Northern Technical University
Technical Engineering College / Mosul
Dept. of Power Mechanics Techniques Engineering
Class: 4th

Measurements
introduction

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Measurement and Instrumentation

Measurement

- Measurement is the art of assigning a specific value to a physical variable detected by a sensor.

- A measurement tells us about a property of something. It might tell us how heavy an object is, or how hot, or how long it is.

- The result of a measurement is normally in two parts: a number and a unit of measurement, e.g. ‘How long is it? ... 2 metres.’

Measurement and Instrumentation

<u>An example is weighing system</u>

- ➢the measurand is the weight of some object
- ➢the measurement is the number of units (ton, kilogram, gram, etc.) that represent the weight.

How do we measure

Measurand

Standard

Compare

Result

Comparison Device

Measurement and Instrumentation

☐ Examples of Measurements

What kinds of measurements did you make today?

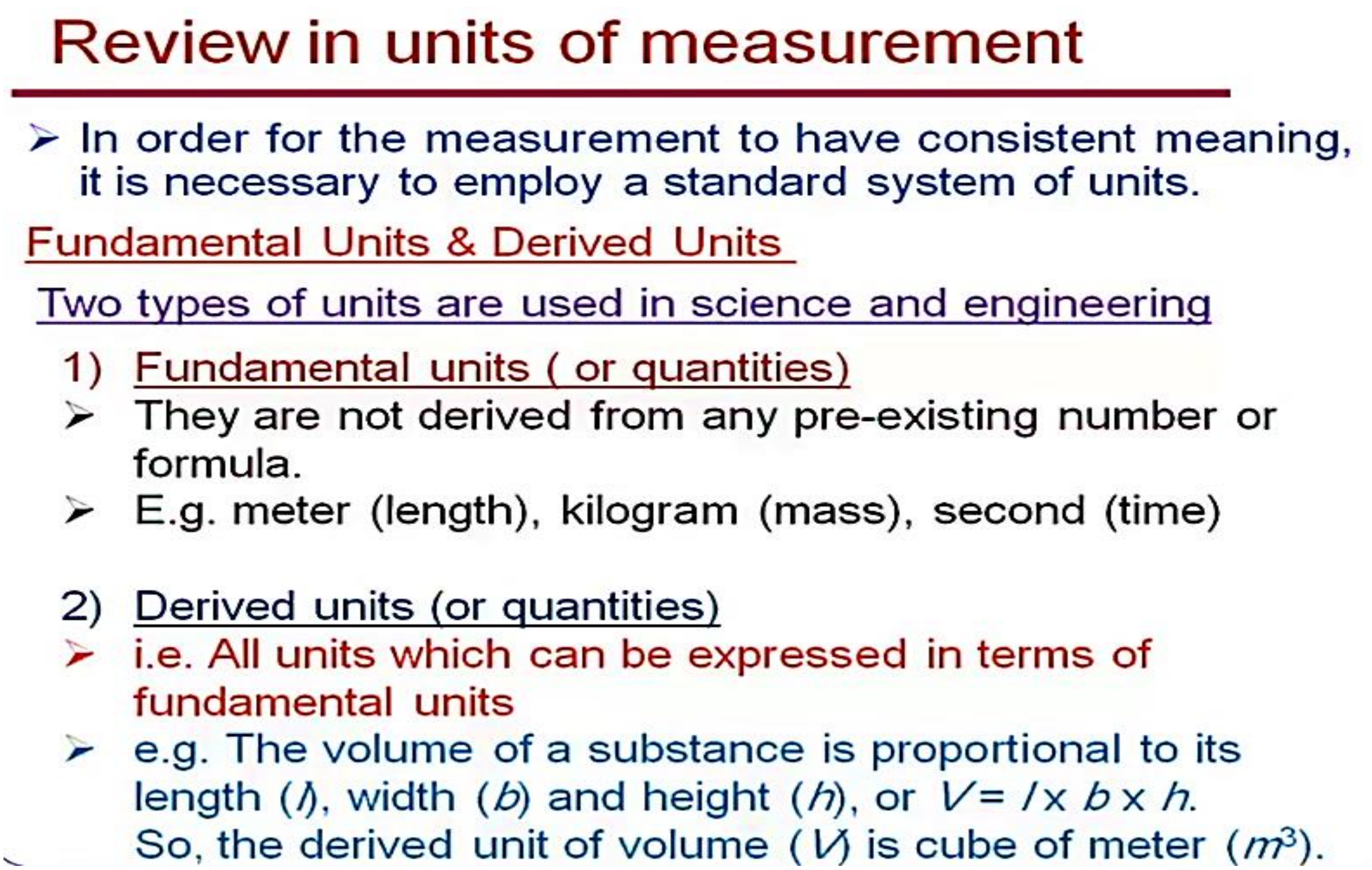
You are making a measurement when you:

- Check your weight.
- Read your watch.
- Take your temperature.
- Read the speed indicator of your car.

Significance of Measurement

☐ Why measurement? **☐ Significance of Measurements**

- To estimate the size/amount of things
- To monitor devices and industrial processes
- To control process and systems
- To verify laws of nature
- To establish standards
- To design and build systems
- To improve the quality of the product
- To improve the efficiency of production



Review in units of measurement

- In order for the measurement to have consistent meaning, it is necessary to employ a standard system of units.

Fundamental Units & Derived Units

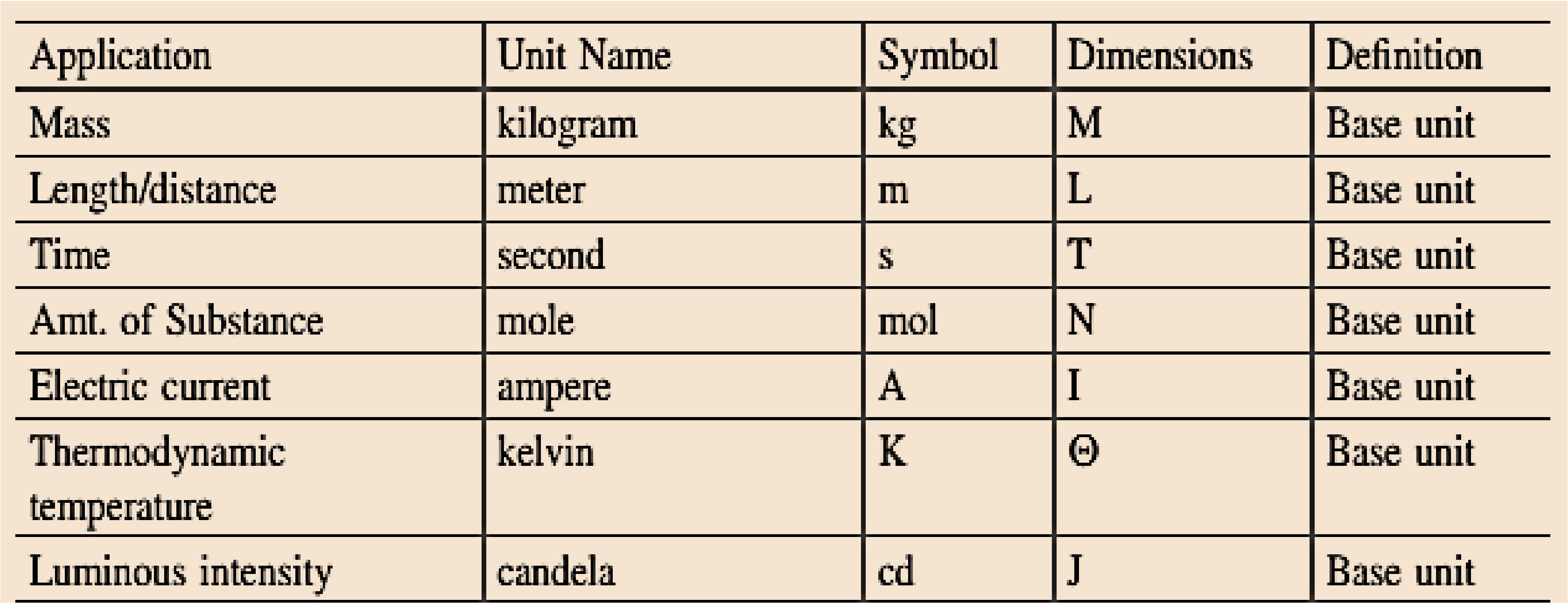
Two types of units are used in science and engineering

1) Fundamental units (or quantities)
   - They are not derived from any pre-existing number or formula.
   - E.g. meter (length), kilogram (mass), second (time)

2) Derived units (or quantities)
   - i.e. All units which can be expressed in terms of fundamental units
   - e.g. The volume of a substance is proportional to its length (l), width (b) and height (h), or $V = l \times b \times h$. So, the derived unit of volume (V) is cube of meter (m^3).

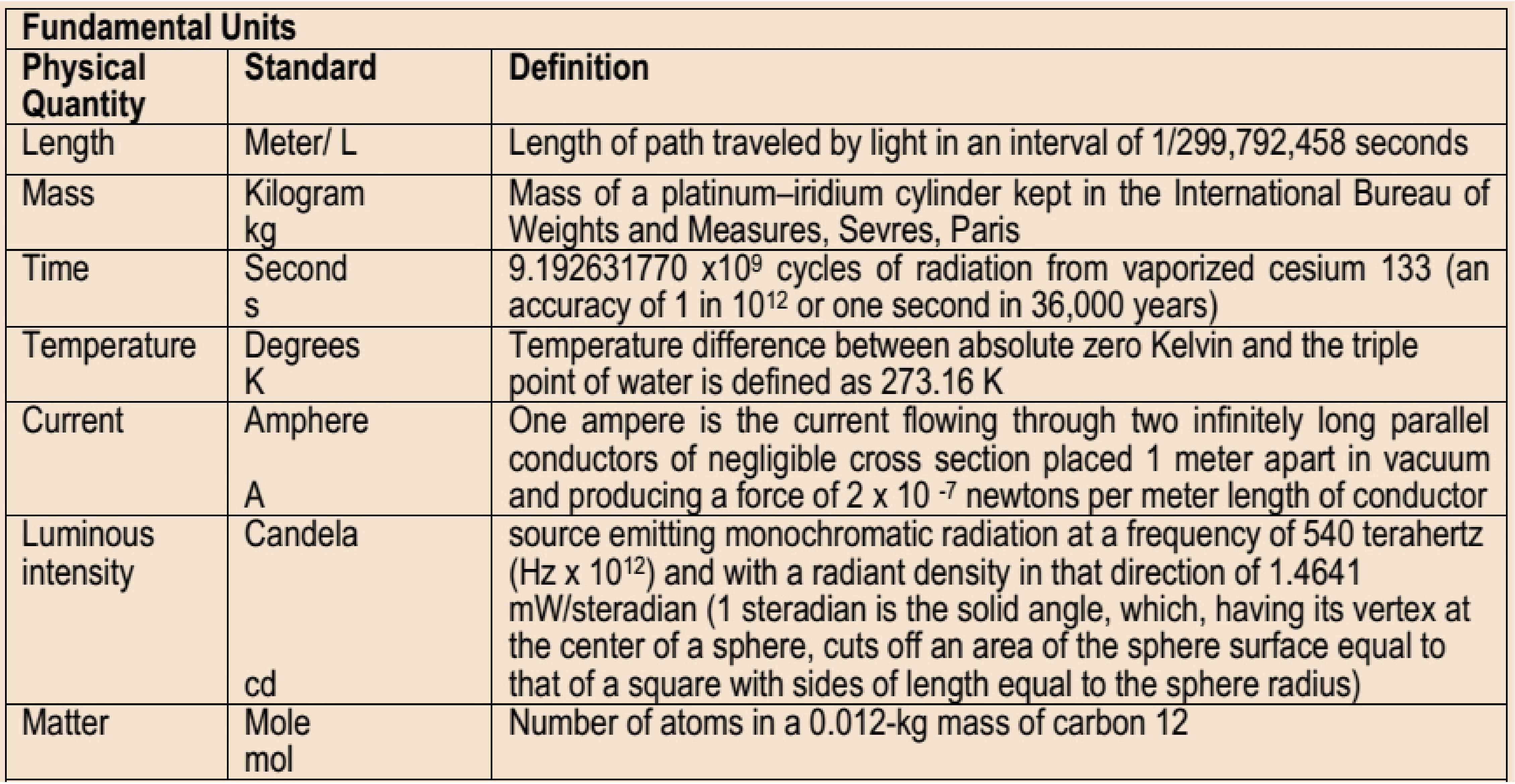
Derived and base SI metric units

Base units



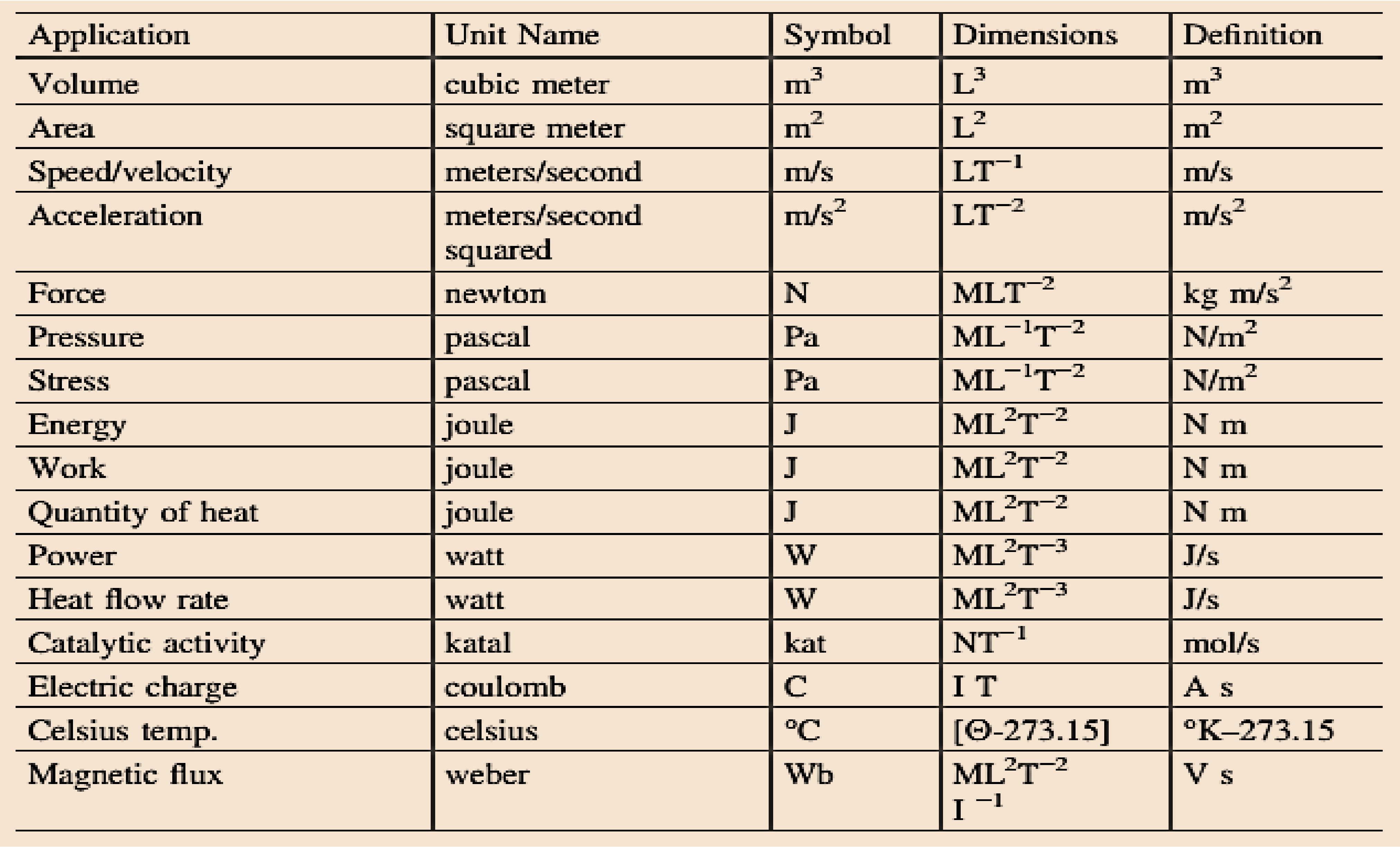
Application	Unit Name	Symbol	Dimensions	Definition
Mass	kilogram	kg	M	Base unit
Length/distance	meter	m	L	Base unit
Time	second	s	T	Base unit
Amt. of Substance	mole	mol	N	Base unit
Electric current	ampere	A	I	Base unit
Thermodynamic temperature	kelvin	K	Θ	Base unit
Luminous intensity	candela	cd	J	Base unit

Definitions of Standard Units



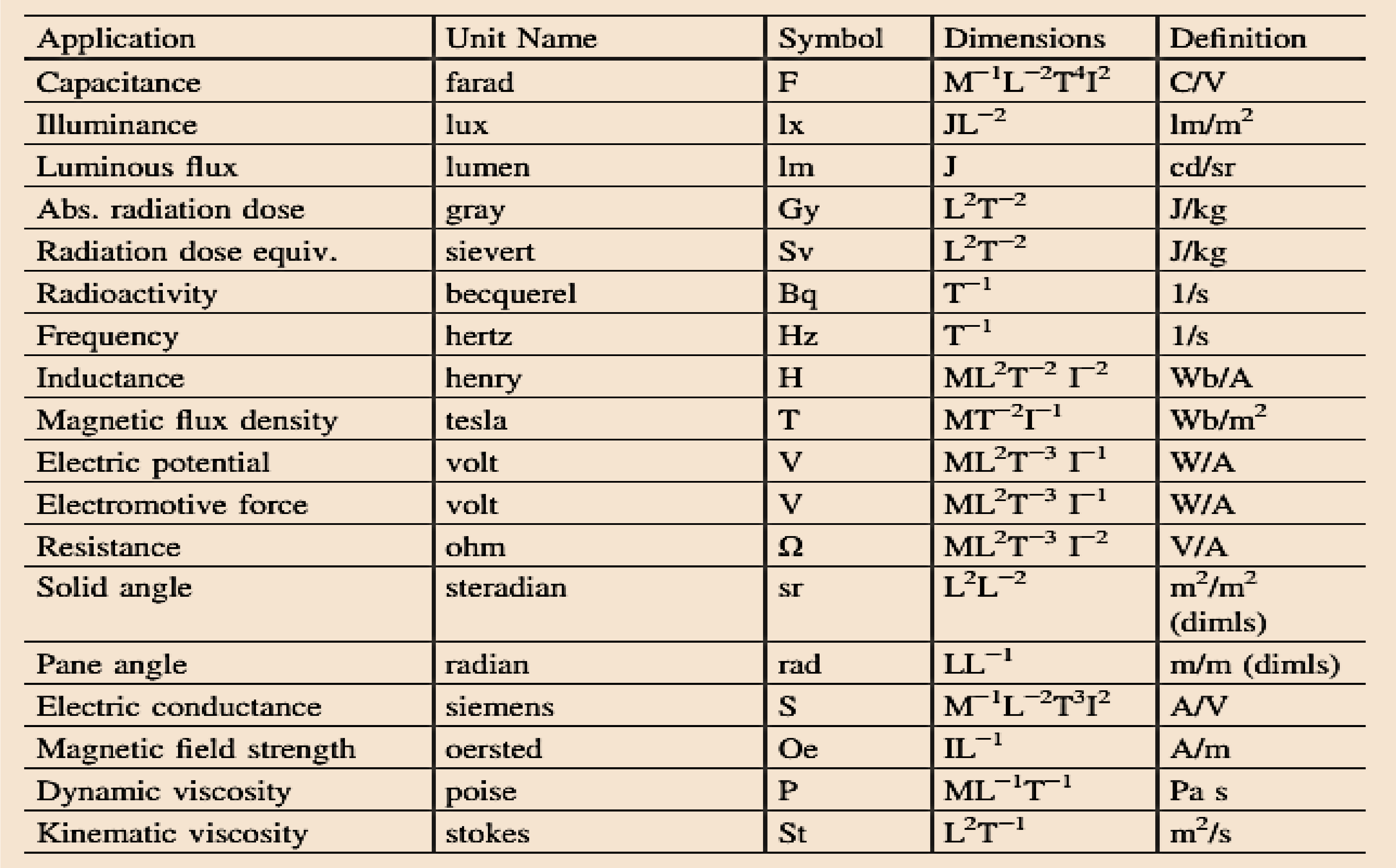
Fundamental Units		
Physical Quantity	**Standard**	**Definition**
Length	Meter/ L	Length of path traveled by light in an interval of 1/299,792,458 seconds
Mass	Kilogram kg	Mass of a platinum–iridium cylinder kept in the International Bureau of Weights and Measures, Sevres, Paris
Time	Second s	9.192631770 x10^9 cycles of radiation from vaporized cesium 133 (an accuracy of 1 in 10^{12} or one second in 36,000 years)
Temperature	Degrees K	Temperature difference between absolute zero Kelvin and the triple point of water is defined as 273.16 K
Current	Amphere A	One ampere is the current flowing through two infinitely long parallel conductors of negligible cross section placed 1 meter apart in vacuum and producing a force of 2 x 10^{-7} newtons per meter length of conductor
Luminous intensity	Candela cd	source emitting monochromatic radiation at a frequency of 540 terahertz (Hz x 10^{12}) and with a radiant density in that direction of 1.4641 mW/steradian (1 steradian is the solid angle, which, having its vertex at the center of a sphere, cuts off an area of the sphere surface equal to that of a square with sides of length equal to the sphere radius)
Matter	Mole mol	Number of atoms in a 0.012-kg mass of carbon 12

