The Republic of Iraq

Ministry of Higher Education and Scientific Research

Northern Technical University

Engineering Technical College / Mosul

Department of Power Mechanics Techniques Engineering

ENGINEERING CONTROL SYSTEMS

For Fourth-Year Students

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Ministry of Higher Education & Scientific Research
Foundation of Technical Education
Technical College/ Mosul
Dept. of Ref. & Air-Cond. Eng. Technology

CONTROL SYSTEMS

4th year students

By
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2025-2026

Control Systems

Topics Description:

- Fundamentals of control systems.
- Analysis and design of control systems using physical system models.
- State variables, steady-state error, time- and frequency responses, control system stability.
- Root locus analysis and controller design – PI, PD, PID, lead-lag compensator.
- Nyquist stability criterion.

Credit Hours: 3

Prerequisite(s) by course: Accepted or better

Prerequisite by topics:

- Signals & Systems,
- Convolution,
- Laplace transform,
- Differential Equation,
- Modulation,
- Time Sampling,
- Feedback

Required Textbook:

- Control Systems Engineering by Nise, 7th edition, Wiley, 9781118170519
- J. Nagrath and M. Gopal, “Control System Engineering”, New Age International Publishers,

Other Materials: Class notes posted on Blackboard

Learning Outcomes:

Students who successfully complete the course will be able to:

1. demonstrate an understanding of the fundamentals of (feedback) control systems.
2. determine and use models of physical systems in forms suitable for use in the analysis and design of control systems.
3. express and solve system equations in state-variable form (state variable models).
4. determine the time and frequency-domain responses of first and second-order systems to step and sinusoidal (and to some extent, ramp) inputs.
5. determine the (absolute) stability of a closed-loop control system
6. apply root-locus technique to analyze and design control systems.
7. communicate design results in written reports.

Course Topics:

- Introduction to Control Systems
- Laplace Transforms
- Transfer Function,
- Stability
- Block Diagrams and Signal Flow Graphs
- Physical Systems Modeling
- Root Locus Analysis
- Time Domain Analysis of Control Systems
- Frequency Domain Analysis of Control Systems
- Control System Design (separated in different topics)

Course Contribution to Undergraduate Program Outcomes:

Control systems course contributes to an achievement of:

- An ability to identify, formulate, and solve complex engineering problems by applying principles of engineering, science, and mathematics
- An ability to recognize ethical and professional responsibilities in engineering situations and make informed judgements, which must consider the impact of engineering solutions in global, economic, environmental, and societal context.
- An ability to acquire and apply new knowledge as needed, using appropriate learning strategies.

General Course Policies

Academic Integrity

- Unless otherwise stated, assignments and examination work are expected to be the sole effort of the student submitting the work.
- Students are expected to follow the Northern Technical University Honor Code and they should report on every instance of a suspected violation.
- Students found responsible for violations of the Code will be subject to academic penalties under the Code in addition to whatever disciplinary sanctions are applied.

Recommended Study Habits

- Read the assigned material before class.
- Bring thoughtful questions to class for discussion.
- Prepare for the exams in study groups.
- Take notes during class discussions and while completing reading assignments.

Deviations

Minor deviations from the syllabus are a normal part of any adaptive teaching and learning process.

Syllabus of Control Systems Engineering

Undergraduate Program / 4th year

First Course 2020 – 2021

Objectives:

- To understand the purpose of a control system, it is useful to examine examples of control systems through the course of history, it is useful to examine examples of control systems through the course of history.
- Discuss the notion of a control system design and the model used in the control system synthesis.
- Understand basic control concepts and basic control actions.
- To introduce state variable representation of physical systems and study the effect of state Feedback.
- To understand the use of transfer function models for analysis physical systems and introduce the control system components.
- To introduce the elements of control system and their modeling using various Techniques.

WEEK	TOPIC
1	Introduction: A History and need of Control Systems, Basic Elements of Control System and Terminology, Open-loop and Closed-loop systems and their comparison, Manual vs. Automatic Control System,
2	Representation of Control Systems, Concept of Feedback, Feedback Control System Characteristics and Performance, Requirements of Ideal Control System,
3	Representation of Control System Components: Study of various types of control system components and their mathematical representation of Mechanical system, Electrical system, Thermal system, and Fluid system,
4	Mathematical representation (Modeling) of Mechanical and Electrical systems,
5	Block Diagrams and Transfer function,
6	Translational Mechanical System Transfer Functions,
7	Comparators for Rotational or Linear Motions
8	Rotational Mechanical System Transfer Functions,
9	Transfer Functions for Systems with Gears,
10-11	System Analogy (Force-Voltage and Force-Current-analogy),
12-13	Block Diagram Reduction Techniques,
14	Signal Flow Graph,
15	Mason’s Rule.

1 / A –Target population:

For students of 4th year.
Technical college.
Department of refrigeration and air conditioning.

1 / B –Rationale:

The refrigeration and air conditioning industry is second biggest industry in the world after automobile industry. The application of refrigeration and air conditioning industry are so numerous as there is hardly any field left without the use of refrigeration or air conditioning.

1 / C –Central Idea:

1. To make the students understanding the meaning of air conditioning and refrigerating.
2. Learning the students to components of air conditioning and refrigerating systems.
3. To make the students can be solution the problems of air conditioning and refrigerating systems.

1 / D –Instructions:

To let the student, study the basic refrigeration and air conditioning operations and study the different types of refrigeration and the use of its tables and charts. The study of wet air specifications, different air conditioning operations and compression refrigeration cycle.

CONTROL SYSTEMS

Introduction

There were several stages in the history of development of technology. After the invention of steam engine and other machinery, the efficiency and the speed of production was improved in the era of Industrial Revolution. However, so many machines claimed to be automatic were those nonstop machines indeed. Owing to the lack of sensitive sensors and processors, those machines were not equipped with good control systems, thus they were only categorized as **open loop control system (OLCS)**. To improve the precision of the control systems, operators were engaged to control the machines. They played an important role as sensors and decision-makers. They compared the inputs with the status needed, then provided feedback and made decision (their brains). Afterwards, they adopted some procedures to stabilize the systems and minimized the errors. Lastly, the outputs were close to the requirements. Therefore, manual operation in the system is a kind of **closed loop control system (CLCS)**.

After the trustable sensors, processors and driving devices were well developed, automatic machinery gradually replaced those manual ones. Under the conditions of clear and repeated procedures, and those procedures which are operated by automatic adjustment system instead, automatic control machinery is more suitable for use. Therefore, those automatic controlled machines are suitable for boring and repeated works. For example, it is better for a temperature sensor involved in the control of the switching on or off the compressor of the air-conditioner.

After the emergence of processors and new models of sensors, manual control systems were gradually and easily replaced by **computer control systems**. Therefore, machinery becomes automatically controlled. For example, a newly developed "internet refrigerator" can automatically order food through internet when it is empty.

Definition of Control System

A **control system** is a **system of devices** or **set of devices**, that manages, commands, directs or regulates the behavior of other device (s) or system (s) using control loops to achieve desire results. It can range from a single home heating controller using a thermostat controlling a domestic boiler to large Industrial control systems which are used for controlling processes or machines.

Internal **control** activities are usually classified into the following **three types**:

1. **Preventive** controls.
2. **Detective** controls.
3. **Corrective** controls.

The **input** to the system is the **reference value**, or set point, for the system output. The **output** is the actual response obtained from a system.

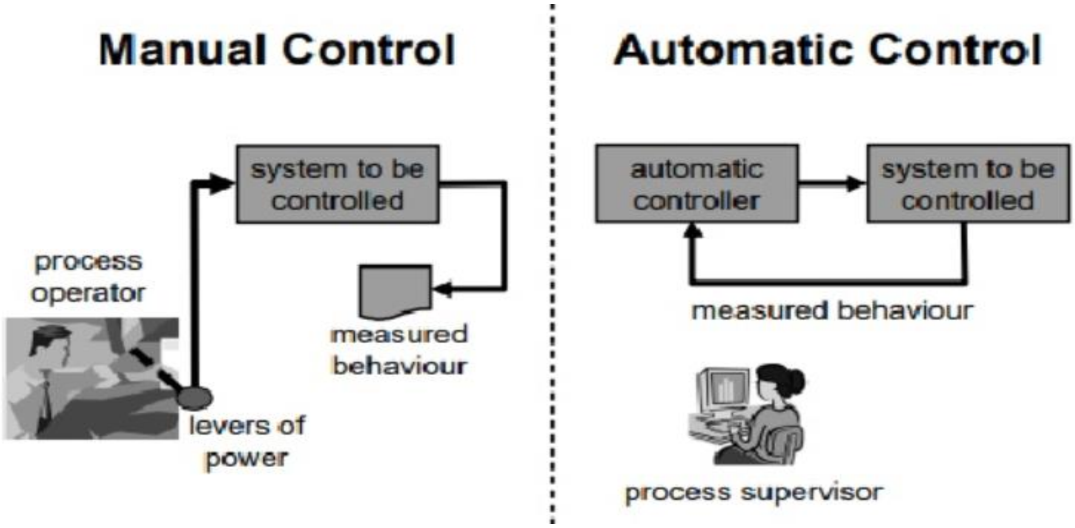
Manual and Automatic Control Systems

Manual Control System:

A person controlling a machine is called **manual control**. Ex: Driving a car

Automatic Control System:

Machines only is called an **automatic control**. Ex: Central air conditioning (or central A/C) is a system in which air is cooled at a central location and distributed to and from rooms by one or more fans and ductwork. The work of the air conditioner compressor is what makes the whole process of air conditioning possible.



Advantages of Automatic Control Systems

The main advantages of Automatic Control Systems are:

- Increased throughput or productivity.
- Improved quality or increased predictability of quality.
- Improved robustness (consistency), of processes or product.
- Increased consistency of output.
- Reduced direct human labor costs and expenses.

Open Loop and Closed Loop Control System

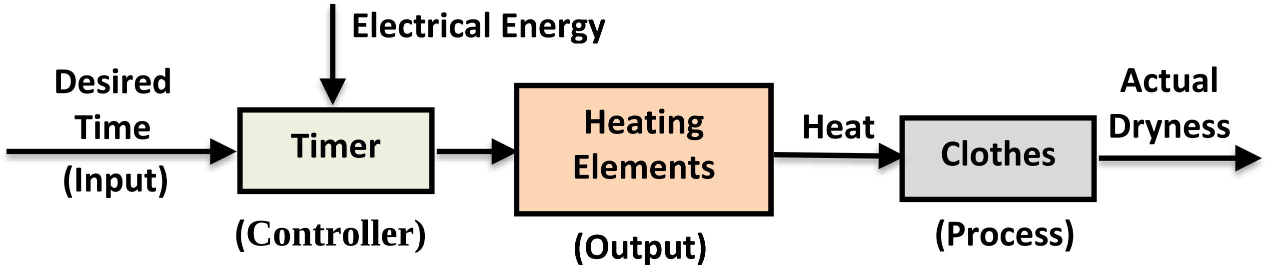
Control action is the quality of a system which is responsible for activating the system in order to produce the output. The open loop and closed loop system are basically the parts of Control system. Depending on weather a control action performed should depends on the status of output, Control System are classified as:

- Open Loop Control System
- Closed Loop Control System

► Open Loop Control System

An Open loop System is also known as the control system **without feedback**, it **does NOT measure the actual output** and there is no correction to make that output conform to the desired output. The components of these system are:

- input,
- processing system and
- output.



Advantages of Open Loop Control System:

- They are simple in constructions,
- They are economical in nature.
- They have stability problem.

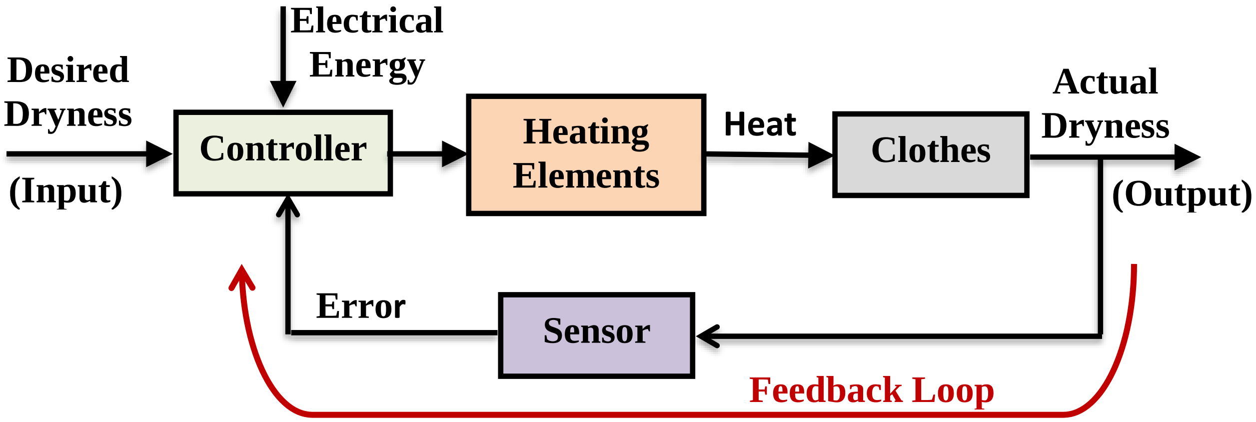
Examples of these systems are:

1. Water sprinkler.
2. Traffic Control System: this system is based upon the timer and not on the density of traffic.

▶ CLOSED LOOP SYSTEM

These systems are also known as the system **with feedback** where the system **includes a sensor to measure the output** and **uses feedback** of the sensed value to compare with the reference element and thus the error occurred at that time is filtered out to achieve desired output. The components of these system are:

- input,
- reference element,
- controller,
- processing system and
- output.



Advantages of Closed loop Control System

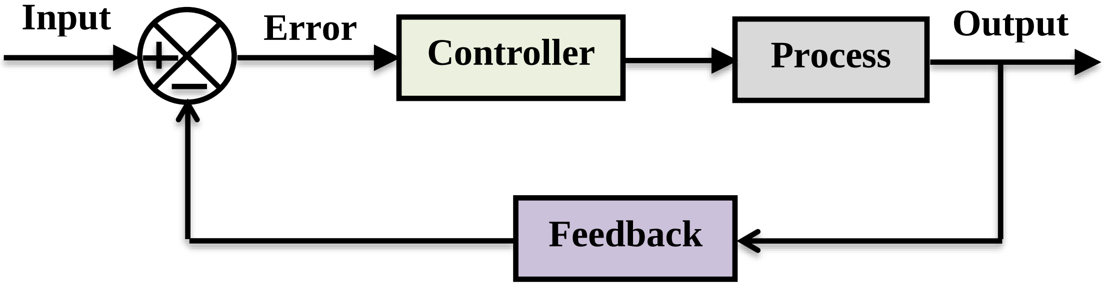
- They reduce the effect of non-linearities.
- They are less economical in nature.
- These systems are highly stable.

Example of these systems are:

- Missile launching system,
- Voltage stabilizer

Feedback Control Systems

A system that **maintains** a prescribed **relationship** between the **output** and some **reference input** by **comparing them** and **using** the **difference** (i.e. **error**) as a **means of control** is called a **feedback control system**. Closed Loop Control Systems utilizes feedback to compare the actual output to the desired output response.



The term "**feedback control system**" is a general term which applies to any system in which the controlled variable is measured and fed back to be compared with the reference input.

Main feature: Feedback and Comparison

- **Sensor –** measures the system output and feeds it back
- **Comparator –** computes the difference between the reference signal and the sensor output to give the controller a measure of the system error.

The Effects of Feedback

- Reduce the error between the actual and the desired value
- Change the stability of the system
- Change the overall system gain
- Change the sensitivity of the system gain
- Change the bandwidth of the system
- Reduce the effect of external disturbances and noise
- Reduce the effect of variations of system parameters

Requirements for a Good Control System

a) **Accuracy:** Accuracy must be very high as error arising should be corrected. Accuracy can be improved by the use of feedback element.

b) **Sensitivity:** A good control system senses quick changes in the output due to an environment, parametric changes, internal and external disturbances.

c) **Noise:** Noise is a unwanted signal and a good control system should be sensitive to these type of disturbances.

d) **Stability:** The stable systems has bounded input and bounded output. A good control system should response to the undesirable changes in the stability.

e) **Bandwidth:** To obtain a good frequency response, bandwidth of a system should be large.

f) **Speed:** A good control system should have heigh speed that is the output of the system should be fast as possible.

g) **Oscillation:** For a good control system oscillation in the output should be constant and must follow Barkhausein's Criteria (Conditions which are required to be satisfied to operate the circuit as an oscillator are called as “Barkhausen criterion” for sustained oscillations. The Barkhausen criteria should be satisfied by an amplifier with positive feedback to ensure the sustained oscillations).

Transfer Function

A control system consists of an output as well as an input signal. The output is **related** to the **input** through a **function** call **transfer function**. The transfer function system is defined as the ratio of the Laplace transform of the output (response function), **C(s)** to the Laplace transform of the input (driving function) **R(s)** under the assumption that all **initial conditions** are **zero**. Thus, the **transfer fun**ction of a system can be **represented** by **Laplace form** by **dividing output Laplace transfer function** to **input Laplace transfer function.**

Or

$$\boldsymbol{TF = G(s) = \frac{\mathcal{L}\{\text{output}\}}{\mathcal{L}\{\text{input}\}} = \frac{C(s)}{R(s)}}$$

That is, transfer function of the system multiplied by input function gives the output function of the system

$$\boldsymbol{G}(\boldsymbol{s}) \cdot \boldsymbol{R}(\boldsymbol{s}) = \boldsymbol{C}(\boldsymbol{s})$$

This function is represented by a block and the complete diagram of control system using these **blocks** which represent transfer function and **arrows** which represent various signals, is collectively known as **block diagram** of a control system.

For any control system there exists a reference input termed as excitation or cause which operates through a transfer operation termed as transfer function and produces an effect resulting in controlled output or response. Thus, the cause and effect relationship between the output and input is related to each other through a transfer function.

Why Input, Output and Other Signals are Represented in Laplace Form in a Control System?

It is not necessary that output and input of a control system are of same category. For example, in electric motors the input is electrical signal whereas the output is mechanical signal since electrical energy required to rotate the motors. Similarly, in an electric generator, the input is mechanical signal and the output is electrical signal since mechanical energy is required to produce electricity in a generator. **But** for **mathematical analysis**, of a system **all kinds of signals** should be **represented** in a **similar form**. This is done by transforming all kinds of signal to their Laplace form.

Representation of Control System Components

The mathematical relationships of control systems are usually represented by block diagrams. These diagrams indicate more realistically the actual processes which are taking place as opposed to a purely abstract mathematical representation. In addition, it is easy to form the overall block diagram for an entire system by merely combining the block diagrams for each component or part of the system.

A **circle** is the symbol which is used to indicate a **comparator**, where the arrowheads pointing toward the circle indicate input quantities (reference input) $(\boldsymbol{r})$. While the arrowhead leading away indicates the output $(\boldsymbol{e})$ which is equal to the actuating signal. For the case in which the controlled variable $(\boldsymbol{c})$ is fed back directly (i.e., for unity-feedback systems), the signal coming from the comparator is $(\boldsymbol{r}-\boldsymbol{c})$, which is equal to the actuating signal $(\boldsymbol{e})$. The **sign** at each input arrowhead **indicates** whether the quantity is to be **added** or **subtracted** as illustrated in **Figure 1.**

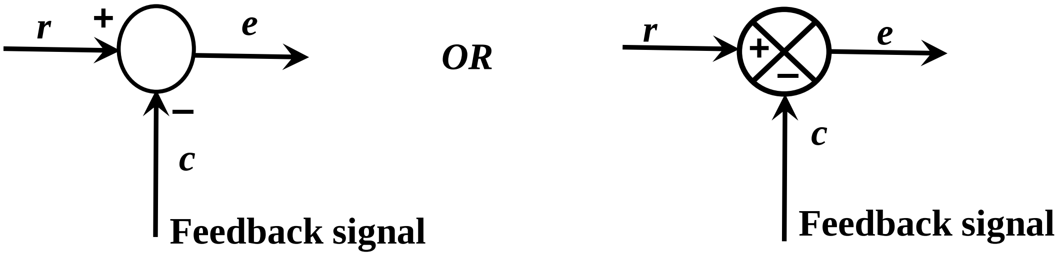


Figure 1. Block diagram of a comparator

The mathematical relationship for this operation is:

$$\boldsymbol{e}=\boldsymbol{r}-\boldsymbol{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{1})$$

The relationship between the actuating signal $\boldsymbol{e}$, which enters the control elements, and the controlled variable $\boldsymbol{c}$, which is the output of the control element, is expressed by the equation:

$$\boldsymbol{c} = \boldsymbol{G}(\boldsymbol{s})\ \boldsymbol{e} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{2})$$

where $\boldsymbol{G}(\boldsymbol{s})$ represents the operation of the control elements. The actual values of $\boldsymbol{G}(\boldsymbol{s})$ can be found for any specific control systems.

A **<u>box</u>** is the **symbol** is used to **represents multiplication**. In this case, the quantity $\boldsymbol{e}$ is multiplied by the transfer function in the box $\boldsymbol{G}$ to obtain the output $\boldsymbol{c}$**.** Thus, **circles** are used to indicate **summing points** and **boxes**, or **blocks** to indicate **multiplication**, any **linear mathematical expression** may be **represented** by **block-diagram notation**.

The block-diagram representation for the **Equation (2)** is shown in **Figure 2**.

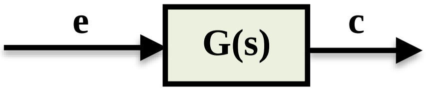


Figure 2. Block diagram of the control system

To complete block diagram for an elementary **<u>unity-feedback control system</u>** is obtained by combining **Figure 1** and **Figure 2** to yield **Figure 3**.

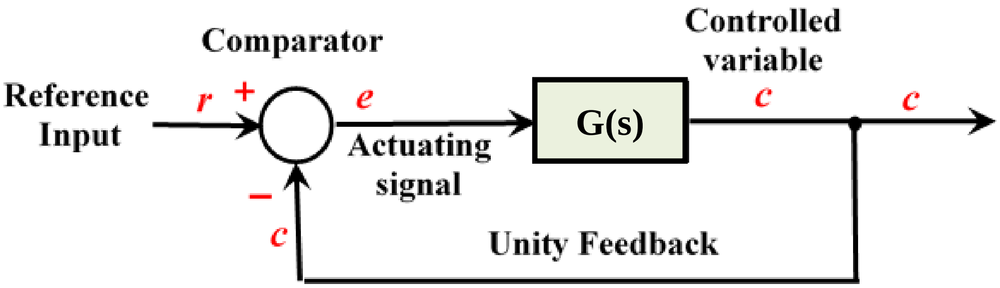


Figure 3.
Block diagram of an elementary unity-feedback control system

When the controlled variable is fed back to the comparator, it is usually necessary to convert the form of the controlled variable to form that is suitable for the comparator. For example, in a temperature control system the controlled temperature is generally converted to a proportional force or position for use in the comparator. This conversion is accomplished by feedback elements $\boldsymbol{G}$. The block-diagram representation for this more general case of a feedback control system is illustrated in **Figure 4**.

