**NORTHERN TECHNICAL UNIVERSITY**

**ENGINEERING TECHNICAL COLLEGE / MOSUL**

**Power Mechanics Engineering Technology**

***Principles of***

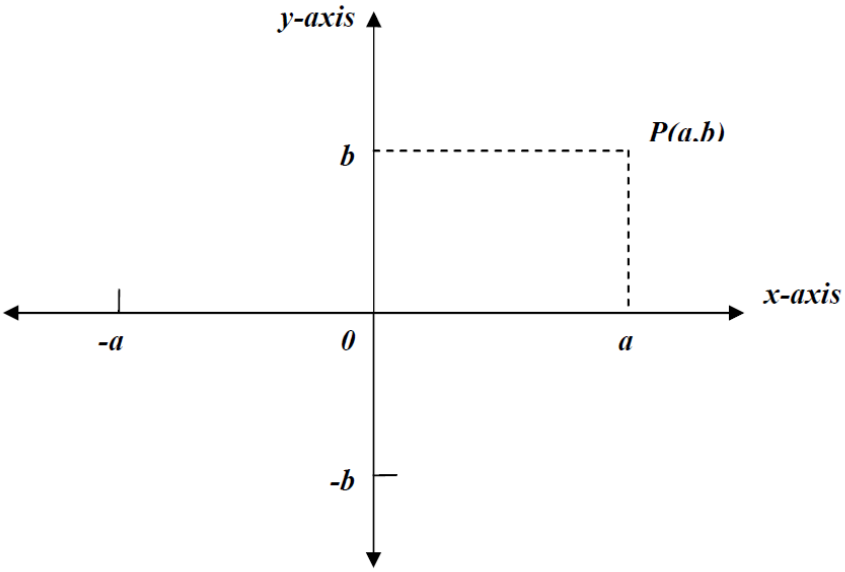
# MATHEMATICS

***For First class students***

# The Rate of Change of a Function

**Coordinates for the plane:**

***Cartesian Coordinate*- Two number lines, one of them horizontal (called *x-axis* ) and the other vertical ( called *y-axis* ). The point where the lines cross is the *origin*. Each line is assumed to represent the real number.**



**The Slope of a line :**

***Increments* –** When a particle moves from one position in the plane to another , the net changes in the particle's coordinates are calculated by subtracting the coordinates of the starting point ( $x_1$ , $y_1$ ) from the coordinates of the stopping point ($x_2$ , $y_2$ ) ,
**i.e. $\Delta x = x_2 - x_1$ , $\Delta y = y_2 - y_1$ .**

**Slopes of no vertical lines :**

**Let L be a no vertical line in the plane , Let $P_1(x_1 , y_1)$ and $P_2(x_2 , y_2)$ be two points on L. Then the slope m is :**

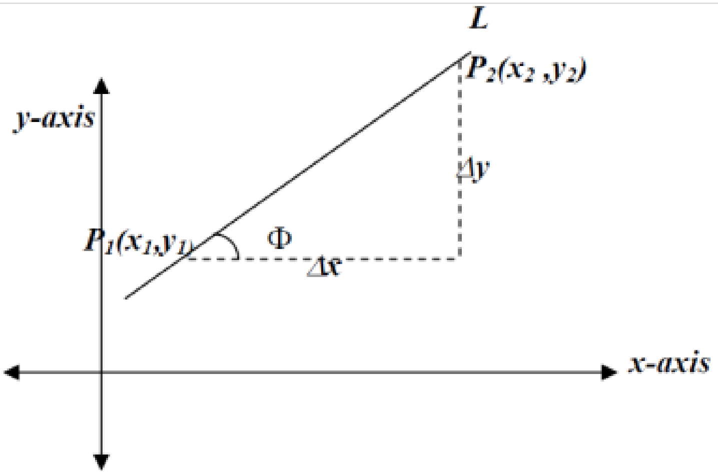
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \qquad where \quad \Delta x \neq 0$$

- **For the two perpendicular lines $L_1$ and $L_2$: $m_1 . m_2 = -1$ .**
- **For the parallel two lines: $m_1 = m_2 = 1$**

**Angles of Inclination:**
The angle of inclination of a line that crosses the *x-axis* is the smallest angle we get when we measure counter clock from the *x-axis* around the point of intersection. The slope of a line is the tangent of the line angle of inclination.

**$m = \tan\theta$ where $\theta$ is the angle of inclination.**

- The angle of inclination of a horizontal line is taken to be $0^o$.
- Parallel lines have equal angle of inclination.



**Example: Find the slope of the line determined by two points A(2,1) and B(-1,3) and find the common slope of the line perpendicular to AB.**

**Solution:**

Slope of AB is:

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-1-2} = -\frac{2}{3}$$

Slope of line perpendicular to AB is $-\frac{1}{m_{AB}} = \frac{3}{2}$ :

**Equations for lines : An equation for a line is an equation that is satisfied by the coordinates of the points that lies on the line and is not satisfied by the coordinates of the points that lie elsewhere .**

Vertical lines : Every vertical line *L* has to cross the x-axis at some point. $x = a$, $y = 0$.

Horizontal lines : The standard equation for the horizontal line through the point ( 0 , b ) is : $y = b$, $x = 0$ .

Non vertical and non horizontal lines : That point – slope equation of the line through the point ( $x_1$ , $y_1$ ) with slope m is :

$$y - y_1 = m ( x - x_1 )$$

The distance from a point to a line : To calculate the distance d between the point P($x_1$ , $y_1$ ) and Q( $x_2$ , $y_2$ ) is :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Example: Write an equation for the line that passes through point :**
a) P( -1 , 3 ) with slope m = -2 .
b) P1( -2 , 0 ) and P2 ( 2 , -2 ).

**Solution:**

a) $y - y_1 = m ( x - x_1 ) \rightarrow y - 3 = -2 ( x - (-1)) \rightarrow y + 2x = 1$

b)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-0}{2-(-2)} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1) \Rightarrow \quad y - 0 = -\frac{1}{2}(x - (-2)) \Rightarrow \quad 2y + x + 2 = 0$$

**Example: Find the slope of the line : 3x + 4y = 12 .**

**Solution:**

$$y = -\frac{3}{4}x + 3 \Rightarrow \quad the \quad slope \quad is \quad m = -\frac{3}{4}$$

**Example Find :**
**a) an equation for the line through P( 2 ,1 ) parallel to L: y = x + 2 .**
**b) an equation for the line through P perpendicular to L .**
**c) the distance from P to L .**

**Functions :** Function is any rule that assigns to each element in one set some element from another set :

$$y = f(x)$$

## Limits and continuity:

**Limits : The limit of F( t ) as t approaches C is the number L.**

$$\lim_{t \to C} F(t) = L$$

**The limit of a function F( t ) as t→C never depend on what happens when t = C .**

**Right hand limit :** $$\lim_{t \to C^{+}} F(t) = L$$

**Left hand limit :** $$\lim_{t \to C^{-}} F(t) = L$$

**Note that – A function F( t ) has a limit at point C if and only if the right hand and the left hand limits at C exist and equal . In symbols:**

$$\lim_{t \to C} F(t) = L \quad \Leftrightarrow \quad \lim_{t \to C^{+}} F(t) = L \quad and \quad \lim_{t \to C^{-}} F(t) = L$$