Derived units



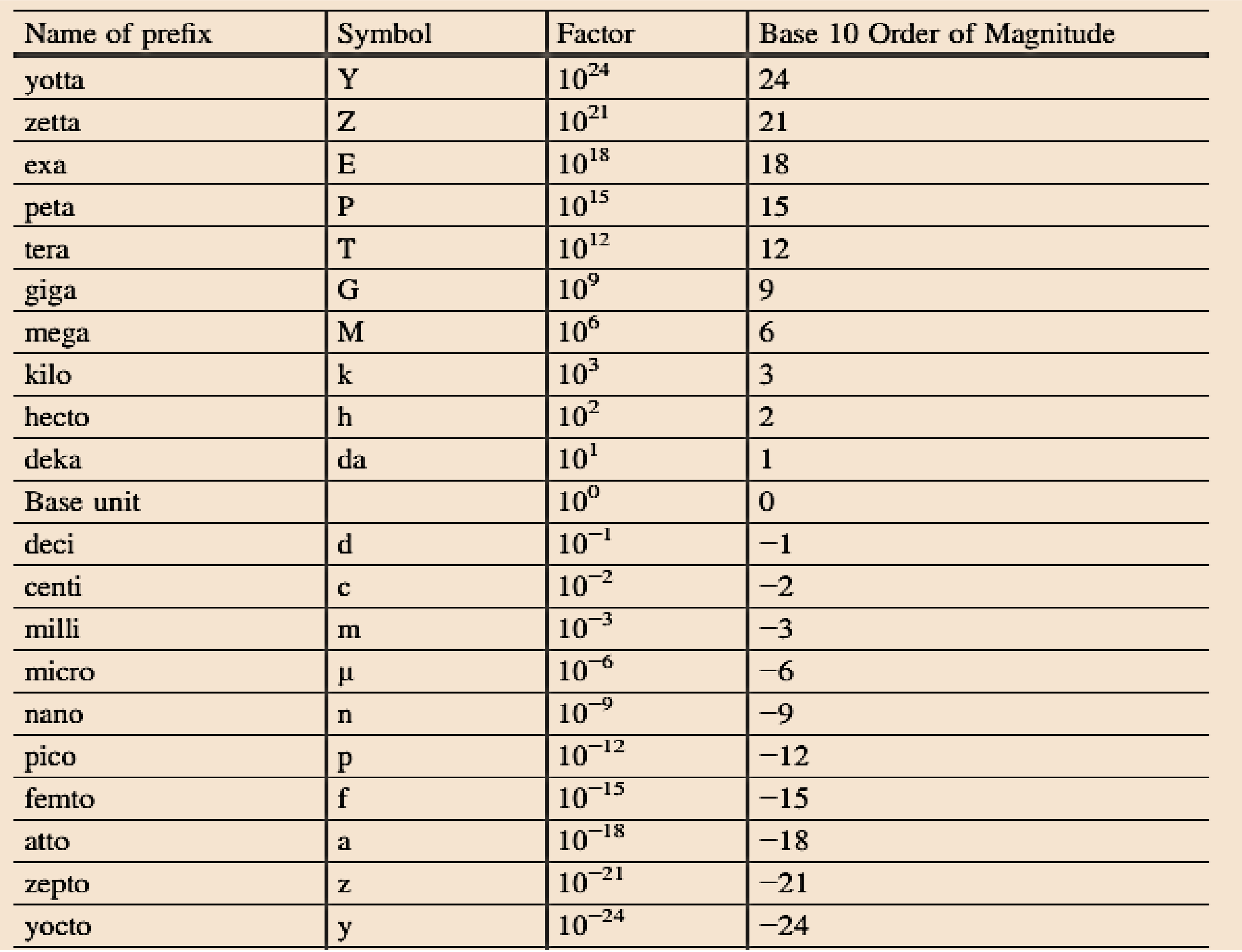
Application	Unit Name	Symbol	Dimensions	Definition
Volume	cubic meter	m^3	L^3	m^3
Area	square meter	m^2	L^2	m^2
Speed/velocity	meters/second	m/s	LT^{-1}	m/s
Acceleration	meters/second squared	m/s^2	LT^{-2}	m/s^2
Force	newton	N	MLT^{-2}	$kg\ m/s^2$
Pressure	pascal	Pa	$ML^{-1}T^{-2}$	N/m^2
Stress	pascal	Pa	$ML^{-1}T^{-2}$	N/m^2
Energy	joule	J	ML^2T^{-2}	N m
Work	joule	J	ML^2T^{-2}	N m
Quantity of heat	joule	J	ML^2T^{-2}	N m
Power	watt	W	ML^2T^{-3}	J/s
Heat flow rate	watt	W	ML^2T^{-3}	J/s
Catalytic activity	katal	kat	NT^{-1}	mol/s
Electric charge	coulomb	C	I T	A s
Celsius temp.	celsius	°C	[Θ-273.15]	°K–273.15
Magnetic flux	weber	Wb	$ML^2T^{-2}I^{-1}$	V s

Derived units (continued)

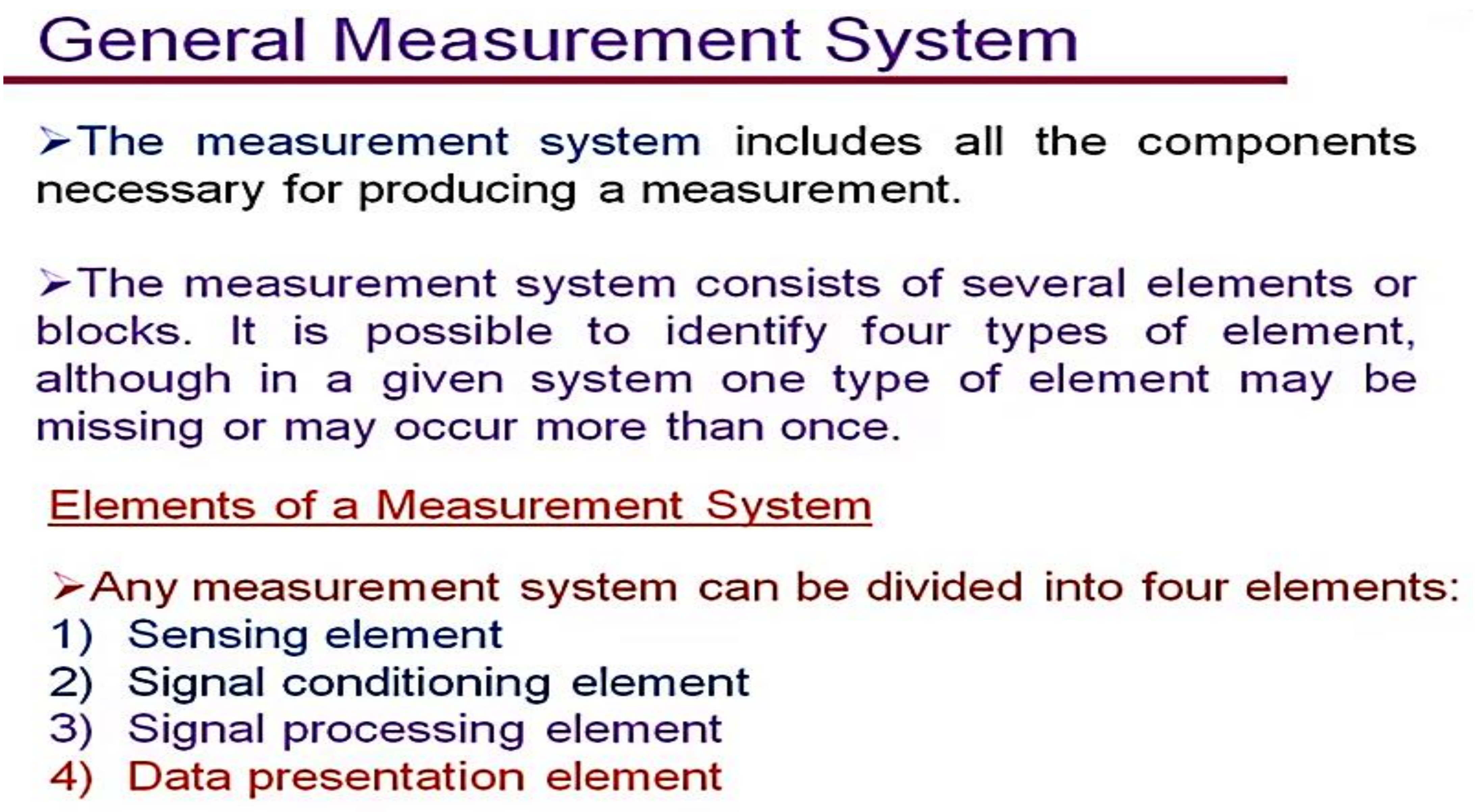


Application	Unit Name	Symbol	Dimensions	Definition
Capacitance	farad	F	$M^{-1}L^{-2}T^{4}I^{2}$	C/V
Illuminance	lux	lx	JL^{-2}	lm/m^2
Luminous flux	lumen	lm	J	cd/sr
Abs. radiation dose	gray	Gy	$L^{2}T^{-2}$	J/kg
Radiation dose equiv.	sievert	Sv	$L^{2}T^{-2}$	J/kg
Radioactivity	becquerel	Bq	T^{-1}	1/s
Frequency	hertz	Hz	T^{-1}	1/s
Inductance	henry	H	$ML^{2}T^{-2}\ I^{-2}$	Wb/A
Magnetic flux density	tesla	T	$MT^{-2}I^{-1}$	Wb/m^2
Electric potential	volt	V	$ML^{2}T^{-3}\ I^{-1}$	W/A
Electromotive force	volt	V	$ML^{2}T^{-3}\ I^{-1}$	W/A
Resistance	ohm	Ω	$ML^{2}T^{-3}\ I^{-2}$	V/A
Solid angle	steradian	sr	$L^{2}L^{-2}$	m^2/m^2 (dimls)
Pane angle	radian	rad	LL^{-1}	m/m (dimls)
Electric conductance	siemens	S	$M^{-1}L^{-2}T^{3}I^{2}$	A/V
Magnetic field strength	oersted	Oe	IL^{-1}	A/m
Dynamic viscosity	poise	P	$ML^{-1}T^{-1}$	Pa s
Kinematic viscosity	stokes	St	$L^{2}T^{-1}$	m^2/s

Prefixes for SI system units



Name of prefix	Symbol	Factor	Base 10 Order of Magnitude
yotta	Y	10^{24}	24
zetta	Z	10^{21}	21
exa	E	10^{18}	18
peta	P	10^{15}	15
tera	T	10^{12}	12
giga	G	10^{9}	9
mega	M	10^{6}	6
kilo	k	10^{3}	3
hecto	h	10^{2}	2
deka	da	10^{1}	1
Base unit		10^{0}	0
deci	d	10^{-1}	−1
centi	c	10^{-2}	−2
milli	m	10^{-3}	−3
micro	µ	10^{-6}	−6
nano	n	10^{-9}	−9
pico	p	10^{-12}	−12
femto	f	10^{-15}	−15
atto	a	10^{-18}	−18
zepto	z	10^{-21}	−21
yocto	y	10^{-24}	−24



General Measurement System

- The measurement system includes all the components necessary for producing a measurement.
- The measurement system consists of several elements or blocks. It is possible to identify four types of element, although in a given system one type of element may be missing or may occur more than once.

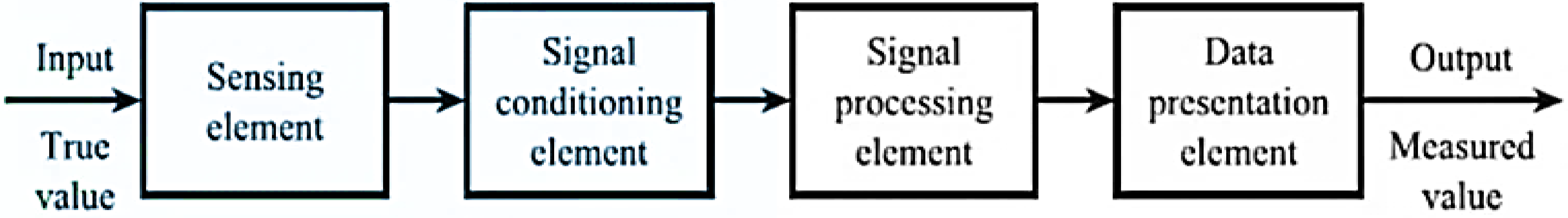
Elements of a Measurement System

- Any measurement system can be divided into four elements:

1) Sensing element
2) Signal conditioning element
3) Signal processing element
4) Data presentation element

General Measurement System

Elements of a Measurement System

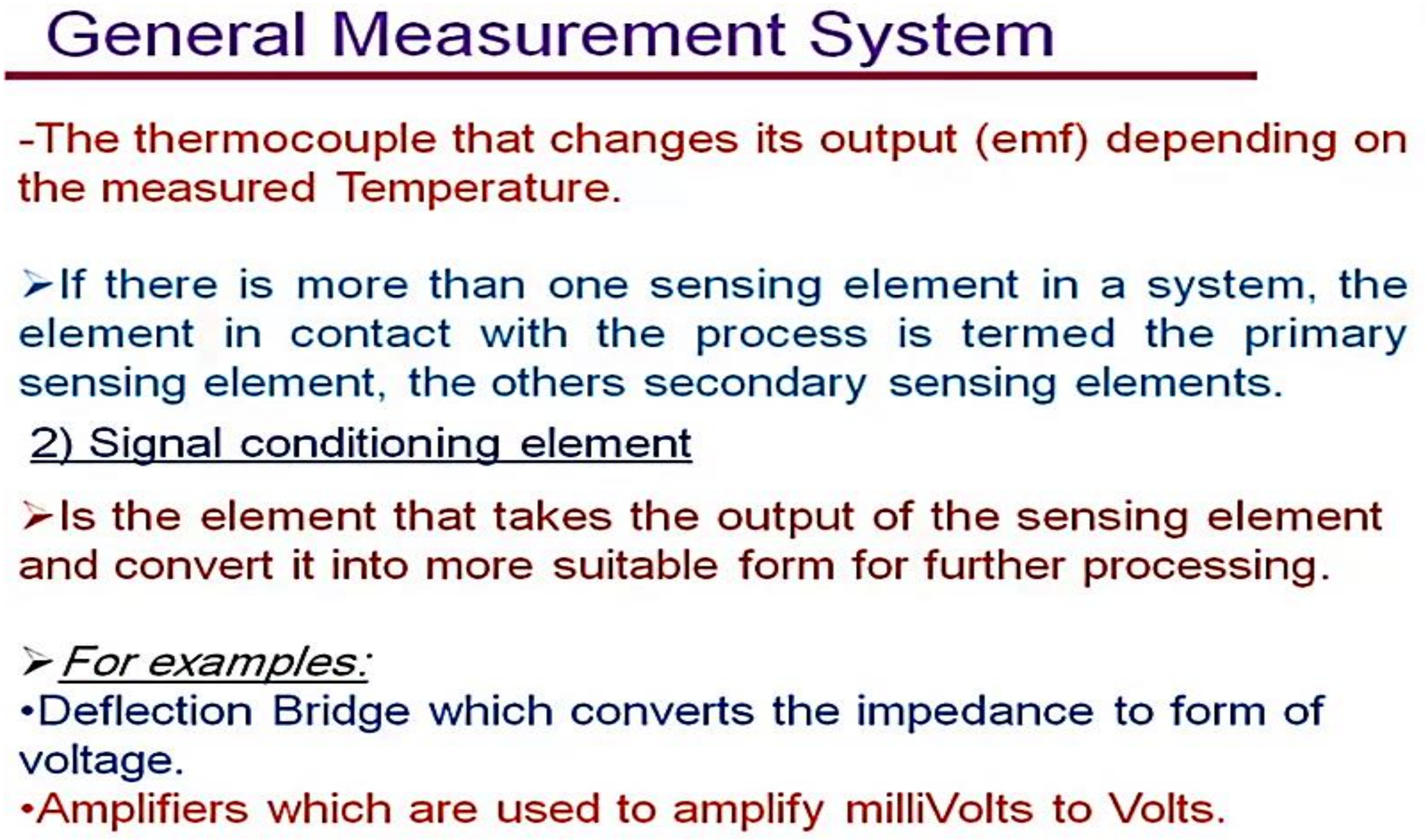


General Structure of Measuring System

1) Sensing element

- Is the element in direct contact with the process or variable being measured.
- *For examples:*

-The strain gauges where its resistance depends on the measured mechanical strain.



General Measurement System

-The thermocouple that changes its output (emf) depending on the measured Temperature.

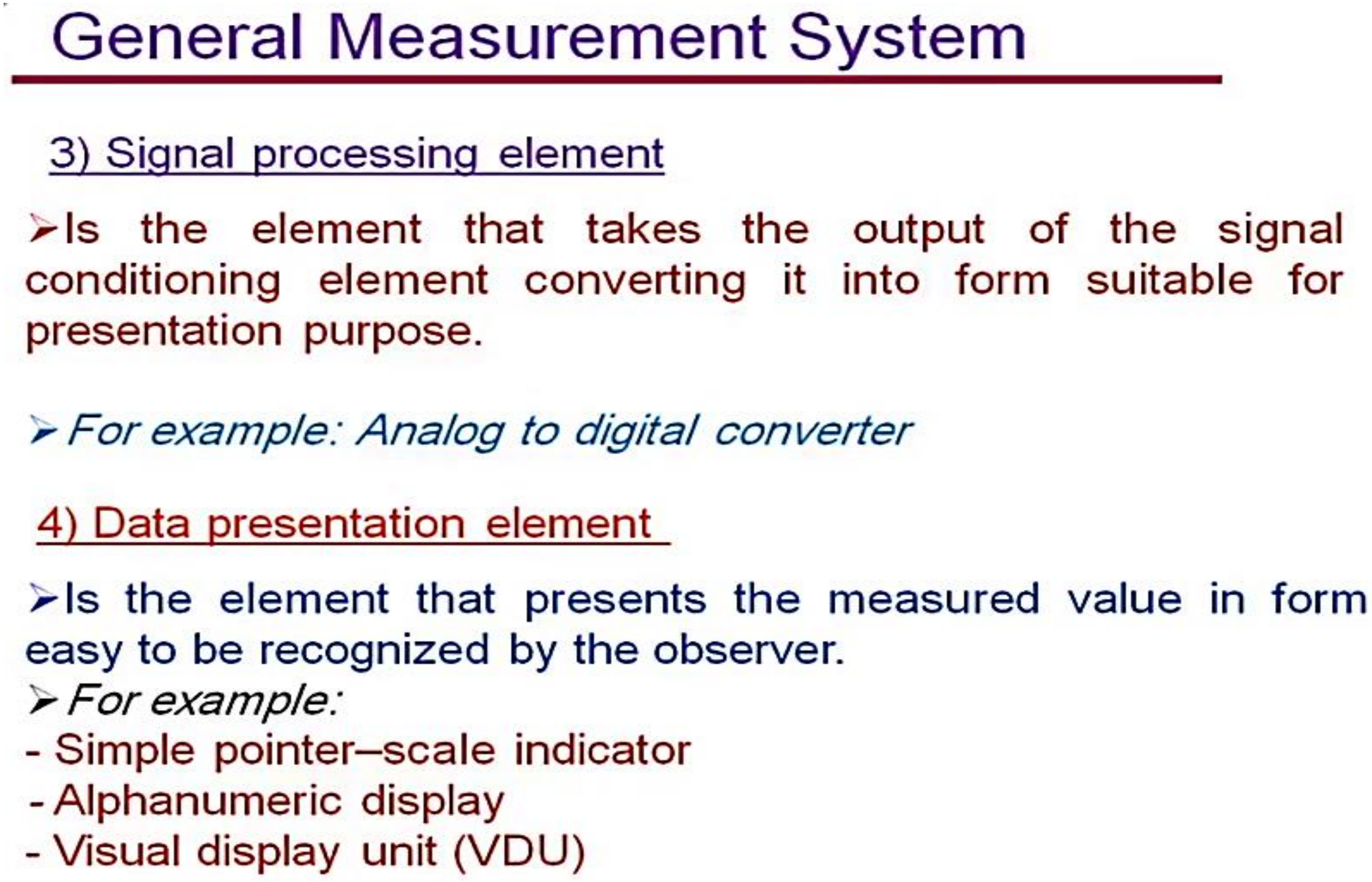
- If there is more than one sensing element in a system, the element in contact with the process is termed the primary sensing element, the others secondary sensing elements.

2) Signal conditioning element

- Is the element that takes the output of the sensing element and convert it into more suitable form for further processing.

- *For examples:*

- Deflection Bridge which converts the impedance to form of voltage.
- Amplifiers which are used to amplify milliVolts to Volts.



General Measurement System

3) Signal processing element

- Is the element that takes the output of the signal conditioning element converting it into form suitable for presentation purpose.

- *For example: Analog to digital converter*

4) Data presentation element

- Is the element that presents the measured value in form easy to be recognized by the observer.
- *For example:*

- Simple pointer–scale indicator
- Alphanumeric display
- Visual display unit (VDU)

Examples of measurement systems

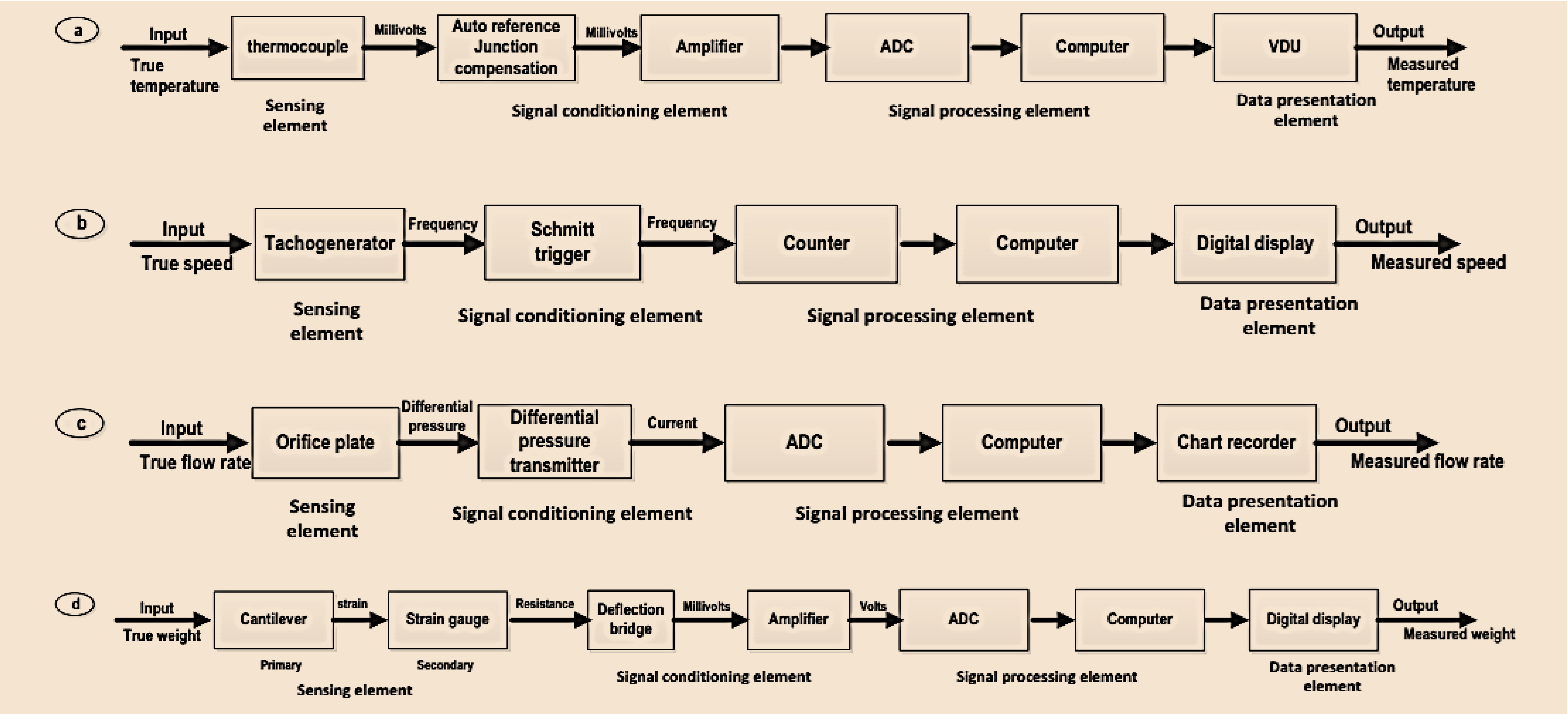
Figure below shows some typical examples of measurement systems. (a) shows a temperature system with a thermocouple sensing element; this gives a millivolt output. Signal conditioning consists of a circuit to compensate for changes in reference junction temperature, and an amplifier. The voltage signal is converted into digital form using an analogue-to-digital converter, the computer corrects for sensor non-linearity, and the measured value is displayed on a VDU.