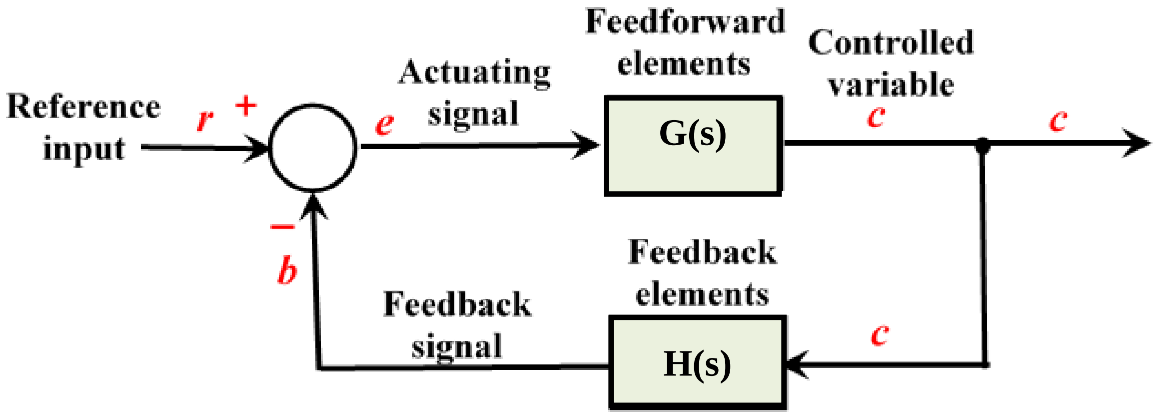


Figure 4.
Block diagram of an elementary feedback control system

The signal which is fed back is:

$$\boldsymbol{b = H(s)c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{3})$$

The elements represented by $\boldsymbol{H}$ are called the **feedback elements** because they are located in the feedback portion of the control system. The control elements represented by $\boldsymbol{G(s)}$ are called the **feedforward elements** because of their location in the feedforward portion of the loop. The actuating signal $\boldsymbol{e}$ is now:

$$\boldsymbol{e = r - b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{4})$$

This actuating signal $\boldsymbol{e}$ is a measure of indication of the error.

The term “**feedback control system**” is a general term which applies to any system in which the controlled variable is measured and fed back to be compared with the reference input.

To investigate the performance of the control system, it is necessary to obtain the mathematical relationship ***G*****(s)** relating the controlled variable ***c*** and the actuating signal ***e*** of the feedforward elements. This is accomplished by first obtaining the mathematical representation for each component between the actuating signal and the controlled variable and the expressing each of these equations as a block diagram. The combination of the block diagrams for each component yields the desired representation for ***G*****(*s*)**. The value of ***H*****(s)** is obtained by applying same technique to the components in the feedback portion of the control.

The quantity ***G*****(s)** could be obtained by writing the mathematical equation describing the operation of each component between ***e*** and ***c*** and then combining these individual equations algebraically to obtain the overall relationship between ***e*** and ***c***. However, for all but the simplest systems, this procedure proves cumbersome because of the interaction between the various components in a typical control system. In addition, the block diagram method gives one a better understanding of the system because of its visual representation.

Definition of Transfer Function

❖ Mechanical System

○ Mechanical System Components

- **Spring:** The load deflection characteristics for a mechanical spring are shown in **Figure 5**.

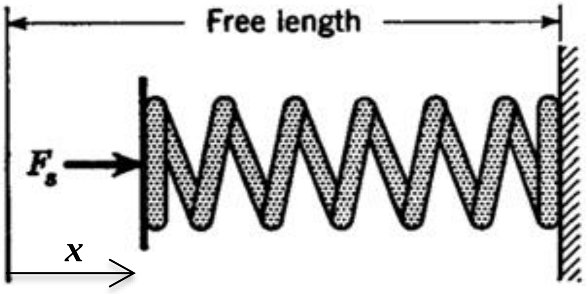


Figure 5.
Spring characteristics

The spring force F_s required to compress a spring x in from its free length is given by the equation:

$$F_s = Kx \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (5)$$

Where K, the spring rate, is a constant which is equal to the slope of the curve of the load F_s versus deflection x. The input to a spring is usually the force F_s and the output is the deflection x**.** Generally, we define the transfer function as:

$$TF(s) = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}} = \frac{x}{F_s} = \frac{x}{Kx} = \frac{1}{K} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$$

We refer to the $(TF)^{-1}$ of a single element as **the Impedance $Z(s)$**:

$$Z(s) = (TF)^{-1} = K \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (7)$$

so that the block diagram representation for the **Equation (6)** is as shown in **Figure 6**.

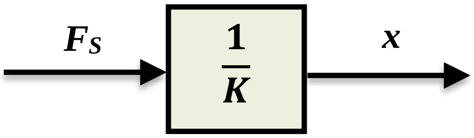


Figure 6. Block diagram of **Equation 6**

- **<u>Damper</u>:** (a device for damping shock or vibration)

  For a various damper illustrated in **Figure 7**, the force F_d required to move one end of the dashpot at a velocity **V** relative to the other end is equal to the product of the damping coefficient D and the velocity. That is:

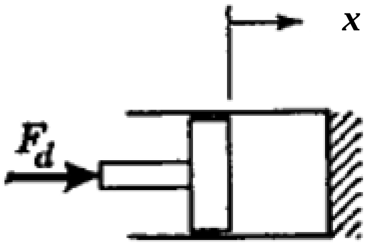


Figure 7.
Linear viscous damper

$$\boldsymbol{F_d = DV = D\frac{dx}{dt} = D\dot{x}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(\mathbf{8})$$

With the force $\boldsymbol{F_d}$ as the input and the displacement $\boldsymbol{x}$ as the output, the transfer function can be defined as:

$$\boldsymbol{TF(s) = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}} = \frac{x}{F_d} = \frac{x}{DV}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\quad(\mathbf{9})$$

so that the block diagram representation for **Equation (9)** is shown **in Figure 8**.

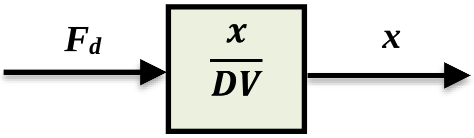


Figure 8. Block diagram of **Equation (9)**

The Impedance $\boldsymbol{Z(s)}$ as the $(\boldsymbol{TF})^{-\mathbf{1}}$ of a single element is:

$$\boldsymbol{Z(s) = (TF)^{-1} = \frac{DV}{x}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots(\mathbf{10})$$

- **<u>Mass</u>:** By Newton`s first law of motion, it follows that the summation of the external forces $\boldsymbol{\Sigma F_e}$ acting on a mass is equal to the product of the mass and acceleration:

$$\boldsymbol{F = \sum F_e = ma = m\frac{d^2X}{dt^2} = m\,\ddot{x}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots\ldots\ldots..\ldots...(\mathbf{11})$$

With the external forces $\Sigma \boldsymbol{F_e}$ as the **input** and the **displacement** $\boldsymbol{x}$ as the **output**, the transfer function can be defined as:

$$\boldsymbol{TF(s)} = \frac{\boldsymbol{\mathcal{L}\{output\}}}{\boldsymbol{\mathcal{L}\{input\}}} = \frac{\boldsymbol{x}}{\boldsymbol{F}} = \frac{\boldsymbol{x}}{\boldsymbol{m\,\ddot{x}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\boldsymbol{12})$$

This is represented diagrammatically in **Figure 9**.

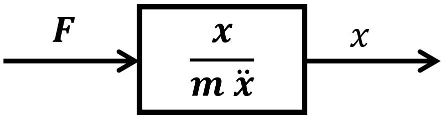


Figure 9. Block diagram of **Equation (12)**

The Impedance $\boldsymbol{Z(s)}$ **as** the $(\boldsymbol{TF})^{-\boldsymbol{1}}$ of a single element is:

$$\boldsymbol{Z(s)} = (\boldsymbol{TF})^{-\boldsymbol{1}} = \frac{\boldsymbol{m\,\ddot{x}}}{\boldsymbol{x}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\boldsymbol{13})$$

- **Mass, Spring, and Damper Combination**

For the combination of mass, spring, and damper shown in **Figure 10**, the spring force and damper force are opposed to, or resist, the motion caused by the applied load $\boldsymbol{F}$.

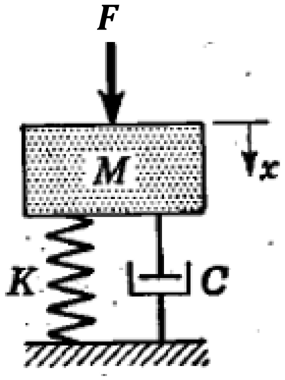


Figure 10. Combination of mass, spring, and

The summation of the forces acting on the mass is:

$$\sum F_e = F + mg - F_s - F_d = ma = m\,\ddot{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots....(14)$$

or

$$F = m\,\ddot{x} + D\dot{x} + Kx - mg \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots\ldots\ldots..\ldots\ldots\ldots..(15)$$

For control work, it is more convenient to make measurements with respect to some initial or reference operating point x_i.

In case of the system is at rest or equilibrium at the reference operating point (i.e., when

$$m\,\ddot{x} = D\dot{x} = x = 0,$$

then the value of the applied force is:

$$F_i = Kx_i - mg \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ldots\ldots\ldots..(16)$$

Which is constant in the preceding equation, gives:

$$F - F_i = f = m\,\ddot{x} + D\dot{x} + Kx \ldots\ldots\ldots..\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots....(17)$$

Where f is the change in applies force from the value F_i required for equilibrium at the reference operating point.

Equation (17) is a general equation describing the dynamic behavior of the system although x and f are measured from reference operating point. It is not necessary that the system be initially at this reference operating point or that the system be initially at rest. It is usually much easier to obtain the equation of operating with respect to some convenient reference point rather than using absolute values. When absolute values are desired, it is an easy matter to add the reference value to the variation. The block diagram for **Equation (17)** is shown in **Figure 11**.

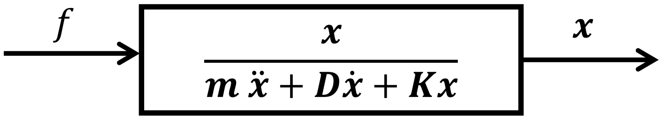


Figure 11. Block diagram of **Equation (17)**

✪ Rotational Mechanical Components

A torsional spring is characterized by the equation:

$$\boldsymbol{T_s = K_s\theta} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (\mathbf{18})$$

Where $\boldsymbol{T_s}$**:** is the **torque** tending to **twist spring**,

$\boldsymbol{K_s}$: torsional spring rate, and

$\boldsymbol{\theta}$: angular displacement of spring.

A well – known example of a torsional spring is a shaft shown in Figure 12.

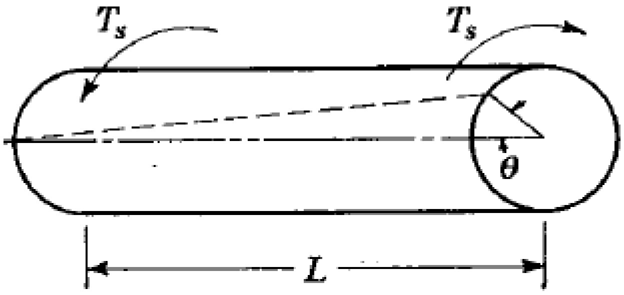


Figure 12.
Shaft acting as a torsional spring

The right end of the shaft is displaced an angle $\boldsymbol{\theta}$ with respect to the left end because of the **twisting torque** $\boldsymbol{T_s}$. For a straight shaft, the torsional spring rate is:

$$\boldsymbol{K_s = \frac{\pi D^4 G}{32L}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \dots \quad (\mathbf{19})$$

Where, $\boldsymbol{G}$: is the modulus of elasticity in shear,

$\boldsymbol{D}$: diameter of shaft, and

$\boldsymbol{L}$: length of shaft.

A disk rotating in a viscous medium and supported by a shaft is shown in **Figure 13**.

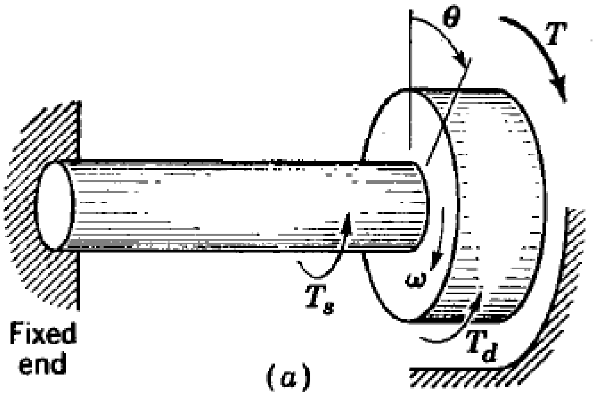


Figure 13.
Torsional inertia spring damper combination

The **torque** $\boldsymbol{T_d}$ required to overcome viscous of a rotating member is:

$$\boldsymbol{T_d = D_v\omega = D_v \frac{d\theta}{dt} = D_v\dot{\theta}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (\mathbf{20})$$

Where $\boldsymbol{D_v}$ is the **coefficient of viscous friction**, and $\boldsymbol{\omega}$ **angular velocity**.

The applied torque tending to rotate the disk is $\boldsymbol{T}$. The **shaft torque** and **viscous friction** oppose the motion so that:

$$\sum \boldsymbol{T_e = T - T_s - T_d = J\alpha = J\,\ddot{\theta}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \dots\dots .. (\mathbf{21})$$

Where $\sum \boldsymbol{T_e}$: is the summation of external torque acting on the disk.

$\boldsymbol{J}$: shaft inertia

$\boldsymbol{\alpha}$: angular acceleration

The substitution of $\boldsymbol{T_s}$ from **Equation (18)** and $\boldsymbol{T_d}$ from **Equation (20)** into **Equation (21)** yields:

$$\boldsymbol{T = J\,\ddot{\theta} + D_v\dot{\theta} + K_s\theta} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . . . \dots .. (\mathbf{22})$$

The block diagram representation for this system is shown in **Figure 14**.

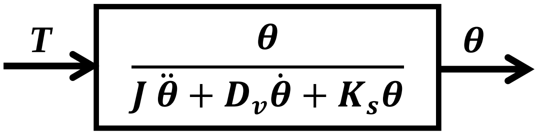


Figure 14. The block diagram representation for torsional inertia spring damper combination acting as a torsional spring.

✷ Combination of Mechanical Elements

Often, many elements are connected in either a series or a parallel arrangement. Much simplification in arriving at the equation for such systems is afforded by the use of the theorem for series and theorem for parallel combinations.

◊ Series Combination of Mechanical Elements

A series arrangement of linear mechanical elements is shown in **Figure 15**.

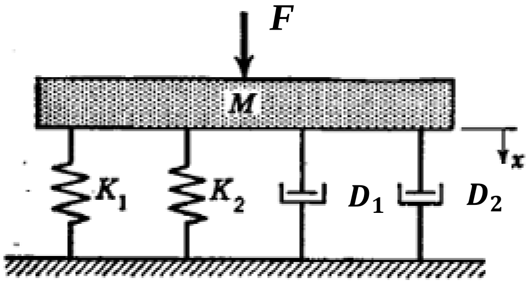


Figure 15: Mechanical elements in series Combination

In general, it is better to use the **equivalent "grounded-chair" representation** for **a mass**, as shown in **Figure 16** rather than the more common representation of **Figure 15**.

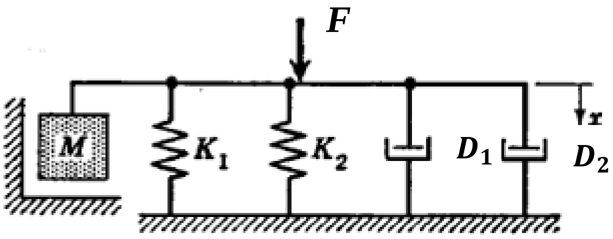


Figure 16: The **equivalent "grounded-chair" representation** for **a mass**.

The fact that the mass is in series with the other elements is more readily seen in **Figure 16** than from **Figure 15**. In determining inertia force, the acceleration of the mass is always taken with respect to the earth. Thus, providing the grounded chair to indicate motion relative to ground is a more justifiable representation than **Figure 15**, which shows better the actual physical arrangement of the elements in the system.

For **series mechanical elements**, the force $\boldsymbol{F}$ is equal to the **summation of the forces acting** on **each individual component**, and **each element** undergoes the **same displacement**. Thus,

$$\boldsymbol{F = m\,\ddot{x} + K_1x + K_2x + D_1\dot{x} + D_2\dot{x}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \boldsymbol{(23)}$$

Where $\boldsymbol{x}$ and $\boldsymbol{F}$ are measured from a convenient reference operating point.

The equivalent representation for this system is shown in **Figure 17**.

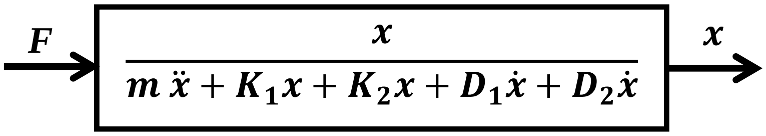


Figure 17. The equivalent representation for torsional inertia spring damper combination acting as a torsional spring.

◊ **<u>Parallel Combination of Mechanical Elements</u>**

A parallel combination of mechanical elements is shown in **Figure 18**.

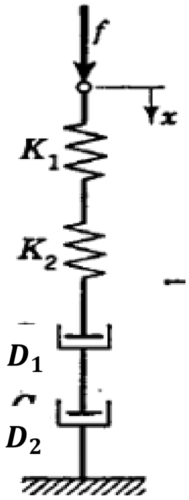


Figure 18: Mechanical elements in parallel combination

For parallel combination, the force $\boldsymbol{F}$ is transmitted through each element. In addition, **the deflection** is seen to be the **sum** of the **individual deflection** of **each element**. Thus

$$\boldsymbol{x} = \frac{\boldsymbol{F}}{\boldsymbol{K_1}} + \frac{\boldsymbol{F}}{\boldsymbol{K_2}} + \frac{\boldsymbol{F}}{\boldsymbol{D_1 p}} + \frac{\boldsymbol{F}}{\boldsymbol{D_2 p}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (\boldsymbol{24})$$

or

$$\boldsymbol{F} = \frac{\mathbf{1}}{\frac{\mathbf{1}}{\boldsymbol{K_1}} + \frac{\mathbf{1}}{\boldsymbol{K_2}} + \frac{\mathbf{1}}{\boldsymbol{C_1 p}} + \frac{\mathbf{1}}{\boldsymbol{C_2 p}}} \boldsymbol{x} = \boldsymbol{Z_p x} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \dots (\boldsymbol{25})$$

The equivalent impedance $\boldsymbol{Z_p}$ for mechanical elements in parallel is:

$$\boldsymbol{Z_p} = \frac{\mathbf{1}}{\frac{\mathbf{1}}{\boldsymbol{K_1}} + \frac{\mathbf{1}}{\boldsymbol{K_2}} + \frac{\mathbf{1}}{\boldsymbol{C_1 p}} + \frac{\mathbf{1}}{\boldsymbol{C_2 p}}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \dots\dots (\boldsymbol{26})$$

The equivalent representation is shown in **Figure 19,**

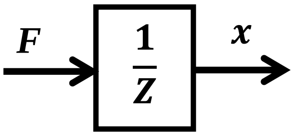


Figure 19. The equivalent representation for parallel combination

A necessary condition for **parallel elements** is that the **force** be **transmitted** through **each element**. Spring and dampers satisfy this condition because the force in the same on both sides. However, this is not the case for a mass such as that in **Figure 20**, because the difference in forces acting on both sides of a mass is utilized in acceleration.

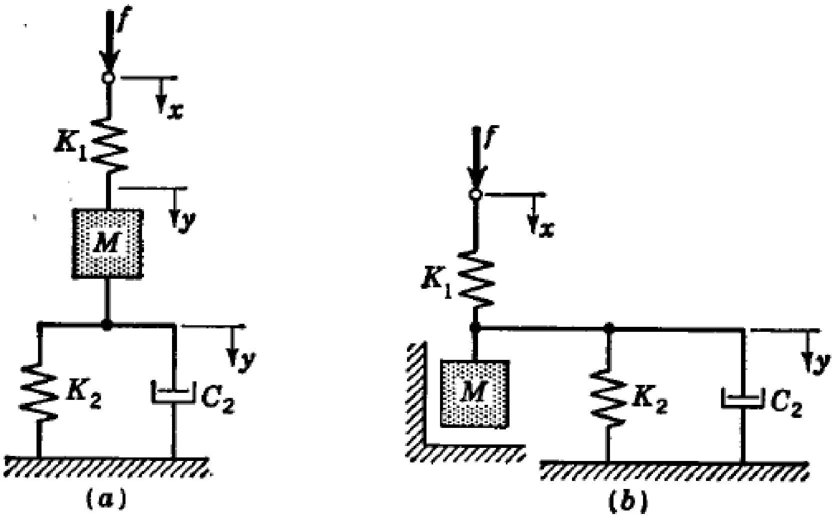


Figure 20.
Mechanical system

Thus, **a mass** located between other elements **cannot be** in **parallel** with them. A mass can be in **parallel only if it is the last element**, as shown in **Figure 21**.

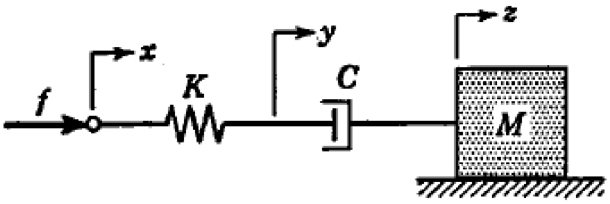


Figure 21.
Parallel mass – spring - damper combination.

For this system, the displacement $\boldsymbol{x}$ is:

$$\boldsymbol{x = (x-y) + (y-z) + z = \left(\frac{1}{K} + \frac{1}{Cp} + \frac{1}{mp^2}\right) f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\boldsymbol{27})$$

For rotational mechanical components, Parallel and series rules may also be developed by applying same techniques.

Illustrative Example: For the mass – spring – damper combination drawn in **Figure 20a** determine the equation relating $\boldsymbol{f}$ and $\boldsymbol{x}$, the equation relating $\boldsymbol{f}$ and $\boldsymbol{y}$, the equation relating $\boldsymbol{x}$ and $\boldsymbol{y}$.

Solution:

The first step is to draw the equivalent grounded – chair system, in which the motion of the mass with respect to ground is clearly indicated as pictorial in **Figure 20b.**

The spring $\boldsymbol{K_1}$ is in parallel with the series combination of $\boldsymbol{M}$, $\boldsymbol{K_2}$, and $\boldsymbol{C_2}$. Thus,

$$\boldsymbol{f = Z_p x = \frac{1}{\frac{1}{K_1} + \frac{1}{Z_s}} x = \frac{K_1 Z_s}{K_1 + Z_s} x} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots\dots (\boldsymbol{28})$$

Where, according to the series arrangement of $\boldsymbol{M}$, $\boldsymbol{K_2}$, and $\boldsymbol{C_2}$; $\boldsymbol{Z_s}$ equals:

$$\boldsymbol{Z_s = mp^2 + C_2 p + K_2} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots\dots\dots\dots (\boldsymbol{29})$$

Thus, from Equation (44) yields,

$$\boldsymbol{\frac{f}{Z_s} = \frac{K_1}{K_1 + Z_s} x = \frac{K_1}{mp^2 + C_2 p + K_1 + K_2} x} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots (\boldsymbol{30})$$

The force $\boldsymbol{f}$ is transmitted through the spring $\boldsymbol{K_1}$ and acts upon the series combination of $\boldsymbol{M}$, $\boldsymbol{K_2}$, and $\boldsymbol{C_2}$. Thus, the equation of motion for this part of the system which relates $\boldsymbol{f}$ and $\boldsymbol{y}$ is:

$$\boldsymbol{f = Z_s y \ \Rightarrow y = \frac{f}{Z_s} = \frac{f}{mp^2 + C_2 p + K_2}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots (\boldsymbol{31})$$

Equating Equations (30) and (31) yields the desired relationship between $\boldsymbol{x}$ and $\boldsymbol{y}$. That is,

$$\boldsymbol{y = \frac{K_1}{mp^2 + C_2 p + K_1 + K_2} x} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (\boldsymbol{32})$$

A comparison of corresponding terms in Equations for series and parallel mechanical and electrical systems show that the differential equation of operation for each system has the same form. The terms which possess similar positions are called **analogous quantities**.