**The limit combinations theorems :**

$$1) \quad \lim\left[F_1(t) \mp F_2(t)\right] = \lim F_1(t) \mp \lim F_2(t)$$

$$2) \quad \lim\left[F_1(t) * F_2(t)\right] = \lim F_1(t) * \lim F_2(t)$$

$$3) \quad \lim \frac{F_1(t)}{F_2(t)} = \frac{\lim F_1(t)}{\lim F_2(t)} \qquad where \quad \lim F_2(t) \neq 0$$

$$4) \quad \lim\left[k * F_1(t)\right] = k * \lim F_1(t) \qquad \forall k$$

$$5) \quad \lim_{\theta \to 0} \frac{Sin\theta}{\theta} = 1$$

*provided that* $\theta$ *is measured in radius*

**Infinity as a limit :**

**1.The limit of the function f( x ) as x approaches infinity is the number L:**

$$\lim_{x\to\infty} f(x) = L$$

**2. The limit of f( x ) as x approaches negative infinity is the number L :**

$$\lim_{x\to-\infty} f(x) = L$$

**The following facts are sometimes abbreviated by saying:**

**a) As x approaches 0 from the right, 1/x tends to ∞.**

**b) As x approaches 0 from the left, 1/x tends to -∞.**

**c) As x tends to ∞, 1/x approaches 0 .**

**d) As x tends to -∞, 1/x approaches 0 .**

**Continuity:**

**The continuity test: The function y = f ( x ) is continuous at x = C if and only if all three of the following statements are true :**

1) $f(C)$ **exist** ( $C$ **is in the domain of** $f$ ) .
2) $\lim_{x\to C} f(x)$ **exists** ( $f$ **has a limit as** $x\to C$ ) .
3) $\lim_{x\to C} f(x) = f(C)$ ( **the limit equals the function value** ) .

**Example: Find the following:**

1. $$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2}$$

2. $$\lim_{x\to a} \frac{x^3-a^3}{x^4-a^4}$$

3. $$\lim_{x\to\infty} \frac{3x^3+5x^2-7}{10x^3-11x+5}$$

4. $$\lim_{x\to -1^-} \frac{1}{x+1}$$

**Solution:**

1. $$\lim_{x\to 0} \frac{5x^3+8x^2}{3x^4-16x^2} = \lim_{x\to 0} \frac{5x+8}{3x^2-16} = \frac{0+8}{0-16} = -\frac{1}{2}$$

2. $$\lim_{x\to a} \frac{x^3-a^3}{x^4-a^4} = \lim_{x\to a} \frac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)(x^2+a^2)} = \frac{a^2+a^2+a^2}{(a+a)(a^2+a^2)} = \frac{3}{4a}$$

3. $$\lim_{x\to \infty} \frac{3x^3+5x^2-7}{10x^3-11x+5} = \lim_{x\to \infty} \frac{3+\frac{5}{x}-\frac{7}{x^3}}{10-\frac{11}{x^2}+\frac{5}{x^3}} = \frac{3}{10}$$

**Example: Test continuity for the following function:**

$$f(x) = \left\{ \begin{array}{cc} x^2-1 & -1 \le x < 0 \\ 2x & 0 \le x < 1 \\ 1 & x = 1 \\ -2x+4 & 1 < x \le 2 \\ 0 & 2 < x \le 3 \end{array} \right\}$$

**Solution:**

We test the continuity at midpoints $x = 0 , 1 , 2$ and endpoints $x = -1 , 3$ .

$$At \quad x=0 \Rightarrow \qquad f(0)=2*0=0$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(x^2-1)=-1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0} 2x=0 \neq \lim_{x\to 0^-} f(x)$$

$$Since \quad \lim_{x\to 0} f(x) \ doesn't \ exist$$

$$Hence \quad the \ function \ discontinuous \ at \ x=0$$

$$At \quad x=1 \Rightarrow \quad f(1)=1$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1} 2x=2$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(-2x+4)=2=\lim_{x\to 1^-} f(x)=\lim_{x\to 1} f(x)$$

$$Since \quad \lim_{x\to 1} f(x) \neq f(1)$$

$$Hence \quad the \ function \ is \ discontinuous \ at \ x=1$$

$$At \quad x=2 \Rightarrow \quad f(2)=-2*2+4=0$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(-2x+4)=0$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2} 0=0=\lim_{x\to 2^-} f(x)=\lim_{x\to 2} f(x)$$

$$Since \quad \lim_{x\to 2} f(x)=f(2)=0$$

$$Hence \quad the \ function \ is \ continuous \ at \ x=2$$

$$At \quad x=-1 \Rightarrow \quad f(-1)=(-1)^2-1=0$$

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1}(x^2-1)=0=f(-1)$$

$$Hence \quad the \ function \ is \ continuous \ at \ x=-1$$

$$At \quad x=3 \Rightarrow \quad f(3)=0$$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 0=0=f(3)$$

$$Hence \quad the \ function \ is \ continuous \ at \ x=3$$

# Transcendental Function

## 1. Exponential and Logarithm functions:

Exponential functions: If a is a positive number and x is any number, we define the exponential function as:

$$y = a^x$$

The properties of the exponential functions are:

1. If $a > 0 \leftrightarrow a^x > 0$ .
2. $a^x \cdot a^y = a^{x+y}$ .
3. $a^x / a^y = a^{x-y}$ .
4. $(a^x)^y = a^{x.y}$ .
5. $(a \cdot b)^x = a^x \cdot b^x$ .
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$ .
7. $a^{-x} = 1/a^x$ and $a^x = 1/a^{-x}$ .
8. $a^x = a^y \leftrightarrow x = y$ .
9. $a^0 = 1$ ,

   $a^{\infty} = \infty$ , $a^{-\infty} = 0$ , where $a > 1$ .

   $a^{\infty} = 0$ , $a^{-\infty} = \infty$ , where $a < 1$ .

**Logarithm function: If a is any positive number other than 1 , then the logarithm of x to the base a denoted by :**

$$y = \log_a x \qquad \text{where} \quad x > 0$$

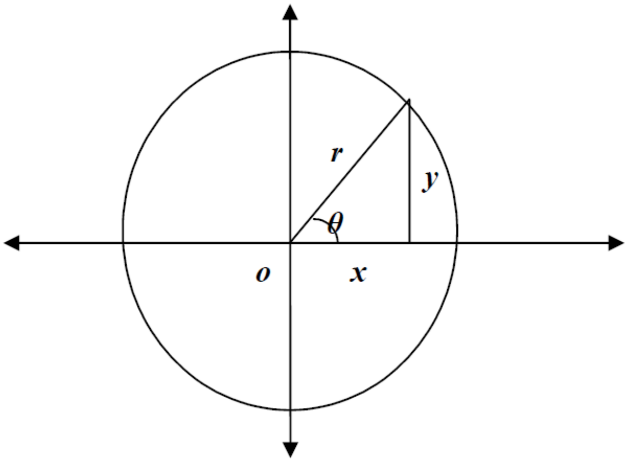
At a = e = 2.7182828… , we get the natural logarithm and denoted by:
**y = lnx**

Let x , y > 0 then the properties of logarithm functions are :

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$ .
2. $\log_e x = \ln x$ .
3. $\log_a x = \ln x / \ln a$ .

4. $\ln (x.y) = \ln x + \ln y$ .
5. $\ln ( x / y ) = \ln x - \ln y$ .
6. $\ln x^n = n. \ln x$ .
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$ .
8. $a^x = e^{x. \ln a}$ .
9. $e^{\ln x} = x$ .