(b) the speed of rotation of an engine is sensed by an electromagnetic tachogenerator which gives an A.C. output signal with frequency proportional to speed. The Schmitt trigger converts the sine wave into sharp-edged pulses which are then counted over a fixed time interval. The digital count is transferred to a computer which calculates frequency and speed, and the speed is presented on a digital display.

The flow system of (c) has an orifice plate sensing element; this gives a differential pressure output. The differential pressure transmitter converts this into a current signal and therefore combines both sensing and signal conditioning stages. The ADC converts the current into digital form and the computer calculates the flow rate, which is obtained as a permanent record on a chart recorder.

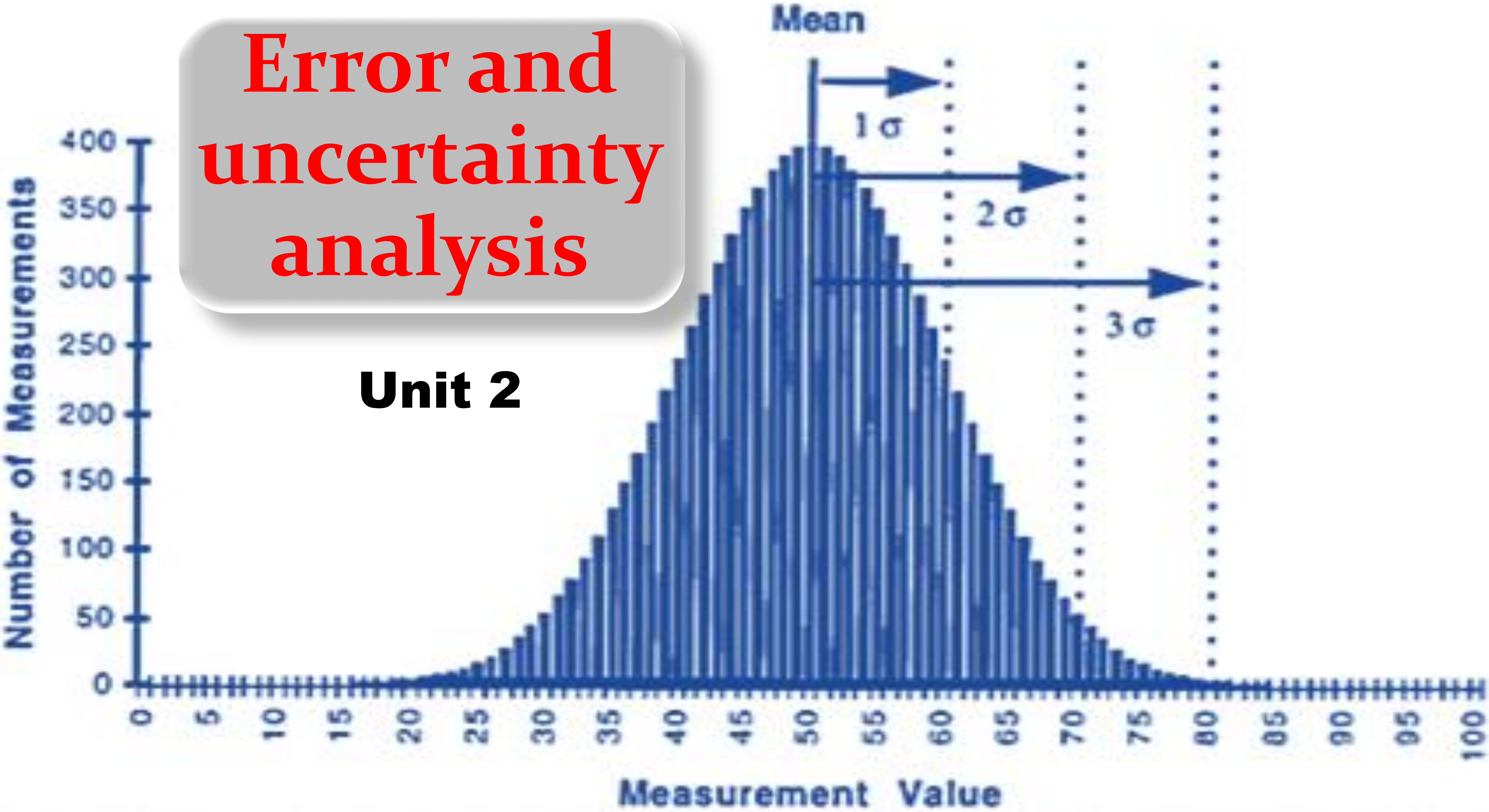
The weight system of (d) has two sensing elements: the primary element is a cantilever which converts weight into strain; the strain gauge converts this into a change in electrical resistance and acts as a secondary sensor. There are two signal conditioning elements: the deflection bridge converts the resistance change into millivolts and the amplifier converts millivolts into volts. The computer corrects for non-linearity in the cantilever and the weight is presented on a digital display.

Examples of measurement systems



Error and uncertainty analysis

Unit 2



Topics

- Errors in Measurements
- Accuracy and precision
- Sources of errors
- Types of errors
- Uncertainty analysis and Propagation of Error

Errors in Measurements

Introduction

- The purpose of any measurement is to describe some physical property of an object or a system quantitatively, ex: length, temperature, pressure,…

- All measured values are inaccurate to some degree. In fact, it is impossible to find the true value of a physical quantity.

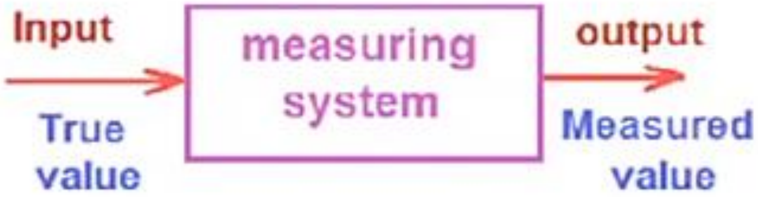
- If the measured value is very close to the true value, we call it to be a very accurate measuring system.

- But before using the measured data for further use, one must have some idea how accurate is the measured data. So error analysis is an integral part of measurement.

Errors in Measurements

❑ **Error:**

➢ Error is the difference between the measured value and the true value of the variable being measured.



❑ **True value:**

➢ Exact value of a variable (never known)

❑ **Measured value:**

➢ Value of a variable as indicated by measurement system.

Errors in Measurements

- The difference between the measured value (V_m) and the true value (V_t) of the quantity represents static error or absolute error of measurement (E_s), i.e.

$$\boxed{E_s = V_m - V_t}$$

☐ Relative Error (E_r):

- Estimate of error based on a reference value used in place of a known “true value” of a variable.

$$\boxed{E_r = \frac{|E_s|}{reference\ \ value} \times 100}$$

Errors in Measurements

- The error may be either positive or negative.

- For positive static errors the instrument reads high and for negative static errors the instrument reads low.

- From experimentalist's view point, static correction or simply correction (C_s) is more important than the static error.

☐ static correction (C_s):

- The difference between the true value and the measured value of a quantity.

$$\boxed{C_s = V_t - V_m = -E_s}$$

Errors in Measurements

- Example 1:

➢ A voltmeter reads 112.68 V. If the true value of the voltage is 112.6 V, determine the following:

i) The static error

ii) The static correction for the voltmeter

Solution :

i) The static error

$$\Theta E_s = V_m - V_t = 112.68 - 112.6 = +0.08V$$

ii) The static correction for the voltmeter

$$\Theta C_s = V_t - V_m = -E_s = -0.08V$$

Errors in Measurements

❑Example 2:

- A thermometer reads 92.35 °C and the static correction given in the correction curves is -0.07 °C. Determine the true value of the temperature.

Solution :

The true value (V_t)

$$\Theta E_s = V_m - V_t$$

$$\therefore C_s = V_t - V_m$$

$$\therefore V_t = V_m + C_s = 92.35 - 0.07 = 92.28^\circ C$$

Accuracy and precision

- The terms precision and accuracy are used in connection with the performance of the instrument.

☐ Accuracy:

- Closeness of agreement between the measured value and the true value.
- Accuracy is the ability of an instrument to show the exact reading.
- Less the error more accurate is the measurement.

Accuracy and precision

❑ Precision:

- **Precision** is the repeatability of the measuring process.
- Repeatability is the ability of a measurement system to indicate the same value on repeated measurements for a specific input value.
- Precision refers to the degree of agreement within a group of measurements or instruments.

Accuracy and precision

- Distinction between Accuracy and Precision

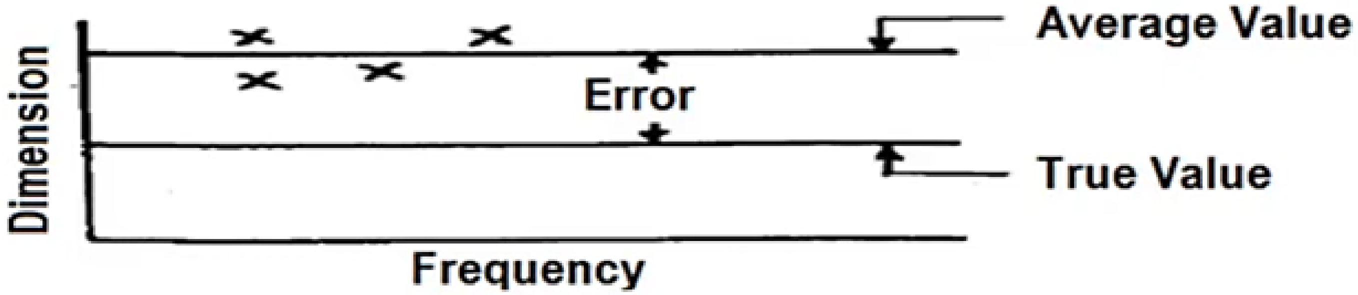
Low Accuracy
High Precision

High Accuracy
Low Precision

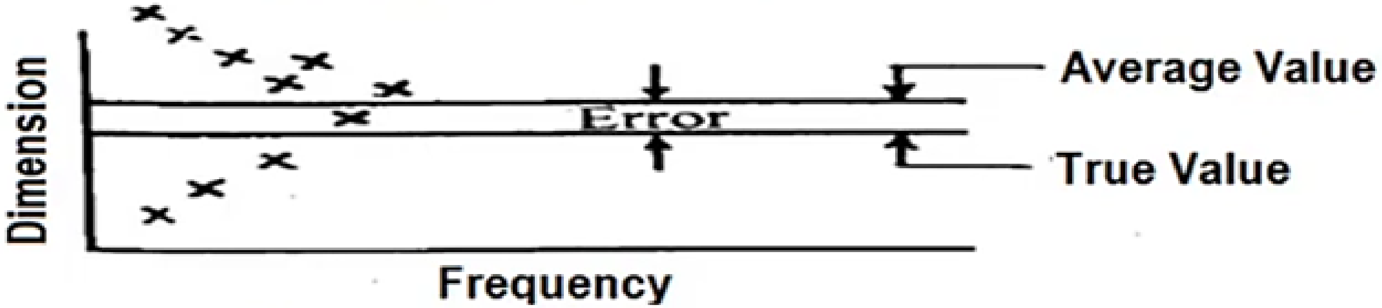
High Accuracy
High Precision

Accuracy and precision

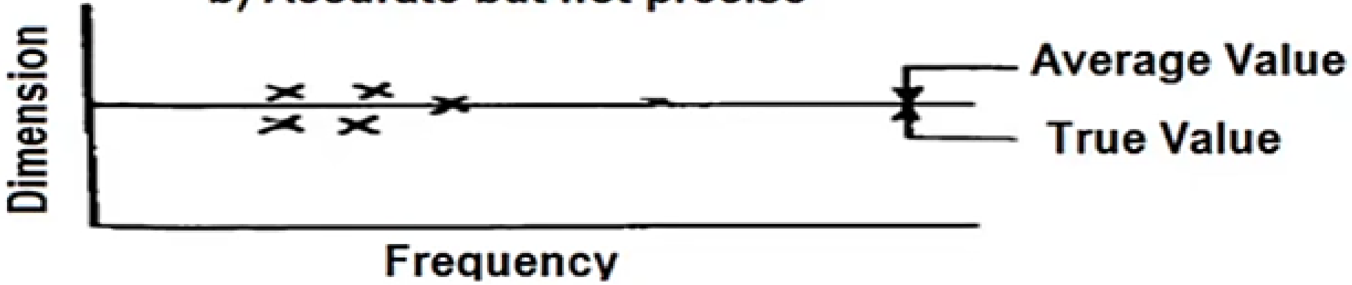
➢ Distinction between Accuracy and Precision



a) Precise but not accurate



b) Accurate but not precise



c) Accurate and precise

Accuracy and precision

❑Example 3:

- Two pressure gauges (pressure gauge A and B) have a full scale accuracy of ± 5%. Sensor (A) has a range of 0-1 bar and Sensor (B) 0-10 bar. Which gauge is more suitable to be used if the reading is 0.9 bar?

Solution :

Sensor A :

$$\text{Equipment max error} = \pm \frac{5}{100} \times 1 \text{ bar} = \pm 0.05 \text{ bar}$$

$$\text{Equipment accuracy at 0.9 bar (\%)} = \pm \frac{0.05 \text{ bar}}{0.9 \text{ bar}} \times 100 = \pm 5.6\%$$

Accuracy and precision

Solution :

Sensor B :

$$\text{Equipment max error} = \pm \frac{5}{100} \times 10 \text{ bar} = \pm 0.5 \text{ bar}$$

$$\text{Equipment accuracy at 0.9 bar (\%)} = \pm \frac{0.5 \text{ bar}}{0.9 \text{ bar}} \times 100 = \pm 55\%$$

Conclusion :

- Sensor A is more suitable to use at a reading of 0.9 bar because the error percentage (± 5.6%) is smaller compared to the percentage error of Sensor B (± 55%).

Sources of errors

There are different sources of errors include:

1.Defect in instrument.

2. Instrument calibration

-Due to frequent use of a measuring instrument and also of aging, the instrument may go out of calibration. Which leads to systematic error. Therefore the instrument should be sent for calibration at frequent intervals.

3. Instrument reproducibility

- Even if the instrument has been calibrated under a set of conditions, there may still be error in measurement due to the difference in the calibration conditions.

Sources of errors

4. Imperfection in design of instrument.
5. Environmental effects.
6. Error due to properties of object.
7. Error due to surface finish of object.
8. Observational error.

Types of errors

- The errors in instrument readings may be classified in to three categories as:

1. Gross errors
2. Systematic errors
3. Random errors.

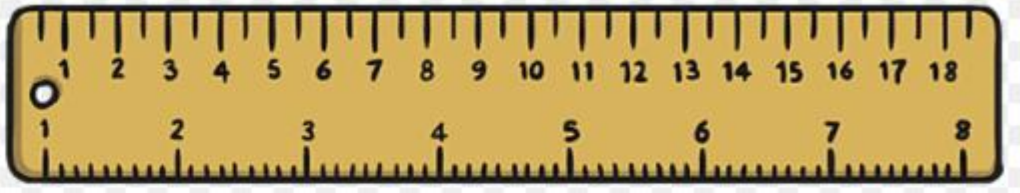
1) Gross errors

- These are basically human errors caused by the operator or person using the instrument.

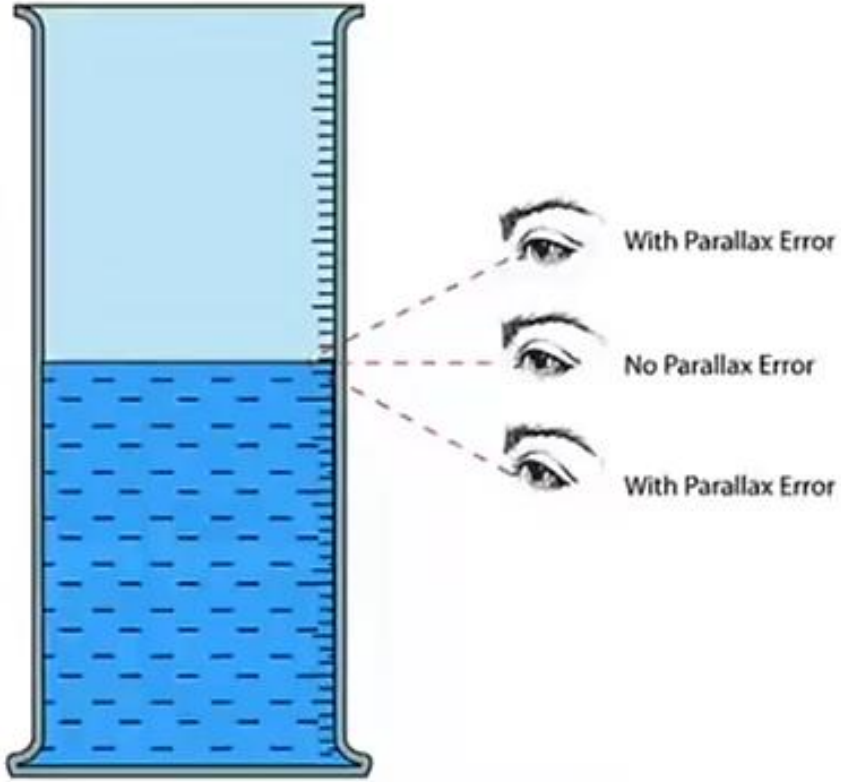
- The instrument may be good and may not give any error but still the measurement may go wrong due to the operator.

Types of errors

1) Gross errors



- The different types of gross errors are:



1. Observational errors.
2. Reading with parallax error.
3. Incorrect adjustments of zero and full-scale adjustments.
4. Improper applications of instruments (i.e. using a 0–100 V voltmeter to measure 0.1 V, etc)

Types of errors

2) Systematic errors (Bias errors)

- The portion of the absolute error that remains constant during repeated measurements (i.e. measured values have similar deviation from correct value).

- May result from shortcomings of the instruments, such as defective or worn parts, and effects of the environment on the equipment or the user.

- They are sometimes called bias due to error in one direction- high or low.

- Can be result from mis-calibrated device and loading errors.

- This kind of error can be eliminated by calibration.

Types of errors

3) Random errors (precision error)

- Random errors are distinguishable by their lack of consistency (i.e. measured values deviate randomly around mean value).

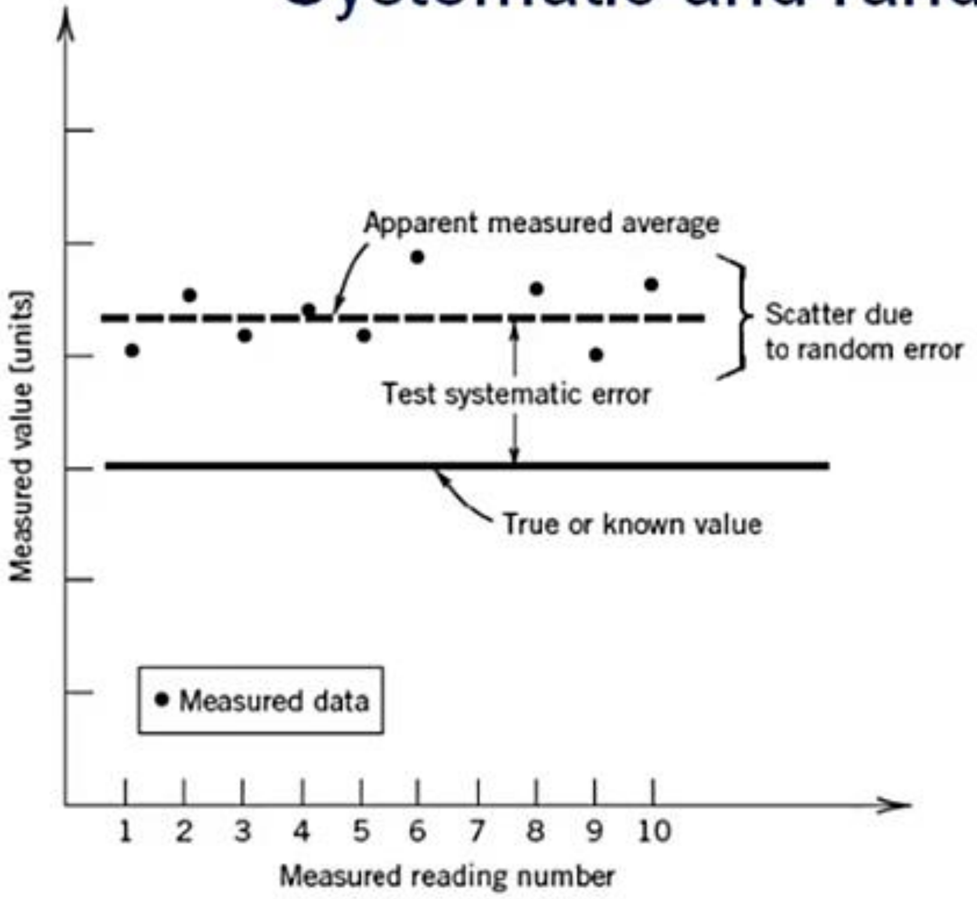
- The variations in temperature, vibrations of external medium, etc. cause errors in the instrument.

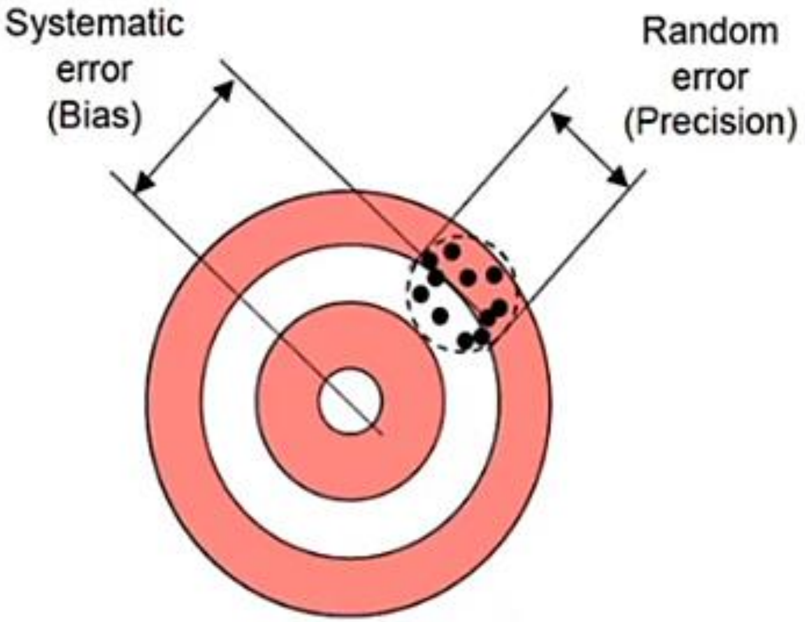
- Errors of this type are normally of limited duration and are inherent to specific environment.