❖ Electrical System

o Electrical System Components

The three basic components of electrical circuits are; resistor ($\boldsymbol{R}$), inductor ($\boldsymbol{L}$), and capacitor ($\boldsymbol{C}$), and the equations for the voltage drop $\boldsymbol{E_R}$ are:

- For a resistor ($\boldsymbol{R}$), the voltage drop $\boldsymbol{E_R}$ is given by the equation:

$$\boldsymbol{E_R = RI} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathbf{33})$$

Where the **resistor** ($\boldsymbol{R}$) in **Ohms** and **current** ($\boldsymbol{I}$) in **Amperes**.

- For an **indicator** ($\boldsymbol{L}$), the voltage drop $\boldsymbol{E_L}$ is given by the equation:

$$\boldsymbol{E_L = L\frac{dI}{dt}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathbf{34})$$

Where ($\boldsymbol{L}$) is the **inductance** in **Henrys**.

Similarly, the voltage drop across a **capacitor** $\boldsymbol{E_C}$ is given by:

$$\boldsymbol{E_L = \frac{1}{C}\frac{dI}{dt}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathbf{35})$$

Where ($\boldsymbol{C}$) is the **capacitance** in **Farads**.

The diagrammatic representations of Equations (33), (34), and (35) are drawn in **Figure 22**.

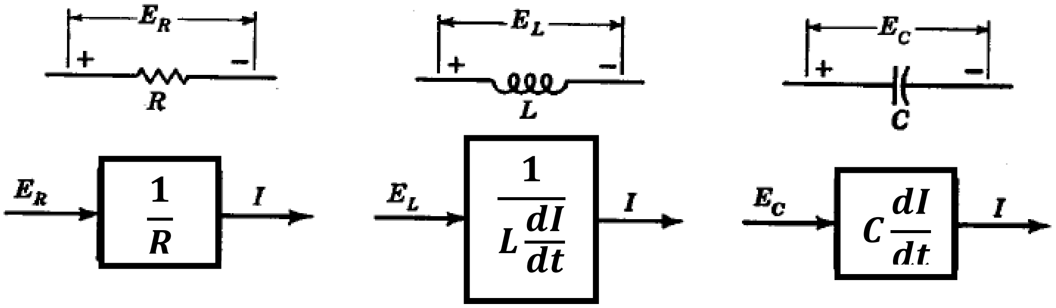


Figure 22.
Representation for resistor, inductor, and capacitor.

1.4.1 Series Combination of Electrical Components

For the **series RLC circuit** shown in **Figure 23a**.

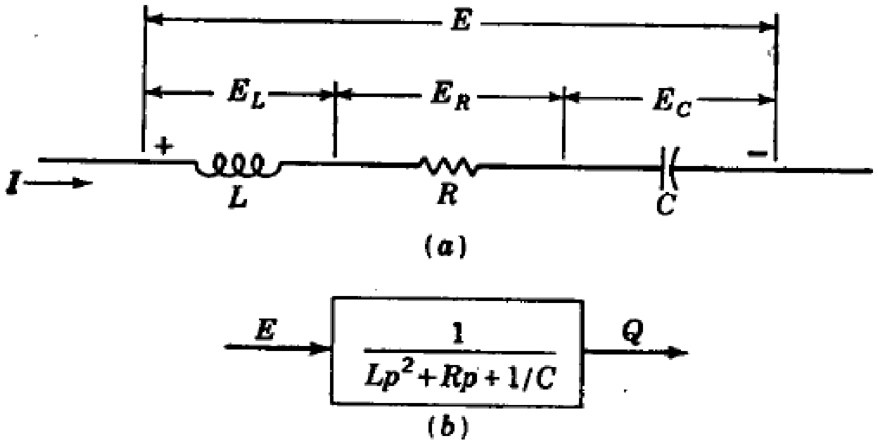


Figure 23.

General RLC Series combination

The total voltage drop is the sum of the voltage drop across the inductor E_L, plus that across the resistor E_R, and that across that capacitor E_C:

$$E = E_L + E_R + E_C = \left(Lp + R + \frac{1}{Cp}\right) I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$$

The charge Q is the time ntegral of the current, that is,

$$Q = \left(\frac{1}{p}\right) I \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

By noting that,

$$LpI = Lp^2 \left(\frac{I}{p}\right) = Lp^2 Q \,, \qquad RI = Rp \left(\frac{I}{p}\right) = RpQ \,, \qquad \frac{1}{C}\left(\frac{I}{p}\right) = \left(\frac{I}{C}\right) Q$$

Equation (36) becomes,

$$E = \left(Lp^2 + Rp + \frac{1}{C}\right) Q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

The overall block diagram representation for this RLC circuit is drawn in **Figure 23b**.

For the general series circuit drawn in **Figure 24a**, the total voltage drop $\boldsymbol{E}$ is the sum of the individual voltage drop across each element and the same current $\boldsymbol{I}$ flows through each element.

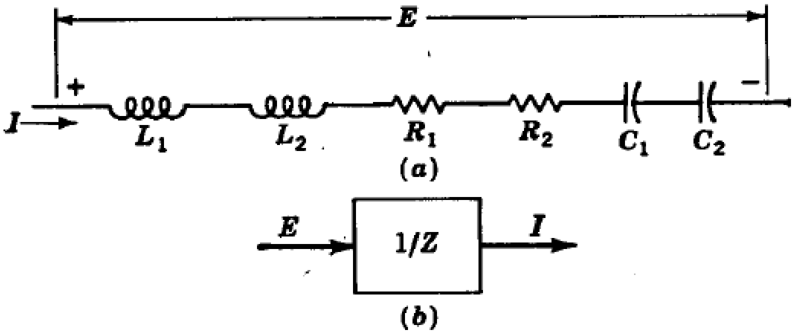


Figure 24.
Series circuit

$$\boldsymbol{E = \left(L_1 p + L_2 p + R_1 + R_2 + \frac{1}{C_1 p} + \frac{1}{C_2 p}\right) I = ZI} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots....(\boldsymbol{39})$$

Thus, the equivalent impedance $\boldsymbol{Z}$ for elements in series is:

$$\boldsymbol{Z = L_1 p + L_2 p + R_1 + R_2 + \frac{1}{C_1 p} + \frac{1}{C_2 p}} \quad \ldots\ldots\ldots..\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots..(\boldsymbol{40})$$

So that the equivalent representation is drawn in **Figure 24b**.

1.4.2 Parallel Combination of Electrical Components

General parallel combination of **RLC circuit** is drawn in **Figure 25a**.

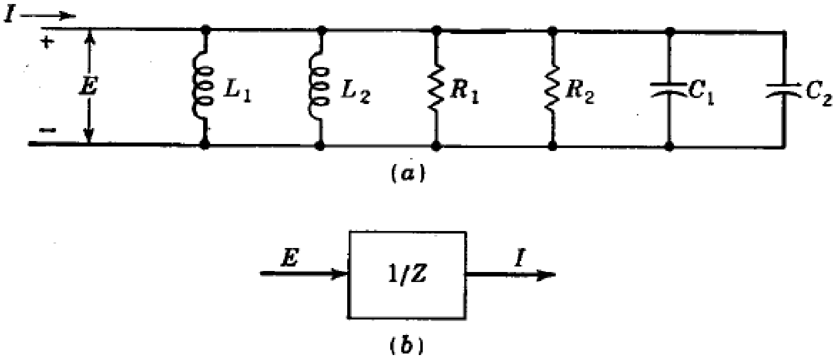


Figure 25.
General RLC parallel combination

The distinguishing features of a parallel combination are that the voltage drop $\boldsymbol{E}$ across each element is the same, and the total current $\boldsymbol{I}$ flowing into the system is the sum of the currents flowing through each element. Thus

$$\boldsymbol{I} = \frac{\boldsymbol{E}}{\boldsymbol{L_1 p}} + \frac{\boldsymbol{E}}{\boldsymbol{L_2 p}} + \frac{\boldsymbol{E}}{\boldsymbol{R_1}} + \frac{\boldsymbol{E}}{\boldsymbol{R_2}} + \frac{\boldsymbol{E}}{\frac{\boldsymbol{1}}{\boldsymbol{C_1 p}}} + \frac{\boldsymbol{E}}{\frac{\boldsymbol{1}}{\boldsymbol{C_2 p}}} \quad \dots\dots\dots..\dots\dots\dots\dots\dots\dots\dots\dots..\dots..(\boldsymbol{41})$$

or

$$\boldsymbol{E} = \frac{\boldsymbol{1}}{\frac{\boldsymbol{1}}{\boldsymbol{L_1 p}} + \frac{\boldsymbol{1}}{\boldsymbol{L_2 p}} + \frac{\boldsymbol{1}}{\boldsymbol{R_1}} + \frac{\boldsymbol{1}}{\boldsymbol{R_2}} + \boldsymbol{C_1 p} + \boldsymbol{C_2 p}} \boldsymbol{I} = \boldsymbol{ZI} \quad \dots\dots\dots..\dots\dots\dots\dots\dots..\dots..(\boldsymbol{42})$$

The equivalent impedance $\boldsymbol{Z}$ for elements in parallel is:

$$\boldsymbol{Z} = \frac{\boldsymbol{1}}{\frac{\boldsymbol{1}}{\boldsymbol{L_1 p}} + \frac{\boldsymbol{1}}{\boldsymbol{L_2 p}} + \frac{\boldsymbol{1}}{\boldsymbol{R_1}} + \frac{\boldsymbol{1}}{\boldsymbol{R_2}} + \boldsymbol{C_1 p} + \boldsymbol{C_2 p}} \quad \dots\dots\dots..\dots\dots\dots\dots\dots\dots\dots\dots....(\boldsymbol{43})$$

So that the equivalent representation is drawn in **Figure 25b**.

Block Diagrams and Transfer Functions

The combination of block diagrams and transfer functions is a powerful way to represent control systems. Transfer functions relating different signals in the system can be derived by purely algebraic manipulations of the transfer functions of the blocks using ***block diagram algebra***. To show how this can be done, we will begin with simple combinations of systems

Consider a system that is a cascade combination of systems with the transfer functions $\boldsymbol{G_1(s)}$ and $\boldsymbol{G_2(s)}$**,** as shown in **Figure 26**.

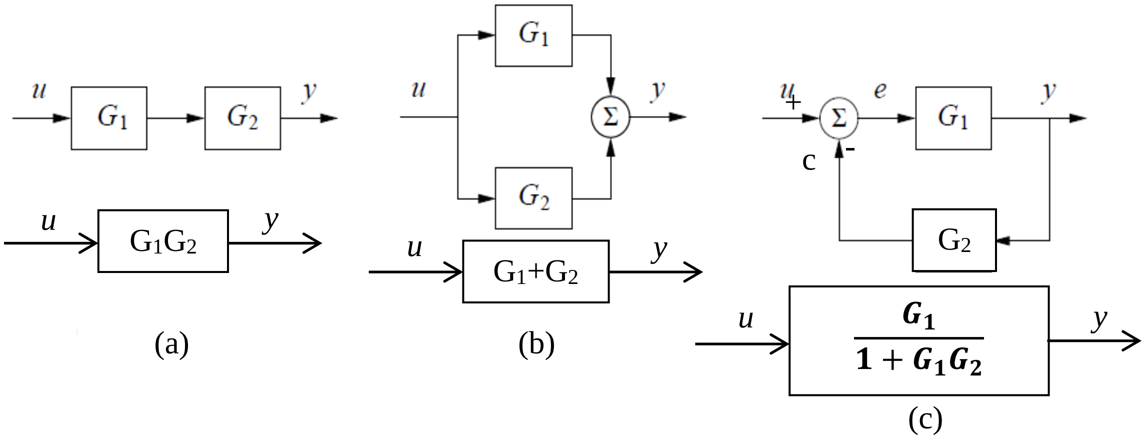


Figure 26
Interconnections of linear systems.
Series (a), parallel (b) and feedback (c) connections are shown.
The transfer functions for the composite systems can be derived by algebraic manipulations assuming exponential functions for all signals.

Let the input of the system be $\boldsymbol{u = e^{st}}$. The pure exponential output of the first block is the exponential signal $\boldsymbol{G_1 u}$, which is also the input to the second system as shown in **Figure 26a**. The pure exponential output of the second system is $\boldsymbol{y}$:

$$\boldsymbol{y = G_2(G_1 u) = (G_2 G_1)u}$$

The **transfer function** of the **series connection** is thus $\boldsymbol{G = G_2 G_1}$, i.e., the product of the transfer functions. The order of the individual transfer functions is due to the fact that we place the input signal on the right-hand side of this expression, hence we first multiply by $\boldsymbol{G_1}$ and then by $\boldsymbol{G_2}$. Unfortunately, this has the opposite ordering from the diagrams that we use, where we typically have the signal flow from left to right, so one needs to be careful. The ordering is important if either $\boldsymbol{G_1}$ or $\boldsymbol{G_2}$ is a vector-valued transfer function, as we shall see in some examples.

Consider next a parallel connection of systems with the transfer functions $\boldsymbol{G_1}$ and $\boldsymbol{G_2}$, as shown in **Figure 26b**. Letting $\boldsymbol{u = e^{st}}$ be the input to the system, the pure exponential output of the first system is then,

$$\boldsymbol{y_1 = G_1 u}$$

and the output of the second system is:

$$\boldsymbol{y_2 = G_2 u}$$

The pure exponential output of the parallel connection is thus:

$$\boldsymbol{y = G_1 u + G_2 u = (G_1 + G_2) u}$$

and the transfer function for a parallel connection is:

$$\boldsymbol{G = G_1 + G_2}$$

Finally, consider a feedback connection of systems with the transfer functions $\boldsymbol{G_1}$ and $\boldsymbol{G_2}$, as shown in **Figure 26c**. Let $\boldsymbol{u = e^{st}}$ be the input to the system, $\boldsymbol{y}$ be the pure exponential output, and $\boldsymbol{e}$ be the pure exponential part of the intermediate signal given by the sum of $\boldsymbol{u}$ and the output of the second block. Writing the relations for the different blocks and the summation unit, we find:

$$\boldsymbol{y = G_1 e, \qquad e = u - G_2 y}$$

Elimination of $\boldsymbol{e}$ gives:

$$\boldsymbol{y = G_1(u + G_2 y) \implies (1 + G_1 G_2) y = G_1 u \implies y = \frac{G_1}{1 + G_1 G_2}\, u}$$

The transfer function of the feedback connection is thus:

$$\boldsymbol{G = \frac{G_1}{1 + G_1 G_2}}$$

These three basic interconnections can be used as the basis for computing transfer functions for more complicated systems.

Control System Transfer Functions

Consider the system in **Figure 27**.

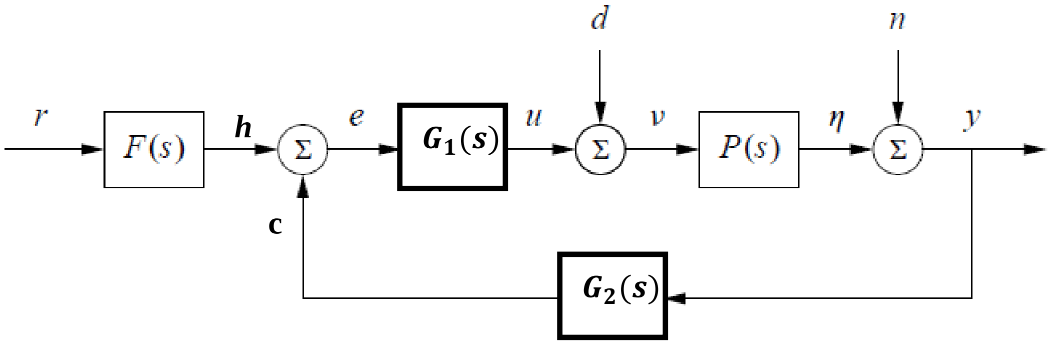


Figure 27

Block diagram of a feedback system.

The inputs to the system are the **reference signal *r***, the **process disturbance *d*** and the **measurement noise *n***. The remaining signals in the system can all be chosen as possible outputs, and transfer functions can be used to relate the system inputs to the other labeled signals.

The system has three blocks representing a process ***P***, a feedback controller ***C*** and a feedforward controller ***F***. Together, ***C*** and ***F*** define the **control law** for the system. There are **three external signals**:

- the **reference** (or **command signal**) ***r***,
- the **load disturbance *d***, and
- the **measurement noise *n***.

A typical problem is to find out how the **error *e*** is related to the **signals *r***, ***d*** and ***n***.

To derive the relevant **transfer functions**, we assume that **all signals** are **exponential signals**, drop the arguments of signals and transfer functions and trace the signals around the loop. We begin with the signal in which we are interested, in this case the **control error $\boldsymbol{e}$**, given by

$$\boldsymbol{e = h - c = rF - yG_2}$$

The **signal $\boldsymbol{y}$** is the **sum** of $\boldsymbol{n}$ and $\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is the output of the process:

$$\boldsymbol{y = n + \eta, \quad \eta = P(d + u), \qquad u = eG_1}$$

Combining these equations gives:

$$\boldsymbol{e = rF - (n + \eta)G_2 = Fr - [n + P(d + u)]G_2 = Fr - [n + P(d + eG_1)]G_2}$$

and hence

$$\boldsymbol{e = rF - [n + Pd + PeG_1]G_2 = rF - nG_2 - PdG_2 - PeG_1G_2}$$

$$\boldsymbol{e(1 + PG_1G_2) = rF - nG_2 - PdG_2}$$

Finally, solving this equation for $\boldsymbol{e}$ gives:

$$\boldsymbol{e = \frac{F}{1 + PG_1G_2}r - \frac{G_2}{1 + PG_1G_2}n - \frac{PG_2}{1 + PG_1G_2}d = G_{er}r + G_{en}n + G_{ed}d}$$

and the **error $\boldsymbol{e}$** is thus the **sum** of **three terms**, depending on the **reference $\boldsymbol{r}$**, the **measurement noise $\boldsymbol{n}$** and the **load disturbance $\boldsymbol{d}$**. The functions:

$$\boldsymbol{G_{er} = \frac{F}{1 + PG_1G_2}\ , \quad G_{en} = \frac{-G_2}{1 + PG_1G_2}\ , G_{ed} = \frac{-PG_2}{1 + PG_1G_2}}$$

are **transfer functions** from **reference $\boldsymbol{r}$**, **noise $\boldsymbol{n}$** and **disturbance $\boldsymbol{d}$** to the **error $\boldsymbol{e}$**.

We can also derive transfer functions by manipulating the block diagrams directly, as illustrated in Figure below.

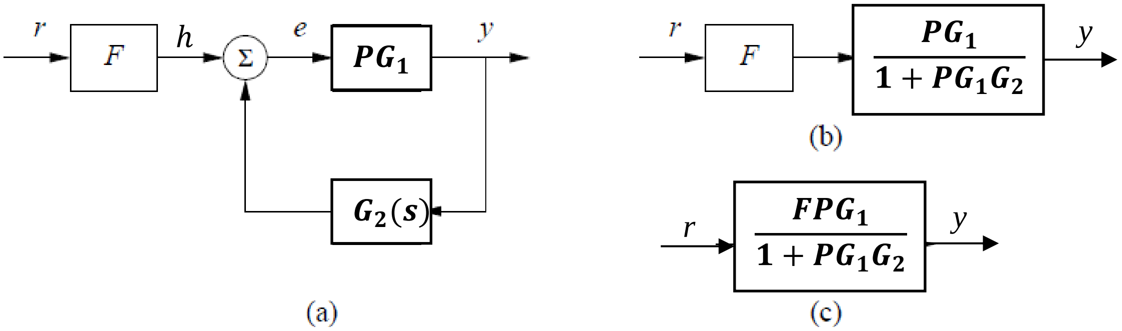


Figure 28

Example of block diagram algebra.

The results from multiplying the process and controller transfer functions (from previous figure) are shown in **Figure (a)**. Replacing the feedback loop with its transfer function equivalent yields **Figure (b)**, and finally multiplying the two remaining blocks gives the reference to output representation in **Figure (c).**

Suppose we wish to compute the transfer function between the reference $\boldsymbol{r}$ previous and the output $\boldsymbol{y}$. We begin by combining the process and controller blocks in previous figure to obtain the diagram in **Figure 28a**. We can now eliminate the feedback loop using the algebra for a feedback interconnection (**Figure 28b**) and then use the series interconnection rule to obtain:

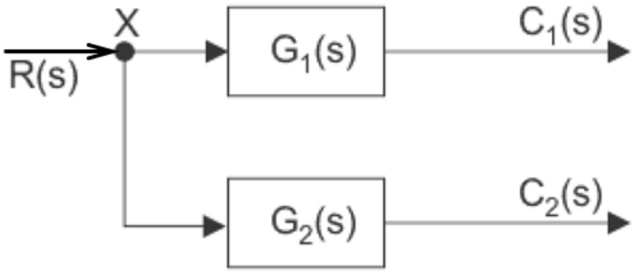
$$\boldsymbol{G_{yr}} = \frac{\boldsymbol{FPG_1}}{\boldsymbol{1 + PG_1G_2}}$$

Similar manipulations can be used to obtain the other transfer functions.

The derivation illustrates an effective way to manipulate the equations to obtain the relations between inputs and outputs in a feedback system. The general idea is to start with the signal of interest and to trace signals around the feedback loop until coming back to the signal we started with. Notice, for example, that all terms in above equation have the same denominators and that the numerators are the blocks that one passes through when going directly from input to output (ignoring the feedback). This type of rule can be used to compute transfer functions by inspection, although for systems with multiple feedback loops it can be tricky to compute them without writing down the algebra explicitly.

Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points. In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.