2. **Trigonometric functions:** When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :



$$Sin\theta = \frac{y}{r} = \frac{1}{csc\,\theta} \quad , \quad Cos\theta = \frac{x}{r} = \frac{1}{sec\,\theta} \quad , \quad tan\,\theta = \frac{y}{x} = \frac{1}{Cot\theta} = \frac{Sin\theta}{Cos\theta}$$

**The following are some properties of these functions:**

1. $$Sin^2\theta + Cos^2\theta = 1$$

2. $$1 + tan^2\,\theta = sec^2\,\theta \quad and \quad 1 + Cot^2\theta = csc^2\,\theta$$

3. $$Sin(\theta \mp \beta) = Sin\,\theta.Cos\beta \mp Cos\theta.Sin\beta$$

4. $$Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\,\theta.Sin\beta$$

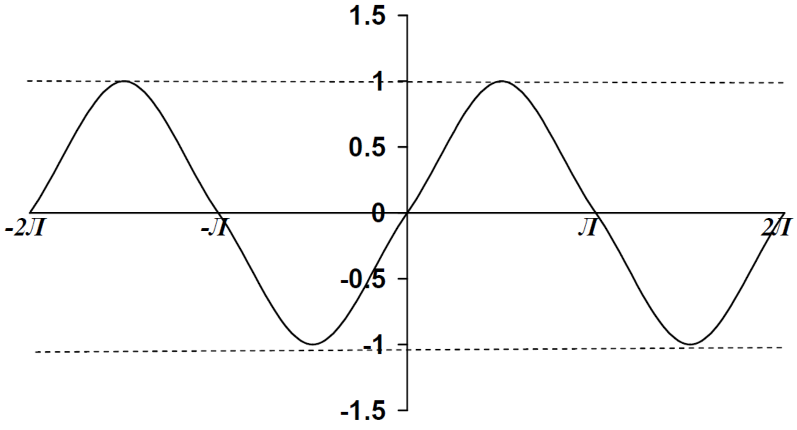
5. $$Sin\,2\theta = 2Sin\,\theta.Cos\theta \;\; and \;\; Cos\,2\theta = Cos^2\theta - Sin^2\theta$$

6. $$Cos^2\theta = \frac{1 + Cos\,2\theta}{2} \;\; and \;\; Sin^2\theta = \frac{1 - Cos\,2\theta}{2}$$

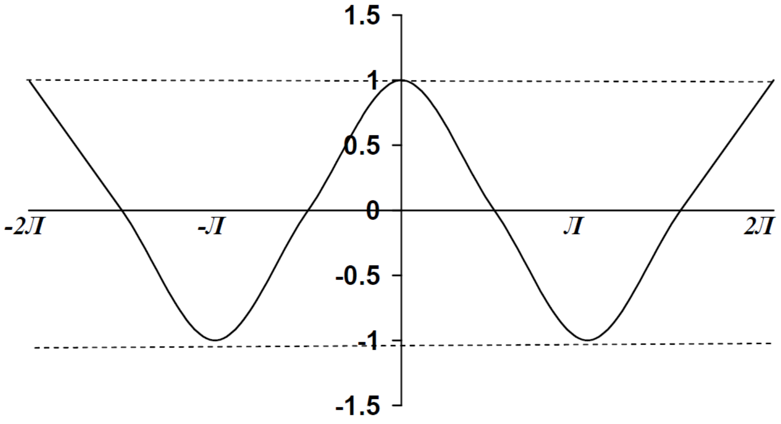
7. $$Sin(-\theta) = -Sin\theta \;\; and \;\; Cos(-\theta) = Cos\theta \;\; and \;\; tan(-\theta) = -\,tan\,\theta$$