- *Random errors can never be corrected, they can only be reduce.*

Types of errors

Systematic and random error





Types of errors

Overall error (u)

- Overall error represents the combination of all known errors.

$$u_c = [e_1^2 + e_2^2 + e_3^2 + \text{------} + e_M^2]^{1/2}$$

- Example: an instrument with known hysteresis, linearity, and sensitivity errors has the following total instrument error.

$$u_c = [e_h^2 + e_L^2 + e_K^2]^{1/2}$$

Types of errors

❑Example 4:

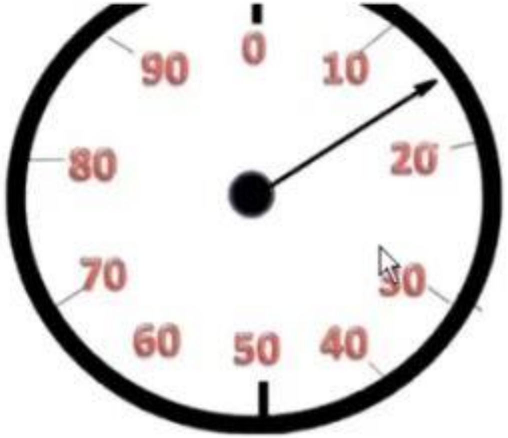
➢ A speedometer with resolution 5 mph (8 kph) and accuracy within ±4% reading. Find: the design-stage uncertainty (u_d) at 60 mph (90 kph) .

Solution :

$$u_c = [e_1^2 + e_2^2 + e_3^2 + \text{------} + e_M^2]^{1/2}$$

$$\therefore u_c = u_d = \pm[e_R^2 + e_A^2]^{1/2}$$

$$u_R = \pm[\frac{1}{2} \times \text{Re}\,solution]$$



Types of errors

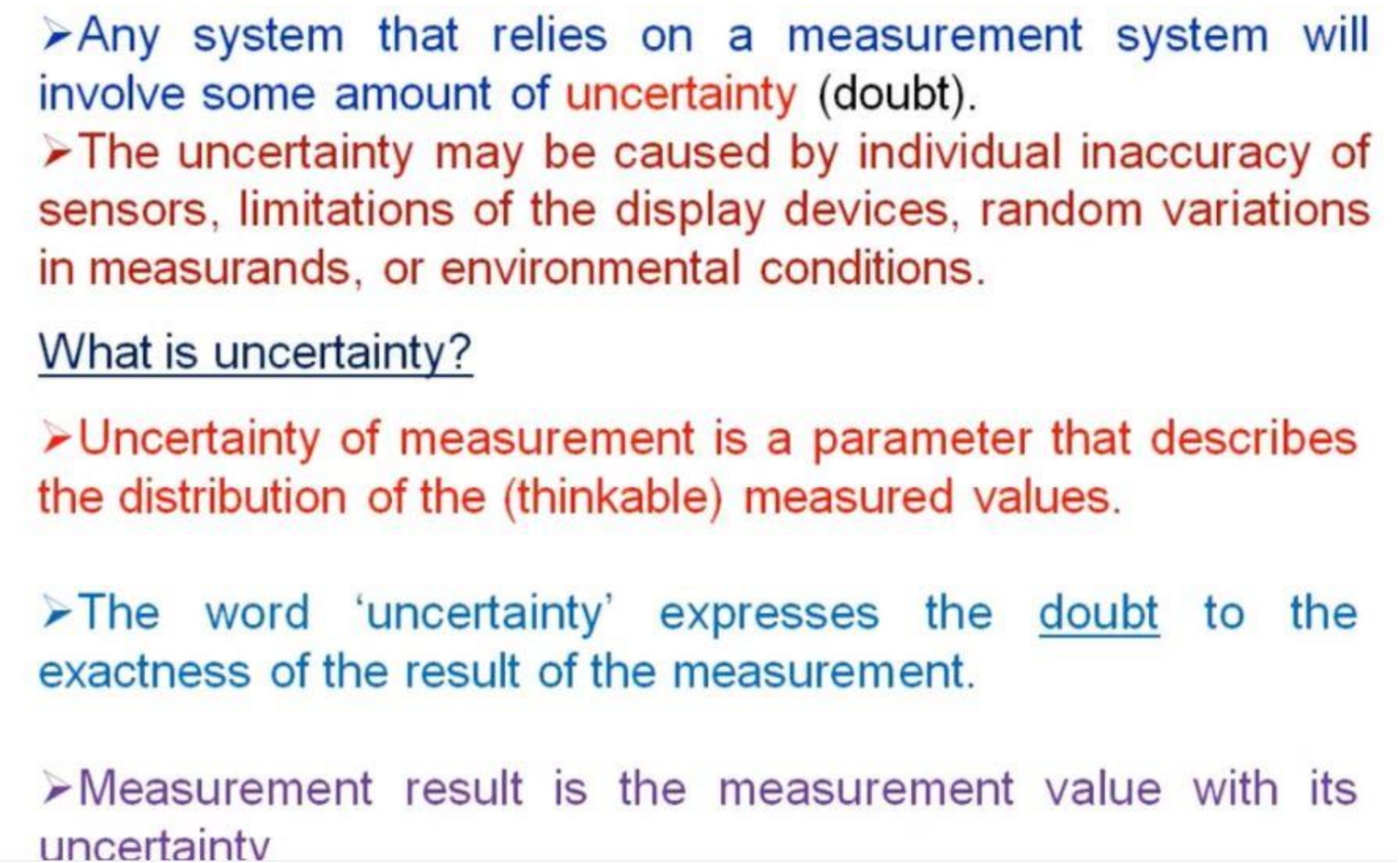
Solution :

$$\therefore u_R = \pm[\frac{1}{2} \times 5] = \pm 2.5 \quad mph$$

$$\Theta\ u_A = \pm 4\%\ reading \quad = \pm \frac{4}{100} \times 60 \ = \pm 2.4\, mph$$

$$\therefore u_c = u_d = \pm[(2.5)^2 + (2.4)^2]^{1/2} = \pm 3.5\, mph$$

Uncertainty analysis and propagation of error



- Any system that relies on a measurement system will involve some amount of uncertainty (doubt).
- The uncertainty may be caused by individual inaccuracy of sensors, limitations of the display devices, random variations in measurands, or environmental conditions.

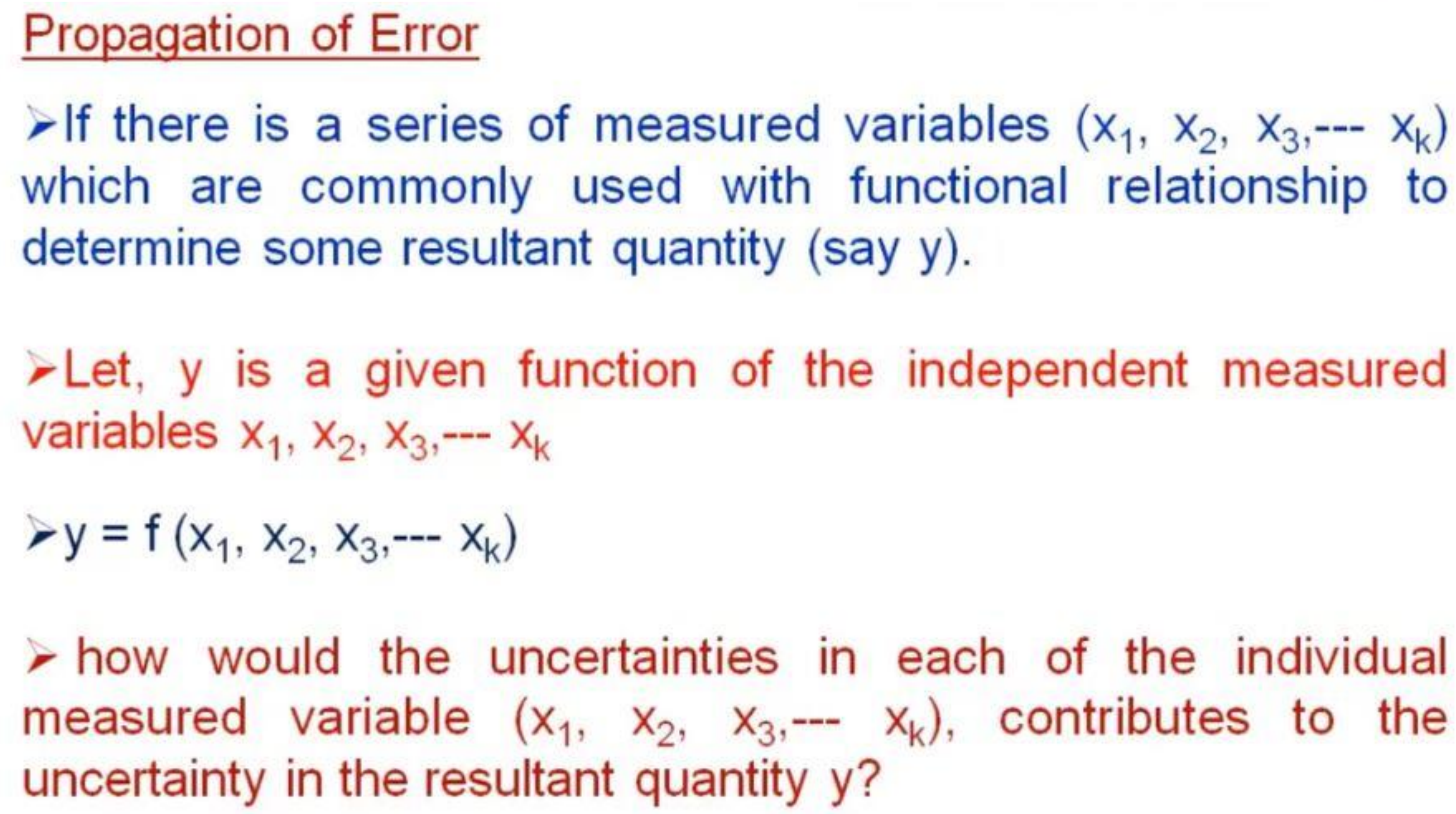
What is uncertainty?

- Uncertainty of measurement is a parameter that describes the distribution of the (thinkable) measured values.

- The word 'uncertainty' expresses the doubt to the exactness of the result of the measurement.

- Measurement result is the measurement value with its uncertainty

Uncertainty analysis and propagation of error



Propagation of Error

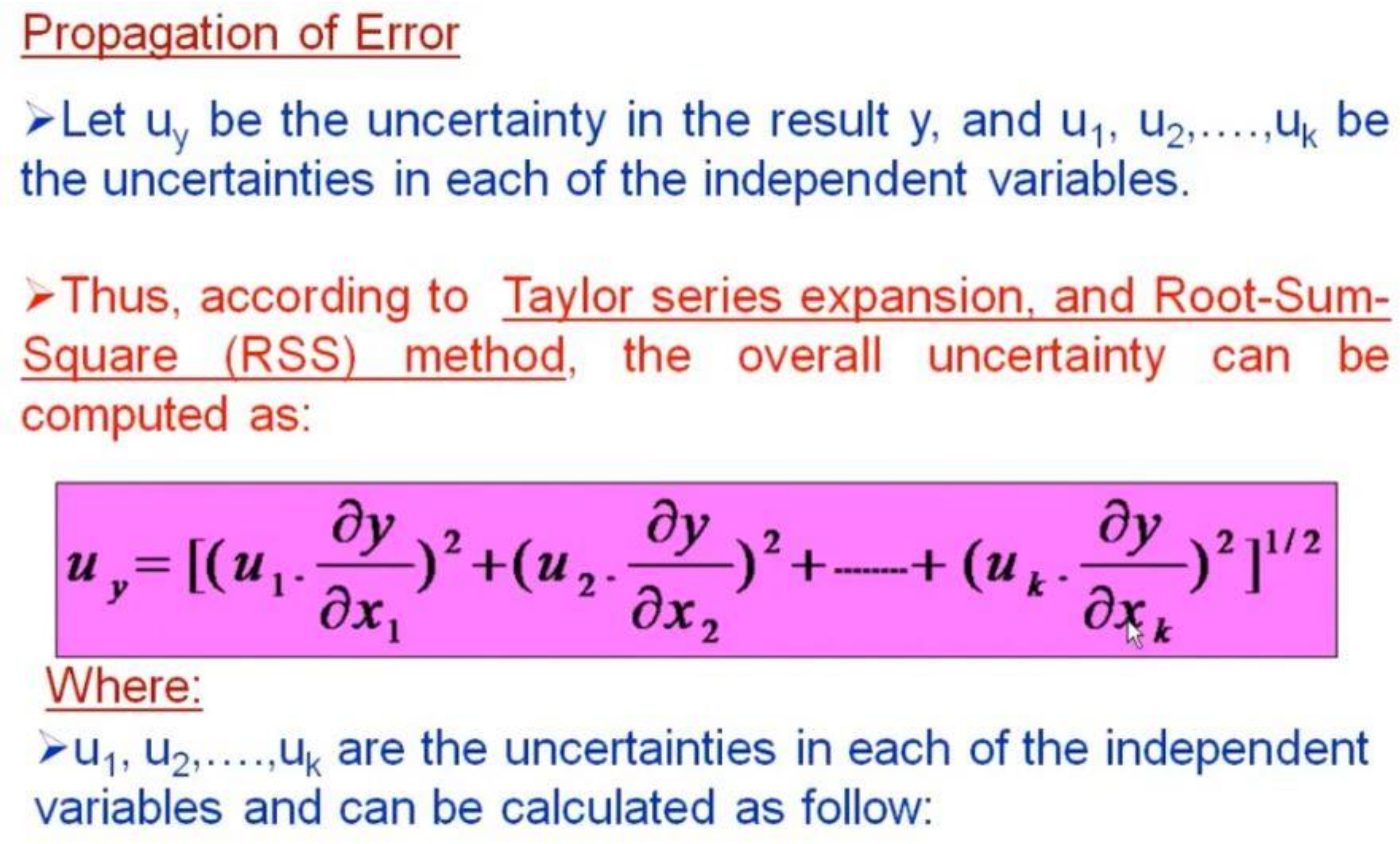
- If there is a series of measured variables (x_1, x_2, x_3,--- x_k) which are commonly used with functional relationship to determine some resultant quantity (say y).

- Let, y is a given function of the independent measured variables x_1, x_2, x_3,--- x_k

- $y = f(x_1, x_2, x_3, \text{---}\ x_k)$

- how would the uncertainties in each of the individual measured variable (x_1, x_2, x_3,--- x_k), contributes to the uncertainty in the resultant quantity y?

Uncertainty analysis and propagation of error



Propagation of Error

- Let u_y be the uncertainty in the result y, and $u_1, u_2, \ldots, u_k$ be the uncertainties in each of the independent variables.

- Thus, according to Taylor series expansion, and Root-Sum-Square (RSS) method, the overall uncertainty can be computed as:

$$\boxed{u_y = \left[\left(u_1 \cdot \frac{\partial y}{\partial x_1}\right)^2 + \left(u_2 \cdot \frac{\partial y}{\partial x_2}\right)^2 + \cdots + \left(u_k \cdot \frac{\partial y}{\partial x_k}\right)^2\right]^{1/2}}$$

Where:

- $u_1, u_2, \ldots, u_k$ are the uncertainties in each of the independent variables and can be calculated as follow:

Uncertainty analysis and propagation of error

$$u_i = \sigma = \sqrt{\frac{\sum_1^n (x_i - \bar{x})^2}{n-1}}$$

$$\bar{x} = \frac{\sum_1^n x_i}{n}$$

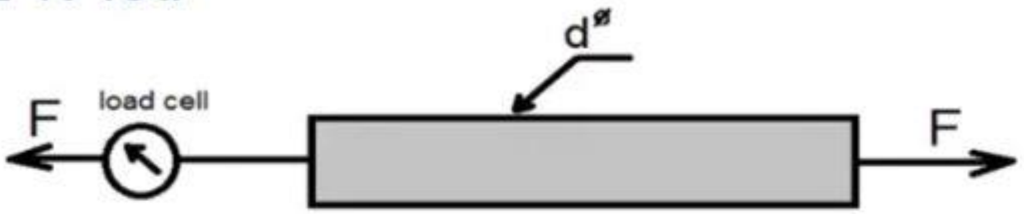
- The Root-Sum-Square (RSS) method of combining uncertainties is based on the following assumptions:

1. Individual uncertainties are following the Normal distribution
2. Each individual uncertainty is estimated at the same probability level
3. All individual uncertainties are consistent in units

Uncertainty analysis and propagation of error

❑Example 5:

➤ Calculate the uncertainty (u_σ) in stress (σ) for the shown steel rod having a length of L = 50 mm, and diameter (d mm), if the indicated reading of the load cell (0 : 2500 N capacity), F = 1500 N ± 0.2 % fsd



Measurements for the rod diameters (d mm) were as follows:

d: 5.21 5.18 5.25 5.26 5.22 5.19 5.23 5.24 5.18 5.26

Uncertainty analysis and propagation of error

Solution :

$$\sigma = \frac{F}{A} = \frac{F}{\frac{\pi}{4} d^2} = \frac{4F}{\pi d^2}$$

$$\bar{d} = \frac{\sum_1^n d_i}{n} = \frac{\sum_1^{10} d_i}{10} = \frac{52.22}{10} = 5.222 \ mm$$

$$u_d = \sqrt{\frac{\sum_1^n (d_i - \bar{d})^2}{n-1}} = \pm 0.0311 \ mm$$

F = 1500 N ± 0.2 % fsd

u_F = ± (0.2/100)x2500 = ±5 N

Uncertainty analysis and propagation of error

To find overall uncertainty:

$$u_\sigma = [(u_F \cdot \frac{\partial \sigma}{\partial F})^2 + (u_d \cdot \frac{\partial \sigma}{\partial d})^2]^{1/2}$$

$$\frac{\partial \sigma}{\partial F} = \frac{4}{\pi \bar{d}^2} = \frac{4}{\pi (5.222)^2} = 0.047 \; mm^{-2}$$

$$\frac{\partial \sigma}{\partial d} = \frac{4 \bar{F}}{\pi} x - 2 \bar{d}^{-3} = \frac{-8 x 1500}{\pi (5.222)^3} = -26.823 \; N / mm^3$$

$$\therefore u_\sigma = [(5x0.047)^2 + (0.0311x - 26.823)^2]^{1/2} = \pm 70.03 N / mm^2$$

3. Statistical analysis of experimental data

Statistical analysis and best estimate from replicate data:

- ❒ Let a certain quantity X be measured repeatedly to get

$$X_i,\ i=1,n \qquad \textbf{(1)}$$

- ❒ Because of random errors these are all *different*.
- ❒ How do we find the best estimate X_b for the true value of X?
- ❒ It is reasonable to assume that the best value be such that the measurements are as precise as they can be!
- ❒ In other words, the experimenter is confident that he has conducted the measurements with the best care and he is like the skilled shooter in the target practice example presented earlier!
- ❒ Thus, we minimize the variance with respect to the best estimate X_b of X.
- ❒ Thus we minimize:

$$S = \sum_{i=1}^{n} \left[X_i - X_b \right]^2 \qquad \textbf{(2)}$$

- ❒ This requires that:

$$\frac{\partial S}{\partial X_b} = 2\sum_{i=1}^{n}[X_i - X_b]\,(-1) = 0$$

$$\text{or } X_b = \frac{\sum_{i=1}^{n} X_i}{n} \qquad (3)$$

- ❒ The best estimate is thus nothing but the mean of all the individual measurements!

Error distribution:

When a quantity is measured **repeatedly** it is expected that it will be distributed around the best value according to some **distribution**. Many times the random errors may be distributed as a **normal distribution**. If μ and σ are, respectively, the mean and the standard deviation, then, the probability density is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \qquad (4)$$

The probability that the error around the mean is ($x-\mu$) is the area under the probability density function between ($x-\mu$)+dx and ($x-\mu$) represented by the product of the probability density and dx. The probability that the error is anywhere between -∞ and x is thus given by the following integral:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\left[\frac{v-\mu}{\sigma}\right]^2} dv \qquad (5)$$

This is referred to as the **cumulative probability**. It is noted that if $x \to \infty$ the integral tends to 1. Thus the probability that the error is of all possible magnitudes (between -∞ and +∞) is unity! The integral is **symmetrical** with respect to x=μ as may be easily verified. The above integral is in fact the **error integral** that is a tabulated function. A plot of f(x) and F(x) is given in Figure 5.

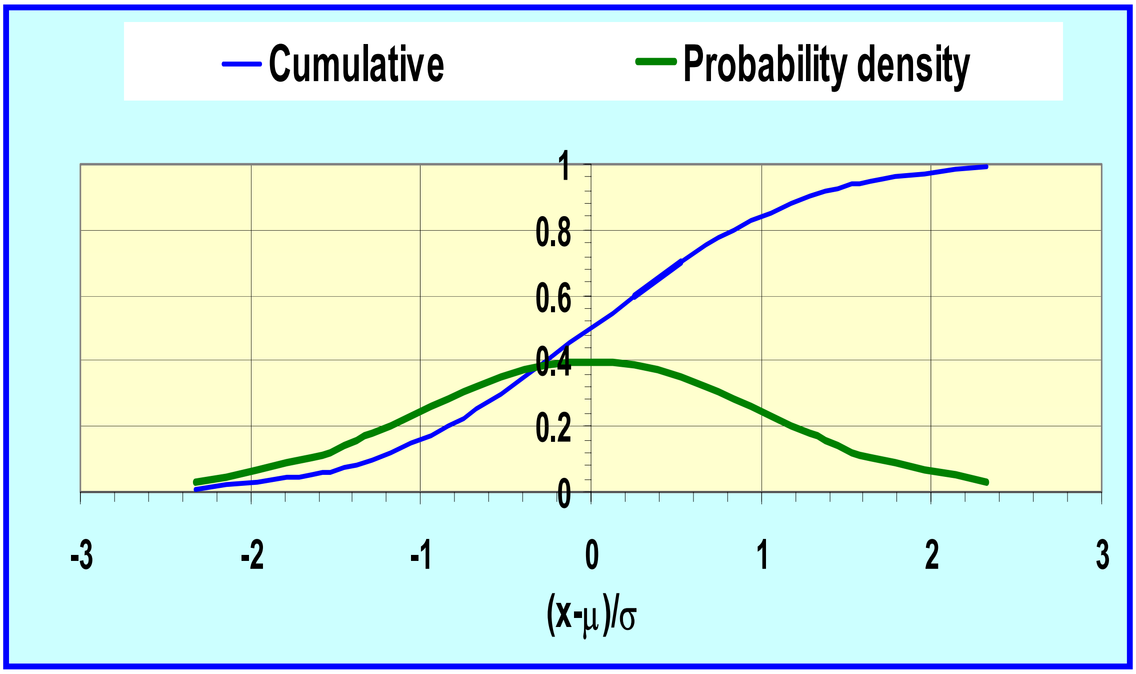


Figure 5 Normal distribution and its integral

Many times we are interested in finding out the chances of error lying between two values in the form ±**p**σ. This is referred to as the "confidence interval" and the corresponding cumulative probability specifies the chances of the error occurring within the confidence interval. Table 1 gives the confidence intervals that are useful in practice:

Table 1

Confidence intervals according to normal distribution

Cumulative Probability	**0**	**0.95**	**0.99**	**0.999**
Interval p	**0**	**±1.96**	**±2.58**	**±3.29**

The table indicates that error of magnitude greater than ±3.29σ is very unlikely to occur. In most applications we specify **±1.96**σ as the error bounds based on **95%** confidence.