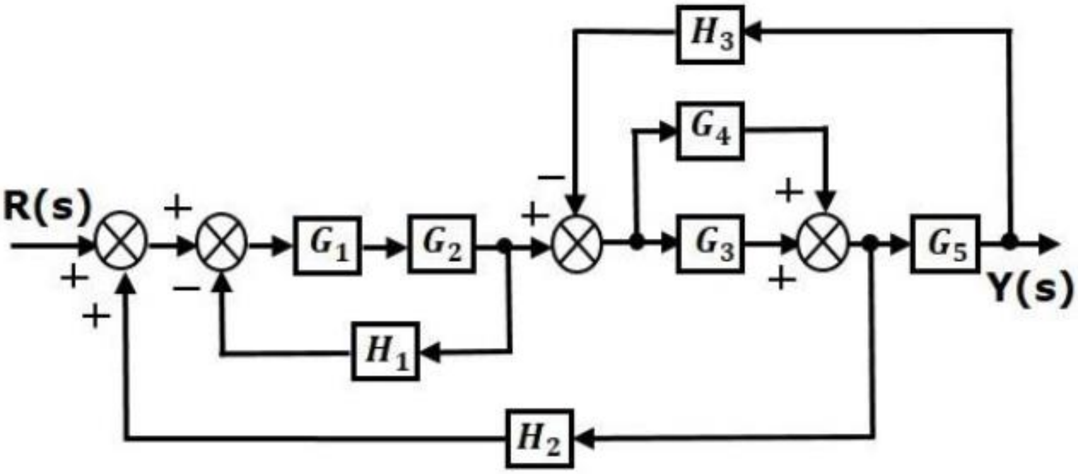
Block Diagram Reduction Rules

Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

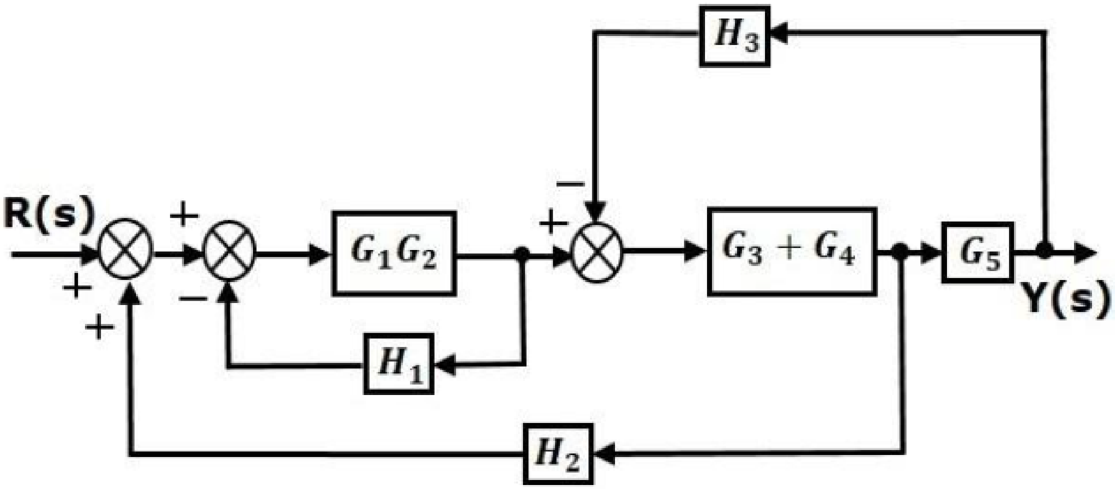
- **Rule 1** − Check for the blocks connected in series and simplify.
- **Rule 2** − Check for the blocks connected in parallel and simplify.
- **Rule 3** − Check for the blocks connected in feedback loop and simplify.
- **Rule 4** − If there is difficulty with take-off point while simplifying, shift it towards right.
- **Rule 5** − If there is difficulty with summing point while simplifying, shift it towards left.
- **Rule 6** − Repeat the above steps till you get the simplified form, i.e., single block.

Note − The transfer function present in this single block is the transfer function of the overall block diagram.

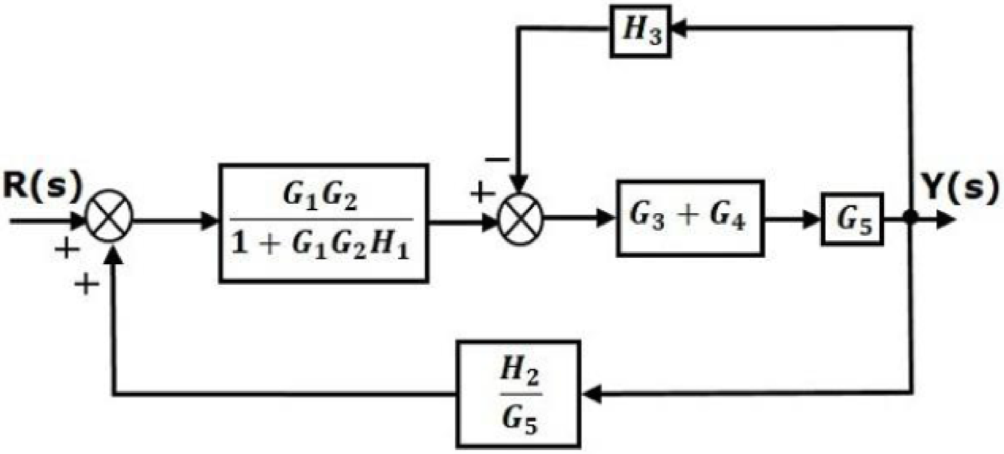
<u>Example</u>: Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.



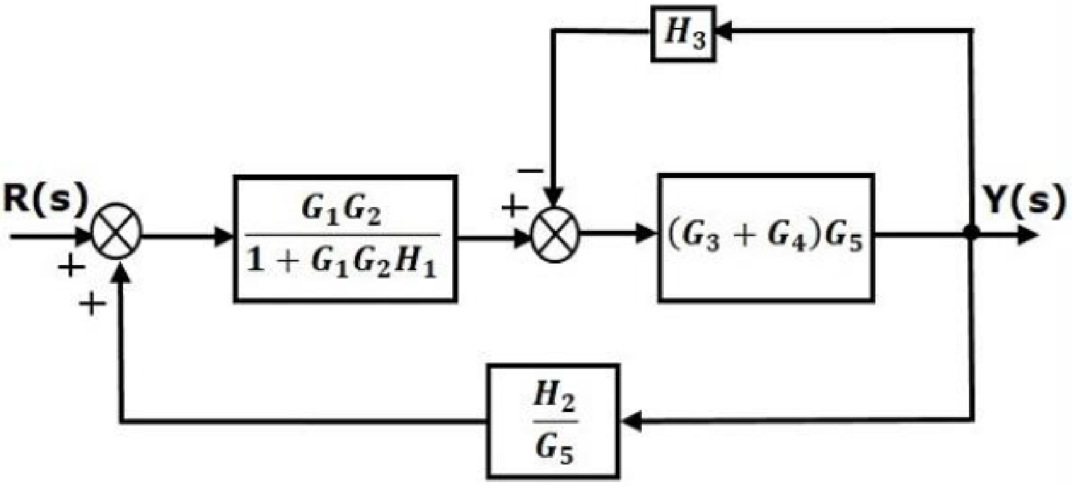
<u>Step 1</u> − Use **<u>Rule 1</u>** for **blocks G1** and **G2**. Use **<u>Rule 2</u>** for **blocks G3** and **G4**. The modified block diagram is shown in the following figure.



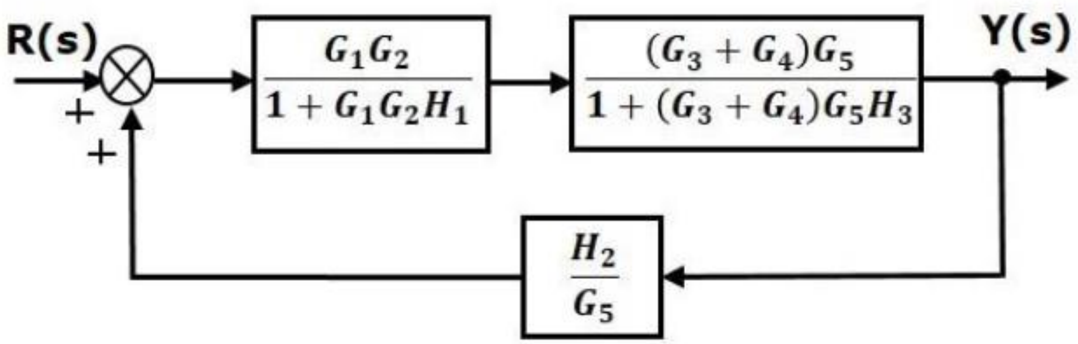
Step 2 − Use **Rule 3** for **blocks G1G2** and **H1**. Use **Rule 4** for **shifting take-off point** after the **block G5**. The modified block diagram is shown in the following figure.



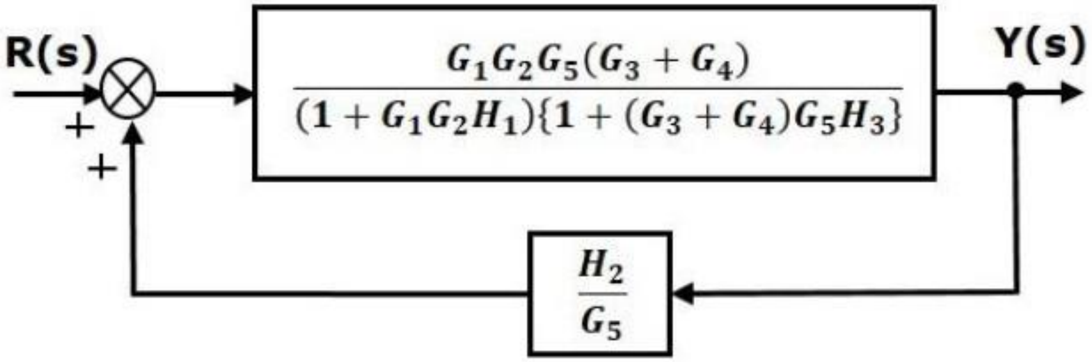
Step 3 − Use **Rule 1** for **blocks (G3+G4)** and **G5**. The modified block diagram is shown in the following figure.



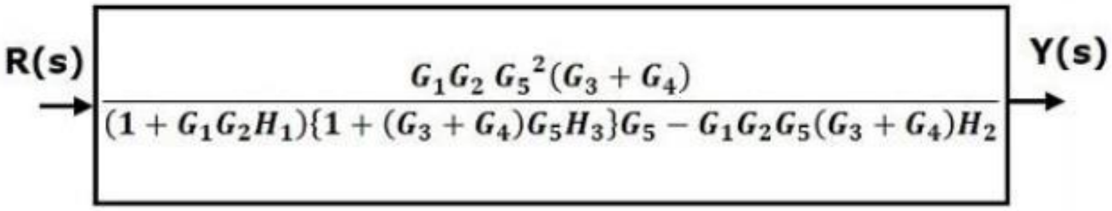
Step 4 − Use **<u>Rule 3</u>** for **blocks (G3+G4) G5** and **H3**. The modified block diagram is shown in the following figure.



Step 5 − Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.



Step 6 − Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



Therefore, the transfer function of the system is:

$$\boldsymbol{TF = \frac{Y(s)}{X(s)} =}$$

$$\boldsymbol{= \frac{G_1 G_2 G_5^2 (G_3 + G_4)}{(1 + G_1 G_2) H_1 [1 + (G_3 + G_4) G_5 H_3] G_5 - G_1 G_2 G_5 (G_3 + G_4) H_2}}$$

Note − Follow these steps in order to calculate the transfer function of the block diagram having multiple inputs.

- ◊ **Step 1** − Find the transfer function of block diagram by considering one input at a time and make the remaining inputs as zero.
- ◊ **Step 2** − Repeat step 1 for remaining inputs.
- ◊ **Step 3** − Get the overall transfer function by adding all those transfer functions.

The block diagram reduction process takes more time for complicated systems. Because, we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

Control Systems - Signal Flow Graphs

Signal flow graph is a graphical representation of algebraic equations.

Basic Elements of Signal Flow Graph

Nodes and **branches** are the **basic elements** of **signal flow graph**.

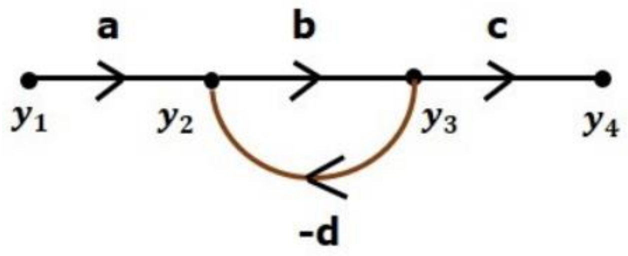
❖ Node

Node is a point which represents either a variable or a signal. There are three types of nodes:

- **Input Node** − It is a node, which has only outgoing branches.
- **Output Node** − It is a node, which has only incoming branches.
- **Mixed Node** − It is a node, which has both incoming and outgoing branches.

Example

Let us consider the following signal flow graph to identify these nodes.



- The **nodes** present in this signal flow graph are **y_1, y_2, y_3** and **y_4**.
- **y_1** and **y_4** are the **input node** and **output node** respectively.
- **y_2** and **y_3** are **mixed nodes**.

❖ **Branch**

Branch is a line segment which joins two nodes. It has both gain and direction. For example, there are four branches in the above signal flow graph. These branches have gains of **a**, **b**, **c** and **– d** .

Construction of Signal Flow Graph

Let us construct a signal flow graph by considering the following algebraic equations:

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

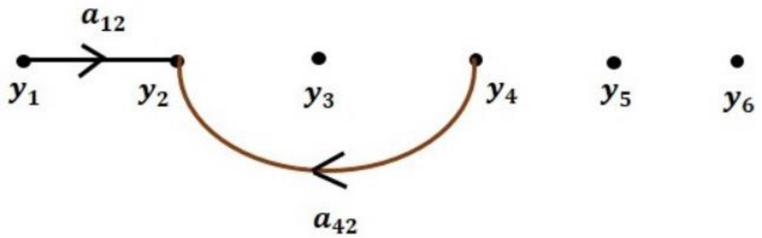
$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

There will be six **nodes** (y_1, y_2, y_3, y_4, y_5 and y_6) and eight **branches** in this signal flow graph. The gains of the branches are a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{42}, a_{53} and a_{35}.

To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below :

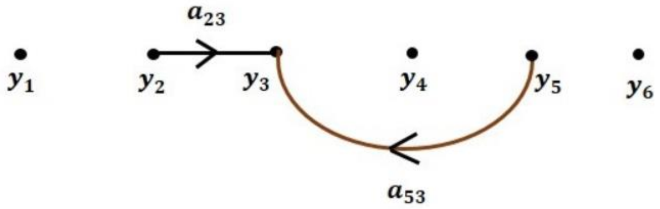
Step 1 − Signal flow graph for $y_2 = a_{12}y_1 + a_{42}y_4$ is shown in the following figure.



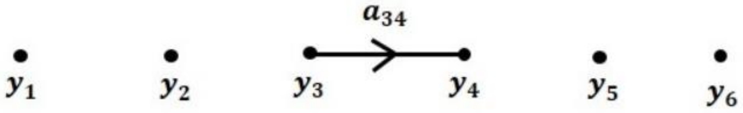
Step 2 − Signal flow graph for

$$y_3 = a_{23}y_2 + a_{53}y_5$$

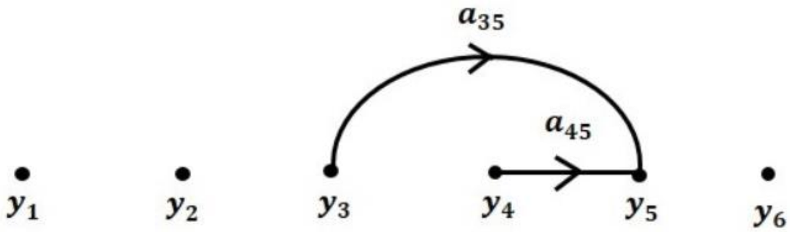
is shown in the following figure.



Step 3 − Signal flow graph for $y_4 = a_{34}y_3$ is shown in the following figure.

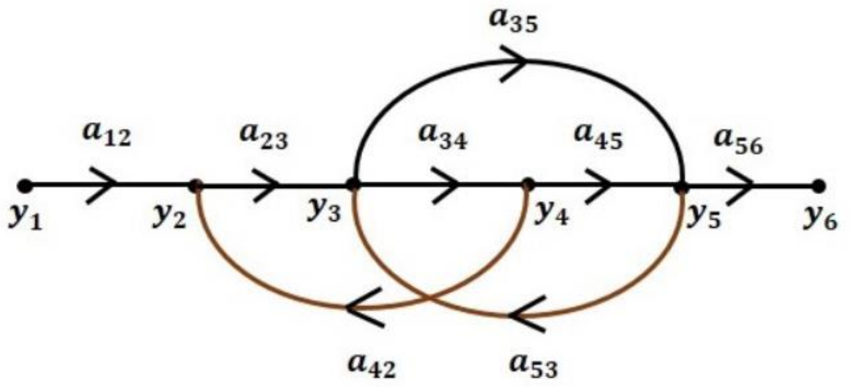


Step 4 − Signal flow graph for $y_5 = a_{45}y_4 + a_{35}y_3$ is shown in the following figure.



Step 5 – Signal flow graph for $y_6 = a_{56}y_5$ is shown in the following figure.

Step 6 – Signal flow graph of overall system is shown in the following figure.

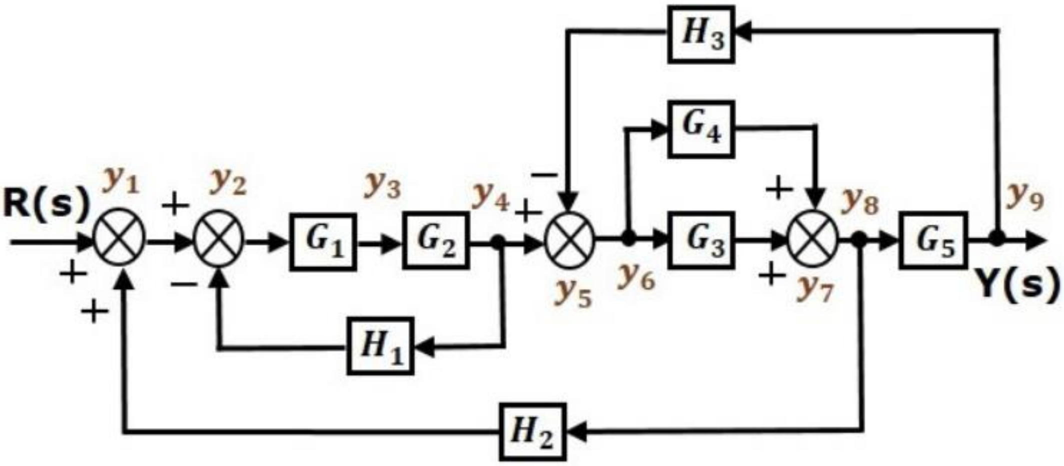


Conversion of Block Diagrams into Signal Flow Graphs

Follow these steps for converting a block diagram into its equivalent signal flow graph.

- Represent all the **signals**, **variables**, **summing points** and **take-off points** of **block diagram** as **nodes** in **signal flow graph**.
- Represent the **blocks** of **block diagram** as **branches** in **signal flow graph**.
- Represent the **transfer functions** inside the blocks of block diagram as **gains** of the branches in signal flow graph.
- **Connect** the **nodes** as **per the block diagram**. If there is **connection** between **two nodes** (but there is **no block** in between), then represent the **gain** of the branch as **one**. **For example**, between **summing points**, between **summing point** and **takeoff point**, between **input** and **summing point**, between **take-off point** and **output**.

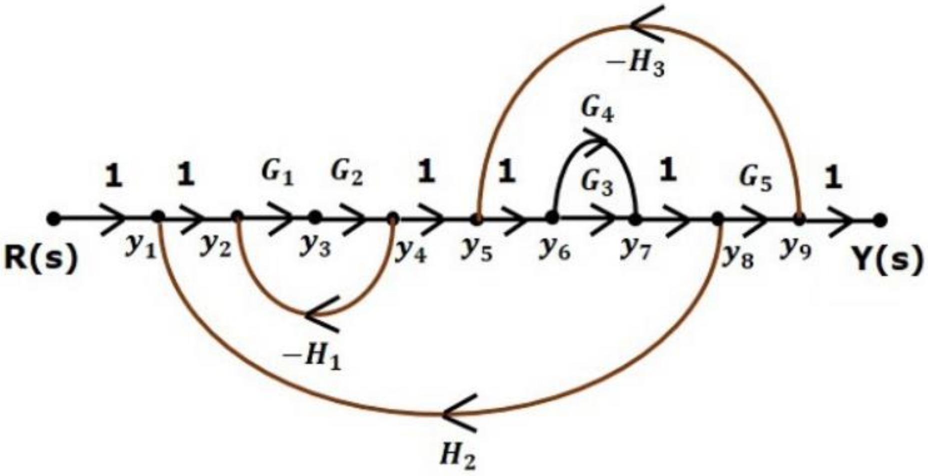
Example: convert the following block diagram into its equivalent signal flow graph.



Represent the input signal **R(s)** and output signal **Y(s)** of block diagram as **input node R(s)** and **output node Y(s)** of signal flow graph.

Just for reference, the **remaining nodes (y_1 to y_9)** are labelled in the block diagram. There are **nine nodes** other than input and output nodes. That is **four nodes** for **four summing points**, **four nodes** for **four take-off points** and **one node** for the variable between blocks ***G_1*** and **G_2**.

The following figure shows the equivalent signal flow graph.



With the help of **Mason's gain formula,** we can calculate the transfer function of this signal flow graph. This is the advantage of signal flow graphs. Here, we no need to simplify (reduce) the signal flow graphs for calculating the transfer function.

Mason's Gain Formula

Suppose there are 'N' forward paths in a signal flow graph. The **gain** between the input and the output nodes of a signal flow graph is nothing but the **transfer function** of the system. It can be calculated by using **Mason's gain formula**.

Mason's gain formula is

$$\boldsymbol{TF} = \frac{\boldsymbol{Y}(\boldsymbol{s})}{\boldsymbol{R}(\boldsymbol{s})} = \frac{\sum_{i=1}^{N} \boldsymbol{P}_i \Delta_i}{\Delta}$$

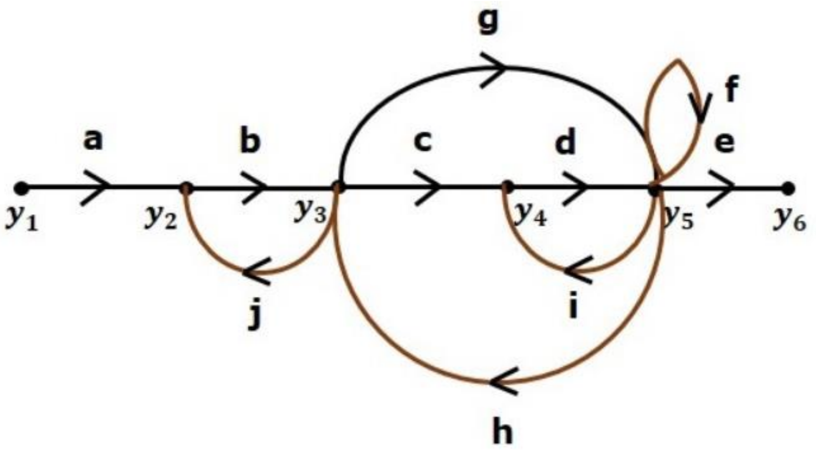
Where,

- $\boldsymbol{Y}(\boldsymbol{s})$ is the output node,
- $\boldsymbol{R}(\boldsymbol{s})$ is the input node,
- $\boldsymbol{TF}$ is the transfer function or gain between $\boldsymbol{Y}(\boldsymbol{s})$ and $\boldsymbol{R}(\boldsymbol{s})$**,**
- $\boldsymbol{P}$ is the $\boldsymbol{i^{th}}$ forward path gain.

$$\begin{aligned}\Delta = \mathbf{1} - (&\textbf{sum of all individual loop gains}) + \\ &+(\textbf{sum of gain products of all possible two nontouching loops}) - \\ &-(\textbf{sum of gain products of all possible three nontouching loops}) + \\ &+ \cdots\end{aligned}$$

Δ_i is obtained from Δ by removing the loops which are touching the $\boldsymbol{i^{th}}$ forward path

Consider the following signal flow graph in order to understand the basic terminology involved here.