**Graphs of the trigonometric functions are:**

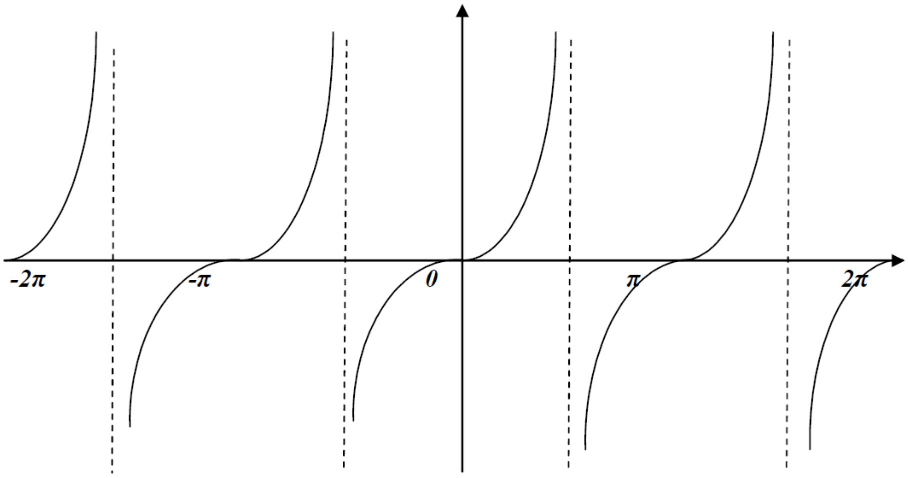
1. **y = sinx**



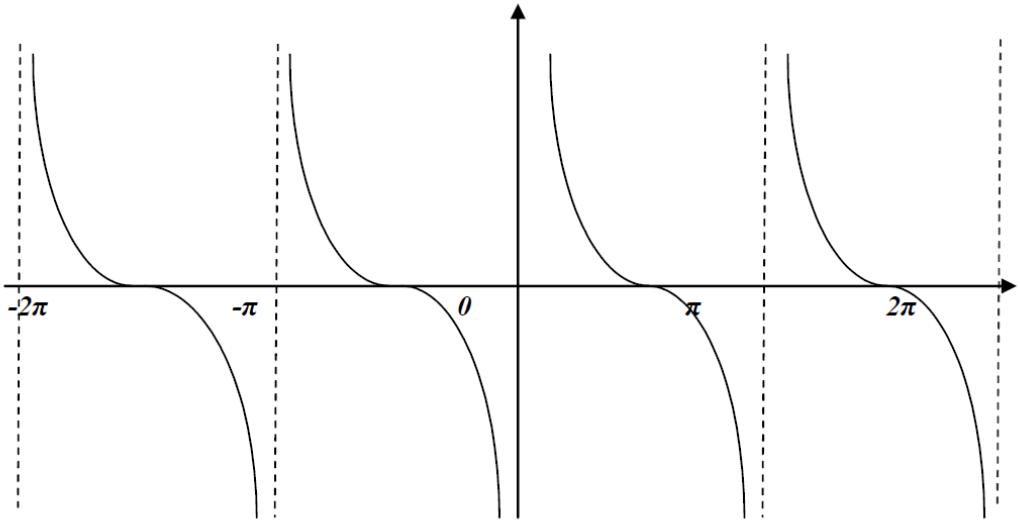
2. **y = cosx**



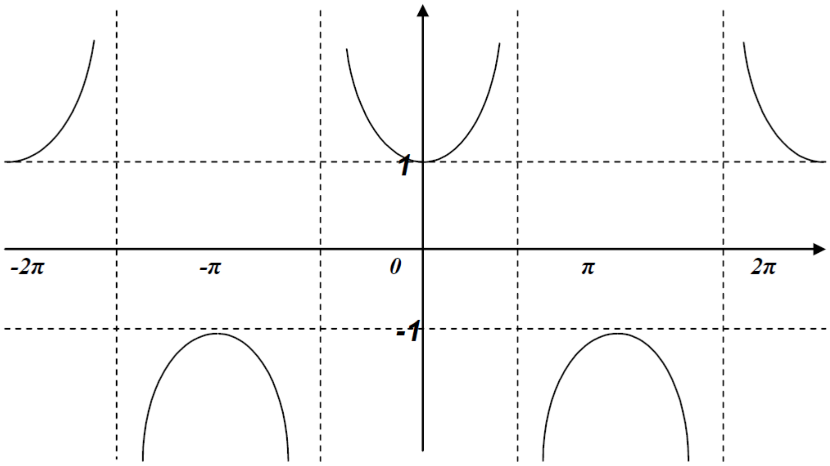
**3. y = tanx**



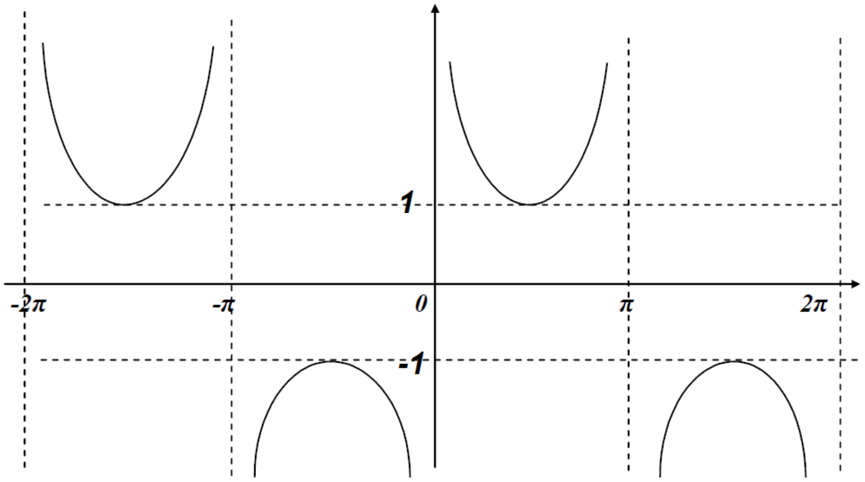
**4. y = cotx**



**5. y = secx**



**6. y = cscx**



**Example:** Solve the following equations , for values of θ from $0^o$ to $360^o$ inclusive .

a) $\tan\theta = 2\,\text{Sin}\,\theta$, b) $1 + \text{Cos}\,\theta = 2\,\text{Sin}^2\theta$

**Solution:**

**a)**

$$tan\,\theta = 2Sin\,\theta \Rightarrow \frac{Sin\,\theta}{Cos\,\theta} = 2Sin\,\theta$$

$$\Rightarrow Sin\,\theta(1 - 2Cos\,\theta) = 0$$

$$either\ \ Sin\,\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o$$

$$or\quad Cos\,\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

**b)**

$$1 + Cos\,\theta = 2.Sin^2\theta \Rightarrow 1 + Cos\,\theta = 2(1 - Cos^2\theta)$$

$$\Rightarrow (2Cos\,\theta - 1)(Cos\,\theta + 1) = 0$$

$$either\qquad Cos\,\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

$$or\qquad Cos\,\theta = -1 \Rightarrow \theta = 180^o$$

**Example: If tan θ = 7/24, find without using tables the values of Secθ and Sinθ.**

**Solution:**

$$tan\,\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$Sec\,\theta = \frac{r}{x} = \frac{25}{24}\qquad and\qquad Sin\,\theta = \frac{y}{r} = \frac{7}{25}$$

**Example: Prove the following identities:**

$$a)\quad Csc\theta + tan\,\theta.Sec\theta = Csc\theta.Sec^2\theta$$

$$b)\quad Cos^4\theta - Sin^4\theta = Cos^2\theta - Sin^2\theta$$

**Solution:**

$$a)\quad L.H.S. = Csc\theta + tan\,\theta.Sec\theta = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos\theta}.\frac{1}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Sin\theta.Cos^2\theta} = \frac{1}{Sin\theta}.\frac{1}{Cos^2\theta} = Csc\theta.Sec^2\theta = R.H.S.$$

$$b)\quad L.H.S. = Cos^4\theta - Sin^4\theta = (Cos^2\theta - Sin^2\theta).(Cos^2\theta + Sin^2\theta)$$

$$= Cos^2\theta - Sin^2\theta = R.H.S.$$

**Example:** Simplify $\frac{1}{\sqrt{x^2 - a^2}}$ $\quad when \quad x = a.Csc\theta$ .

**Solution:**

$$\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{a^2 Csc^2\theta - a^2}} = \frac{1}{a\sqrt{Cot^2\theta}} = \frac{1}{a}\,tan\,\theta$$

3. **The inverse trigonometric functions : The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles :**

$$y = Sinx \quad \Leftrightarrow \quad x = Sin^{-1} y$$

**The following are some properties of the inverse trigonometric functions :**

1. $Sin^{-1}(-x) = -Sin^{-1} x$
2. $Cos^{-1}(-x) = \pi - Cos^{-1} x$
3. $Sin^{-1} x + Cos^{-1} x = \dfrac{\pi}{2}$
4. $tan^{-1}(-x) = -tan^{-1} x$
5. $Cot^{-1} x = \dfrac{\pi}{2} - tan^{-1} x$
6. $Sec^{-1} x = Cos^{-1} \dfrac{1}{x}$
7. $Csc^{-1} x = Sin^{-1} \dfrac{1}{x}$
8. $Sec^{-1}(-x) = \pi - Sec^{-1} x$

$$and \;\; noted \;\; that \;\; (Sinx)^{-1} = \frac{1}{Sinx} = Cscx \neq Sin^{-1} x$$

**Example:** Given that $\alpha = Sin^{-1}\frac{\sqrt{3}}{2}$ , find :

$Csc\alpha \ , Cos\alpha \ , Sec\alpha \ , tan\,\alpha \ , and \ , Cot\alpha$

**Solution:**

$$\alpha = Sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow Sin\alpha = \frac{\sqrt{3}}{2} = \frac{x}{y} \Rightarrow r = \sqrt{4-3} = 1$$

$$Csc\alpha = \frac{2}{\sqrt{3}} \ , Cos\alpha = \frac{1}{2} \ , Sec\alpha = 2 \ , tan\,\alpha = \sqrt{3} \ , Cot\alpha = \frac{1}{\sqrt{3}}$$

**Example: Evaluate the following expressions:**

$$a)\ Sec(Cos^{-1}\frac{1}{2}) \quad b)\ Sin^{-1}1 - Sin^{-1}(-1) \quad c)\ Cos^{-1}(-Sin\frac{\pi}{6})$$

**Solution:**

$$a)\ Sec(Cos^{-1}\frac{1}{2}) = Sec\frac{\pi}{3} = 2$$

$$b)\ Sin^{-1}1 - Sin^{-1}(-1) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$$

$$c)\ Cos^{-1}(-Sin\frac{\pi}{6}) = Cos^{-1}(-\frac{1}{2}) = \frac{2}{3}\pi$$

**Example : Prove the following:**

$$Sec^{-1}x = Cos^{-1}\frac{1}{x}$$

**Solution:**

$$Let\ \ y = Sec^{-1}x \Rightarrow x = Secy \Rightarrow x = \frac{1}{Cosy}$$

$$\Rightarrow y = Cos^{-1}\frac{1}{x} \Rightarrow Sec^{-1}x = Cos^{-1}\frac{1}{x}$$

4. **Hyperbolic functions : Hyperbolic functions are used to describe the motions of waves in elastic solids ; the shapes of electric power lines ; temperature distributions in metal fins that cool pipes …etc. The hyperbolic sine (Sinh) and hyperbolic cosine (Cosh) are defined by the following equations :**

1. $Sinhu = \frac{1}{2}(e^{u} - e^{-u})$ and $Coshu = \frac{1}{2}(e^{u} + e^{-u})$
2. $\tanh u = \frac{Sinhu}{Coshu} = \frac{e^{u} - e^{-u}}{e^{u} + e^{-u}}$ and $Cothu = \frac{Coshu}{Sinhu} = \frac{e^{u} + e^{-u}}{e^{u} - e^{-u}}$
3. $Sechu = \frac{1}{Coshu} = \frac{2}{e^{u} + e^{-u}}$ and $Cschu = \frac{1}{Sinhu} = \frac{2}{e^{u} - e^{-u}}$
4. $Cosh^{2}u - Sinh^{2}u = 1$
5. $\tanh^{2} u + Sech^{2}u = 1$ and $Coth^{2}u - Csch^{2}u = 1$
6. $Coshu + Sinhu = e^{u}$ and $Coshu - Sinhu = e^{-u}$
7. $Cosh(-u) = Coshu$ and $Sinh(-u) = -Sinhu$
8. $Cosh0 = 1$ and $Sinh0 = 0$
9. $Sinh(x + y) = Sinhx.Coshy + Coshx.Sinhy$
10. $Cosh(x + y) = Coshx.Coshy + Sinhx.Sinhy$
11. $Sinh2x = 2.Sinhx.Coshx$
12. $Cosh2x = Cosh^{2}x + Sinh^{2}x$
13. $Cosh^{2}x = \frac{Cosh2x + 1}{2}$ and $Sinh^{2}x = \frac{Cosh2x - 1}{2}$

# Derivatives

Let y = f ( x ) be a function of x . If the limit:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\Delta x}{\Delta y}$$

Exists and is finite, we call this limit the derivative of f at x and say that f is differentiable at x.

