \pm \frac{1}{2}\sqrt{(80 - 30)^2 + 4(60)^2}$$

$$\sigma_{max,min} = 55 \pm 65$$

$$\sigma_{max} = 55 + 65 = 120\ MPa$$

$$\sigma_{min} = 55 - 65 = -10\ MPa\ \ (compressive)$$

Maximum shear stress:

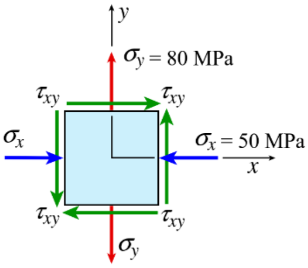
$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$\tau_{max} = \frac{1}{2}\sqrt{(80 - 30)^2 + 4(60)^2} = 65\ MPa$$

Example 25:

A material is subjected to two mutually perpendicular direct stresses of 80 MN/m^2 tensile and 50 MN/m^2 compressive, together with a shear stress of 30 MN/m^2. The shear couple acting on planes carrying the 80 MN/m^2 stress is clockwise in effect. Calculate

1. The magnitude and nature of the principal stresses;
2. The magnitude of the maximum shear stresses in the plane of the given stress system.



Solution

the max. and min. stress will be found as follows:

$$\sigma_{max,min} = \frac{1}{2}(-50 + 80) \pm \frac{1}{2}\sqrt{(-50 - 80)^2 + 4(900)^2}$$

$$\sigma_{max} = 86.55\ MPa \quad (tensile)$$

$$\sigma_{min} = -56.55\ MPa \quad (compressive)$$

Maximum shear stress:

$$\tau_{max} = \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2}$$

$$\tau_{max} = \frac{1}{2}\sqrt{(-50 - 80)^2 + 4(900)^2}$$

$$\tau_{max} = 71.6\ MPa$$