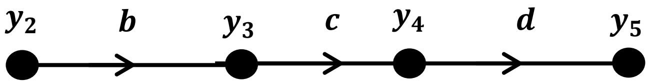


Path

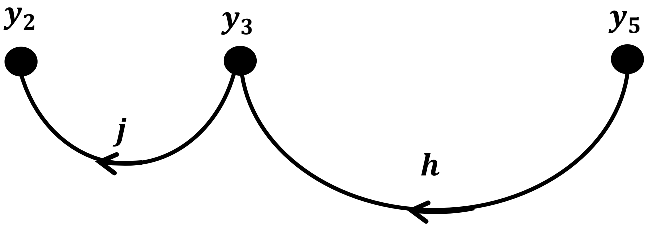
It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.

Examples:

$$y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5$$



and $y_5 \rightarrow y_3 \rightarrow y_2$

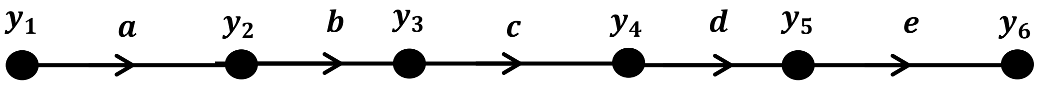


Forward Path

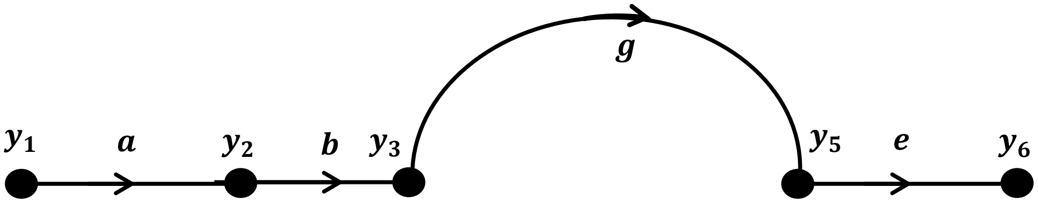
The path that exists **from** the **input node** to the **output node** is known as **forward path**.

Examples:

$$y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$$



and $\boldsymbol{y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6}$



Forward Path Gain

It is obtained by calculating the product of all branch gains of the forward path.

Examples:

$\underline{\boldsymbol{abcde}}$ is the **forward path gain** of:

$$\boldsymbol{y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6}$$

And $\underline{\boldsymbol{abge}}$ is the forward path gain of:

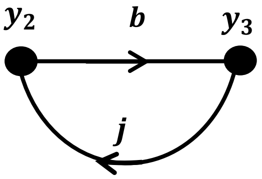
$$\boldsymbol{y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6}$$

Loop

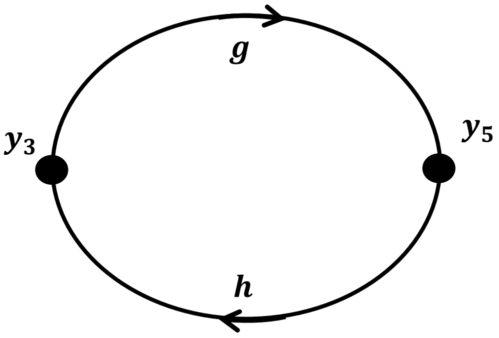
The path that starts from one node and ends at the same node is known as **loop**. Hence, it is a **closed path**.

Examples:

$\boldsymbol{y_2 \rightarrow y_3 \rightarrow y_2}$



And $\boldsymbol{y_3 \rightarrow y_5 \rightarrow y_3}$



Loop Gain

It is obtained by calculating the **product** of **all branch gains** of a **loop**.

Examples:

$\boldsymbol{bj}$ is the loop gain of $y_2 \rightarrow y_3 \rightarrow y_2$

and $\boldsymbol{gh}$ is the loop gain of $y_3 \rightarrow y_5 \rightarrow y_3$

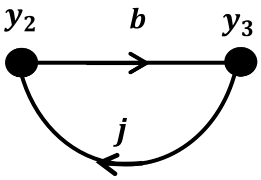
Non-touching Loops

These are the loops, which should not have any common node.

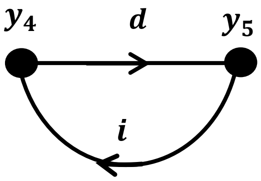
Examples:

The loops,

$y_2 \rightarrow y_3 \rightarrow y_2$



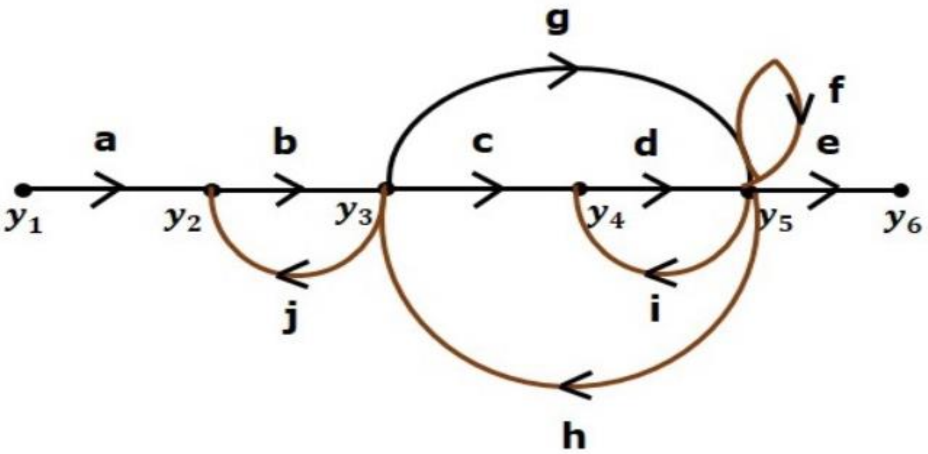
and $y_4 \rightarrow y_5 \rightarrow y_4$



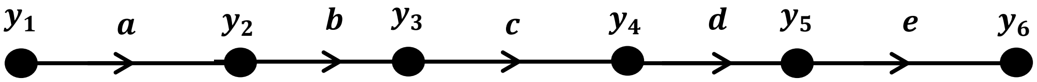
are non-touching.

Calculation of Transfer Function using Mason's Gain Formula

Let us consider the same signal flow graph for finding transfer function.

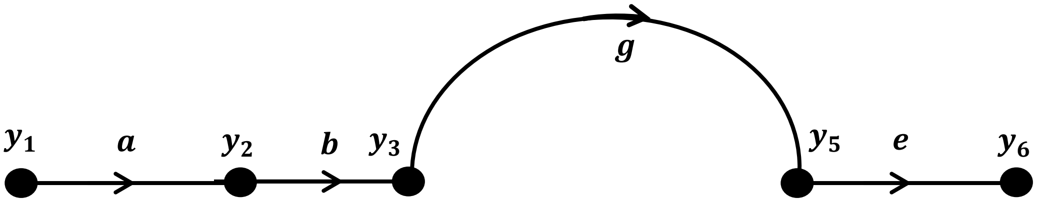


- Number of forward paths, $\boldsymbol{N = 2}$.
- **First forward path** is: $\boldsymbol{y_1 \to y_2 \to y_3 \to y_4 \to y_5 \to y_6}$



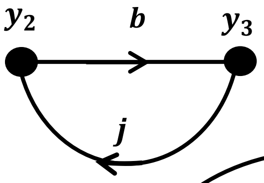
- **First forward path gain**, $\boldsymbol{p_1 = abcde}$.
- **Second forward path** is:

$$\boldsymbol{y_1 \to y_2 \to y_3 \to y_5 \to y_6}$$

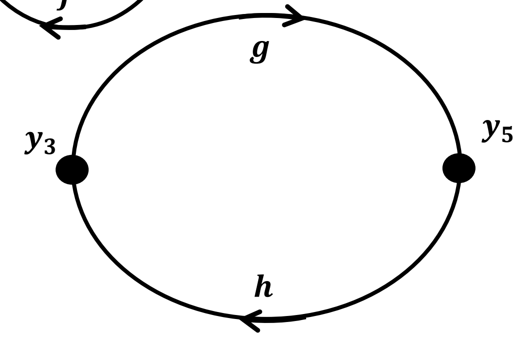


- **Second forward path gain**, $\boldsymbol{p_2 = abge}$.
- **Number of individual loops**, **L = 5**.
- **Loops are:**

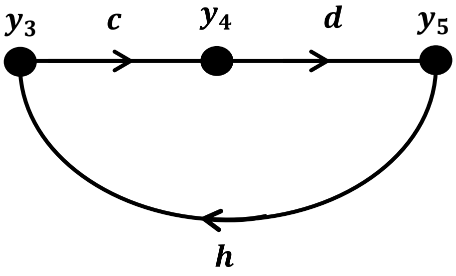
1. $\boldsymbol{L_1 = y_2 \to y_3 \to y_2}$,



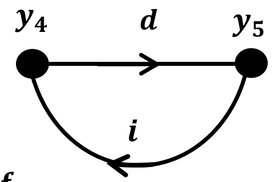
2. $\boldsymbol{L_2 = y_3 \to y_5 \to y_3}$,



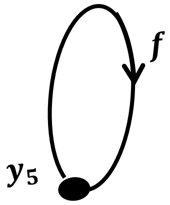
3. $\boldsymbol{L_3 = y_3 \to y_4 \to y_5 \to y_3}$,



4. $L_4 = y_4 \rightarrow y_5 \rightarrow y_4$,



5. $L_5 = y_5 \rightarrow y_5$

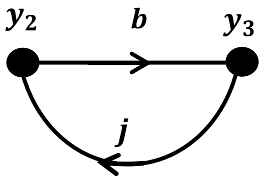


- **Loop gains are:**

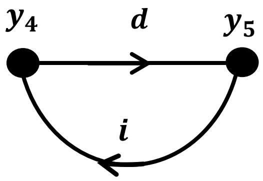
$L_1 = bj$, $L_2 = gh$, $L_3 = cdh$, $L_4 = di$, and $L_5 = f$

- **Number of two non-touching loops** = 2.
- **First non-touching loops pair is:**

$y_2 \rightarrow y_3 \rightarrow y_2$,

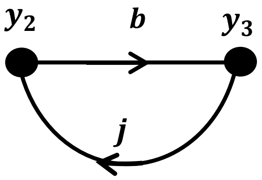


$y_4 \rightarrow y_5 \rightarrow y_4$

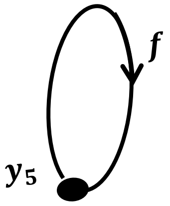


- **Gain product of first non-touching loops pair**, $L_1 L_4 = bjdi$
- **Second non-touching loops pair are:**

$y_2 \rightarrow y_3 \rightarrow y_2$,



and $y_5 \rightarrow y_5$



- **Gain product of second non-touching loops pair** is: $L_1 L_5 = bjf$

  Higher number of (more than two) non-touching loops are not present in this signal flow graph.

We know,

$$\Delta = 1 - (\textbf{sum of all individual loop gains}) +$$
$$+(\textbf{sum of gain products of all possible two nontouching loops}) -$$
$$-(\textbf{sum of gain products of all possible three nontouching loops})$$
$$+ \cdots$$

Substitute the values in the above equation,

$$\Delta = 1 - (bj + gh + cdh + di + f) + (bjdi + bjf) - (0) \implies$$
$$\Delta = 1 - (bj + gh + cdh + di + f) + bjdi + bjf$$

There is no loop which is non-touching to the first forward path. So,

$$\Delta_1 = 1$$

Similarly, $\Delta_2 = 1$. Since, no loop which is non-touching to the second forward path.

Substitute, **N = 2** in **Mason's gain formula**

$$TF = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

Substitute all the necessary values in the above equation.

$$TF = \frac{(abcde)1 + (abge)1}{1 - (bj + gh + cdh + di + f) + bjdi + bjf} =$$
$$= \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Therefore, the transfer function is:

$$TF = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Syllabus of Control Systems

Undergraduate Program / 4th year

Second Course (2020 – 2021)

Northern Technical University

Engineering Technical College/Mosul

Engineering of Refrigeration and Air Conditioning Technologies

Objectives:

- To provide adequate knowledge in the time response of systems and steady state error analysis.
- To accord basic knowledge in obtaining the open loop and closed–loop frequency responses of systems.
- To introduce stability analysis and design of compensators
- To introduce state variable representation of physical systems and study the effect of state Feedback

WEEK	TOPIC
1	Time response analysis of control systems, Transient and Steady State Response Analysis, Various types of standard input signals,
2	Response of first order response to Step, Ramp and Impulse Input,
3	Response of second order system to step input, System specifications. Concept of time constant and its importance in speed response,
4	Systems with Time Delay, Effect of Damping ratio on response of Second Order System.
5	Frequency Response Analysis: Stability Analysis, System Stability and Routh's Stability Criteria,
6	Relative Stability Concepts, Nyquist stability criterion, Polar plots Phase and Gain margin Bode plot attenuation diagram,
7	Stability Analysis using Bode plots, Simplified Bode plot.,
8 - 9	Definition of Root loci, General Rules for constructing Root Locus,

10	Effect of Damping ratio on response of Second Order System.
11	Frequency Response Analysis: Stability Analysis, System Stability and Routh's Stability Criteria,
12	Relative Stability Concepts, Nyquist stability criterion, Polar plots Phase and Gain margin Bode plot attenuation diagram, Relative Stability, Analysis using MATLAB,
13	Stability Analysis using Bode plots, Simplified Bode plot.
14	Analysis using Root Locus Plots,
15	Analysis using Root Locus Plots, Use of MATLAB software in control system.

TEXTBOOK:

1. Control Systems Engineering by Nise, 7th edition, Wiley, 9781118170519
2. J. Nagrath and M. Gopal, "Control System Engineering", New Age International Publishers,

REFERENCES:

1. Control System Engineering, Ogatta, Prentice Hall of India Pvt. Ltd.
2. Automatic Control Systems, Kuo, Golnaraghi, Kunche, Wiley India.
3. Automatic Control Engineering, Francis H. Raven, McGraw Hill.
4. Control Systems- Principles and Design, M. Gopa1, McGraw Hill Education.
5. Feedback Control System, Dr. S.D. Bhide, S. Satyanarayan, N.A. Jalgaonkar: Technova Pub. [Pune] Pvt. Ltd.
6. Control System Engineering, I.I Nagrath, M. Gopal, New Age International Publishers.

Control Systems - Time Response Analysis

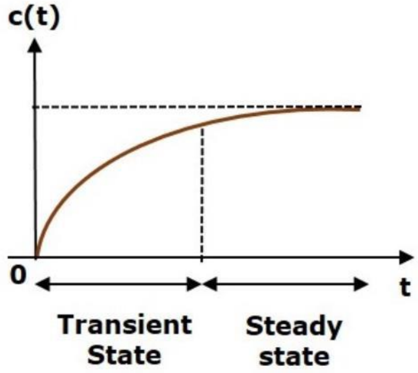
The response of the control systems can be analyzed in both the time domain and the frequency domain.

The time response analysis of control systems

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The **time response** consists of **two parts**:

- **Transient response, and**
- **Steady state response**

The response of control system in time domain is shown in the following figure.



Mathematically, we can write the time response $\boldsymbol{C(t)}$ as:

$$\boldsymbol{C(t) = C_{tr}(t) + C_{ss}(t)}$$

Where,

- $\boldsymbol{c_{tr}(t)}$ is the transient response, and
- $\boldsymbol{c_{ss}(t)}$ is the steady state response.

✷ Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in **transient state** till it goes to a **steady state**. Therefore, the **response of the control system during the transient state** is **known** as **transient response**.

The **transient response** will be **zero** for **large values** of $(\boldsymbol{t})$. Ideally, this value of $(\boldsymbol{t})$ is **infinity** and practically, it is **five times constant**.

Mathematically, we can write it as:

$$\boldsymbol{\lim_{t\to\infty} C_{tr}(t) = 0}$$

✷ Steady state Response

The **part** of the **time response** that **remains even after** the **transient response** has **zero value** for **large values** of (t) is **known** as **steady state response**. This means, the **transient response** will be **zero** even **during** the **steady state**.

Example: **F**ind the **transient** and **steady state terms** of the **time response** of the **control system**:

$$\boldsymbol{C(t) = 10 + 5\, e^{-t}}$$

Solution:

The **second term** $(\boldsymbol{5\, e^{-t}})$ will be **zero** as ($\boldsymbol{t} \to \infty$) **approaches infinity**:

$$\lim_{t\to\infty} C_{tr}(t) = \lim_{t\to\infty} 5\, e^{-t} = 5 \times 0 = 0$$

So, this is the **transient term**, and the **first term** $(\boldsymbol{10})$ **remains even** as ($\boldsymbol{t}$) **approaches infinity**. So, this is the **steady state term**.

❖ Standard Test Signals

The **standard test signals** are:

- o **impulse,**
- o **step,**
- o **ramp** and
- o **parabolic.**

These **signals** are **used to know** the **performance** of the **control systems** using **time response** of the **output**.

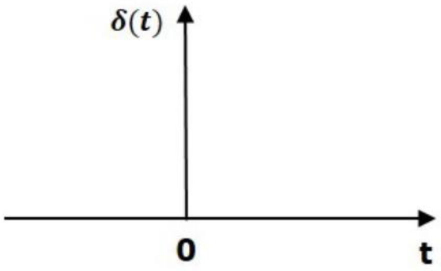
♦ Unit Impulse Signal

A **unit impulse signal** is denoted by $\boldsymbol{\delta(t)}$ and defined as:

$$\boldsymbol{\delta(t) = 0} \quad \text{for} \quad \boldsymbol{t \neq 0} \quad \text{and}$$

$$\int_{-0}^{+0} \boldsymbol{\delta(t)\, dt = 1}$$

The following figure shows the **unit impulse signal**.



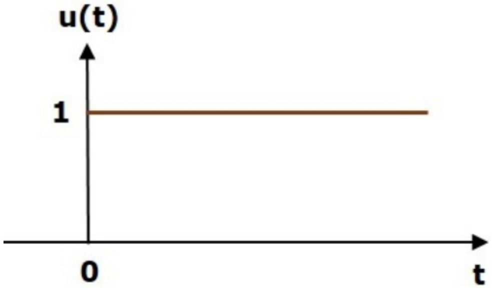
So, the **unit impulse signal exists only** at ($\boldsymbol{t = 0}$). The **area** of this signal **under small interval** of **time** around ($\boldsymbol{t = 0}$) is **one**. The **value** of **unit impulse signal** is **zero** for **all negative values of** $\boldsymbol{t}$.

♦ <u>Unit Step Signal</u>

A **unit step signal** symbolized as $\boldsymbol{u(t)}$ is defined as:

$$\boldsymbol{u(t) = 1} \;;\; \boldsymbol{t \geq 0} \quad \text{and}$$
$$\boldsymbol{u(t) = 0} \;;\; \boldsymbol{t < 0}$$

The figure below shows the **<u>unit step signal</u>**.



So, the **<u>unit step signal</u> exists** for all **<u>positive values</u>** of $\boldsymbol{t}$ ($\boldsymbol{t > 0}$) including **<u>zero</u>**. And **its value** is **<u>one</u>** during **this interval** ($\boldsymbol{0 - 1}$). The **value** of the **<u>unit step signal</u>** is **<u>zero</u>** for all **negative values** of $\boldsymbol{t}$ ($\boldsymbol{t < 0}$).

♦ <u>Unit Ramp Signal</u>

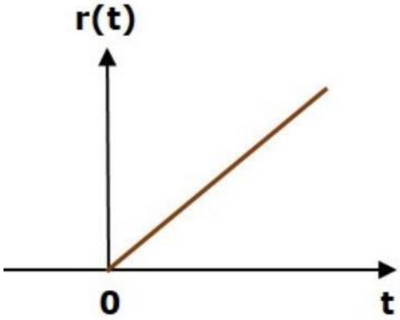
A **<u>unit ramp signal</u>** denoted by $\boldsymbol{r(t)}$ is defined as:

$$\boldsymbol{r(t) = t} \;;\; \boldsymbol{t \geq 0} \quad \text{and}$$
$$\boldsymbol{r(t) = 0} \;;\; \boldsymbol{t < 0}$$

We can write **unit ramp signal**, $\boldsymbol{r(t)}$ in terms of **unit step signal**, $\boldsymbol{u(t)}$ as:

$$\boldsymbol{r(t) = t\, u(t)}$$

The figure below shows unit ramp signal.



So, the unit ramp signal exists for **all positive values** of ($\boldsymbol{t}$) including **zero**. And its value **increases <u>linearly</u>** with respect to ($\boldsymbol{t}$) during this interval. The **value** of **unit ramp signal** is **zero** for **all negative values** of ($\boldsymbol{t}$).

◆ **Unit Parabolic Signal**

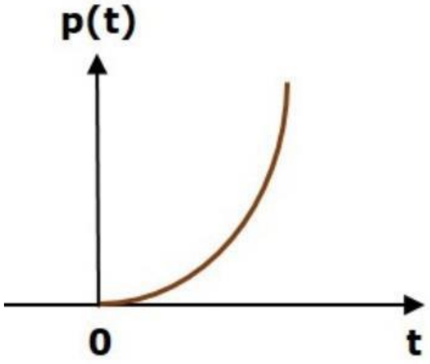
A **unit parabolic signal**, symbolized as $\boldsymbol{p(t)}$ is defined as:

$$\boldsymbol{p(t)=\frac{t^2}{2}}\ ;\ \boldsymbol{t\geq 0},\ \text{and}$$
$$\boldsymbol{p(t)=0}\ ;\ \boldsymbol{t<0}$$

We can write **unit parabolic signal**, $\boldsymbol{p(t)}$ in terms of the **unit step signal**, $\boldsymbol{u(t)}$ as:

$$\boldsymbol{p(t)=\frac{t^2}{2}\,u(t)}$$

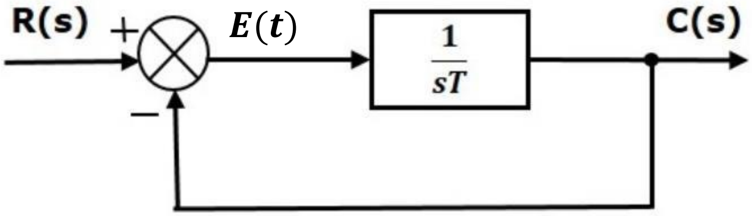
The figure below shows the **unit parabolic signal**.



So, the **unit parabolic signal exists** for **all the positive values** of $(\boldsymbol{t})$ including **zero**. And **its value** increases **non-linearly** with **respect** to $(\boldsymbol{t})$ during this interval. The **value** of the **unit parabolic signal** is **zero** for all the **negative values** of $(\boldsymbol{t})$

Response of the First Order System

Consider the following **block diagram** of the **closed loop control system**. Here, an open loop transfer function, $\boldsymbol{\frac{1}{sT}}$ is connected with a **unity negative feedback**.