**Example: Find the derivative of the function** $(x) = \frac{1}{\sqrt{2x+3}}$ **.**

**Solution:**

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{2(x+\Delta x)+3}} - \frac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} \cdot \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

**Rules of derivatives: Let c and n are constants, u , v and w are differentiable functions of x :**

1. $$\frac{d}{dx}c=0$$
2. $$\frac{d}{dx}u^n=nu^{n-1}\frac{du}{dx}\Rightarrow\frac{d}{dx}\left(\frac{1}{u}\right)=-\frac{1}{u^2}\frac{du}{dx}$$
3. $$\frac{d}{dx}cu=c\frac{du}{dx}$$
4. $$\frac{d}{dx}(u\mp v)=\frac{du}{dx}\mp\frac{dv}{dx}\quad;\frac{d}{dx}(u\mp v\mp w)=\frac{du}{dx}\mp\frac{dv}{dx}\mp\frac{dw}{dx}$$
5. $$\frac{d}{dx}(u.v)=u.\frac{dv}{dx}+v\frac{du}{dx}$$

   and $$\frac{d}{dx}(u.v.w)=u.v\frac{dw}{dx}+u.w\frac{dv}{dx}+v.w\frac{du}{dx}$$
6. $$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\qquad where\quad v\neq 0$$

**Example: Find $\frac{dy}{dx}$ for the following functions:**

a) $y=(x^2+1)^5$

b) $y=[(5-x)(4-2x)]^2$

c) $y=(2x^3-3x^2+6x)^{-5}$

**Solution:**

$$a)\quad \frac{dy}{dx}=5(x^2+1)^4.2x=10x(x^2+1)^4$$

$$b)\quad \frac{dy}{dx}=2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$$

$$=8(5-x)(2-x)(2x-7)$$

$$c)\quad \frac{dy}{dx}=-5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$$

$$=-30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$$

**The Chain Rule:**

1. **Suppose that h = $g_o f$ is the composite of the differentiable functions y = g( t ) and x = f( t ) , then h is a differentiable functions of x whose derivative at each value of x is :**

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

2. **If y is a differentiable function of t and t is differentiable function of x , then y is a differentiable function of x :**

$$y=g(t)\ \ and\ \ t=f(x)\Rightarrow\frac{dy}{dx}=\frac{dy}{dt}*\frac{dt}{dx}$$

**Example: Use the chain rule to express dy / dx in terms of x and y :**

$$a)\quad y=\frac{t^2}{t^2+1}\qquad and\qquad t=\sqrt{2x+1}$$

$$b)\quad y=\frac{1}{t^2+1}\qquad and\qquad x=\sqrt{4t+1}$$

**Solution:**

$$a)\quad y=\frac{t^2}{t^2+1}\Rightarrow\frac{dy}{dt}=\frac{2t(t^2+1)-2t.t^2}{(t^2+1)^2}=\frac{2t}{(t^2+1)^2}$$

$$t=(2x+1)^{\frac{1}{2}}\Rightarrow\frac{dt}{dx}=\frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2=\frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}=\frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}}=\frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}}=\frac{1}{2(x+1)^2}$$

$$b)\quad y=(t^2+1)^{-1}\Rightarrow\frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow\frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}..x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

**Higher derivatives: If a function y = f(x) possesses a derivative at every point of some interval, we may form the function f '(x) and talk about its derivate, if it has one. The procedure is formally identical with that used before, that is:**

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$$

This derivative is called the second derivative of y with respect to x. It is written in a number of ways, for example

$$y'' , f''(x) , or \ \frac{d^2 f(x)}{dx^2} .$$

In the same manner we may define third and higher derivatives, using similar notations. The *nth* derivative may be written:

$$y^{(n)}, f^{(n)}(x), \frac{d^n y}{dx^n} .$$

**Example: Find all derivatives of the following function:**

$$y = 3x^3 - 4x^2 + 7x + 10$$

**Solution:**

$$\frac{dy}{dx} = 9x^2 - 8x + 7 \quad , \quad \frac{d^2 y}{dx^2} = 18x - 8$$

$$\frac{d^3 y}{dx^3} = 18 \quad , \quad \frac{d^4 y}{dx^4} = 0 = \frac{d^5 y}{dx^5} = ....$$

**Example: Find the third derivative of the following function:**

$$y = \frac{1}{x} + \sqrt{x^3}$$

**Solution:**

$$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{3}{2} x^{\frac{1}{2}}$$

$$\frac{d^2 y}{dx^2} = \frac{2}{x^3} + \frac{3}{4} x^{-\frac{1}{2}}$$

$$\frac{d^3 y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8} x^{-\frac{3}{2}} \qquad \Rightarrow \frac{d^3 y}{dx^3} = -\frac{6}{x^4} - \frac{3}{8\sqrt{x^3}}$$

**Implicit Differentiation: If the formula for f is an algebraic combination of powers of x and y. To calculate the derivatives of these implicitly defined functions, we simply differentiate both sides of the defining equation with respect to x.**

**Example: Find dy/dx for the following functions:**

$$a)\ x^2.y^2 = x^2 + y^2$$

$$b)\ (x+y)^3 + (x-y)^3 = x^4 + y^4$$

**Solution:**

$$a)\ x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2y - y}$$

$$b)\ 3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

**Exponential functions: If u is any differentiable function of x, then:**

$$\frac{d}{dx}a^u = a^u.\ln a.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}e^u = e^u.\frac{du}{dx}$$

**Example: Find dy/dx for the following functions:**

a) $y = 2^{3x}$

b) $y = e^{\sqrt{1+5x^2}}$

**Solution:**

a) $y = 2^{3x} \Rightarrow \dfrac{dy}{dx} = 2^{3x} * 3\,ln\,2$

b) $y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \dfrac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}}\,\dfrac{1}{2}(1+5x^2)^{-\frac{1}{2}}.10x = e^{\sqrt{1+5x^2}}\,\dfrac{5x}{\sqrt{1+5x^2}}$

**Logarithm functions: If u is any differentiable function of x, then:**

$$\frac{d}{dx}\log_a u = \frac{1}{u.\ln a}.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}$$

**Example: Find dy/dx for the following functions:**

a) $y = log_{10} e^{x}$

b) $y + ln(xy) = 1$

**Solution:**

a) $y = log_{10} e^{x} \Rightarrow y = x \, log_{10} e \Rightarrow \dfrac{dy}{dx} = log_{10} e = \dfrac{ln\, e}{ln\, 10} = \dfrac{1}{ln\, 10}$

b) $y + lnx + lny = 1 \Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x} + \dfrac{1}{y} . \dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x(y+1)}$