Example 1

- Resistance of a certain resistor is measured repeatedly to obtain the following data.

No.	1	2	3	4	5	6	7	8	9
R, kΩ	1.22	1.23	1.26	1.21	1.22	1.22	1.22	1.24	1.19

- What is the best estimate for the resistance? What is the error with 95% confidence?
- Best estimate is the mean of the data.

$$\bar{R} = \frac{1.22\times 4 + 1.23 + 1.26 + 1.21 + 1.24 + 1.19}{9}$$
$$= 1.223 \approx 1.22 \text{ k}\Omega$$

- Standard deviation of the error σ:

$$\text{Variance } = \frac{1}{9}\sum_{1}^{9}\left[\text{Ri} - \bar{R}\right]^2$$
$$= 3.33\times 10^{-4}$$

Hence:

$$\sigma = \sqrt{3.33\times 10^{-4}}$$
$$= 0.183 \approx 0.02 \text{ k}\Omega$$

- Error with 95% confidence :

$$\text{Error}_{95\%} = 1.96\sigma = 1.96\times 0.0183$$
$$= 0.036 \approx 0.04 \text{ k}\Omega$$

Example 2

Thickness of a metal sheet (in mm) is measured repeatedly to obtain the following replicate data. What is the best estimate for the sheet thickness? What is the variance of the distribution of errors with respect to the best value? Specify an error estimate to the mean value based on 99% confidence.

Experiment No.	1	2	3	4	5	6
t, mm	0.202	0.198	0.197	0.215	0.199	0.194
Experiment No.	7	8	9	10	11	12
t, mm	0.204	0.198	0.194	0.195	0.201	0.202

- The best estimate for the metal sheet thickness is the mean of the 12 measured values. This is given by

$$t_b = \bar{t} = \frac{\sum_{1}^{12} t_i}{12} = \frac{\begin{bmatrix} 0.202+0.198+0.197+0.215+0.199+0.194+0.204 \\ +0.198+0.194+0.195+0.201+0.202 \end{bmatrix}}{12} = 0.2 \text{ mm}$$

- The variance with respect to the mean or the best value is given by (on substituting $\bar{t}$ for t_b) as

$$\sigma^2{}_b = \frac{\sum_{1}^{12} \left[t_i - \bar{t}\right]^2}{12} = \frac{\sum_{1}^{12} t_i}{12} - \bar{t}^2$$

- $$= \frac{\begin{bmatrix} 0.202^2 + 0.198^2 + 0.197^2 + 0.215^2 + 0.199^2 + 0.194^2 + 0.204^2 \\ +0.198^2 + 0.194^2 + 0.195^2 + 0.201^2 + 0.202^2 \end{bmatrix}}{12} - 0.2 \text{ mm}$$

$$= 3.04 \times 10^{-5} \text{mm}^2$$

- The corresponding standard deviation is given by

$$\sigma_b = \sqrt{3.04\times10^{-5}} = 0.0055 \approx 0.006 \text{ mm}$$

- The corresponding error estimate based on 99% confidence is

$$\text{Error} = \pm 2.58\sigma_b = \pm 2.58\times 0.0055 \approx \pm 0.014 \text{ mm}$$

Principle of Least Squares

Earlier we have dealt with the method of obtaining the best estimate from replicate data based on **minimization of variance**. No mathematical proof was given as a basis for this. We shall now look at the above afresh, in the light of the error distribution that has been presented above.

Consider a set of replicate data x_i. Let the best estimate for the measured quantity be x_b. The probability for a certain value x_i within the interval $x_i, x_i + dx_i$ to occur in the measured data is given by the relation

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_b - x_i)^2}{2\sigma^2}} dx_i \tag{6}$$

The probability that the particular values of measured data are obtained in replicate measurements must be given by the compound probability given by

$$p = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_{i=1}^{n} e^{-\frac{(x_b - x_i)^2}{2\sigma^2}} dx_i = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} e^{-\sum_{i=1}^{n}\frac{(x_b - x_i)^2}{2\sigma^2}} \prod_{i=1}^{n} dx_i \tag{7}$$

The reason the set of data was obtained as replicate data is that it was the **most probable**! Since the intervals dx_i are arbitrary, the above will have to be maximized by the proper choice of x_b and σ such that the exponential factor is a maximum. Thus we have to choose x_b and σ such that

$$p' = \frac{1}{\sigma^n} e^{-\sum_{i=1}^{n} \frac{(x_b - x_i)^2}{2\sigma^2}} \qquad \textbf{(8)}$$

has the largest possible value. As usual we set the derivatives $\frac{\partial p'}{\partial x_b} = \frac{\partial p'}{\partial \sigma} = 0$ to get the values of the two parameters x_b and σ. We have:

$$\frac{\partial p'}{\partial x_b} = -\frac{1}{2\sigma^{n+2}} e^{\sum_{i=1}^{n} \frac{(x_i - x_b)^2}{2\sigma^2}} \underbrace{\sum_{i=1}^{n} 2(x_i - x_b)(-1)}_{\text{This part should go to zero}} = 0 \qquad \textbf{(9)}$$

Or

$$\sum_{i=1}^{n} (x_i - x_b) = 0 \text{ or } x_b = \sum_{i=n}^{n} x_i = \overline{x} \qquad \textbf{(10)}$$

It is clear thus that the best value is nothing but the mean of the values! We also have:

$$\frac{\partial p'}{\partial \sigma} = \underbrace{\left[-\frac{n}{\sigma^{n+1}} + \frac{1}{\sigma^{n+3}} \sum_{i=0}^{n} (x_i - x_b)^2 \right]}_{\text{This part should go to Zero}} e^{\sum_{i=1}^{n} \frac{(x_i - x_b)^2}{2\sigma^2}} = 0 \qquad \textbf{(11)}$$

Or

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - x_b)^2}{n} \qquad \textbf{(12)}$$

This last expression indicates that the parameter σ^2 is nothing but the variance of the data with respect to the mean! Thus the best values of the measured quantity and its spread is based on the minimization of the squares of errors with respect to the mean. This embodies what is referred to as the **“Principle of Least Squares”**.

Propagation of errors:

Replicate data collected by measuring a single quantity repeatedly enables us to calculate the best value and characterize the spread by the variance with respect to the best value, using the principle of least squares. Now we look at the case of a **derived quantity** that is estimated from the measurement of several **primary** quantities. The question that needs to be answered is the following:

"A derived quantity Q is estimated using a formula that involves the primary quantities. $a_1, a_2, ...a_n$ Each one of these is available in terms of the respective best values $\bar{a}_1, \bar{a}_2, ...\bar{a}_n$ and the respective standard deviations $\sigma_1, \sigma_2 ....\sigma_n$. What is the best estimate for Q and what is the corresponding standard deviation σ_Q?"

We have, by definition

$$Q = Q(a_1, a_2, ...a_n) \qquad \textbf{(13)}$$

It is obvious that the best value of Q should correspond to that obtained by using the **best values** for the a's. Thus, the best estimate for Q given by $\bar{Q}$ as

$$\bar{Q} = Q(\bar{a}_1, \bar{a}_2, ...\bar{a}_n) \qquad \textbf{(14)}$$

Again, by definition, we should have:

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left(Q_i - \bar{Q}\right)^2 \qquad \textbf{(15)}$$

The subscript i indicates the experiment number and the ith estimate of Q is given by

$$Q_i = Q\left(a_{1i}, a_{2i}, ...a_{ni}\right) \qquad \textbf{(16)}$$

If we assume that the spread in values are small compared to the mean or the best values (this is what one would expect from a well conducted experiment), the difference between the ith estimate and the best value may be written using a Taylor expansion around the best value as

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=2}^{N}\left(\frac{\partial Q}{\partial a_1}\Delta a_{1i} + \frac{\partial Q}{\partial a_2}\Delta a_{2i} + \ldots\ldots + \frac{\partial Q}{\partial a_2}\Delta a_{ni}\right)^2 \qquad \textbf{(17)}$$

where the partial derivatives are all evaluated at the best values for the a's. If the a's are all **independent** of one another then the errors in these are unrelated to one another and hence the cross terms. $\sum_{i=1}^{N}\Delta a_{mi}\Delta a_{ki} = 0 \text{ for } m \neq k$ Thus equation (17) may be rewritten as

$$\sigma_Q^2 = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{\partial Q}{\partial a_1}\Delta a_{1i}\right)^2 + \left(\frac{\partial Q}{\partial a_2}\Delta a_{2i}\right)^2 + \ldots\ldots. + \left(\frac{\partial Q}{\partial a_n}\Delta a_{ni}\right)^2\right] \qquad \textbf{(18)}$$

Noting that $\sum_{i=1}^{N}\left(\Delta a_{ji}\right)^2 = N\sigma_j^2$ we may recast the above equation in the form

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial a_1}\right)^2 \sigma_1^2 + \left(\frac{\partial Q}{\partial a_2}\right)^2 \sigma_2^2 + \ldots\ldots. + \left(\frac{\partial Q}{\partial a_n}\right)^2 \sigma_n^2 \qquad \textbf{(19)}$$

Equation (19) is the error propagation formula. It may also be recast in the form

$$\sigma_Q = \sqrt{\left(\frac{\partial Q}{\partial a_1}\right)^2 \sigma_1^2 + \left(\frac{\partial Q}{\partial a_2}\right)^2 \sigma_2^2 + \ldots\ldots. + \left(\frac{\partial Q}{\partial a_n}\right)^2 \sigma_n^2} \qquad \textbf{(20)}$$

Example 3

The volume of a sphere is estimated by measuring its diameter by vernier calipers. In a certain case the diameter has been measured as D = 0.0502 ± 0.00005 m. Determine the volume and specify a suitable uncertainty for the same.

Nominal volume of sphere:

$$V = \pi \frac{D^3}{6} = 3.14159 \times \frac{0.0502^3}{6} = 6.624 \times 10^{\ 5} m^3$$

- The error in the measured diameter is specified as:

$$\Delta D = \pm 0.00005 m$$

- The influence coefficient is defined as

$$I_D = \frac{\partial V}{\partial D} = \pi \frac{D^2}{2} = 3.14159 \times \frac{0.0502^2}{2} = 3.958 \times 10^{-3}\ m^2$$

- Using the error propagation formula, we have

$$\Delta V = I_D \Delta D = 3.958 \times 10^{-3} \times 0.00005 = 1.979 \times 10^{\ 7}\ m^3$$

- Thus

$$V = 6.624 \times 10^{-5} \pm 1.979 \times 10^{-7}\ m^3$$

Definition & Equations & Examples

1. Mean

Definition: The mean is the average of a set of values, calculated by summing all the values and dividing by the number of values.

2. Standard Deviation

Definition: Standard deviation is a measure of the amount of variation or dispersion in a set of values. It shows how much the values deviate from the mean.

3. Variance

Definition: Variance is the square of the standard deviation. It provides a measure of how far the values in a set are from the mean.

4. Confidence Interval

Definition: A confidence interval is a range of values, derived from the data, that is likely to contain the true value of the parameter being measured, with a certain level of confidence (e.g., 95% confidence).

5. Normal Distribution

Definition: A normal distribution is a probability distribution where data is symmetrically distributed around the mean, forming a bell-shaped curve. Most of the data points are near the mean, and fewer are found as you move away from the mean.

6. Least Squares

Definition: The least squares method is a statistical technique used to find the best-fitting curve by minimizing the sum of the squares of the differences between observed values and the values predicted by the model.

7. Error Propagation

Definition: Error propagation refers to the process of determining the uncertainty in a result that comes from the uncertainties in the measurements that were used to calculate that result.

$$w_R = \left[\left(\frac{\partial R}{\partial x_1} w_1\right)^2 + \left(\frac{\partial R}{\partial x_2} w_2\right)^2 + \cdots + \left(\frac{\partial R}{\partial x_n} w_n\right)^2\right]^{1/2} \tag{3.2}$$

$$\frac{w_R}{R} = \left[\sum \left(\frac{a_i w_{x_i}}{x_i}\right)^2\right]^{1/2} \tag{3.2a}$$

$$w_R = \left\{\sum \left[\left(\frac{\partial R}{\partial x_i}\right)^2 w_{x_i}^2\right]\right\}^{1/2}$$

$$= \left[\sum (a_i w_{x_i})^2\right]^{1/2} \tag{3.2b}$$

Example 3.1

UNCERTAINTY OF RESISTANCE OF A COPPER WIRE. The resistance of a certain size of copper wire is given as

$$R = R_0[1 + \alpha(T - 20)]$$

where $R_0 = 6\ \Omega \pm 0.3$ percent is the resistance at 20°C, $\alpha = 0.004°\text{C}^{-1} \pm 1$ percent is the temperature coefficient of resistance, and the temperature of the wire is $T = 30 \pm 1°\text{C}$. Calculate the resistance of the wire and its uncertainty.

Solution

The nominal resistance is

$$R = (6)[1 + (0.004)(30 - 20)] = 6.24\ \Omega$$

The uncertainty in this value is calculated by applying Eq. (3.2). The various terms are

$$\frac{\partial R}{\partial R_0} = 1 + \alpha(T - 20) = 1 + (0.004)(30 - 20) = 1.04$$

$$\frac{\partial R}{\partial \alpha} = R_0(T - 20) = (6)(30 - 20) = 60$$

$$\frac{\partial R}{\partial T} = R_0\alpha = (6)(0.004) = 0.024$$

$$w_{R_0} = (6)(0.003) = 0.018\ \Omega$$

$$w_\alpha = (0.004)(0.01) = 4 \times 10^{-5}°\text{C}^{-1}$$

$$w_T = 1°\text{C}$$

Thus, the uncertainty in the resistance is

$$
\begin{aligned}
W_R &= [(1.04)^2(0.018)^2 + (60)^2(4 \times 10^{-5})^2 + (0.024)^2(1)^2]^{1/2} \\
&= 0.0305\ \Omega \quad \text{or} \quad 0.49\%
\end{aligned}
$$

Example 3.2

UNCERTAINTY IN POWER MEASUREMENT. The two resistors R and R_s are connected in series as shown in the accompanying figure. The voltage drops across each resistor are measured as

$$
\begin{aligned}
E &= 10\,\text{V} \pm 0.1\,\text{V}\ (1\%) \\
E_s &= 1.2\,\text{V} \pm 0.005\,\text{V}\ (0.467\%)
\end{aligned}
$$

along with a value of

$$R_s = 0.0066\ \Omega \pm 1/4\%$$

From these measurements determine the power dissipated in resistor R and its uncertainty.

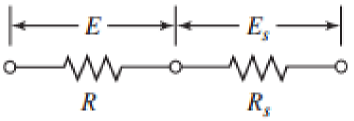


Figure Example 3.2

Solution

The power dissipated in resistor R is

$$P = EI$$

The current through both resistors is $I = E_s/R_s$ so that

$$P = \frac{EE_s}{R_s} \qquad \textbf{[a]}$$

The nominal value of the power is therefore

$$P = (10)(1.2)/(0.0066) = 1818.2\ \text{W}$$

The relationship for the power given in Eq. (a) is a product function, so the fractional uncertainty in the power may be determined from Eq. (3.2a). We have

$$a_E = 1 \qquad a_{E_s} = 1 \qquad \text{and} \qquad a_{R_s} = -1$$

so that

$$
\begin{aligned}
\frac{w_P}{P} &= \left[\left(\frac{a_E w_E}{E}\right)^2 + \left(\frac{a_{E_s} w_{E_s}}{E_s}\right)^2 + \left(\frac{a_{R_s} w_{R_s}}{R_s}\right)^2\right]^{1/2} \\
&= \left[(1)^2\left(\frac{0.1}{10}\right)^2 + (1)^2\left(\frac{0.005}{1.2}\right)^2 + (-1)^2(0.0025)^2\right]^{1/2} = 0.0111
\end{aligned}
$$

Then

$$w_P = (0.0111)(1818.2) = 20.18\ \text{W}$$

Example 3.3 **SELECTION OF MEASUREMENT METHOD.** A resistor has a nominal stated value of 10 Ω ± 1 percent. A voltage is impressed on the resistor, and the power dissipation is to be calculated in two different ways: (1) from $P = E^2/R$ and (2) from $P = EI$. In (1) only a voltage measurement will be made, while both current and voltage will be measured in (2). Calculate the uncertainty in the power determination in each case when the measured values of E and I are

$$E = 100\text{ V} \pm 1\% \qquad \text{(for both cases)}$$
$$I = 10\text{ A} \pm 1\%$$

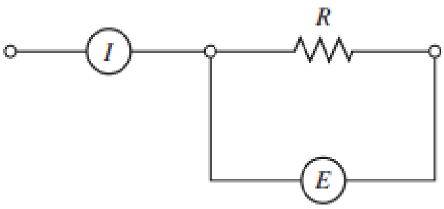


Figure Example 3.3 Power measurement across a resistor.

Solution

The schematic is shown in the accompanying figure. For the first case we have

$$\frac{\partial P}{\partial E} = \frac{2E}{R} \qquad \frac{\partial P}{\partial R} = -\frac{E^2}{R^2}$$

and we apply Eq. (3.2) to give

$$w_P = \left[\left(\frac{2E}{R}\right)^2 w_E^2 + \left(-\frac{E^2}{R^2}\right)^2 w_R^2\right]^{1/2} \qquad \textbf{[a]}$$

Dividing by $P = E^2/R$ gives

$$\frac{w_P}{P} = \left[4\left(\frac{w_E}{E}\right)^2 + \left(\frac{w_R}{R}\right)^2\right]^{1/2} \qquad \textbf{[b]}$$

Inserting the numerical values for uncertainty gives

$$\frac{w_P}{P} = [4(0.01)^2 + (0.01)^2]^{1/2} = 2.236\%$$

For the second case we have

$$\frac{\partial P}{\partial E} = I \qquad \frac{\partial P}{\partial I} = E$$

and after similar algebraic manipulation we obtain

$$\frac{w_P}{P} = \left[\left(\frac{w_E}{E}\right)^2 + \left(\frac{w_I}{I}\right)^2\right]^{1/2} \qquad \textbf{[c]}$$

Inserting the numerical values of uncertainty yields

$$\frac{w_P}{P} = [(0.01)^2 + (0.01)^2]^{1/2} = 1.414\%$$

Comment

The second method of power determination provides considerably less uncertainty than the first method, even though the primary uncertainties in each quantity are the same. In this example the utility of the uncertainty analysis is that it affords the individual a basis for *selection of a measurement method* to produce a result with less uncertainty.

Example 3.4

INSTRUMENT SELECTION. The power measurement in Example 3.2 is to be conducted by measuring voltage and current across the resistor with the circuit shown in the accompanying figure. The voltmeter has an internal resistance R_m, and the value of R is known only approximately. Calculate the nominal value of the power dissipated in R and the uncertainty for the following conditions:

$$R = 100\,\Omega \qquad \text{(not known exactly)}$$
$$R_m = 1000\,\Omega \pm 5\%$$
$$I = 5\text{ A} \pm 1\%$$
$$E = 500\text{ V} \pm 1\%$$

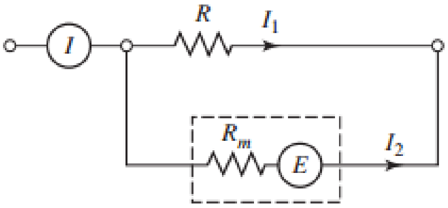


Figure Example 3.4 Effect of meter impedance on measurement.

Solution

A current balance on the circuit yields

$$I_1 + I_2 = I$$
$$\frac{E}{R} + \frac{E}{R_m} = I$$

and

$$I_1 = I - \frac{E}{R_m} \qquad \textbf{[a]}$$

The power dissipated in the resistor is

$$P = EI_1 = EI - \frac{E^2}{R_m} \qquad \textbf{[b]}$$

The nominal value of the power is thus calculated as

$$P = (500)(5) - \frac{500^2}{1000} = 2250\text{ W}$$

In terms of known quantities the power has the functional form $P = f(E, I, R_m)$, and so we form the derivatives

$$\frac{\partial P}{\partial E} = I - \frac{2E}{R_m} \qquad \frac{\partial P}{\partial I} = E$$
$$\frac{\partial P}{\partial R_m} = \frac{E^2}{R_m^2}$$

The uncertainty for the power is now written as

$$w_P = \left[\left(I - \frac{2E}{R_m}\right)^2 w_E^2 + E^2 w_I^2 + \left(\frac{E^2}{R_m^2}\right)^2 w_{R_m}^2\right]^{1/2} \qquad \textbf{[c]}$$

Inserting the appropriate numerical values gives

$$w_P = \left[\left(5 - \frac{1000}{1000}\right)^2 5^2 + (25 \times 10^4)(25 \times 10^{-4}) + \left(25 \times \frac{10^4}{10^6}\right)^2 (2500)\right]^{1/2}$$

$$= [16 + 25 + 6.25]^{1/2}(5)$$

$$= 34.4 \text{ W}$$

or

$$\frac{w_P}{P} = \frac{34.4}{2250} = 1.53\%$$

In order of influence on the final uncertainty in the power we have

1. Uncertainty of current determination
2. Uncertainty of voltage measurement
3. Uncertainty of knowledge of internal resistance of voltmeter

Comment

There are other conclusions we can draw from this example. The relative influence of the experimental quantities on the overall power determination is noted above. But this listing may be a bit misleading in that it implies that the uncertainty of the meter impedance does not have a large effect on the final uncertainty in the power determination. This results from the fact that $R_m \gg R$ ($R_m = 10R$). If the meter impedance were lower, say, 200 Ω, we would find that it was a dominant factor in the overall uncertainty. For a very *high* meter impedance there would be little influence, even with a very inaccurate knowledge of the exact value of R_m. Thus, we are led to the simple conclusion that we need not worry too much about the precise value of the internal impedance of the meter as long as it is very large compared with the resistance we are measuring the voltage across. This fact should influence *instrument selection* for a particular application.