We know that the **transfer function** of the **closed loop control system** has **unity negative feedback** as,

$$\boldsymbol{TF=\frac{C(t)}{R(t)}=\frac{G(s)}{1+G(s)}}$$

Substitute, $\boldsymbol{G(s)} = \frac{\mathbf{1}}{\boldsymbol{sT}}$ in the above equation, we have

$$\boldsymbol{TF} = \frac{\frac{\mathbf{1}}{\boldsymbol{sT}}}{\mathbf{1} + \frac{\mathbf{1}}{\boldsymbol{sT}}} = \frac{\frac{\mathbf{1}}{\boldsymbol{sT}}}{\frac{\boldsymbol{sT} + \mathbf{1}}{\boldsymbol{sT}}} = \frac{\mathbf{1}}{\boldsymbol{sT} + \mathbf{1}}$$

The **power** of ($\boldsymbol{s}$) is **one** in the **denominator term**. Hence, the above **transfer function** is of the **first order** and the **system** is said to be the **first order system**.

We can re-write the above equation as:

$$\boldsymbol{C(s)} = \left(\frac{\mathbf{1}}{\boldsymbol{sT} + \mathbf{1}}\right)\boldsymbol{R(s)}$$

Where,

- $\boldsymbol{C(s)}$ is the Laplace transform of the output signal $\boldsymbol{C(t)}$,
- $\boldsymbol{R(s)}$is the Laplace transform of the input signal $\boldsymbol{R(t)}$, and
- $\boldsymbol{T}$ is the time constant.

Following these steps to get the response (output) of the first order system in the time domain.

- ◊ Taking the Laplace transform of the input signal $\boldsymbol{R(t)}$,
- ◊ Considering the equation, $\boldsymbol{C(s)} = \left(\frac{\mathbf{1}}{\boldsymbol{sT+1}}\right)\boldsymbol{R(s)}$
- ◊ Substituting $\boldsymbol{R(s)}$ value in the above equation.
- ◊ Doing partial fractions of $\boldsymbol{C(s)}$ if required.
- ◊ Applying inverse Laplace transform to $\boldsymbol{C(s)}$.

Let us now find out the responses of the **first order system** for **each input**, one by one. The **name** of the **response** is given as **per** the **name** of the **input signal**. For example, the **response** of the **system** for an **impulse input** is called as **impulse response**.

➢ Impulse Response of First Order System

Suppose the **unit impulse signal** $\boldsymbol{\delta(t)}$ as an **input** to the **first order system**. So,

$$\boldsymbol{R(t)} = \boldsymbol{\delta(t)}$$

Applying Laplace transform on both the sides we have,

$$\boldsymbol{R(s)} = \mathbf{1}$$

Substituting $\boldsymbol{R(s)} = \mathbf{1}$ in the equation, $\boldsymbol{C(s)} = \left(\frac{\mathbf{1}}{\boldsymbol{sT+1}}\right)\boldsymbol{R(s)}$**,** we get

$$\boldsymbol{C(s)} = \left(\frac{\mathbf{1}}{\boldsymbol{sT} + \mathbf{1}}\right)(\mathbf{1}) = \frac{\mathbf{1}}{\boldsymbol{sT} + \mathbf{1}}$$

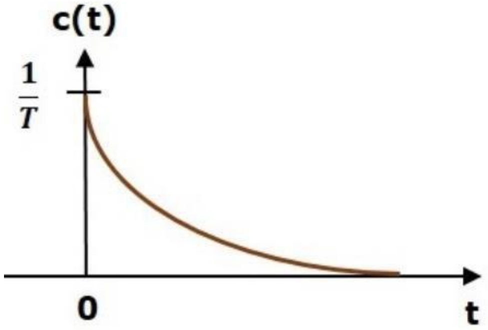
Rearranging the above equation in one of the standard forms of Laplace transforms.

$$\boldsymbol{C(s)} = \frac{\mathbf{1}}{\boldsymbol{T}\left(\boldsymbol{s} + \frac{\mathbf{1}}{\boldsymbol{T}}\right)} = \frac{\mathbf{1}}{\boldsymbol{T}}\left(\frac{\mathbf{1}}{\boldsymbol{s} + \frac{\mathbf{1}}{\boldsymbol{T}}}\right)$$

Applying inverse Laplace transform on both sides, we get

$$\left[\ \frac{\mathbf{1}}{\boldsymbol{s+a}} \quad \Leftrightarrow \quad \boldsymbol{e^{-at}\,u(t)} \quad \Rightarrow \quad \boldsymbol{C(t)} = \frac{\mathbf{1}}{\boldsymbol{T}}\boldsymbol{e^{\left(-\frac{1}{T}t\right)}\,u(t)}\ \right]$$

The **unit impulse response** is shown in the following figure.



The **unit impulse response**, $\boldsymbol{C(t)}$ is an **exponential decaying signal** for **positive values** of $(\boldsymbol{t})$ and it is **zero** for **negative values** of $(\boldsymbol{t})$.

➢ Step Response of First Order System

Consider the **unit step signal** $\boldsymbol{u(t)}$ as an **input** to **first order system**. So,

$$\boldsymbol{R(t) = u(t) = 1}$$

Appling Laplace transform on both the sides, $\boldsymbol{R(s) = C(1)} = \frac{\mathbf{1}}{\boldsymbol{s}}$

Substituting $\boldsymbol{R(s)} = \frac{\mathbf{1}}{\boldsymbol{s}}$ in the equation, $\boldsymbol{C(s)} = \left(\frac{\mathbf{1}}{\boldsymbol{sT+1}}\right)\boldsymbol{R(s)},$ we have

$$\boldsymbol{C(s)} = \left(\frac{\mathbf{1}}{\boldsymbol{sT+1}}\right)\left(\frac{\mathbf{1}}{\boldsymbol{s}}\right) = \frac{\mathbf{1}}{\boldsymbol{s(sT+1)}}$$

Doing partial fractions of $\boldsymbol{C(s)}$.

$$\boldsymbol{C(s)} = \frac{\mathbf{1}}{\boldsymbol{s(sT+1)}} = \frac{\boldsymbol{A}}{\boldsymbol{s}} + \frac{\boldsymbol{B}}{\boldsymbol{sT+1}} = \frac{\boldsymbol{A(sT+1)+Bs}}{\boldsymbol{s(sT+1)}} = \frac{\boldsymbol{A+s(AT+B)}}{\boldsymbol{s(sT+1)}}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$\boldsymbol{1 = A + s(AT+B)}$$

By equating the constant terms on both the sides, you will get $\boldsymbol{A} = \mathbf{1}$.

Substituting, $\boldsymbol{A}$=$\mathbf{1}$ and equate the coefficient of the $(\boldsymbol{s})$ terms on both the sides.

$$\boldsymbol{0 = T + B \quad \Rightarrow \quad B = -T}$$

Substituting, $\boldsymbol{A} = \mathbf{1}$ and $\boldsymbol{B} = -\boldsymbol{T}$ in partial fraction expansion of $\boldsymbol{C(s)}$,

$$\boldsymbol{C(s)} = \frac{\mathbf{1}}{\boldsymbol{s}} - \frac{\boldsymbol{T}}{\boldsymbol{sT+1}} = \frac{\mathbf{1}}{\boldsymbol{s}} - \frac{\boldsymbol{T}}{\boldsymbol{T\left(s+\frac{1}{T}\right)}} = \frac{\mathbf{1}}{\boldsymbol{s}} - \frac{\mathbf{1}}{\boldsymbol{s+\frac{1}{T}}}$$

Applying inverse Laplace transform on both the sides. We have

$$\left[\ \frac{\mathbf{1}}{\boldsymbol{s}} \quad \Leftrightarrow \quad \boldsymbol{C(t) = 1} \ \text{ and } \ \frac{\mathbf{1}}{\boldsymbol{s+a}} \quad \Leftrightarrow \quad \boldsymbol{e^{-at}\,u(t)} \quad \Rightarrow \quad \boldsymbol{C(t) = e^{\left(-\frac{1}{T}t\right)}}\ \right]$$

$$\Rightarrow \quad \boldsymbol{C(t)} = \left[\boldsymbol{1 - e^{\left(-\frac{1}{T}t\right)}}\right]\boldsymbol{u(t)}$$

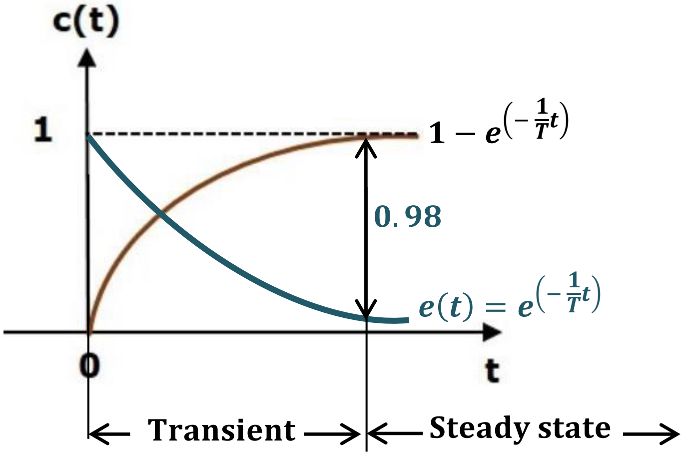
The **unit step response**, $\boldsymbol{C(t)}$ **has** both the **transient** and the **steady state terms**, where the **transient term** is:

$$C_{tr}(t) = -\,e^{\left(-\frac{1}{T}t\right)}u(t)$$

And the **steady state term** is:

$$C_{ss}(t) = u(t)$$

The following figure shows the unit step response.



The value of the **unit step response,** $\boldsymbol{C(t)}$ = **0** at ($\boldsymbol{t}$ = **0**) and for **negative values** of $\boldsymbol{t}$. It is **gradually increasing** from **zero value** and finally reaches **to one** in **steady state**. So, the **steady state value depends** on the **magnitude** of the **input**.

Error signal:

$$e(t) = r(t) - c(t) = 1 - \left(1 - e^{\left(-\frac{t}{T}\right)}\right) = e^{\left(-\frac{t}{T}\right)}$$

Steady state Error:

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} e^{\left(-\frac{t}{T}\right)} = 0$$

$\boldsymbol{T}$ – **Time constant** ralated to the **speed** of the **system response**

▶ If $\boldsymbol{T}$ is **small** – The **response** of the **system** will be **faster**

▶ If $\boldsymbol{T}$ is **large** – The **response** will be **slow** and the **system** is called **sluggish**

To find the **slope** of the **function,** we need to **calculate** the **first derivative** as follows:

$$\left.\frac{dC(t)}{dt}\right|_{t=0} = \frac{1}{T}e^{\left(-\frac{t}{T}\right)} = \frac{1}{T}e^{0} = \frac{1}{T}$$

63.2% of its **final value** → **T (one time constant)**

86.4% of its **final value** → **2T**

98% of its **final value** → **4T – Settling time**

$$e(t) = e^{\left(-\frac{t}{T}\right)} \;\rightarrow\; e_{ss}(t) = \lim_{t\to\infty} e^{\left(-\frac{t}{T}\right)} = 0$$

➢ Ramp Response of First Order System

Consider the **unit ramp signal** $\boldsymbol{r(t)}$ as an **input** to the **first order system**. So,

$$\boldsymbol{R(t) = r(t) = t\,u(t)}$$

Applying Laplace transform on both the sides.

$$\boldsymbol{R(s) = \frac{1}{s^2}}$$

Substituting $\boldsymbol{R(s) = \frac{1}{s^2}}$ in the equation: $\boldsymbol{C(s)} = \left(\frac{1}{sT+1}\right)\boldsymbol{R(s)}$, we have,

$$\boldsymbol{C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}}$$

Doing partial fractions of $\boldsymbol{C(s)}$.

$$\boldsymbol{C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}}$$

$$\Rightarrow\ \boldsymbol{C(s) = \frac{A + s(AT+B) + s^2(BT+C)}{s^2(sT+1)}}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$\boldsymbol{1 = A + s(AT+B) + s^2(BT+C)}$$

By equating the constant terms on both the sides, you will get $\boldsymbol{A} = \boldsymbol{1}$.
Substituting $\boldsymbol{A} = \boldsymbol{1}$ and equate the coefficient of the $\boldsymbol{s}$ terms on both the sides. We have

$$\boldsymbol{0 = AT + B \ \Rightarrow\ B = -T}$$

Similarly, substituting $\boldsymbol{B} = -\boldsymbol{T}$ and equate the coefficient of $\boldsymbol{s^2}$ terms on both the sides. we get,

$$\boldsymbol{0 = BT + C \ \ \Rightarrow\ \ C = T^2}$$

Substitute $\boldsymbol{A} = \boldsymbol{1}$, $\boldsymbol{B} = -\boldsymbol{T}$, and $\boldsymbol{C} = \boldsymbol{T^2}$ in the partial fraction expansion of $\boldsymbol{C(s)}$. We have

$$\Rightarrow\ \boldsymbol{C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{s+\frac{1}{T}}}$$

Applying **inverse Laplace transform** on both the sides. We get

$$\boldsymbol{C(t) = \left[t - T + Te^{\left(-\frac{t}{T}\right)}\right] u(t)}$$

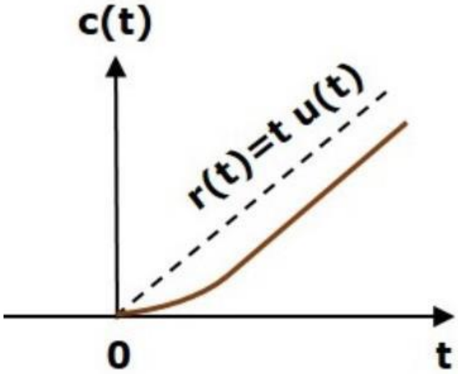
The **unit ramp response**, $C(t)$ **has** both the **transient** and the **steady state terms**, where the **transient term** is:

$$C_{tr}(t) = Te^{\left(-\frac{t}{T}\right)}\, u(t)$$

And the **steady state term** is:

$$C_{ss}(t) = (t - T)\, u(t)$$

The figure below shows the unit ramp response.



The **unit ramp response**, $C(t)$ **follows** the **unit ramp input signal** for **all positive values** of (t). But there is a **deviation** of (T) **units** from the **input signal** as shown in figure.

➢ Parabolic Response of First Order System

Suppose the **unit parabolic signal** $p(t)$ as an **input** to the **first order system**. So,

$$R(t) = \frac{t^2}{2}\, u(t)$$

Applying Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Substituting $R(s) = \frac{1}{s^3}$ in the equation: $C(s) = \left(\frac{1}{sT+1}\right) R(s)$, we have

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Doing partial fractions of $C(s)$. We have

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

$$= \frac{A(sT+1) + Bs(sT+1) + Cs^2(sT+1) + Ds^3}{s^3(sT+1)}$$

$$= \frac{A + AsT + Bs + Bs^2T + Cs^2 + Cs^3T + Ds^3}{s^3(sT+1)}$$

$$= \frac{A + s(AT+B) + s^2(BT+C) + s^3(CT+D)}{s^3(sT+1)}$$

By equating the constant terms on both the sides, you will get $\boldsymbol{A} = \mathbf{1}$.
Substituting $\boldsymbol{A} = \mathbf{1}$ and equate the coefficient of the $\boldsymbol{s}$ terms on both the sides. We have

$$\mathbf{0} = \boldsymbol{AT} + \boldsymbol{B} \;\; \Rightarrow \;\; \boldsymbol{B} = -\boldsymbol{T}$$

Similarly, substituting $\boldsymbol{B} = -\boldsymbol{T}$ and equate the coefficient of $\boldsymbol{s}^2$ terms on both the sides. we get,

$$\mathbf{0} = \boldsymbol{BT} + \boldsymbol{C} \;\; \Rightarrow \;\; \boldsymbol{C} = \boldsymbol{T}^2$$

Similarly, substituting $\boldsymbol{C} = \boldsymbol{T}^2$ and equate the coefficient of $\boldsymbol{s}^3$ terms on both the sides. we get,

$$\mathbf{0} = \boldsymbol{CT} + \boldsymbol{D} \;\; \Rightarrow \;\; \boldsymbol{D} = -\boldsymbol{T}^3$$

Substituting $\boldsymbol{A} = \mathbf{1}$, $\boldsymbol{B} = -\boldsymbol{T}$, $\boldsymbol{C} = \boldsymbol{T}^2$, and $\boldsymbol{D} = -\boldsymbol{T}^3$ in the partial fraction expansion of $\boldsymbol{C(s)}$. We have

$$\boldsymbol{C(s)} = \frac{\mathbf{1}}{\boldsymbol{s^2(sT+1)}} = \frac{\mathbf{1}}{\boldsymbol{s^3}} - \frac{\boldsymbol{T}}{\boldsymbol{s^2}} + \frac{\boldsymbol{T^2}}{\boldsymbol{s}} - \frac{\boldsymbol{T^3}}{\boldsymbol{sT+1}} = \frac{\mathbf{1}}{\boldsymbol{s^3}} - \frac{\boldsymbol{T}}{\boldsymbol{s^2}} + \frac{\boldsymbol{T^2}}{\boldsymbol{s}} - \frac{\boldsymbol{T^3}}{\boldsymbol{T}\left(\boldsymbol{s}+\frac{\mathbf{1}}{\boldsymbol{T}}\right)}$$

$$\Rightarrow \qquad \boldsymbol{C(s)} = \frac{\mathbf{1}}{\boldsymbol{s^3}} - \frac{\boldsymbol{T}}{\boldsymbol{s^2}} + \frac{\boldsymbol{T^2}}{\boldsymbol{s}} - \frac{\boldsymbol{T^2}}{\boldsymbol{s}+\frac{\mathbf{1}}{\boldsymbol{T}}}$$

Applying **inverse Laplace transform** on both the sides. We get

$$\boldsymbol{C(t)} = \left[\frac{\boldsymbol{t^2}}{\mathbf{2}} - \boldsymbol{Tt} + \boldsymbol{T^2} - \boldsymbol{T^2 e^{\left(-\frac{t}{T}\right)}}\right] \boldsymbol{u(t)}$$

The **<u>unit parabolic response</u>**, $\boldsymbol{C(t)}$ **has** both the **<u>transient</u>** and **<u>steady state terms</u>**, where The **<u>transient term</u>** is:

$$\boldsymbol{C_{tr}(t)} = -\boldsymbol{T^2 e^{\left(-\frac{t}{T}\right)}}\, \boldsymbol{u(t)}$$

and **<u>steady state term</u>** is:

$$\boldsymbol{C_{ss}(t)} = \left(\frac{\boldsymbol{t^2}}{\mathbf{2}} - \boldsymbol{Tt} + \boldsymbol{T^2}\right) \boldsymbol{u(t)}$$

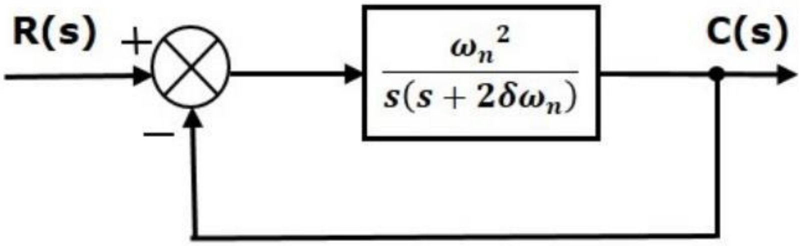
From these responses, we can conclude that the **<u>first order control systems</u>** are **<u>not stable</u>** with the **<u>ramp</u>** and **<u>parabolic inputs</u>** because these **<u>responses</u>** go on **<u>increasing</u>** even at **<u>infinite amount</u>** of **<u>time</u>**. The **<u>first order control systems</u>** are **<u>stable</u>** with **<u>impulse</u>** and **<u>step inputs</u>** because these **<u>responses</u>** have **<u>bounded output</u>**. But the **impulse response doesn't have steady state term**. So, the **<u>step signal</u>** is **<u>widely used</u>** in the **<u>time domain</u>** for **<u>analyzing</u>** the **<u>control systems</u>** from **<u>their responses</u>**.

Response of Second Order System

Consider the following block diagram of closed loop control system. Here, an open loop transfer function,

$$\frac{\omega_n^2}{s(s+2\delta\omega_n)}$$

is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as:

$$TF = \frac{C(t)}{R(t)} = \frac{G(s)}{1+G(s)}$$

Substituting,

$$G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$$

we have

$$TF = \frac{\frac{\omega_n^2}{s(s+2\delta\omega_n)}}{1+\frac{\omega_n^2}{s(s+2\delta\omega_n)}} = \frac{\frac{\omega_n^2}{s(s+2\delta\omega_n)}}{\frac{s(s+2\delta\omega_n)+\omega_n^2}{s(s+2\delta\omega_n)}} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The **power** of 's' is **<u>two</u>** in the **<u>denominator term</u>**. Hence, the above **<u>transfer function</u>** is of the **<u>second order</u>** and the **<u>system</u>** is said to be the **<u>second order system</u>**

The characteristic equation is:

$$s^2+2\delta\omega_n s+\omega_n^2 = 0$$

The roots of characteristic equation are:

$$s = \frac{-2\delta\omega_n \mp \sqrt{(2\delta\omega_n)^2-4\omega_n^2}}{2} = \frac{-2\left(\delta\omega_n \mp \omega_n\sqrt{\delta^2-1}\right)}{2}$$

$$\Rightarrow \quad s = -\delta\omega_n \mp \sqrt{\delta^2-1}$$

- The two roots are imaginary when $\boldsymbol{\delta} = \mathbf{0}$.
- The two roots are real and equal when $\boldsymbol{\delta} = \mathbf{1}$.
- The two roots are real but not equal when $\boldsymbol{\delta} > \mathbf{1}$.
- The two roots are complex conjugate when $\mathbf{0} < \boldsymbol{\delta} < \mathbf{1}$.

We can write $\boldsymbol{C(s)}$ equation as:

$$\boldsymbol{C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)}$$

Where,

- $\boldsymbol{C(s)}$ is the Laplace transform of the output signal, $\boldsymbol{C(t)}$
- $\boldsymbol{R(s)}$ is the Laplace transform of the input signal, $\boldsymbol{r(t)}$
- $\boldsymbol{\omega_n}$ is the natural frequency
- $\boldsymbol{\delta}$ is the damping ratio.

Following these steps to get the response (output) of the second order system in the time domain.

- Take Laplace transform of the input signal, $\boldsymbol{r(t)}$.
- Consider the equation $\boldsymbol{C(s) = \left(\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}\right) R(s)}$
- Substituting $\boldsymbol{R(s)}$ value in the above equation.
- Doing partial fractions of $\boldsymbol{C(s)}$ if required.
- Applying inverse Laplace transform to $\boldsymbol{C(s)}$.