**Trigonometric functions: If u is any differentiable function of x, then:**

1) $\dfrac{d}{dx} sinu = cosu . \dfrac{du}{dx}$

2) $\dfrac{d}{dx} cosu = -\,sin\, u . \dfrac{du}{dx}$

3) $\dfrac{d}{dx} tanu = sec^{2} u . \dfrac{du}{dx}$

4) $\dfrac{d}{dx} cotu = -csc^{2} u . \dfrac{du}{dx}$

5) $\dfrac{d}{dx} secu = secu . tanu . \dfrac{du}{dx}$

6) $\dfrac{d}{dx} cscu = -cscu . cotu . \dfrac{du}{dx}$

**Example: Find dy/dx for the following functions:**

a) $y = tan(3x^{2})$

b) $y = sec^4 x - tan^4 x$

**Solution:**

a) $\dfrac{dy}{dx} = sec^2(3x^2).6x = 6x.sec^2(3x^2)$

b) $\dfrac{dy}{dx} = 4sec^3 x.secx.tanx - 4.tan^3 x.sec^2 x = 4tanx.sec^2 x$

**Example: Prove following:**

$$\frac{d}{dx} tan\, u = sec^2 u.\frac{du}{dx}$$

**Solution:**

$$L.H.S. = \frac{d}{dx} tanu = \frac{d}{dx}\frac{sinu}{cosu} = \frac{cosu.cosu.\frac{du}{dx} - sinu.(-sinu)\frac{du}{dx}}{cos^2 u}$$

$$= \frac{cos^2 u + sin^2 u}{cos^2 u}.\frac{du}{dx} = \frac{1}{cos^2 u}.\frac{du}{dx} = sec^2 u.\frac{du}{dx} = R.H.S.$$

**The inverse trigonometric functions: If u is any differentiable function of x, then:**

1) $$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$$

2) $$\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$$

3) $$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

4) $$\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$$

5) $$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$$

6) $$\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$$

**Example: Find dy/dx for the following functions:**

a) $$y = \cot^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$$

b) $$y = x.\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$$

**Solution:**

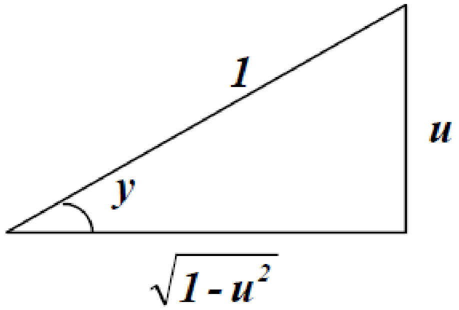
a) $$\frac{dy}{dx} = -\frac{1}{1+\left(\frac{2}{x}\right)^2} 2.\left(-\frac{1}{x^2}\right) + \frac{1}{1+\left(\frac{x}{2}\right)^2}.\frac{1}{2} = \frac{4}{4+x^2}$$

b) $$\frac{dy}{dx} = x\frac{-2}{\sqrt{1-4x^2}} + \cos^{-1} 2x - \frac{1}{4}.\frac{-8x}{\sqrt{1-4x^2}} = \cos^{-1} 2x$$

**Example: Prove the following:**

$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

**Solution:**



$$Let\ \ y = \sin^{-1} u \Rightarrow u = \sin y \Rightarrow \frac{du}{dx} = \cos y.\frac{dy}{dx} = \sqrt{1-u^2}\,\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \Rightarrow \frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

**The derivatives of functions like $u^v$: Where u and v are differentiable functions of x , are found by logarithmic differentiation :**

$$Let\ \ y = u^v \Rightarrow \ln y = v.\ln u$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{v}{u}.\frac{du}{dx} + \ln u.\frac{dv}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{v}{u}.\frac{du}{dx} + \ln u.\frac{dv}{dx}\right]$$

$$\frac{d}{dx}u^v = u^v.\left[\frac{v}{u}.\frac{du}{dx} + \ln u.\frac{dv}{dx}\right]$$

**Example: Find dy/dx for the following:**

$$y = x^{\cos x}$$

**Solution:**

$$y = x^{\cos x} \Rightarrow \ln y = \cos x.\ln x \Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{\cos x}{x} + \ln x.(-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\cos x}{x} - \sin x.\ln x\right]$$

# Integration

## Indefinite integrals:

The set of all anti derivatives of a function is called indefinite integral of the function. Assume *u* and *v* denote differentiable functions of *x*, and *a*, *n*, and *c* are constants, then the integration formulas are:

$$1)\ \int du = u(x) + c$$

$$2)\ \int a \cdot u(x)dx = a\int u(x)dx$$

$$3)\ \int \big(u(x) \mp v(x)\big)dx = \int u(x)\,dx \mp \int v(x)\,dx$$

$$4)\ \int u^{n}du = \frac{u^{n+1}}{n+1} + c \quad when \quad n \neq -1 \quad \& \quad \int u^{-1}du = \int \frac{1}{u}du = \ln u + c$$

$$5)\ \int a^{u}du = \frac{a^{u}}{\ln a} + c \quad \Rightarrow \quad \int e^{u}du = e^{u} + c$$

**Example: Evaluate the following integrals:**

**1.** $\int 3x^{2}\ dx$

**2.** $\int \frac{e^{x}}{1+3e^{x}}dx$

**3.** $\int 2^{-4x}\,dx$

**Solution:**

1. $\int 3x^2\,dx = 3\int x^2 dx = 3\dfrac{x^3}{3} + c = x^3 + c$

2. $\displaystyle\int \frac{e^x}{1+3e^x}\,dx = \frac{1}{3}\int 3e^x(1+3e^x)^{-1}\,dx = \frac{1}{3}\ln(1+3e^x) + c$

3. $\displaystyle\int 2^{-4x}dx = -\frac{1}{4}\int 2^{-4x}\cdot(-4dx) = -\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2} + c$

**Integrals of trigonometric functions:**
**The integration formulas for the trigonometric functions are:**

| | |
|---|---|
| $\int \sin u \cdot du = -\cos u + c$ | $\int \cos u \cdot du = \sin u + c$ |
| $\int \tan u \cdot du = -\ln\lvert\cos u\rvert + c$ | $\int \cot u \cdot du = \ln\lvert\sin u\rvert + c$ |
| $\int \sec u \cdot du = \ln\lvert\sec u + \tan u\rvert + c$ | $\int \csc u \cdot du = -\ln\lvert\csc u + \cot u\rvert + c$ |
| $\int \sec^2 u \cdot du = \tan u + c$ | $\int \csc^2 u \cdot du = -\cot u + c$ |
| $\int \sec u \cdot \tan u \cdot du = \sec u + c$ | $\int \csc u \cdot \cot u \cdot du = -\csc u + c$ |

**Example: Evaluate the following integrals:**

1) $\int \cos(3\theta - 1)\,d\theta$

2) $\int \tan^3(5x)\cdot \sec^2(5x)\ dx$

3) $\int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx$

**Solution:**

1) $$\frac{1}{3}\int 3\cos(3\theta - 1)\,d\theta = \frac{1}{3}\sin(3\theta - 1) + c$$

2) $$\frac{1}{5}\int \tan^3 5x \cdot \left(5\sec^2 5x\ dx\right) = \frac{1}{5}\cdot\frac{\tan^4 5x}{4} + c = \frac{1}{20}\tan^4 5x + c$$

3) $$\int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\csc^2\sqrt{x} - 1}{\sqrt{x}}\,dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

## Integrals of inverse trigonometric functions:

**The integration formulas for the inverse trigonometric functions are:**

- $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \qquad ; \qquad \forall u^2 < a^2$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

-

- $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \quad ; \quad \forall u^2 > a^2$

**Example: Evaluate the following integrals:**

1) $\int \frac{dx}{x\sqrt{4x^2-1}}$

2) $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

**Solution:**

1) $\int \frac{2\ dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$

2) $\int e^{\sin^{-1}x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1}x} + c$

## Integration by parts:

**The formula for integration by parts comes from the product rule:**

$$d(u\cdot v)=u\cdot dv+v\cdot du \quad\Rightarrow\quad u\cdot dv=d(u\cdot v)-v\cdot du$$

**and integrated to give:** $\int u\,dv=\int d(u\cdot v)-\int v\,du$

**then the integration by parts formula is:-**

$$\int u\,dv=u\cdot v-\int v\,du$$

## Rule for choosing u and dv is:

For u: choose something that becomes simpler when differentiated.
For dv: choose something whose integral is simple.
It is not always possible to follow this rule, but when we can.