Example 3.5

WAYS TO REDUCE UNCERTAINTIES. A certain obstruction-type flowmeter (orifice, venturi, nozzle), shown in the accompanying figure, is used to measure the flow of air at low velocities. The relation describing the flow rate is

$$\dot{m} = CA\left[\frac{2g_c p_1}{RT_1}(p_1 - p_2)\right]^{1/2} \qquad \textbf{[a]}$$

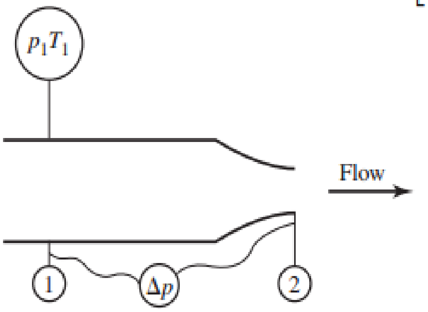


Figure Example 3.5 Uncertainty in a flowmeter.

where C = empirical-discharge coefficient
A = flow area
p_1 and p_2 = upstream and downstream pressures, respectively
T_1 = upstream temperature
R = gas constant for air

Calculate the percent uncertainty in the mass flow rate for the following conditions:

$$C = 0.92 \pm 0.005 \qquad \text{(from calibration data)}$$

$$p_1 = 25 \text{ psia } \pm 0.5 \text{ psia}$$

$$T_1 = 70°\text{F} \pm 2°\text{F} \qquad T_1 = 530°\text{R}$$

$$\Delta p = p_1 - p_2 = 1.4 \text{ psia} \pm 0.005 \text{ psia} \qquad \text{(measured directly)}$$

$$A = 1.0 \text{ in}^2 \pm 0.001 \text{ in}^2$$

Solution

In this example the flow rate is a function of several variables, each subject to an uncertainty.

$$\dot{m} = f(C, A, p_1, \Delta p, T_1) \tag{b}$$

Thus, we form the derivatives

$$\begin{aligned}
\frac{\partial \dot{m}}{\partial C} &= A\left(\frac{2g_c p_1}{RT_1}\Delta p\right)^{1/2} \\
\frac{\partial \dot{m}}{\partial A} &= C\left(\frac{2g_c p_1}{RT_1}\Delta p\right)^{1/2} \\
\frac{\partial \dot{m}}{\partial p_1} &= 0.5CA\left(\frac{2g_c}{RT_1}\Delta p\right)^{1/2} p_1^{-1/2} \\
\frac{\partial \dot{m}}{\partial \Delta p} &= 0.5CA\left(\frac{2g_c p_1}{RT_1}\right)^{1/2} \Delta p^{-1/2} \\
\frac{\partial \dot{m}}{\partial T_1} &= -0.5CA\left(\frac{2g_c p_1}{R}\Delta p\right)^{1/2} T_1^{-3/2}
\end{aligned} \tag{c}$$

The uncertainty in the mass flow rate may now be calculated by assembling these derivatives in accordance with Eq. (3.2). Designating this assembly as Eq. (*c*) and then dividing by Eq. (*a*) gives

$$\frac{w_{\dot{m}}}{\dot{m}} = \left[\left(\frac{w_C}{C}\right)^2 + \left(\frac{w_A}{A}\right)^2 + \frac{1}{4}\left(\frac{w_{p_1}}{p_1}\right)^2 + \frac{1}{4}\left(\frac{w_{\Delta p}}{\Delta p}\right)^2 + \frac{1}{4}\left(\frac{w_{T_1}}{T_1}\right)^2\right]^{1/2} \tag{d}$$

We may now insert the numerical values for the quantities to obtain the percent uncertainty in the mass flow rate.

$$\begin{aligned}
\frac{w_{\dot{m}}}{\dot{m}} &= \left[\left(\frac{0.005}{0.92}\right)^2 + \left(\frac{0.001}{1.0}\right)^2 + \frac{1}{4}\left(\frac{0.5}{25}\right)^2 + \frac{1}{4}\left(\frac{0.005}{1.4}\right)^2 + \frac{1}{4}\left(\frac{2}{530}\right)^2\right]^{1/2} \\
&= [29.5 \times 10^{-6} + 1.0 \times 10^{-6} + 1.0 \times 10^{-4} + 3.19 \times 10^{-6} + 3.57 \times 10^{-6}]^{1/2} \\
&= [1.373 \times 10^{-4}]^{1/2} = 1.172\%
\end{aligned} \tag{e}$$

3.6 STATISTICAL ANALYSIS OF EXPERIMENTAL DATA

arithmetic mean

$$x_m = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad [3.3]$$

The *deviation* d_i for each reading is defined by

$$d_i = x_i - x_m \qquad [3.4]$$

The average of the absolute values of the deviations is given by

$$|\bar{d}_i| = \frac{1}{n}\sum_{i=1}^{n} |d_i| = \frac{1}{n}\sum_{i=1}^{n} |x_i - x_m| \qquad [3.6]$$

The *standard deviation* or *root-mean-square deviation* is defined by

$$\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - x_m)^2\right]^{1/2} \qquad [3.7]$$

and the square of the standard deviation σ^2 is called the *variance*. This is sometimes called the *population* or *biased* standard deviation because it strictly applies only when a large number of samples is taken to describe the population.

In many circumstances the engineer will not be able to collect as many data points as necessary to describe the underlying population. Generally speaking, it is desired to have at least 20 measurements in order to obtain reliable estimates of standard deviation and general validity of the data. For small sets of data an *unbiased* or *sample standard deviation* is defined by

$$\sigma = \left[\frac{\sum_{i=1}^{n}(x_i - x_m)^2}{n-1}\right]^{1/2} \qquad [3.8]$$

Example 3.7

CALCULATION OF POPULATION VARIABLES. The following readings are taken of a certain physical length. Compute the mean reading, standard deviation, variance, and average of the absolute value of the deviation, using the "biased" basis:

Reading	x, cm
1	5.30
2	5.73
3	6.77
4	5.26
5	4.33
6	5.45
7	6.09
8	5.64
9	5.81
10	5.75

Solution

The mean value is given by

$$x_m = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{10}(56.13) = 5.613 \text{ cm}$$

The other quantities are computed with the aid of the following table:

Reading	$d_i = x_i - x_m$	$(x_i - x_m)^2 \times 10^2$
1	−0.313	9.797
2	0.117	1.369
3	1.157	133.865
4	−0.353	12.461
5	−1.283	164.609
6	−0.163	2.657
7	0.477	22.753
8	0.027	0.0729
9	0.197	3.881
10	0.137	1.877

$$\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - x_m)^2\right]^{1/2} = \left[\frac{1}{10}(3.533)\right]^{1/2} = 0.5944 \text{ cm}$$

$$\sigma^2 = 0.3533 \text{ cm}^2$$

$$|\bar{d}_i| = \frac{1}{n}\sum_{i=1}^{n}|d_i| = \frac{1}{n}\sum_{i=1}^{n}|x_i - x_m|$$

$$= \tfrac{1}{10}(4.224) = 0.4224 \text{ cm}$$

Example 3.8

SAMPLE STANDARD DEVIATION. Calculate the best estimate of standard deviation for the data of Example 3.7 based on the "sample" or unbiased basis.

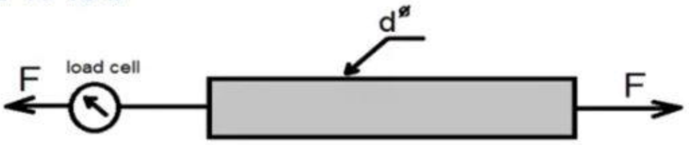
Solution

The calculation gives

$$\sigma = \left[\frac{1}{10-1}(3.536)\right]^{1/2} = (0.3929)^{1/2} = 0.627 \text{ cm}$$

❑Example 5:

➢ Calculate the uncertainty (u_σ) in stress (σ) for the shown steel rod having a length of L = 50 mm, and diameter (d mm), if the indicated reading of the load cell (0 : 2500 N capacity), F = 1500 N ± 0.2 % fsd



Measurements for the rod diameters (d mm) were as follows:

d: 5.21 5.18 5.25 5.26 5.22 5.19 5.23 5.24 5.18 5.26

Solution :

$$\sigma = \frac{F}{A} = \frac{F}{\frac{\pi}{4} d^2} = \frac{4F}{\pi d^2}$$

$$\bar{d} = \frac{\sum_1^n d_i}{n} = \frac{\sum_1^{10} d_i}{10} = \frac{52.22}{10} = 5.222 \ mm$$

$$u_d = \sqrt{\frac{\sum_1^n (d_i - \bar{d})^2}{n-1}} = \pm 0.0311 \ mm$$

F = 1500 N ± 0.2 % fsd

$u_F = \pm (0.2/100) x 2500 = \pm 5$ N

To find overall uncertainty:

$$u_\sigma = [(u_F . \frac{\partial \sigma}{\partial F})^2 + (u_d . \frac{\partial \sigma}{\partial d})^2]^{1/2}$$

$$\frac{\partial \sigma}{\partial F} = \frac{4}{\pi \bar{d}^2} = \frac{4}{\pi (5.222)^2} = 0.047 \ mm^{-2}$$

$$\frac{\partial \sigma}{\partial d} = \frac{4\bar{F}}{\pi} x - 2\bar{d}^{-3} = \frac{-8x1500}{\pi (5.222)^3} = -26.823 \ N / mm^3$$

$$\therefore u_\sigma = [(5x0.047)^2 + (0.0311x - 26.823)^2]^{1/2} = \pm 70.03 N / mm^2$$

Homework

1- A certain resistor draws 110.2 V and 5.3 A. The uncertainties in the measurements are ±0.2 V and ±0.06 A, respectively. Calculate the power dissipated in the resistor and the uncertainty in the power.

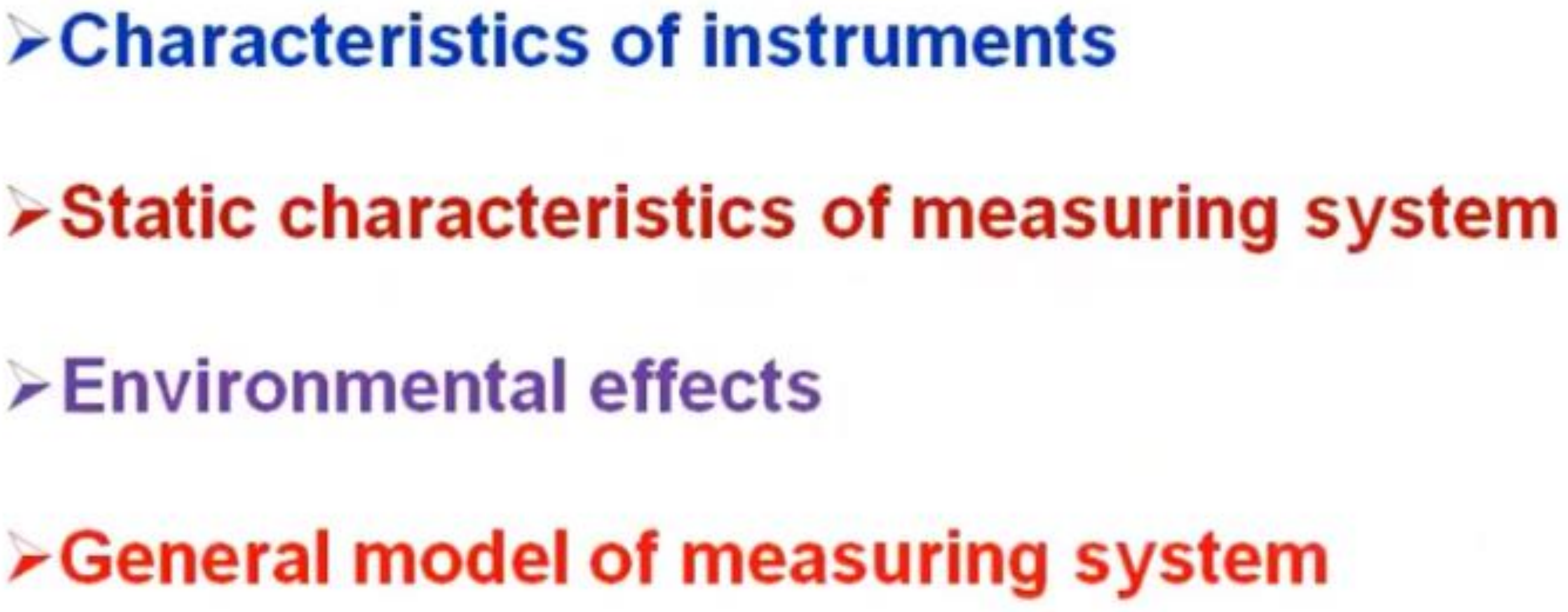
2- A small plot of land has measured dimensions of 50.0 by 150.0 ft. The uncertainty in the 50-ft dimension is ±0.01 ft. Calculate the uncertainty with which the 150-ft dimension must be measured to ensure that the total uncertainty in the area is not greater than 150 percent of that value it would have if the 150-ft dimension were exact.

3- A thermocouple is used to measure the temperature of a known standard maintained at 100°C. After converting the electrical signal to temperature the readings are: 101.1, 99.8, 99.9, 100.2, 100.5, 99.6, 100.9, 99.7, 100.1, and 100.3. Using whatever criteria seem appropriate, make some statements about the calibration of the thermocouple.

4- Seven students are asked to make a measurement of the thickness of a steel block with a micrometer. The actual thickness of the block is known very accurately as 2.000 cm. The seven measurements are: 2.002, 2.001, 1.999, 1.997, 1.998, 2.003, and 2.003 cm. Comment on these measurements using whatever criteria you think appropriate.

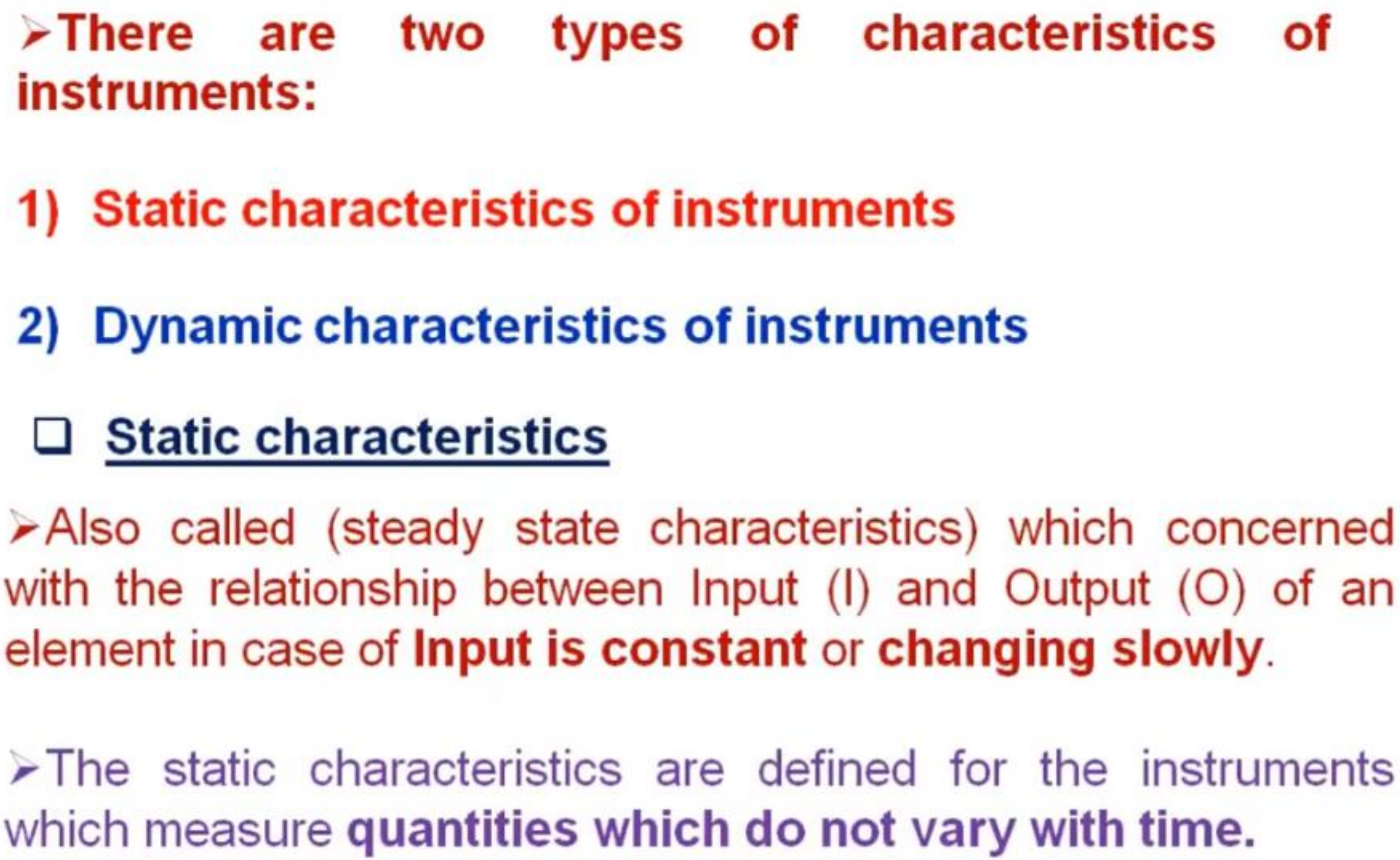
Lecture - 4

Topics



- Characteristics of instruments
- Static characteristics of measuring system
- Environmental effects
- General model of measuring system

Characteristics of instruments



➢ **There are two types of characteristics of instruments:**

1) **Static characteristics of instruments**

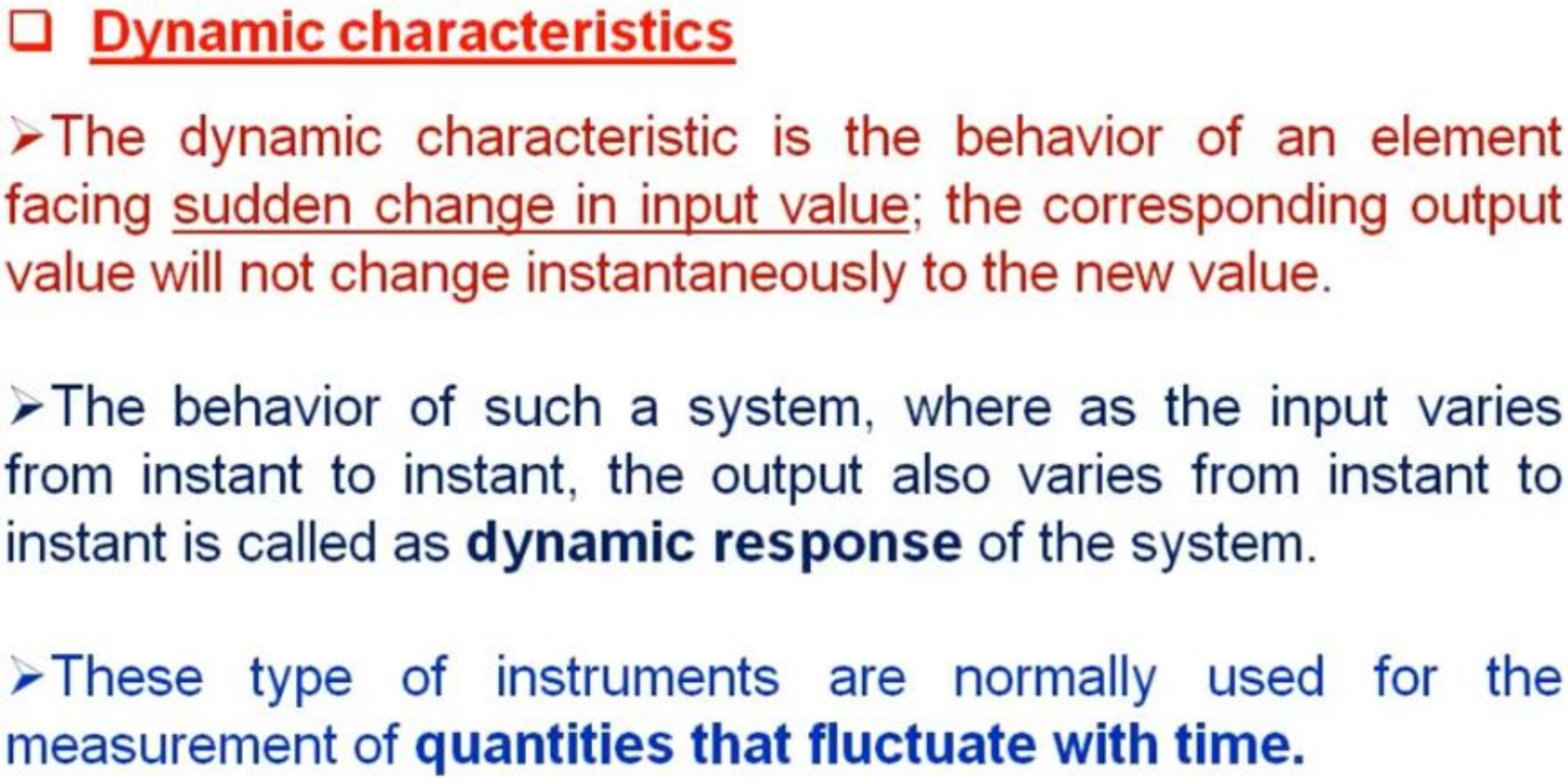
2) **Dynamic characteristics of instruments**

☐ **Static characteristics**

➢ Also called (steady state characteristics) which concerned with the relationship between Input (I) and Output (O) of an element in case of **Input is constant** or **changing slowly**.

➢ The static characteristics are defined for the instruments which measure **quantities which do not vary with time.**

Characteristics of instruments



Dynamic characteristics

- The dynamic characteristic is the behavior of an element facing sudden change in input value; the corresponding output value will not change instantaneously to the new value.

- The behavior of such a system, where as the input varies from instant to instant, the output also varies from instant to instant is called as **dynamic response** of the system.

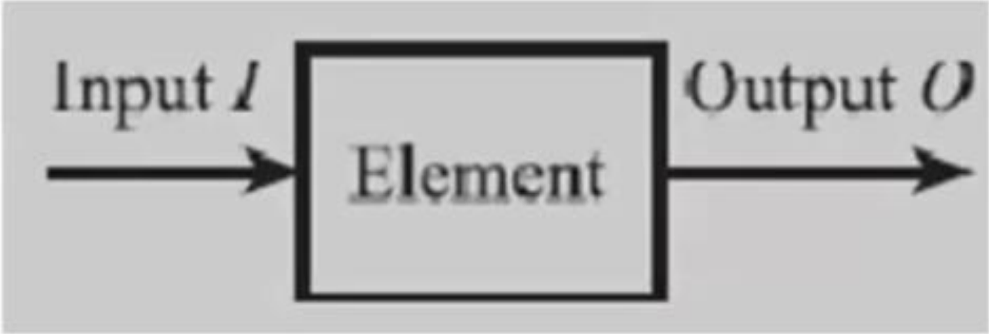
- These type of instruments are normally used for the measurement of **quantities that fluctuate with time.**

Static Characteristics

- The static characteristics of instruments are related **with steady state response.**

- The relationship between the output and the input when the input does not change, or the input is changing at a slow rate.

- Determination of static characteristics is mostly done by calibration.