✷ Step Response of Second Order System

Consider the **unit step signal** $\boldsymbol{u(t)}$as **an input** to the **second order system**, so

$$\boldsymbol{R(s) = u(t) = 1}$$

Laplace transform of the **unit step signal** is:

$$\boldsymbol{R(s) = \frac{1}{s}}$$

We know the transfer function of the second order closed loop control system is,

$$\boldsymbol{TF = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}}$$

▶ Case 1: $\boldsymbol{\delta} = \mathbf{0}$

Substituting, $\boldsymbol{\delta} = \mathbf{0}$ in the equation for transfer function, we have

$$\boldsymbol{TF} = \frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{\omega_n^2}}{\boldsymbol{s^2 + \omega_n^2}} \Longrightarrow \boldsymbol{C(s)} = \left(\frac{\boldsymbol{\omega_n^2}}{\boldsymbol{s^2 + \omega_n^2}}\right) \boldsymbol{R(s)}$$

Substituting, $\boldsymbol{R(s)} = \frac{\mathbf{1}}{\boldsymbol{s}}$, in the above equation we get,

$$\boldsymbol{C(s)} = \left(\frac{\boldsymbol{\omega_n^2}}{\boldsymbol{s^2 + \omega_n^2}}\right)\left(\frac{\mathbf{1}}{\boldsymbol{s}}\right) = \frac{\boldsymbol{\omega_n^2}}{\boldsymbol{s\,(s^2 + \omega_n^2)}}$$

Applying **inverse Laplace transform** on both the sides, we have

$$\boldsymbol{C(t) = [1 - cos(\omega_n t)]\; u(t)}$$

So, the **unit step response** of the **second order system** when $\boldsymbol{\delta} = \mathbf{0}$ will be a **continuous time signal** with **constant amplitude** and **frequency**.

▶ Case 2: $\boldsymbol{\delta} = \mathbf{1}$

Substituting, $\boldsymbol{\delta} = \mathbf{1}$ in the transfer function, we have

$$\boldsymbol{TF} = \frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{\omega_n^2}}{\boldsymbol{s^2 + 2\omega_n s + \omega_n^2}} = \frac{\boldsymbol{\omega_n^2}}{\boldsymbol{(s + \omega_n)^2}}$$

$$\Longrightarrow \boldsymbol{C(s)} = \left[\frac{\boldsymbol{\omega_n^2}}{\boldsymbol{(s + \omega_n)^2}}\right] \boldsymbol{R(s)}$$

Substitute, $\boldsymbol{R(s)} = \frac{\mathbf{1}}{\boldsymbol{s}}$, we get

$$\boldsymbol{C(s)} = \left[\frac{\boldsymbol{\omega_n^2}}{\boldsymbol{(s + \omega_n)^2}}\right]\left(\frac{\mathbf{1}}{\boldsymbol{s}}\right) = \frac{\boldsymbol{\omega_n^2}}{\boldsymbol{s\,(s + \omega_n)^2}}$$

Do partial fractions of $\boldsymbol{C(s)}$, we have

$$\begin{aligned}\boldsymbol{C(s)} &= \frac{\boldsymbol{\omega_n^2}}{\boldsymbol{s\,(s + \omega_n)^2}} = \frac{\boldsymbol{A}}{\boldsymbol{s}} + \frac{\boldsymbol{B}}{\boldsymbol{(s + \omega_n)}} + \frac{\boldsymbol{C}}{\boldsymbol{(s + \omega_n)^2}} \\ &= \frac{\boldsymbol{A(s + \omega_n)^2 + B\,s\,(s + \omega_n) + C\,s}}{\boldsymbol{s(s + \omega_n)^2}} \\ &= \frac{\boldsymbol{A\,s^2 + 2\,A\,s\,\omega_n + A\,\omega_n^2 + B\,s^2 + Bs\omega_n + C\,s}}{\boldsymbol{s(s + \omega_n)^2}}\end{aligned}$$

$$\Longrightarrow \boldsymbol{C(s)} = \frac{\boldsymbol{A\,\omega_n^2 + s^2(A + B) + s\,\omega_n\,(2A\omega_n + B\omega_n + C)}}{\boldsymbol{s(s + \omega_n)^2}}$$

By equating the constant terms on both the sides, you will get

$$\boldsymbol{\omega_n^2 = A\,\omega_n^2 \quad \Rightarrow \quad A = 1}$$

Substituting $\boldsymbol{A}$ = $\mathbf{1}$ and equate the coefficient of the $\boldsymbol{s^2}$ terms on both the sides. We have

$$\boldsymbol{0 = (A + B) \;\; \Longrightarrow \;\; B = -1}$$

Similarly, substituting $\boldsymbol{A}$ = $\mathbf{1}$ and $\boldsymbol{B}$ = − 1 and equate the coefficient of $\boldsymbol{\omega_n}$ terms on both the sides. we get,

$$\boldsymbol{0 = 2A\omega_n + B\omega_n + C = 2\omega_n - \omega_n + C \quad \Longrightarrow \quad C = -\,\omega_n}$$

Substituting $\boldsymbol{A}$ = $\mathbf{1}$, $\boldsymbol{B}$ = − 1, and $\boldsymbol{C = -\,\omega_n}$, in the partial fraction expansion of $\boldsymbol{C(s)}$. We have

$$\boldsymbol{C(s) = \frac{\omega_n^2}{s\,(s+\omega_n)^2} = \frac{1}{s} - \frac{1}{(s+\omega_n)} - \frac{\omega_n}{(s+\omega_n)^2}}$$

Applying **inverse Laplace transform** on both the sides, we get

$$\boldsymbol{C(t) = \left[1 - e^{(-\,\omega_n t)} - \;\omega_n t\, e^{(-\,\omega_n t)}\right] u(t)}$$

So, the **unit step response** of the **second order system** will **try to reach** the **step input** in **steady state**.

▶ Case 3: $\boldsymbol{0 < \delta < 1}$

The **denominator term** of the transfer function can be modified as follows:

$$\begin{aligned}\boldsymbol{s^2 + 2\omega_n s + \omega_n^2} &\boldsymbol{= s^2 + 2\omega_n s + \omega_n^2 + \; (\delta\omega_n)^2 - (\delta\omega_n)^2} \\ &\boldsymbol{= [s^2 + 2\omega_n s + (\delta\omega_n)^2] + \omega_n^2 - \delta^2\omega_n^2} \\ &\boldsymbol{= (s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}\end{aligned}$$

The transfer function becomes,

$$\boldsymbol{C(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \left[\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right] R(s)}$$

Substitute, $\boldsymbol{R(s) = \frac{1}{s}}$, we get

$$\boldsymbol{C(s) = \left[\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right]\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s[(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)]}}$$

Doing partial fractions of $\boldsymbol{C(s)}$**,**

$$\begin{aligned}\boldsymbol{C(s)} &\boldsymbol{= \frac{\omega_n^2}{s[(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)]} = \frac{A}{s} + \frac{Bs + C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}} \\ &\boldsymbol{= \frac{A}{s} + \frac{Bs}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} + \frac{C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}}\end{aligned}$$

After simplifying, you will get the values of $\boldsymbol{A}$ = $\mathbf{1}$, $\boldsymbol{B}$ = − $\mathbf{1}$ and $\boldsymbol{C}$ = − $\boldsymbol{2\delta\omega_n}$.

Substituting these values in the above partial fraction expansion of $\boldsymbol{C(s)}$, we have

$$\begin{aligned} C(s) &= \frac{1}{s} - \frac{s}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} \\ &= \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2 + \left(\omega_n\sqrt{1-\delta^2}\right)^2} - \frac{\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} \end{aligned}$$

$$= \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2 + \left(\omega_n\sqrt{1-\delta^2}\right)^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left[\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2 + \left(\omega_n\sqrt{1-\delta^2}\right)^2}\right]$$

Substituting, $\boldsymbol{\omega_d = \omega_n\sqrt{1-\delta^2}}$, we have

$$C(s) = \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left[\frac{\omega_d}{(s+\delta\omega_n)^2 + \omega_d^2}\right]$$

Apply **inverse Laplace transform** on both the sides. We get

$$\begin{aligned} C(t) &= \left[1 - e^{(-\delta\,\omega_n t)}\cos(\,\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}}\, e^{(-\delta\,\omega_n t)}\sin(\,\omega_d t)\right] u(t) \\ &= \left[1 - \left\{\frac{e^{(-\delta\,\omega_n t)}}{\sqrt{1-\delta^2}}\right\}\left\{\sqrt{1-\delta^2}\,\cos(\,\omega_d t) + \delta\,\sin(\,\omega_d t)\right\}\right] u(t) \end{aligned}$$

- ▶ If $\sqrt{1-\delta^2} = \sin\theta \Rightarrow \sin^2\theta = 1-\delta^2 \Rightarrow \delta^2 = 1-\sin^2\theta = \cos^2\theta$

$$\Rightarrow \delta = \cos\theta,$$

Substituting these values in the above equation, we get

$$C(t) = \left[1 - \left\{\frac{e^{(-\delta\,\omega_n t)}}{\sqrt{1-\delta^2}}\right\}\{\sin\theta\cos(\,\omega_d t) + \cos\theta\sin(\,\omega_d t)\}\right] u(t)$$

$$\Rightarrow C(t) = \left[1 - \left\{\frac{e^{(-\delta\,\omega_n t)}}{\sqrt{1-\delta^2}}\right\}\sin(\,\omega_d t + \theta)\right] u(t)$$

So, the **unit step response** of the **second order system** is having **damped oscillations (decreasing amplitude**) when ‘$\boldsymbol{\delta}$’ lies between **zero** and **one**.

▶ Case 4: $\boldsymbol{\delta} > \mathbf{1}$

The **denominator term** of the transfer function can be modified as follows:

$$\begin{aligned} s^2 + 2\delta\omega_n s + \omega_n^2 &= s^2 + 2\delta\omega_n s + \omega_n^2 + (\delta\omega_n)^2 - (\delta\omega_n)^2 \\ &= [s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2] + \omega_n^2 - \delta^2\omega_n^2 \\ &= (s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1) \end{aligned}$$

The **transfer function** becomes,

$$TF = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}$$

$$\Rightarrow \quad C(s) = \left[\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}\right] R(s)$$

Substitute, $R(s) = \frac{1}{s}$, in the above equation, we have

$$\begin{aligned} C(s) &= \left[\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}\right]\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s[(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)]} \\ &= \frac{\omega_n^2}{s\left[(s + \delta\omega_n)^2 - \left(\omega_n\sqrt{\delta^2 - 1}\right)^2\right]} \end{aligned}$$

$$\Rightarrow \quad C(s) = \frac{\omega_n^2}{s\left[\left(s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}\right)\left(s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}\right)\right]}$$

Doing partial fractions of $C(s)$**,**

$$C(s) = \frac{A}{s} + \frac{B}{\left(s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}\right)} + \frac{C}{\left(s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}\right)}$$

After simplifying, we get the values of $A = 1$,

$$\mathrm{B} = \frac{1}{2\left(\delta + \sqrt{\delta^2 - 1}\right)\left(\sqrt{\delta^2 - 1}\right)} \quad \text{and} \quad \mathrm{C} = \frac{1}{2\left(\delta - \sqrt{\delta^2 - 1}\right)\left(\sqrt{\delta^2 - 1}\right)}$$

Substituting these values in the above partial fraction expansion of $C(s)$**,**

$$\begin{aligned} C(s) = \frac{1}{s} &+ \frac{1}{2\left(\delta + \sqrt{\delta^2 - 1}\right)\left(\sqrt{\delta^2 - 1}\right)}\left[\frac{1}{\left(s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}\right)}\right] \\ &- \frac{1}{2\left(\delta - \sqrt{\delta^2 - 1}\right)\left(\sqrt{\delta^2 - 1}\right)}\left[\frac{1}{\left(s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}\right)}\right] \end{aligned}$$

$$= \frac{1}{s} + \frac{(s + \delta\omega_n)}{(s + \delta\omega_n)^2 + \left(\omega_n\sqrt{1-\delta^2}\right)^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left[\frac{\omega_n\sqrt{1-\delta^2}}{(s + \delta\omega_n)^2 + \left(\omega_n\sqrt{1-\delta^2}\right)^2}\right]$$

Applying **inverse Laplace transform** on both the sides, we have

$$C(t) = \left[1 + \left\{\frac{1}{2\left(\delta + \sqrt{\delta^2 - 1}\right)\left(\sqrt{\delta^2 - 1}\right)}\right\} e^{\left(-\delta\,\omega_n + \omega_n\sqrt{\delta^2-1}\right)t} - \left\{\frac{1}{2\left(\delta - \sqrt{\delta^2 - 1}\right)\left(\sqrt{\delta^2 - 1}\right)}\right\} e^{\left(-\delta\,\omega_n - \omega_n\sqrt{\delta^2-1}\right)t}\right] u(t)$$

Since it is **over damped**, the **<u>unit step response</u>** of the **<u>second order system</u>** when $\delta > 1$ will **<u>never reach step input</u>** in the **<u>steady state</u>**.

✷ <u>Impulse Response of Second Order System</u>

The impulse response of the second order system can be obtained by using any one of these **<u>two methods</u>**.

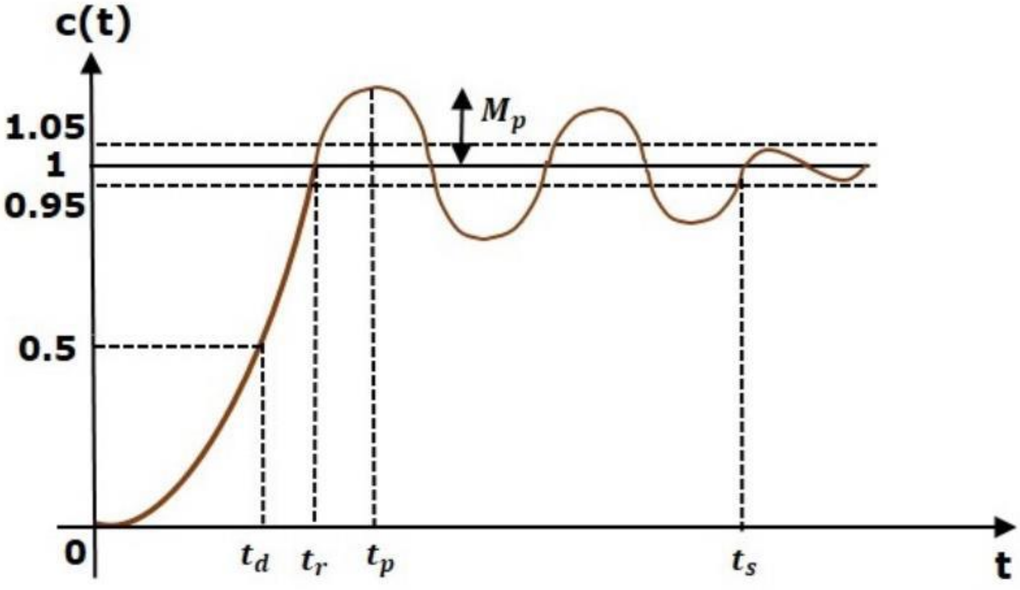
- Follow the procedure involved while deriving step response by considering the value of $R(s)$ = **1 instead of** $R(s) = \frac{1}{s}$**,**
- Doing the **<u>differentiation</u>** of the **step response**.

The following table shows the **impulse response** of the **second order system** for **4 cases** of the damping ratio.

Condition of Damping Ratio	**Impulse Response for** $t \geq 0$
$\delta = 0$	$\omega_n\, sin(\,\omega_n t)$
$\delta = 1$	$\omega_n^2\, t\, e^{(-\,\omega_n t)}$
$0 < \delta < 1$	$\left\{\frac{\omega_n\, e^{(-\,\delta\,\omega_n t)}}{\sqrt{1-\delta^2}}\right\} sin(\,\omega_d t)$
$\delta > 1$	$\left(\frac{\omega_n}{2\sqrt{\delta^2 - 1}}\right)\left[e^{-\left(\delta\,\omega_n - \omega_n\sqrt{\delta^2-1}\right)t} - e^{-\left(\delta\,\omega_n + \omega_n\sqrt{\delta^2-1}\right)t}\right]$

Time Domain Specifications

The **step response** of the **second order system** for the **underdamped case** is shown in the following figure, where all the **time domain specifications** are **represented**.



The **response up** to the **settling time** is known as **transient response** and the **response after** the **settling time** is known as **steady state response**.

Delay Time

It is the **time required** for the **response** to reach **half** of its **final value** from the **zero instant**. It is denoted by t_d.

Suppose the **step response** of the **second order system** for $t \geq 0$, when 'δ' **lies between zero** and **one**.

$$C(t) = 1 - \left[\frac{e^{(-\delta\,\omega_n t)}}{\sqrt{1-\delta^2}}\right] sin(\,\omega_d t + \theta)$$

The **final value** of the **step response** is **one**. Therefore, at $t = t_d$, the **value** of the **step response** will be **0.5**. Substituting, these values in the above equation. We have

$$C(t) = 0.5 = 1 - \left[\frac{e^{(-\delta\,\omega_n t_d)}}{\sqrt{1-\delta^2}}\right] sin(\,\omega_d t_d + \theta)$$

$$\Rightarrow \quad \left[\frac{e^{(-\delta\,\omega_n t_d)}}{\sqrt{1-\delta^2}}\right] sin(\,\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time** t_d as:

$$t_d = \frac{1 + 0.7\,\delta}{\omega_n}$$

- ## Rise Time

It is the **time required** for the **response** to **rise** from **0%** to **100%** of its **final value**. This is applicable for the **under-damped systems**. For the **over - damped systems**, consider the **duration** from **10%** to **90%** of the **final value**. **Rise time** is denoted by t_r. At $t = t_1 = 0$, $C(t) = 0$, and we know that the **final value** of the **step response** is **one**. Therefore, at $t = t_2$, the **value** of **step response** is **one**. Substituting, these values in the following equation:

$$C(t) = 1 - \left[\frac{e^{(-\,\delta\,\omega_n t_d)}}{\sqrt{1-\delta^2}}\right] sin(\,\omega_d t + \theta)$$

Thus, $$C(t) = 1 = 1 - \left[\frac{e^{(-\,\delta\,\omega_n\,t_2)}}{\sqrt{1-\delta^2}}\right] sin(\,\omega_d t_2 + \theta) \quad \Rightarrow$$

$$\left[\frac{e^{(-\,\delta\,\omega_n\,t_2)}}{\sqrt{1-\delta^2}}\right] sin(\,\omega_d t_2 + \theta) = 0 \quad \Rightarrow$$

$$sin(\,\omega_d t_2 + \theta) = 0 \quad \Rightarrow \quad \omega_d t_2 + \theta = \pi \quad \Rightarrow \quad t_2 = \frac{\pi - \theta}{\omega_d}$$

Substituting t_1 and t_2 values in the following **equation** of **rise time**,

$$t_r = t_2 - t_1 = \frac{\pi - \theta}{\omega_d}$$

From above equation, we can conclude that the **rise time** t_r and the **damped frequency** ω_d are **inversely proportional** to **each other**.

- ## Peak Time

It is the **time required** for the **response** to **reach** the **peak value** for the **first time**. It is denoted by t_p. At $t = t_p$, the **first derivate** of the **response** is **zero**.

The **step response** of **second order system** for **under – damped case** is:

$$C(t) = 1 - \left[\frac{e^{(-\delta\, \omega_n\, t)}}{\sqrt{1-\delta^2}} \right] sin(\, \omega_d t + \theta)$$

Differentiating $C(t)$ with respect to 't', we have

$$\begin{aligned} \frac{dC(t)}{dt} &= -\left[\frac{e^{(-\delta\, \omega_n t)}}{\sqrt{1-\delta^2}} \right] (-\delta\, \omega_n) sin(\, \omega_d t + \theta) \\ &\qquad + \omega_d\, cos(\, \omega_d t + \theta) \left[-\left\{ \frac{e^{(-\delta\, \omega_n t)}}{\sqrt{1-\delta^2}} \right\} \right] \\ &= -\left[\frac{e^{(-\delta\, \omega_n t)}}{\sqrt{1-\delta^2}} \right] [\omega_d\, cos(\, \omega_d t + \theta) - \delta\, \omega_n\, sin(\, \omega_d t + \theta)] \end{aligned}$$

Substituting, t = t_p and $\frac{dC(t)}{dt} = 0$ in the above equation. We get

$$0 = \left[\frac{e^{(-\delta\, \omega_n t_p)}}{\sqrt{1-\delta^2}} \right] \left[\omega_d\, cos\left(\, \omega_d t_p + \theta\right) - \delta\, \omega_n\, sin\left(\, \omega_d\, t_p + \theta\right)\right]$$

$$\Rightarrow \quad \omega_n \sqrt{1-\delta^2}\, cos\left(\, \omega_d t_p + \theta\right) - \delta\, \omega_n\, sin\left(\, \omega_d\, t_p + \theta\right) = 0$$

$$\Rightarrow \quad \sqrt{1-\delta^2}\, cos\left(\, \omega_d t_p + \theta\right) - \delta\, sin\left(\, \omega_d\, t_p + \theta\right) = 0$$

$$\Rightarrow \quad sin\, \theta\, cos\left(\, \omega_d t_p + \theta\right) - cos\, \theta\, sin\left(\, \omega_d\, t_p + \theta\right) = 0$$

$$\Rightarrow \quad sin\, \left(\, \theta - \omega_d t_p - \theta\right) = 0 \quad \Rightarrow \quad sin\, \left(-\, \omega_d t_p\right) = 0$$

$$\Rightarrow \quad -\sin\left(\, \omega_d t_p\right) = 0 \quad \Rightarrow \sin\left(\, \omega_d t_p\right) = 0$$

$$\Rightarrow \quad \omega_d t_p = \pi \quad \Rightarrow \quad t_p = \frac{\pi}{\omega_d}$$

From the above equation, we can conclude that the **peak time** t_p and the **damped frequency** ωd are **inversely proportional** to **each other**.

- ## Peak Overshoot

Peak overshoot Mp is defined as the **deviation** of the **response** at **peak time** from the **final value** of **response**. It is also called the **maximum overshoot**.

Mathematically, we can write it as:

$$M_p = C\left(t_p\right) - C(\infty)$$

Where,

- C (t_p) is the **peak value** of the **response**.
- C (∞) is the **final** (**steady state**) value of the **response**.

At t = t_p, the response $C(t)$ is:

$$C(t_p) = 1 - \left[\frac{e^{(-\delta\,\omega_n\,t_p)}}{\sqrt{1-\delta^2}}\right] sin(\,\omega_d t_p + \theta)$$

Substituting, $t_p = \frac{\pi}{\omega_d}$ in the right-hand side of the above equation, we get

$$C(t_p) = 1 - \left[\frac{e^{\left(-\delta\,\omega_n \frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}}\right] sin\left(\,\omega_d \frac{\pi}{\omega_d} + \theta\right) \;\Rightarrow$$

$$C(t_p) = 1 - \left[\frac{e^{\left(-\frac{\delta\,\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}}\right] (-\,sin\,\theta\,)$$

We know that: $sin\,\theta = \sqrt{1-\delta^2}$

so, we will get $C(t_p)$ as: $C(t_p) = 1 + \,e^{\left(-\frac{\delta\,\pi}{\sqrt{1-\delta^2}}\right)}$

Substituting the values of $C\,(t_p)$ and $C\,(\infty)$ in the **peak overshoot equation**, we have

$$M_p = 1 + \,e^{\left(-\frac{\delta\,\pi}{\sqrt{1-\delta^2}}\right)} - 1 \;\Rightarrow\quad M_p = \,e^{\left(-\frac{\delta\,\pi}{\sqrt{1-\delta^2}}\right)}$$

Percentage of **peak overshoot** $\%Mp$ can be calculated by using this formula:

$$\%M_p = \frac{M_p}{C(\infty)} \times 100\%$$

By substituting the values of M_p and $C(\infty)$ in above formula, we will get the **Percentage** of the **peak overshoot** $\%Mp$ as:

$$\%M_p = \left[\,e^{\left(-\frac{\delta\,\pi}{\sqrt{1-\delta^2}}\right)}\right] \times 100\%$$

From the above equation, we can conclude that the **percentage of peak overshoot** $\%Mp$ will **decrease** if the **damping ratio δ increases**.