**Example: Evaluate the following integrals:**

1) $\int xe^{x}\,dx$

2) $\int \frac{x}{\sqrt{x-1}}\,dx$

**Solution:**

1)

$$let \left.\begin{array}{l} u=x \quad\Rightarrow\quad du=dx \\ dv=e^{x}dx \Rightarrow v=e^{x} \end{array}\right\}\Rightarrow \int udv=u\cdot v-\int vdu$$

$$let \left.\begin{array}{l} u=x \quad\Rightarrow\quad du=dx \\ dv=\frac{1}{\sqrt{x-1}}dx \Rightarrow v=2(x-1)^{\frac{1}{2}} \end{array}\right\}\Rightarrow \int udv=u\cdot v-\int vdu$$

2)

$$\int\frac{x}{\sqrt{x-1}}dx=2x\cdot(x-1)^{\frac{1}{2}}-2\int(x-1)^{\frac{1}{2}}dx$$

$$=2x\cdot\sqrt{x-1}-\frac{2(x-1)^{\frac{3}{2}}}{\frac{3}{2}}+c=2x\cdot\sqrt{x-1}-\frac{4}{3}\sqrt{(x-1)^{3}}+c$$

## Trigonometric substitutions:

**Trigonometric substitutions enable us to replace the binomials $a^2 - u^2$ , $a^2 + u^2$ , and $u^2 - a^2$ be single square terms. We can use:**

$$u = a\sin\theta \quad for \quad a^2 - u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

$$u = a\tan\theta \quad for \quad a^2 + u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$$u = a\sec\theta \quad for \quad u^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

**Example: Evaluate the following integrals:**

1) $\displaystyle\int \frac{z^5\,dz}{\sqrt{1+z^2}}$

2) $\displaystyle\int \frac{dt}{\sqrt{25t^2-9}}$

**Solution:**

$$1)\quad let\quad z=\tan\theta\quad\Rightarrow\quad dz=\sec^2\theta\cdot d\theta\qquad\qquad \tan\theta=\frac{z}{1}$$

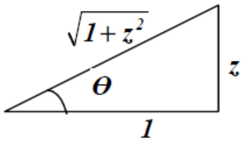
$$\int\frac{z^5\,dz}{\sqrt{1+z^2}}=\int\frac{\tan^5\theta\cdot\sec^2\theta\;d\theta}{\sqrt{1+\tan^2\theta}}=\int\tan^5\theta\cdot\sec\theta\;d\theta$$

$$=\int\tan\theta\cdot\sec\theta\left(\sec^2\theta-1\right)^2\,d\theta$$

$$=\int\sec^4\theta(\tan\theta\cdot\sec\theta\,d\theta)-2\int\sec^2\theta(\tan\theta\cdot\sec\theta\,d\theta)+\int\tan\theta\cdot\sec\theta\,d\theta$$

$$=\frac{1}{5}\sec^5\theta-\frac{2}{3}\sec^3\theta+\sec\theta+c$$

$$=\frac{1}{5}(\sqrt{1+z^2}\,)^5-\frac{2}{3}(\sqrt{1+z^2}\,)^3+\sqrt{1+z^2}+c$$



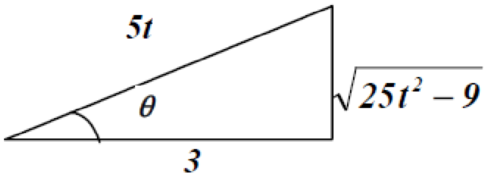
$$2)\qquad let\quad 5t=3\sec\theta\quad\Rightarrow\quad 5dt=3\sec\theta\cdot\tan\theta\,d\theta$$

$$\int\frac{dt}{\sqrt{25t^2-9}}=\int\frac{{}^3\!/_5\sec\theta\cdot\tan\theta\,d\theta}{\sqrt{9\sec^2\theta-9}}=\frac{1}{5}\int\sec\theta\,d\theta$$

$$=\frac{1}{5}\ln\left|\sec\theta+\tan\theta\right|+c$$

$$=\frac{1}{5}\ln\left|\frac{5t}{3}+\frac{\sqrt{25t^2-9}}{3}\right|+c$$

$$=\frac{1}{5}\ln\left|5t+\sqrt{25t^2-9}\right|+c'\qquad where\quad c'=c-\frac{1}{5}\ln 3$$



## Integral involving $ax^2+bx+c$ :

**By using the algebraic process called completing the square, we can convert any quadratic: $ax^2+bx+c$ , $a\neq 0$ to the form:**

**a($u^2 \mp A^2$) we can then use one of the trigonometric substitutions to write the expression as a times a single square term.**

**Example: Evaluate:**

**1)** $$\int \frac{dx}{2x^2+2x+1}$$

**2)** $$\int \frac{dx}{\sqrt{x^2-2x-8}}$$

**Solution:**

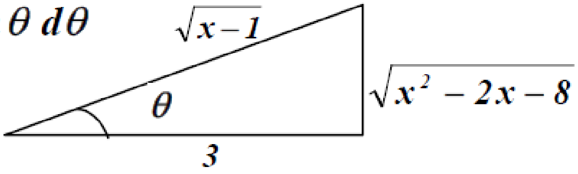
1) $$\int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{dx}{x^2+x+\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$let \quad x+\frac{1}{2} = \frac{1}{2}\tan\theta \quad \Rightarrow \quad dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$\int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\frac{1}{4}\tan^2\theta+\frac{1}{4}} = \int d\theta = \theta + c = \tan^{-1}(2x+1)+c$$

2) $$\int \frac{dx}{\sqrt{x^2-2x-8}} = \int \frac{dx}{\sqrt{(x-1)^2-9}}$$

$$let \quad x-1=3\sec\theta \;\Rightarrow\; dx=3\sec\theta\cdot\tan\theta\, d\theta$$



$$=\int\frac{3\sec\theta\cdot\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}}=\int\sec\theta\, d\theta$$

$$=ln\left|\sec\theta+\tan\theta\right|+c=ln\left|\frac{x-1}{3}+\frac{\sqrt{x^2-2x-8}}{3}\right|+c$$

$$=ln\left|x-1+\sqrt{x^2-2x-8}\right|+c' \qquad where \quad c'=c-ln3$$

## Partial fractions:

Success in separating g( x )/f ( x ) into a sum of partial fractions hinges on two things:

1- The degree of f (x) must be less than the degree of g(x) .
(If this is not case, we first perform a long division, and then work with the remainder term).
2- The factors of g(x) must be known. If these two conditions are met we can carry out the following steps:
Step I - let x - r be a linear factor of g(x). Suppose $(x - r)^m$ is the highest power of (x - r) that divides g( x ) . Then assign the sum of m partial factors to this factor, as follows:

$$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\dots\dots\dots+\frac{A_m}{(x-r)^m}$$

Do this for each distinct linear factor of f ( x ) .