Static Characteristics

- The main static characteristics are:
  - Range
  - Span
  - Sensitivity
  - Linearity
  - Non Linearity
  - Hysteresis
  - Resolution
  - Error bands
  - Reproducibility
  - Environmental Effects

Static Characteristics

❑ Range:

- The range is the minimum and maximum values of a quantity for which an instrument is designed to measure.
- The input range of the element is specified by the minimum and maximum values of the input **(i.e. I_{Min} to I_{Max})**. The output range is specified by the minimum and maximum values of the output **(i.e. O_{Min} to O_{Max})**.

For example:

-For the pressure transducer shown

0 - 10^4 Pa

(Input)

Pressure Transducer

4 - 20 mA

(Output) O

- The input range will be: from 0 to 10^4 Pa

- The output range will be: from 4 to 20 mA

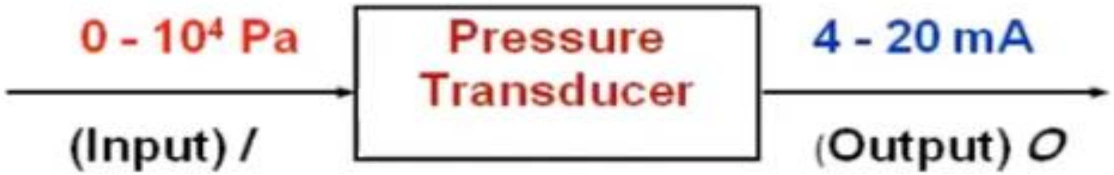
Static Characteristics

❑ Span:

- Span is maximum variation in input or output.
- **Input span = $I_{max} - I_{Min}$**
- **Output span = $O_{max} - O_{Min}$**

For example:

-For the pressure transducer shown



$0 - 10^4$ Pa

(Input) I

Pressure Transducer

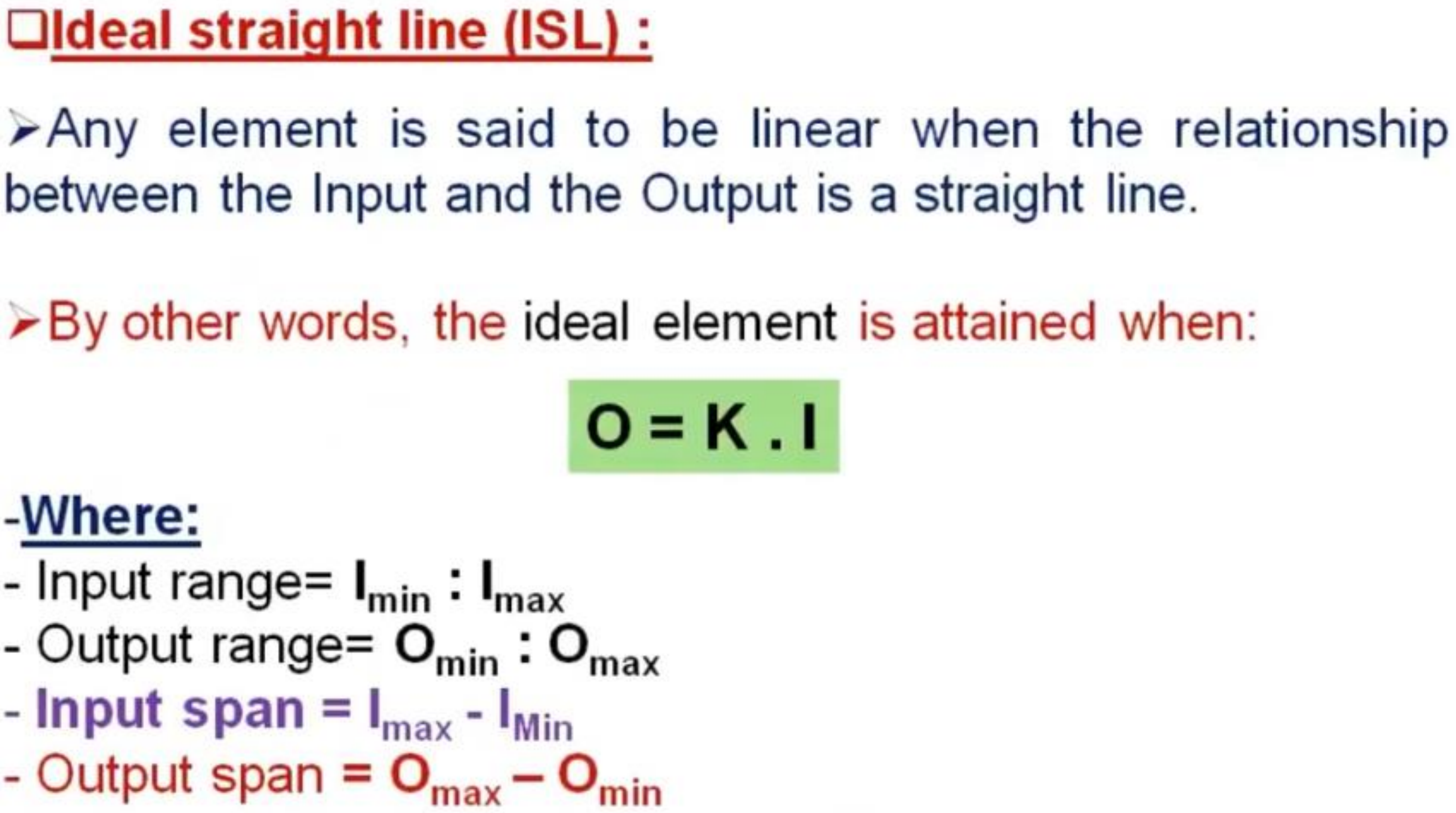
4 - 20 mA

(Output) O

- **Input span = $I_{max} - I_{Min}$** $= 10^4 - 0 = 10^4$ Pa

- **Output span = $O_{max} – O_{min}$** $= 20 – 4 = 16$ mA

Static Characteristics



❑Ideal straight line (ISL) :

➢Any element is said to be linear when the relationship between the Input and the Output is a straight line.

➢By other words, the ideal element is attained when:

$$\boxed{O = K \cdot I}$$

-**Where:**

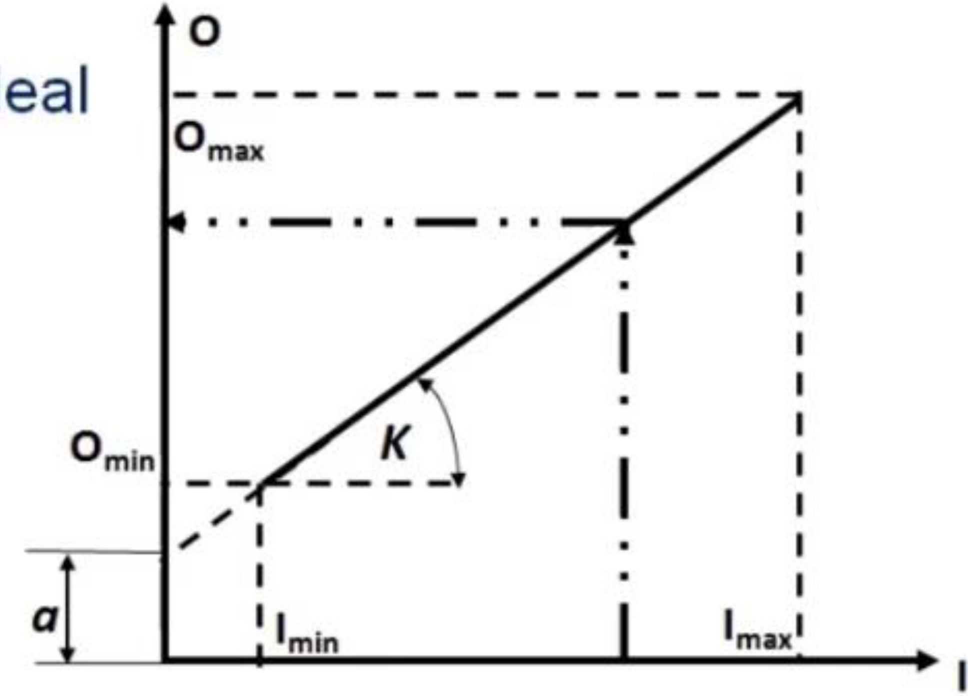
- Input range= $I_{min} : I_{max}$
- Output range= $O_{min} : O_{max}$
- **Input span** = $I_{max} - I_{Min}$
- Output span = $O_{max} - O_{min}$

Static Characteristics

❑ Ideal straight line (ISL) :

- The general form for the ideal straight line will be as follow:

$$O_{ideal} = K.I + a$$



-Where:

- K: Ideal straight line slope.
- a: Ideal straight line Intercept.

$$K = slope = \frac{O_{max} - O_{min}}{I_{max} - I_{min}} = \frac{O - O_{min}}{I - I_{min}}$$

Static Characteristics

Example (1):

- Define Input and Output spans.
- Find the Ideal straight line equation.

Solution:

- Input and Output spans
  - **Input span =** $I_{max} - I_{Min} = 250 - 100 = 150^{\circ}\,C$
  - Output span $= O_{max} - O_{min} = 10 - 4 = 6\ mV$

Static Characteristics

Ideal straight line (ISL) :

1st find the slope:

$$K = slope = \frac{O_{\max} - O_{\min}}{I_{\max} - I_{\min}} = \frac{10-4}{250-100}$$

$$\therefore K = 0.04 mv/^{o}C$$

$$\therefore O = 0.04I + a$$

- To find the intercept use any couple of **(I_{min}, O_{min}) or (I_{max}, O_{max})** for substitution.

Static Characteristics

$$\boxed{\therefore O_{\max} = 0.04 I_{\max} + a}$$

$$\boxed{\therefore 10 = 0.04 \times 250 + a}$$

$$\boxed{\therefore a = 0 mv}$$

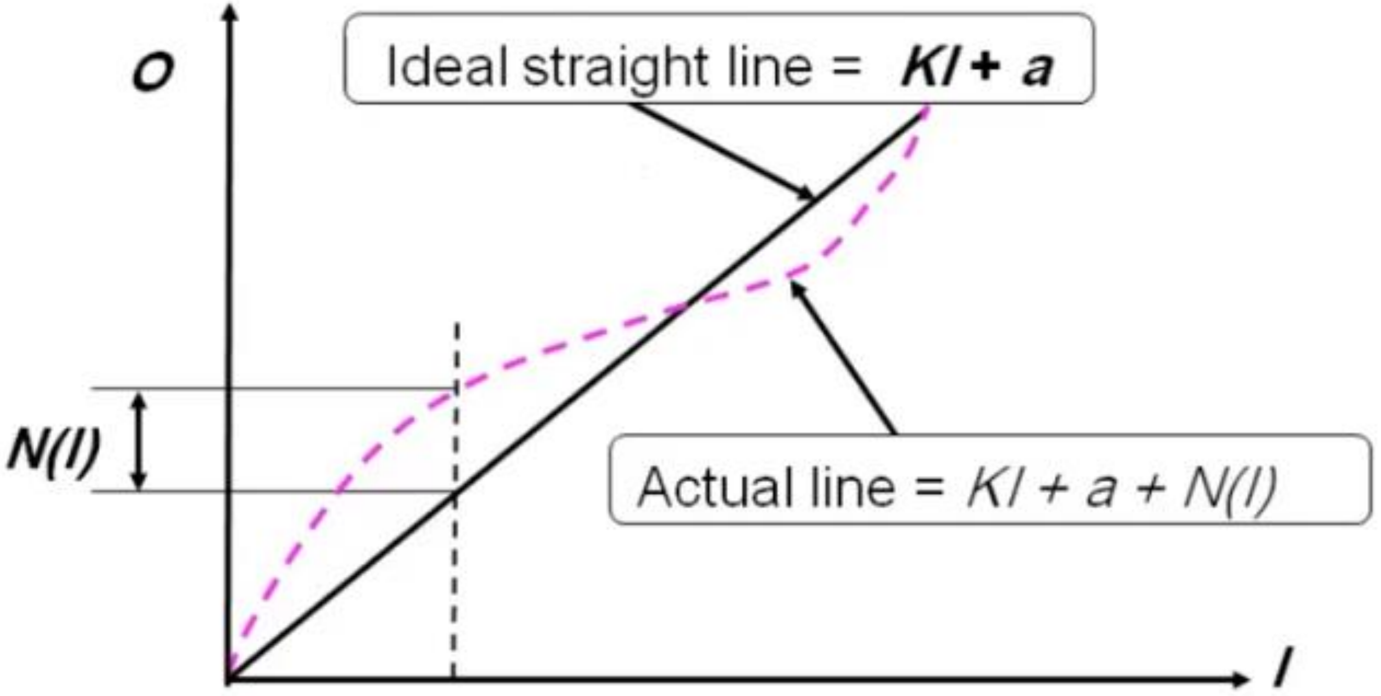
<u>The ideal straight line equation:</u>

$$\boxed{\therefore O = 0.04 I + 0}$$

Static Characteristics

❑ Nonlinearity:

➢In many cases the ideal straight line equation can not represent the Input-Output relation correctly so this relationship is termed Non-linear equation.



Static Characteristics

☐ Nonlinearity:

- N(I) : Is the Non-linearity at input (I).

$$N(I) = actual \quad line - ideal \quad line$$

$$N(I) = O(I) - O_{ideal}(I)$$

$$\therefore N(I) = O(I) - [KI + a]$$

$$\therefore O(I) = KI + a + N(I)$$

Static Characteristics

❑ nonlinearity:

- Non-linearity is often quantified in terms of the maximum non-linearity $(\hat{N})$; expressed as a percentage of full-scale deflection (f.s.d.), i.e. as a percentage of span. Thus:

$$\%N(I) = \frac{\hat{N}}{O_{Max} - O_{Min}} x100$$

Static Characteristics

❑ **Example (2):**

- ➢Thermocouple has an Input-Output relationship of :

$$E(T)=38.74T+3.319\times10^{-2}T^{2}+2.071\times10^{-4}T^{3}$$

- ➢for the range 0 to 400 °C input. *E* = 0 μV and *E* = 20 869 μV outputs.

- ➢Find nonlinearity E(T)

Solution:

$$\Theta\ K=\frac{O_{\max}-O_{\min}}{I_{\max}-I_{\min}}=\frac{20869-0}{400-0}=52.17\,\mu V/{}^{o}C$$

Static Characteristics

Solution:

$$\therefore O_{ideal} = 52.17T + a$$

At $T_{min} = 0$ and $E_{min} = 0$

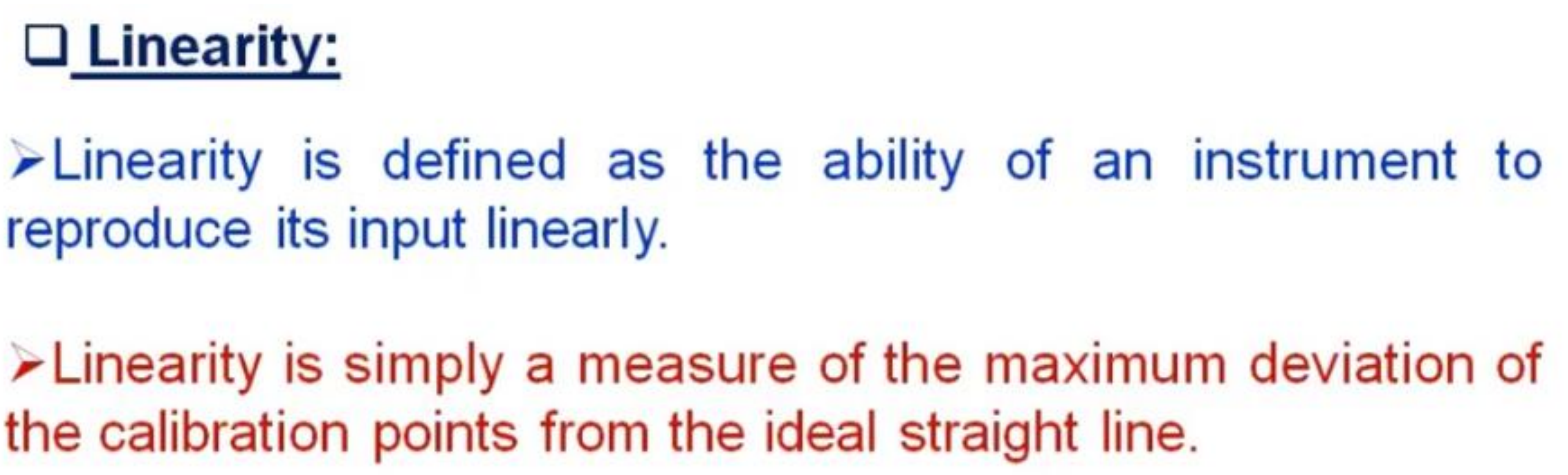
$$\therefore a = 0$$

$$\therefore O_{ideal} = 52.17T$$

$$\Theta\ N(T) = O(T) - [KT + a]$$

$$\therefore N(T) = -13.43T + 3.319 \times 10^{-2} T^{2} + 2.071 \times 10^{-4} T^{3}$$

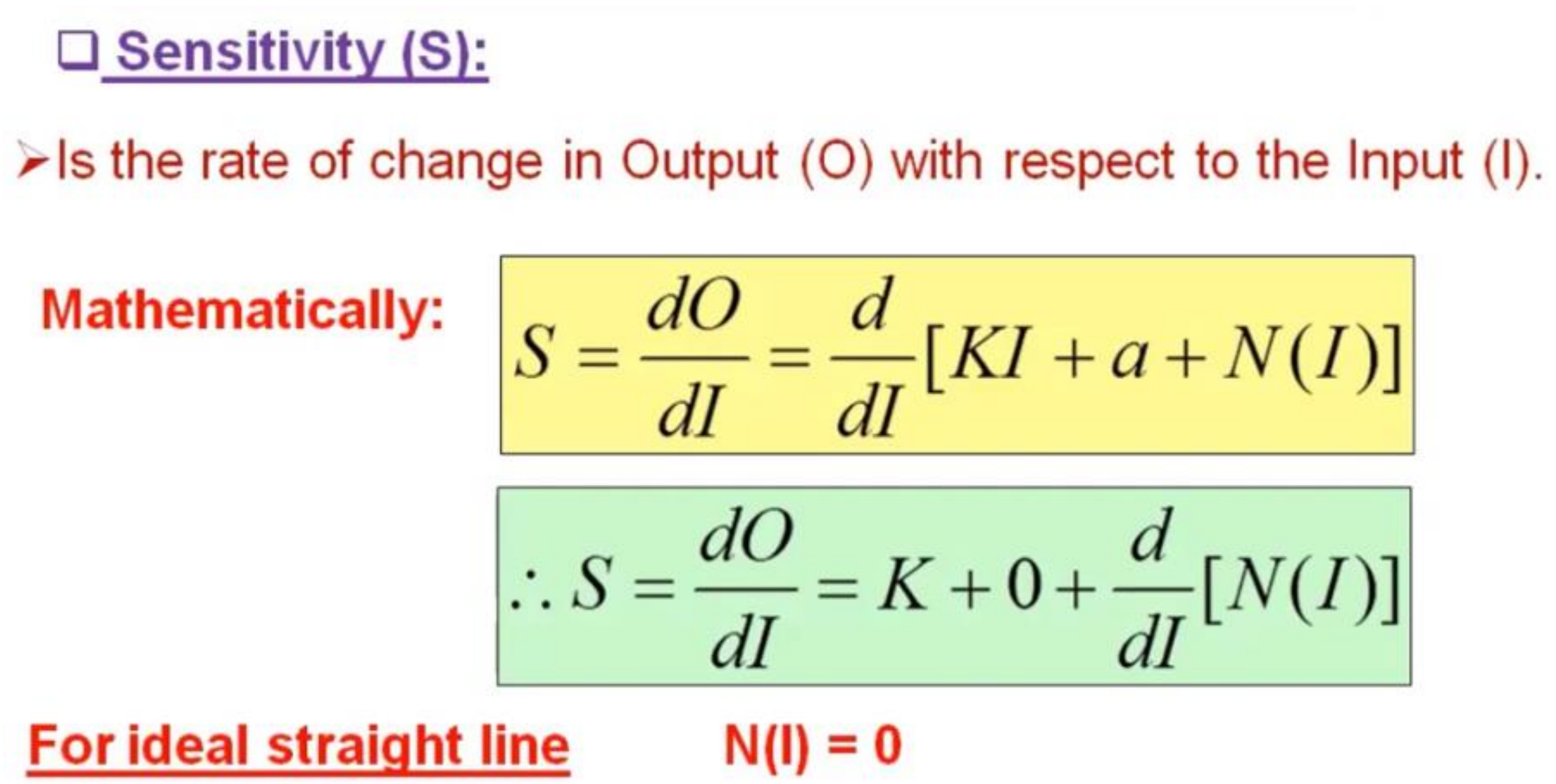
Static Characteristics



❑ **Linearity:**

- Linearity is defined as the ability of an instrument to reproduce its input linearly.
- Linearity is simply a measure of the maximum deviation of the calibration points from the ideal straight line.

Static Characteristics



☐ **Sensitivity (S):**

- ➢ Is the rate of change in Output (O) with respect to the Input (I).