**Step II - let $x^2 + px + q$ be an irreducible quadratic factor of g( x ) . Suppose $(x^2 + px + q)^n$ is the highest power of this factor that divides g( x ) . Then, to this factor, assign the sum of the n partial fractions:**

$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \ldots\ldots\ldots + \frac{B_n x + C_n}{(x^2 + px + q)^n}$$

**Do this for each distinct linear factor of g( x ) .**

**Step III - set the original fraction g(x)/f (x) equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the sums in decreasing powers of x.**

**Step IV - equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.**

**Example: Evaluate the following integrals:**

**1)** $\int \frac{2x + 5}{x^2 - 9}\, dx$

**2)** $\int \frac{x^3 + 4x^2}{x^2 + 4x + 3}\, dx$

**Solution:**

$$1)\ \int \frac{2x+5}{x^2-9}\,dx = \int \frac{2x+5}{(x-3)\cdot(x+3)}\,dx$$

$$\frac{2x+5}{(x-3)\cdot(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \quad \Rightarrow \quad 2x+5 = A(x+3) + B(x-3)$$

$$at \quad x = 3 \quad \Rightarrow \quad 6A = 6+5 \quad \Rightarrow \quad A = \frac{11}{6}$$

$$at \quad x = -3 \quad \Rightarrow \quad -6B = -6+5 \quad \Rightarrow \quad B = \frac{1}{6}$$

$$\int \frac{2x+5}{x^2-9}\,dx = \int \left( \frac{11/6}{x-3} + \frac{1/6}{x+3} \right) dx = \frac{11}{6} ln(x-3) + \frac{1}{6} ln(x+3) + c$$

2)

$$\frac{x^3+4x^2}{x^2+4x+3} = x - \frac{3x}{(x+3)(x+1)}$$

$$\begin{array}{r|l} & \quad x \\ \hline x^2+4x+3 & x^3+4x^2 \\ & \mp x^3 \mp 4x^2 \mp 3x \\ \hline & \qquad\qquad -3x \end{array}$$

$$\frac{3x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \quad \Rightarrow \quad 3x = A(x+1) + B(x+3)$$

$$at \ \ x = -3 \ \Rightarrow A = \frac{9}{2} \quad and \ \ at \ \ x = -1 \ \Rightarrow B = -\frac{3}{2}$$

$$\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int \left( x - \frac{9/2}{x+3} + \frac{3/2}{x+1} \right) dx$$

$$= \frac{x^2}{2} - \frac{9}{2} ln(x+3) - \frac{3}{2} ln(x+1) + c$$

# Application of integrals

## Area between two curves:

Suppose that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ define two functions of $x$ that are continuous for $a \leq x \leq b$ then the area bounded above by the $y_1$ curve, below by $y_2$ curve and on the sides by the vertical lines $x = a$ and $x = b$ is:

$$A = \int_a^b [f_1(x) - f_2(x)]\,dx$$

**Example: Find the area bounded by the *x*-axis and the curve:**

$$y = 2x - x^2$$

Solution:

$$\left.\begin{array}{ll} y = 0 & \ldots\ldots\ldots\ldots(1) \\ y = 2x - x^2 & \ldots\ldots(2) \end{array}\right\} \Rightarrow x(x-2) = 0 \Rightarrow x = 0\,,2$$

The points of the intersection of the curve and the *x*-axis are (0,0) and (2,0) then the area bounded by *x*-axis and the curve is:

$$\int_0^2 (2x - x^2)\,dx = x^2 - \frac{x^3}{3}\Bigg|_0^2 = 4 - \frac{8}{3} - (0-0) = \frac{4}{3}$$

**Example: Find the area bounded by the *y*-axis and the curve:**

$$x = y^2 - y^3$$

**Solution:**

$$\left.\begin{array}{ll} x = 0 & \ldots\ldots\ldots\ldots(1) \\ x = y^2 - y^3 & \ldots\ldots(2) \end{array}\right\} \Rightarrow y^2(1-y) = 0 \Rightarrow y = 0\,,1$$

**Intersection points (0,0), (0,1)**

$$A=\int_0^1 (y^2-y^3)dy=\frac{y^3}{3}-\frac{y^4}{4}\Big|_0^1=\frac{1}{3}-\frac{1}{4}-(0-0)=\frac{1}{12}$$

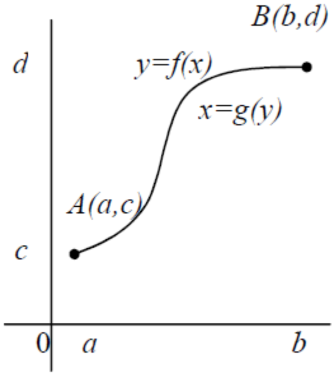
## The length of a plane curve:

The length of the curve y = f ( x ) from point A(a,c) to B(b,d) is:

$$L=\int_a^b \sqrt{1+(\frac{dy}{dx})^2}\ dx$$

If $x$ can be expressed as a function of $y$ then the length is:

$$L=\int_c^d \sqrt{1+(\frac{dx}{dy})^2}\ dy$$



Let the equation of motion be x = g( t ) and y = h( t ) continuously differentiable for t at A) and $t_b$ ( at B), then the length of the curve is:

a (

$$L=\int_{t_a}^{t_b} \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\ dt$$

**Example:**
**following**

$$y=\frac{1}{3}(x^2+2)^{\frac{3}{2}} \quad from \ \ x=0 \quad to \quad x=3$$

**Find the length of the curve:**

**Solution:**

$$y=\frac{1}{3}(x^2+2)^{\frac{3}{2}} \quad \Rightarrow \quad \frac{dy}{dx}=x(x^2+2)^{\frac{1}{2}}$$

$$L=\int_0^3 \sqrt{1+x^2(x^2+2)}\ dx=\int_0^3 (x^2+1)\ dx=\frac{x^3}{3}+x\Big|_0^3=9+3-0=12$$

**Example: Find the distance traveled between t = 0 and t = π/2 a particle P(x,y) whose position at time t is given by:**

**x = a cost + a.t sint and y = a sint – a.t cos t where a is a positive constant.**

**Solution:**

$$x = a\cos t + a\cdot t\sin t \quad \Rightarrow \quad \frac{dx}{dt} = a\cdot t\cos t$$

$$y = a\sin t - a\cdot t\cos t \quad \Rightarrow \quad \frac{dy}{dt} = a\cdot t\sin t$$

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt = \int_0^{\frac{\pi}{2}} \sqrt{a^2\cdot t^2\cos^2 t + a^2\cdot t^2\sin^2 t}\ dt$$

$$= a\int_0^{\frac{\pi}{2}} t\ dt = \frac{a}{2}t^2\Big|_0^{\frac{\pi}{2}} = \frac{a}{2}\left[\frac{\pi^2}{4} - 0\right] = \frac{a}{8}\pi^2$$

## The surface area:

Suppose that the curve y = f (x) is rotated about the *x*-axis. It will generate a surface in space. Then the surface area of the shape is:

$$S = \int_a^b 2\pi\ y\sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

If the curve rotated about the *y*-axis, then the surface area is:

$$S = \int_c^d 2\pi\ x\sqrt{1 + (\frac{dx}{dy})^2}\ dy$$

If the curve sweeps out the surface is given in parametric form with x and y as functions of a third variable t that varies from $t_a$ to $t_b$ then we may compute the surface area from the formula:

$$S = \int_{t_a}^{t_b} 2\pi\, \rho \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Where ρ is the distance from the axis of revolution to the element of arc length and is expressed as a function of t.

**Example: The circle $x^2 + y^2 = r^2$ is revolved about the x-axis. Find the area of the sphere generated.**

**Solution:**

$$y = \sqrt{r^2 - x^2} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}$$

$$S = \int_a^b 2\pi\, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-r}^{r} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = 2\pi\, r \int_{-r}^{r} dx$$

$$= 2\pi\, r\, x \Big|_{-r}^{r} = 2\pi\, r\big(r - (-r)\big) = 4\pi\, r^2$$

**Example: Find the area of the surface generated by rotating the curve**
**$x = t^2$, $y = t$, $0 \le t \le 1$ about the x-axis.**

**Solution:**

$$x = t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2t \qquad and \qquad y = t \quad \Rightarrow \quad \frac{dy}{dt} = 1$$

$$S = \int_{t_a}^{t_b} 2\pi \, \rho \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt = 2\pi \int_0^1 t\sqrt{4t^2 + 1} \, dt$$

$$= \frac{\pi}{4}\left[\frac{(4t^2 + 1)^{\frac{3}{2}}}{3/2}\right]_0^1 = \frac{\pi}{6}\left[5\sqrt{5} - 1\right]$$