Mathematically:

$$S = \frac{dO}{dI} = \frac{d}{dI}[KI + a + N(I)]$$

$$\therefore S = \frac{dO}{dI} = K + 0 + \frac{d}{dI}[N(I)]$$

For ideal straight line N(I) = 0

$$S_{ideal} = \frac{dO}{dI} = K$$

Static Characteristics

❑ Hysteresis

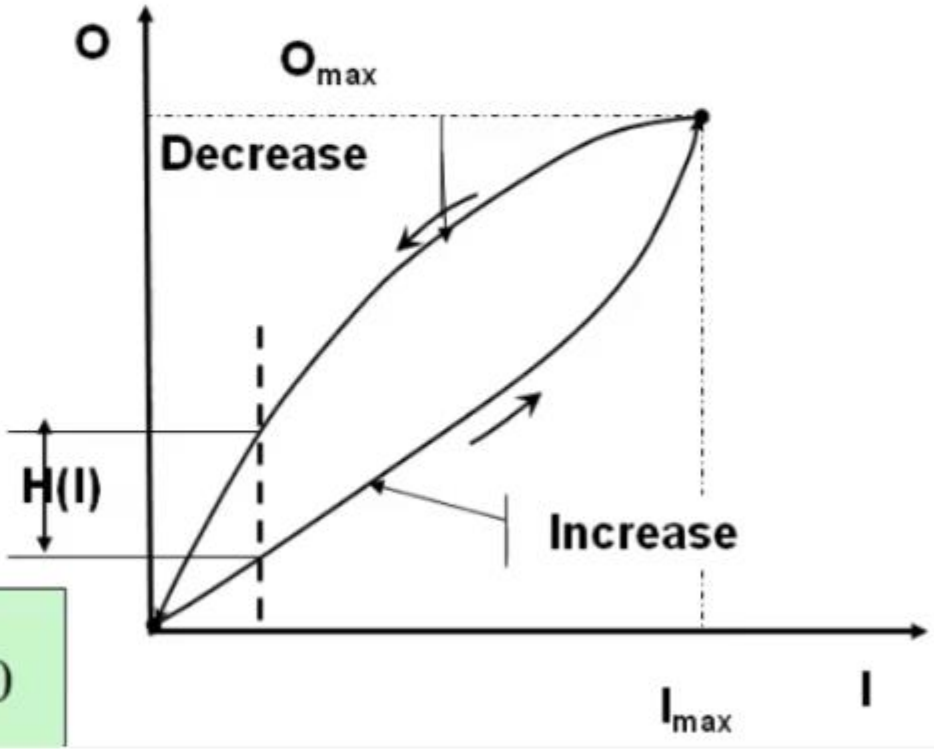
- Is the difference in Output (O) value for same Input value (I) while measuring one time in decreasing sequence and another in increasing sequence.

$$H(I) = O(I)\downarrow - O(I)\uparrow$$

$\hat{H}$: is the max. H (I) … Max. Hysteresis

- Max. Hysteresis as a % of (full scale deflection f.s.d.) is:

$$\% H(I) = \frac{\hat{H}}{O_{Max} - O_{Min}} x100$$



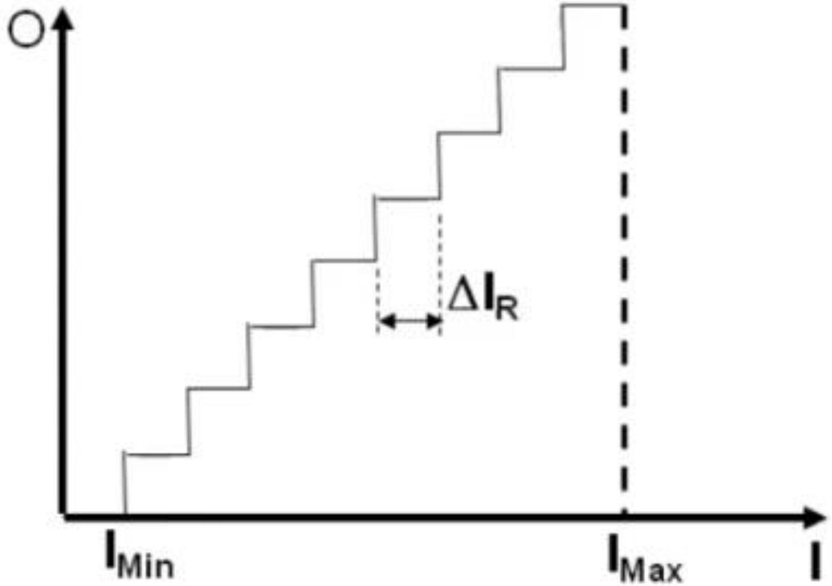
Static Characteristics

❑Resolution:

- Is the Smallest value of Input (I) can produce a change in the Output (O).

- Any element affected by the resolution effect has an Input – Output relation with no smooth line as previous. The resolution effect will be as a stair-like shape.

- ΔI_R is the value of the resolution.



Static Characteristics

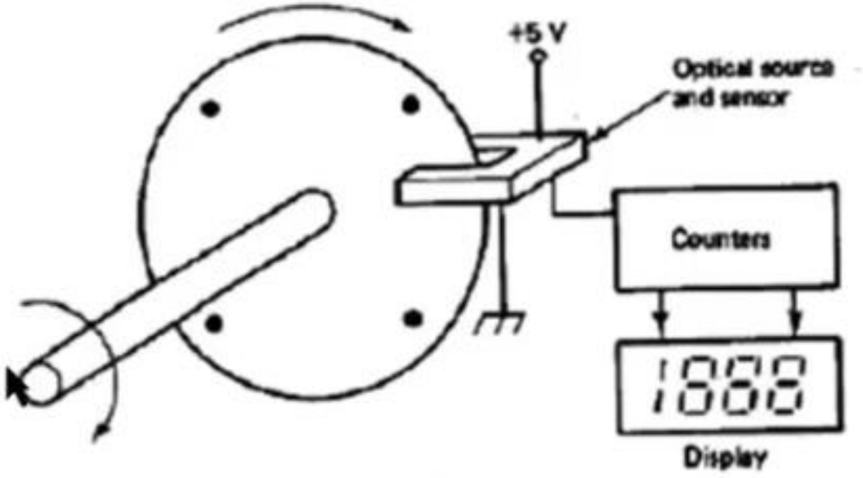
❑Resolution:

- The resolution as a percentage of [F.S.D.]

$$\%\Delta I_R = \frac{\Delta I_R}{I_{Max} - I_{Min}} x100$$

For example:

- For the simple optical encoder shown. Each time the shaft rotates ¼ of a revolution, a pulse will be generated. So, this encoder has a 90° resolution.



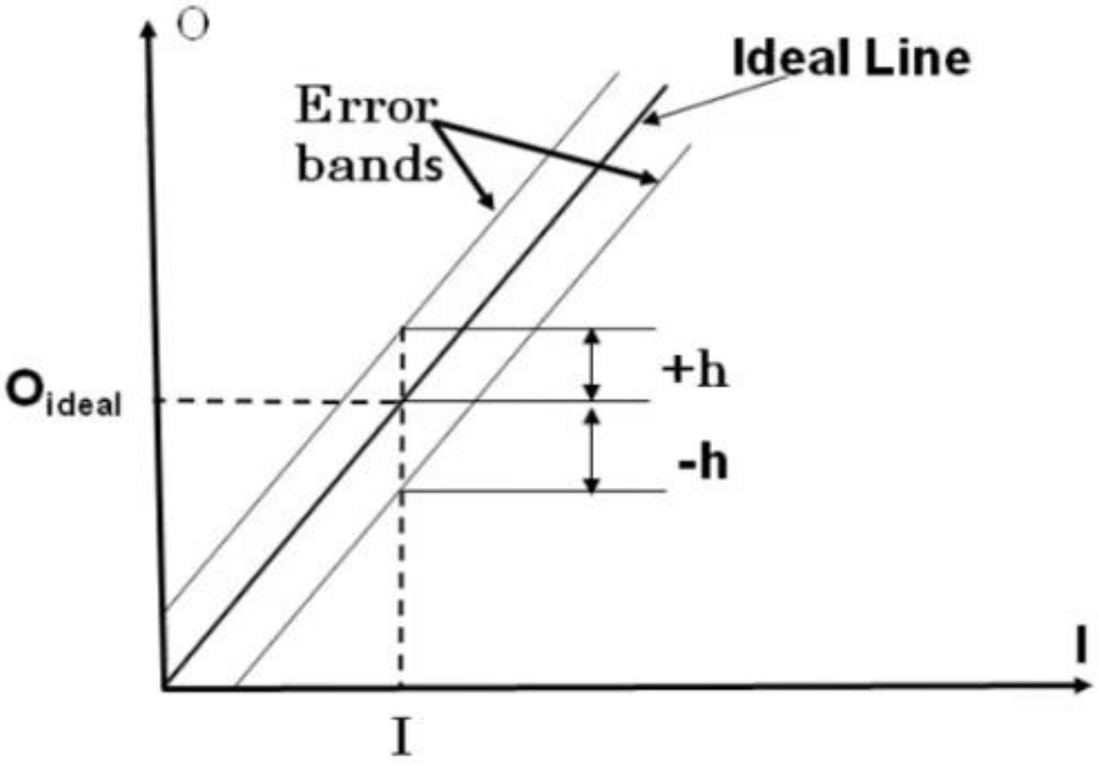
Simple optical encoder

Static Characteristics

❑Error bands:

- Due to the *complex interaction between* the effect of non-linearity, resolution and Hysteresis actions, it is difficult to quantify each individual effect.

- The manufactures of measurement elements defined the performance of such elements in terms of error bands.



Static Characteristics

❑Error bands:

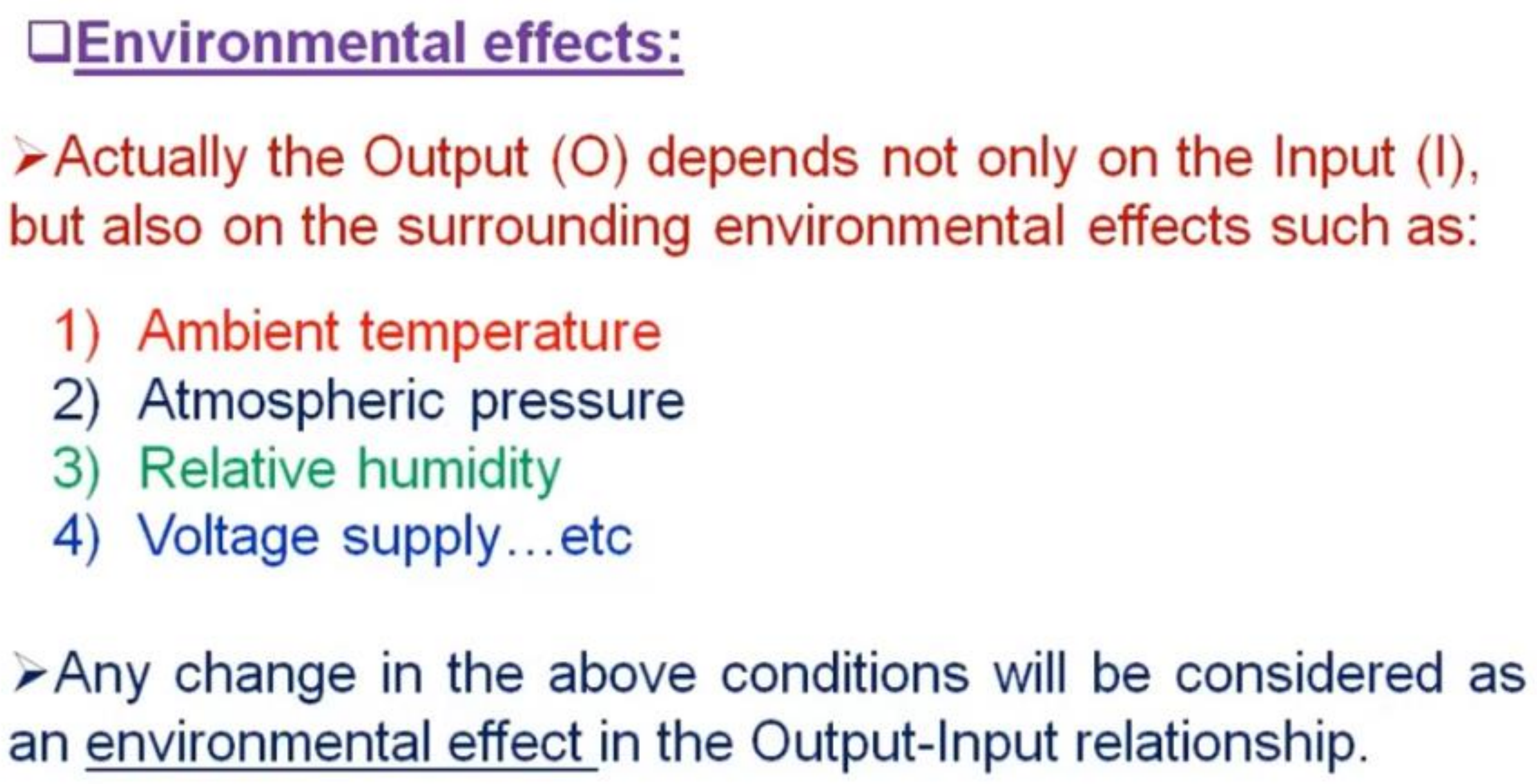
- For any Input (I) the Output actually will be in range of $(\pm h)$ of the Output of its ideal straight-line.

❑Reproducibility

- It is the closeness of agreement among repeated measurements of the output for the same value of the input, **made under the same operating conditions** over a period of time.

- Perfect reproducibility signifies that the given readings that are taken for an input, do not vary with time.

Environmental effects



☐ Environmental effects:

➢ Actually the Output (O) depends not only on the Input (I), but also on the surrounding environmental effects such as:

1) Ambient temperature
2) Atmospheric pressure
3) Relative humidity
4) Voltage supply…etc

➢ Any change in the above conditions will be considered as an environmental effect in the Output-Input relationship.

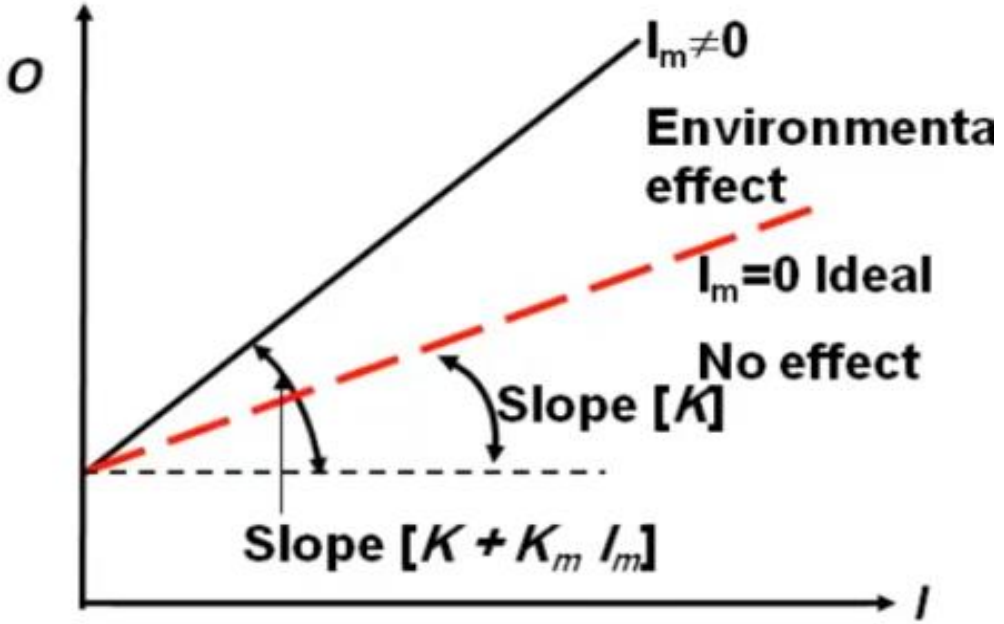
Environmental effects

❑ Environmental effects:

- *The environmental Input has two main types:*

1) modifying input (I_m):

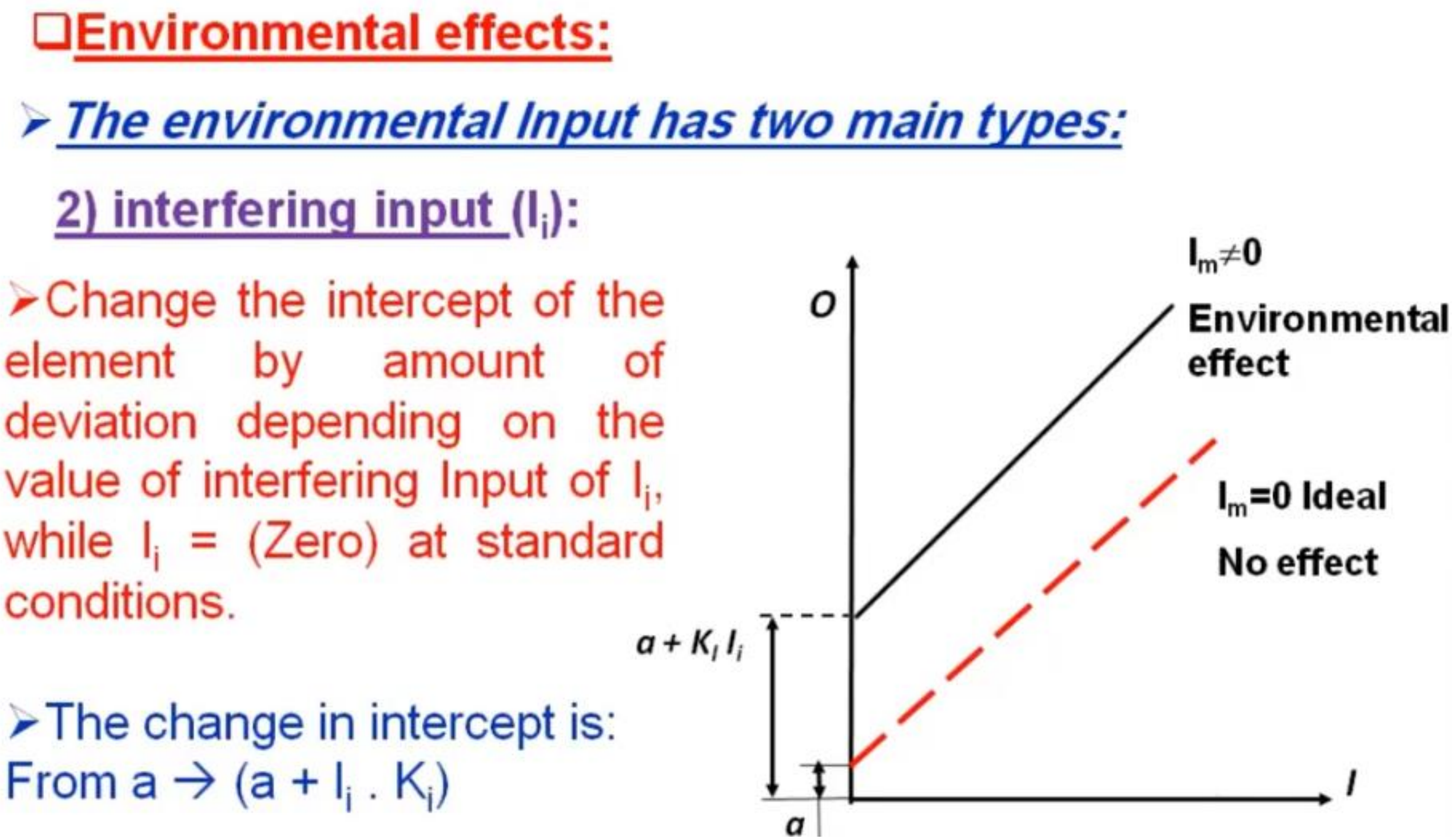
- Change the linear sensitivity of the element by amount of deviation depending on the value of modifying Input of I_m, while I_m = (Zero) at standard conditions.



- The change in linear sensitivity is:

From $S = K \rightarrow S = (K + I_m \cdot K_m)$

Environmental effects



❑Environmental effects:

➢ ***The environmental Input has two main types:***

2) interfering input (I_i):

➢Change the intercept of the element by amount of deviation depending on the value of interfering Input of I_i, while I_i = (Zero) at standard conditions.

➢The change in intercept is: From a → ($a + I_i . K_i$)

General model of measuring system

- If hysteresis and resolution effects are not present in an element but environmental and non-linear effects are, then the steady-state output *O* of the element is in general given by the following equation:

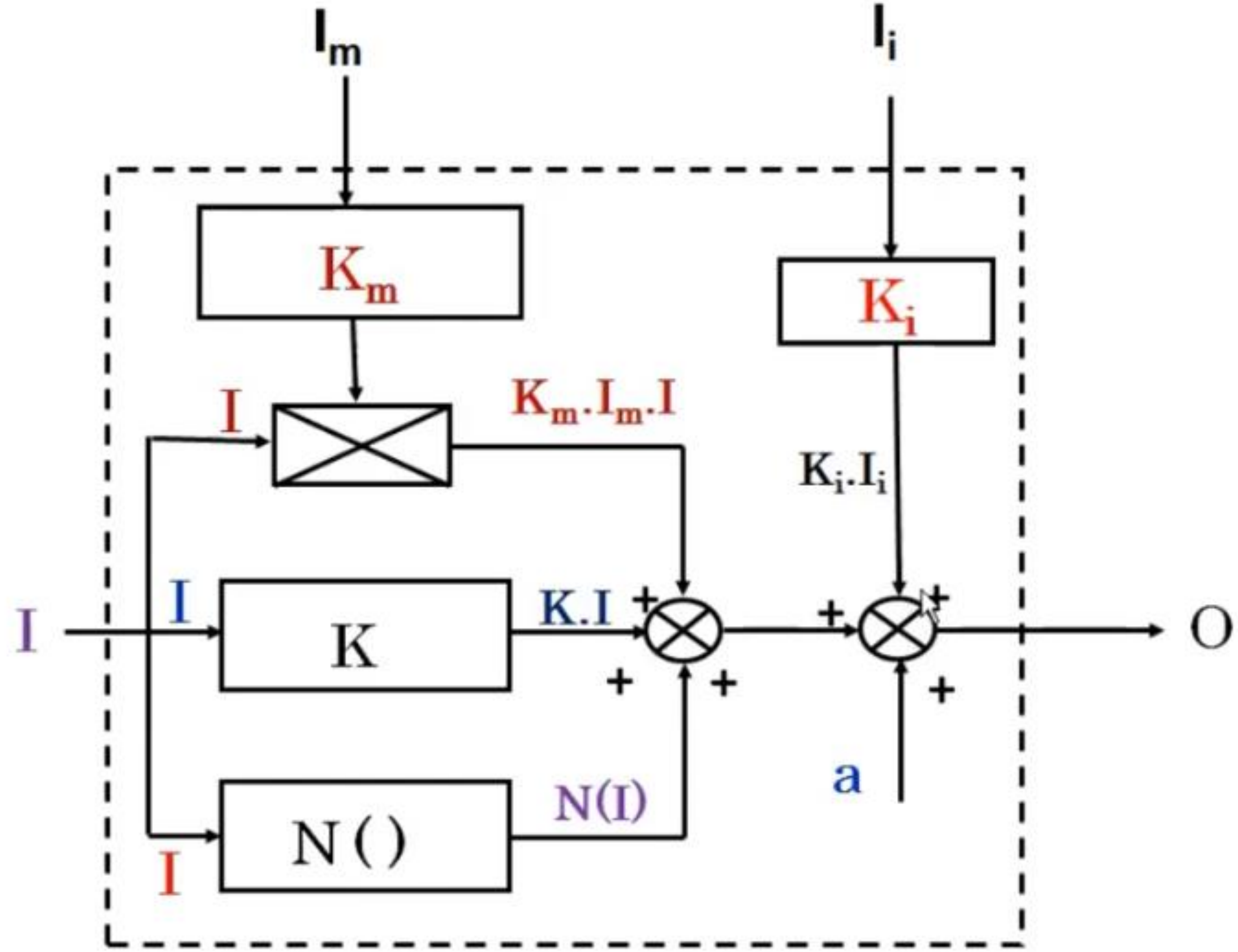
$$O = \underbrace{(K.I) + a}_{\text{Ideal Line}} + \underbrace{N(I)}_{\text{Non-Linearity}} + \underbrace{(K_m.I_m).I}_{\text{Modifying}} + \underbrace{K_i.I_i}_{\text{Interfering}}$$

General model of measuring system

$$O=\underbrace{(K+(K_m.I_m))}_{\text{New Slope}}.I+\underbrace{(a+K_i.I_i)}_{\text{New Intercept}}+\underbrace{N(I)}_{\text{Non-Linearity}}$$

- The Figure below shows this equation in block diagram form to represent the **static** characteristics of an element.

General model of measuring system



General model of element