- ## Settling time

It is the **time required** for the **response** to reach the **steady state** and **stay** within the **specified tolerance bands** around the **final value**. In general, the **tolerance bands** are 2% and 5%. The **settling time** is denoted by t_s.

➢ The **settling time** for **5% tolerance band** is:

$$t_s = \frac{3}{\delta\,\omega_n} = 3\,\tau$$

➢ The **settling time** for **2% tolerance band** is:

$$t_s = \frac{3}{\delta\, \omega_n} = 4\,\tau$$

Where, τ is the **time constant** and is **equal** to $\frac{1}{\delta\, \omega_n}$.

- **Both** the **settling time** t_s and the **time constant** τ are **inversely proportional** to the **damping ratio** δ.
- **Both** the **settling time** t_s and the **time constant** τ are **independent** of the **system gain**. That means even the **system gain changes**, the **settling time** t_s and **time constant** τ will **never change**.

Example 1: Find the **time domain specifications** of a **control system** having the **closed loop transfer function**:

$$\frac{4}{s^2 + 2s + 4}$$

when the **unit step signal** is applied as an **input** to this control system.

Solution:

The **standard form** of the **transfer function** of the **second order closed loop control system** as:

$$\frac{4}{s^2 + 2\delta\, \omega_n s + \omega_n^2}$$

By **equating** these two transfer functions, we will get the **un - damped natural frequency** ω_n=**2 rad/sec** and the **damping ratio** δ = **0.5** as follows:

$$\frac{4}{s^2 + 2s + 4} = \frac{\omega_n^2}{s^2 + 2\delta\, \omega_n s + \omega_n^2}$$

By equating the constant terms on both the sides, you will have

$$s^2 + 2s + 4 = s^2 + 2\delta\, \omega_n s + \omega_n^2$$

$$2 = 2\,\delta\, \omega_n \;\Rightarrow\; \delta = \frac{1}{\omega_n}$$

$$4 = \omega_n^2 \;\Rightarrow\; \omega_n = 2 \;\Rightarrow\; \delta = \frac{1}{2} = 0.5$$

The formula for **damped frequency** ω_d as: $\omega_d = \omega_n\sqrt{1-\delta^2}$

Substituting, ω_n and δ values in the above formula, we get

$$\omega_d = 2\sqrt{1-(0.5)^2} = 1.732\; rad/sec$$

Substitute, $\boldsymbol{\delta}$ value in following relation:

$$\boldsymbol{\delta = cos\theta \quad \Rightarrow \quad \theta = cos^{-1}(\delta) = cos^{-1}(0.5) = \frac{\pi}{3}\ rad}$$

Substituting the above necessary values in the formula of each time domain specification and simplify in order to get the values of time domain specifications for given transfer function. The following table shows the formulae

Time Domain Specification	Formula	Substituting of Values in Formula	Final Value
Delay Time	$t_d = \frac{1 + 0.7\,\delta}{\omega_n}$	$t_d = \frac{1 + 0.7(0.5)}{2}$	$t_d = 0.675\ sec$
Rise Time	$t_r = \frac{\pi - \theta}{\omega_d}$	$t_r = \frac{\pi - \frac{\pi}{3}}{1.732}$	$t_r = 1.207\ sec$
Peak Time	$t_p = \frac{\pi}{\omega_d}$	$t_p = \frac{\pi}{1.732}$	$t_p = 1.813\ sec$
%overshoot	$\%M_p = \left[e^{\left(-\frac{\delta\,\pi}{\sqrt{1-\delta^2}}\right)}\right] \times 100\%$	$\%M_p = \left[e^{\frac{0.5\pi}{\sqrt{1-(0.5)^2}}}\right] \times 100$	$\%M_p = 16.32\%$
Settling Time for 2% Tolerance	$t_s = \frac{4}{\delta\,\omega_n}$	$t_s = \frac{4}{(0.5)(2)}$	$t_s = 4\ sec$

Control Systems – Steady State Errors

The **deviation** of the **output** of control system from **desired response** during **steady state** is known as **steady state error**. It is **represented** as $\boldsymbol{e_{ss}}$. The **steady state error** can be found using the **final value theorem** as follows:

$$\boldsymbol{e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s)}$$

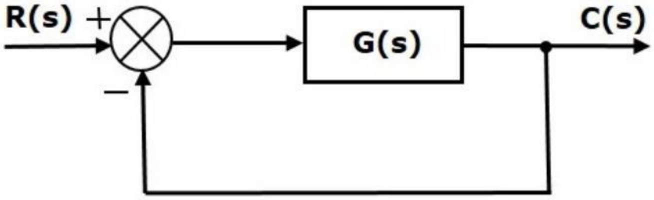
Where, $\boldsymbol{E(s)}$ is the **Laplace transform** of the **error signal** $\boldsymbol{e(t)}$**.**

$$\boldsymbol{E(s) = \frac{R(s)}{1 + G(s)H(s)}}$$

The **steady state errors** for **unity feedback** and **non-unity feedback control systems** one by one can be found as follows:

✸ **Steady State Errors** for **Unity Feedback Systems**

Consider the following **block diagram** of **closed loop control system**, which is **having unity negative feedback**.



Where,

- $\boldsymbol{R(s)}$ is the **Laplace transform** of the **reference Input signal** $\boldsymbol{R(t)}$
- $\boldsymbol{C(s)}$ is the **Laplace transform** of the **output signal** $\boldsymbol{C(t)}$

The **transfer function** of the **unity negative feedback closed loop control system** as:

$$TF = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \quad \Rightarrow \quad C(s) = \frac{G(s)}{1+G(s)} R(s)$$

The **output** of the **summing point** is:

$$E(s) = R(s) - C(s)$$

Substituting $\boldsymbol{C(s)}$ value in the above equation, we get

$$E(s) = R(s) - \frac{G(s)}{1+G(s)} R(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)} = \frac{R(s)}{1+G(s)}$$

Substitute $\boldsymbol{E(s)}$ value in the **steady state error formula, we have**

$$e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)}$$

The following **table shows** the **steady state errors** and the **error constants** for **standard input signals** like **unit step**, **unit ramp** and **unit parabolic signals**.

Input Signal	**Steady State Error** e_{ss}	**Error Constant**
Unit Step Signal	$\frac{1}{1+K_p}$	$K_p = \lim_{s \to 0} G(s)$
Unit Ramp Signal	$\frac{1}{K_v}$	$K_v = \lim_{s \to 0} s\,G(s)$
Unit Parabolic Signal	$\frac{1}{K_a}$	$K_a = \lim_{s \to 0} s^2\,G(s)$

Where,

- $\boldsymbol{K_p}$, are **position error constant**,
- $\boldsymbol{K_v}$, velocity error constant, and
- $\boldsymbol{K_a}$, acceleration error constant.

Note 1: If any of the above **input signals** has the **amplitude** other **than unity**, then **multiply corresponding steady state error** with that **amplitude**.

Note 2: We can't define the **steady state error** for the **unit impulse signal** because, it **exists only at origin**. So, we **can't compare** the **impulse response** with the **unit impulse input** as ***t*** denotes **infinity**.

Example 2: Find the **steady state error** for an **input signal**

$$\boldsymbol{R(t) = \left(5 + 2t + \frac{t^2}{2}\right) u(t)}$$

of unity negative feedback control system with

$$\boldsymbol{G(s) = \frac{5(s+4)}{t^2(s+1)(s+20)}}$$

Solution:

The given **input signal** is a **combination** of **three signals**: **step**, **ramp** and **parabolic**. The following **table** shows the **error constants** and **steady state error** values for these **three signals**.

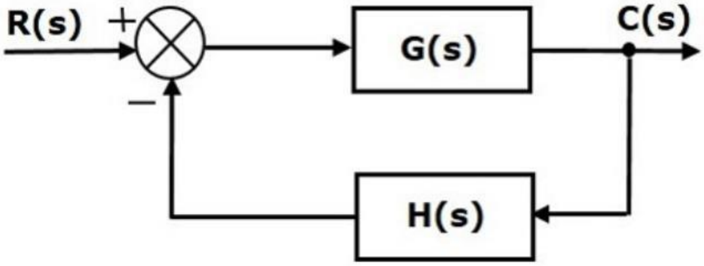
Input Signal	**Error Constant**	**Steady State Error $\boldsymbol{e_{ss}}$**
Unit Step Signal $\boldsymbol{R_1(t) = 5\, u(t)}$	$K_p = \lim_{s\to 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+K_p} = 0$
Unit Ramp Signal $\boldsymbol{R_2(t) = 2\, t\, u(t)}$	$K_v = \lim_{s\to 0} s\, G(s) = \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
Unit Parabolic Signal $\boldsymbol{R_3(t)} = \frac{\boldsymbol{t^2}}{2}\, \boldsymbol{u(t)}$	$K_a = \lim_{s\to 0} s^2\, G(s) = 1$	$e_{ss3} = \frac{1}{K_a} = 1$

The **overall steady state error** can be obtained by **adding** the above **three steady state errors**.

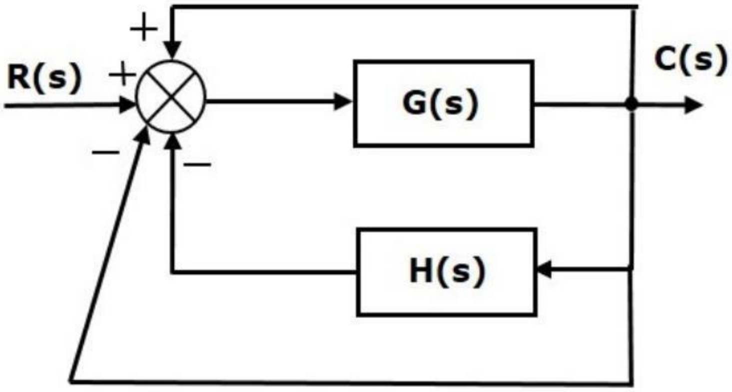
$$\boldsymbol{e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = 0 + 0 + 1 = 1}$$

✷ Steady State Errors for Non-Unity Feedback Systems

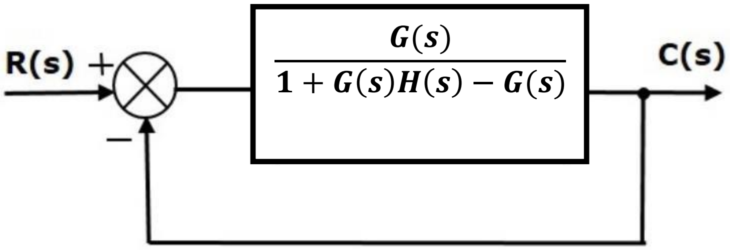
Consider the following **block diagram** of **closed loop control system**, which is **having non-unity negative feedback**.



The **steady state errors** can be **found only** for the **unity feedback systems**. So, need to **convert** the **non-unity feedback system** into **unity feedback system**. For this, include one **unity positive feedback path** and **one unity negative feedback path** in the above block diagram. The **new block diagram** looks like as shown below.



Reduce the **converted block diagram** by **keeping** the **unity negative feedback** as it is. The following is the simplified block diagram.



This **block diagram** **resembles** the **block diagram** of **the unity negative feedback closed loop control system**. Here, the **single block** is having the **transfer function:**

$$\frac{\boldsymbol{G(s)}}{\boldsymbol{1+G(s)\,H(s)-G(s)}}$$

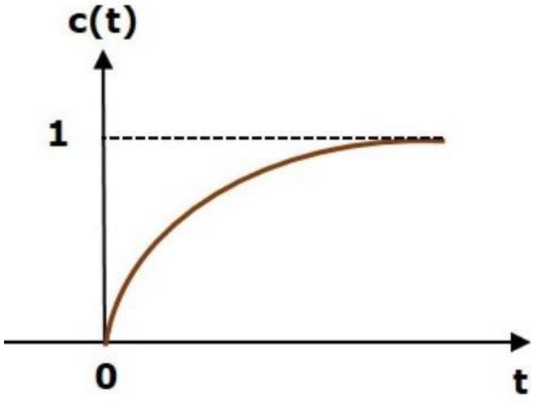
instead of $\boldsymbol{G(s)}$.

Now using **steady state error formula** for the **unity negative feedback systems** to calculate the **steady state errors**.

Note: It is **meaningless** to find the **steady state errors** for **unstable closed loop systems**. So, we have to calculate the **steady state errors** **only** for **closed loop stable systems**. This means we need to **check whether** the **control system** is **stable** or **not** before finding the **steady state errors**.

Control Systems – Stability

A **system** is said to be **stable**, if its **output** is **under control**. **Otherwise**, it is said to be **unstable**. A **stable system** **produces** a **bounded output** for a given **bounded input**. The following figure shows the **response** of a **stable system**.



This is the **response** of **first order control system** for **unit step input**. This **response** has the **values** between **0** and **1**. So, it is **bounded output**. We know that the **unit step signal** has the **value** of **one** for all **positive values** of ***t*** including **zero**. So, it is **bounded input**. Therefore, the **first order control system** is **stable** since both the **input** and the **output** are **bounded**.

Types of Systems based on Stability

Systems can be **classified** based on **stability** to:

- **Absolutely stable system**
- **Conditionally stable system**
- **Marginally stable system**

Absolutely Stable System

If the **system** is **stable** for **all the range** of **system component values**, then it is known as the **absolutely stable system**. The **open loop control system** is **absolutely stable** if all the **poles** of the **open loop transfer function** present in **left half** of **‘*s*’ plane**. Similarly, the **closed loop control system** is **absolutely stable** if all the **poles** of the **closed loop transfer function** present in the **left half** of the ‘***s***’ **plane**.

Conditionally Stable System

If the **system** is **stable** for a **certain range** of **system component values**, then it is known as **conditionally stable** **system**.

Marginally Stable System

If the **system** is **stable** by **producing** an **output signal** with **constant amplitude** and **constant frequency** of **oscillations** for **bounded input**, then it is known as **marginally stable system**. The **open loop control system** is **marginally stable** if **any two poles** of the **open loop transfer function** is **presented** on the **imaginary axis**. Similarly, the **closed loop control system** is **marginally stable** if **any two poles** of the **closed loop transfer function** is **presented** on the **imaginary axis**.

Control Systems - Stability Analysis

To **find** the **stability** of the **closed loop control systems**, and to **discuss** the **stability analysis** in the **‘s’ domain** we will use the **Routh-Hurwitz stability criterion**. In this criterion, we **require** the **characteristic equation**.

Routh-Hurwitz Stability Criterion

Routh-Hurwitz stability criterion is **having** **one necessary condition** and **one sufficient condition** for **stability**. If **any control system** **doesn’t satisfy** the **necessary condition**, then we can say that the **control system** is **unstable**. But, if the **control system** **satisfies** the **necessary condition**, then it **may** or **may not** be **stable**. So, the **sufficient condition** is **helpful** for **knowing whether** the **control system** is **stable** or **not**.

▶ Necessary Condition for Routh-Hurwitz Stability

The **necessary condition** is that the **coefficients** of the **characteristic polynomial** should be **positive**. This implies that all the **roots** of the **characteristic equation** should have **negative real parts**.

Consider the **characteristic equation** of the **order** 'n' is:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n s^0 = 0$$

Note that, there **should not** be **any term** missing in the n^{th} **order characteristic equation**. This means that the n^{th} **order characteristic equation should not** have **any coefficient** that is of **zero value**.

▶ Sufficient Condition for Routh-Hurwitz Stability

The **sufficient condition** is that **all the elements** of the **first column** of the **Routh Array** should **have** the **same sign**. This means that **all the elements** of the **first column** of the **Routh Array** should be either **positive** or **negative**.

○ Routh Array Method

If **all the roots** of the **characteristic equation exist** to the **left half** of the **'s' plane**, then the **control system** is **stable**. If **at least one root** of the **characteristic equation exists** to the **right half** of the **'s' plane**, then the **control system** is **unstable**. So, we **have to find** the **roots** of the **characteristic equation** to **know whether** the **control system** is **stable** or **unstable**. But as the **order** of the **characteristic equation increases**, the process of finding the roots becomes harder. So, to **overcome this problem** there **we have** the **Routh Array Method**. In this method, there is **no need** to **calculate** the **roots** of the **characteristic equation**. **First formulate** the **Routh table** and **find** the **number** of the **sign changes** in the **first column** of the **Routh table**, which **gives** the **number of roots** of **characteristic equation** that **exist** in the **right half** of the **'s' plane** and the **control system** is **unstable**.

Follow this **procedure** for **forming** the **Routh table**.

① **Filling** the **first two rows** of the **Routh array** with the **coefficients** of the **characteristic polynomial** as **mentioned** in the **table below**. **Start** with the **coefficient** of s^n and **continue up** to the **coefficient** of s^0.

② **Filling** the **remaining rows** of the **Routh array** with the **elements** as **mentioned** in the **table below**. **Continue** this process **till** getting the **first column element** of **row** s^0 is a_n. Here, a_n is the **coefficient** of s^0 in **the characteristic polynomial**.

Note: If **any row elements** of the **Routh table** have **some common factor**, then **dividing** the **row elements** with **that factor** for the **simplification** will **be easy**.

The **following table shows** the **Routh array** of the n^{th} **order characteristic polynomial.**

$$a_0\, s^n \ + a_1\, s^{n-1} + a_2\, s^{n-2} + \cdots + a_{n-1}\, s^1 + a_n\, s^0$$

s^n	a_0	a_2	a_4	a_6	$\cdots$	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$	$\cdots$
s^{n-2}	$b_1 \ = \dfrac{a_1\, a_2 - a_3\, a_0}{a_1}$	$b_2 \ = \dfrac{a_1\, a_4 - a_5\, a_0}{a_1}$	$b_3 \ = \dfrac{a_1\, a_6 - a_7\, a_0}{a_1}$	$\cdots$	$\cdots$	$\cdots$
s^{n-3}	$c_1 \ = \dfrac{b_1\, a_3 - b_2\, a_1}{b_1}$	$c_2 \ = \dfrac{b_1\, a_5 - b_3\, a_1}{b_1}$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$				
s^1	$\vdots$					
s^0	a_n					

Example 3: **F**ind the stability of the control system having characteristic equation,

$$s^4 \ + 3s^3 + 3s^2 + 2s + 1 = 0$$

Solution:

Step 1: **Verifyig** the **necessary condition** for the **Routh-Hurwitz stability:**

$$s^4 \ + 3s^3 + 3s^2 + 2s + 1$$

All the coefficients of the **characteristic polynomial**,

$$a_0 \ = 1, a_1 \ = \ 3, a_2 \ = \ 3, a_3 = 2, a_4 \ = 1$$

are **positive**. So, the **control system satisfies** the **necessary condition**.

Step 2: Form the **Routh array** for the **given characteristic polynomial**.

s^4	1	3	1
s^3	3	2	
s^2	$b_1 \ = \dfrac{3 \times 3 - 2 \times 1}{3} = \dfrac{7}{3}$	$b_2 \ = \dfrac{3 \times 1 - 0 \times 1}{3} = 1$	
s^1	$c_1 \ = \dfrac{\frac{7}{3} \times 2 - 1 \times 3}{\frac{7}{3}} = \dfrac{5}{7}$		
s^0	1		

Step 3: **Verify** the **sufficient condition** for the **Routh-Hurwitz stability**:

All the **elements** of the **first column** of the **Routh array** are **positive**. There is **no sign change** in the **first column** of the **Routh array**. So, the **control system** is **stable**.

❖ Special Cases of Routh Array

We may come **across two types** of **situations**, while forming the Routh table. It is difficult to **complete** the **Routh table** from these two situations. The **two special cases** are:

- The **first element** of **any row** of the **Routh array** is **zero**.
- **All** the **elements** of **any row** of the **Routh array** are **zero**.

♦ First Element of any row of the Routh array is zero

If **any row** of the **Routh array contains only** the **first element** as **zero** and **at least one** of the **remaining elements** have **non-zero value**, then **replacing** the **first element** with a **small positive integer**, ϵ. And **then continuing** the **process** of **completing** the **Routh table**. Now, **finding** the **number of sign changes** in the **first column** of the **Routh table by substituting** ϵ tends to **zero**.

Example 4: Find the **stability** of the **control system having characteristic equation**,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

Solution:

Step 1: **Verify** the **necessary condition** for the **Routh-Hurwitz stability**.

All the **coefficients** of the **characteristic polynomial**,

$$s^4 + 2s^3 + s^2 + 2s + 1$$

are **positive**. So, the **control system satisfies** the **necessary condition**.

Step 2: Form the **Routh array** for the **given characteristic polynomial**.

s^4	1	1	1
s^3	$\cancel{2}\ 1$	$\cancel{2}\ 1$	
s^2	$b_1 = \frac{1 \times 1 - 1 \times 1}{3} = 0$	$b_2 = \frac{1 \times 1 - 0 \times 1}{1} = 1$	
s^1			
s^0			

The **row** s^3 **elements** have **2** as the **common factor**. So, **all** these **elements** are **divided** by **2**.

◊ **Special case (i)**:

Only the **first element** of **row s^2** is **zero**. So, **replacing** it by ϵ and **continue** the **process** of **completing** the **Routh table**.

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$c_1 = \frac{\epsilon \times 1 - 1 \times 1}{\epsilon} = \frac{\epsilon - 1}{\epsilon}$		
s^0	1		

Step 3: **Verifying** the **sufficient condition** for the **Routh-Hurwitz stability**.
As ϵ **tends** to **zero**, the **Routh table becomes** like this.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

There are **two sign changes** in the **first column** of **Routh table**. Hence, the **control system** is **unstable**

♦ **All the Elements** of **any row** of the **Routh array** are **zero**

In this case, **two steps have to** be **followed**:

❶ **Writing** the **auxiliary equation**, $A(s)$ of the **row**, which is just **above** the **row** of **zeros**.

❷ **Differentiating** the **auxiliary equation**, $A(s)$ **with respect** to (s). **Filling** the **row** of **zeros** with **these coefficients**.

Example 5: Find the **stability** of the **control system having characteristic equation**,

$$s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0$$

Solution:

Step 1: Verifying the **necessary condition** for the **Routh-Hurwitz stability**.

All the **coefficients** of the **given characteristic polynomial** are **positive**.

So, the **control system satisfies** the **necessary condition**.

Step 2: Forming the **Routh array** for **the given characteristic polynomial**

s^5	1	1	1
s^4	$\cancel{3}\ 1$	$\cancel{3}\ 1$	$\cancel{3}\ 1$
s^3	$b_1 = \frac{1\times 1 - 1\times 1}{1} = 0$	$b_2 = \frac{1\times 1 - 1\times 1}{1} = 0$	
s^2			
s^1			
s^0			

The **row s^4 elements have** the **common factor** of 3. So, **all** these **elements** are **divided** by 3.

◊ **Special case (ii)**:

All the **elements** of **row s^3** are **zero**. So, **writing** the **auxiliary equation**, $A(s)$ of the **row s^4**.

$$A(s) = s^4 + s^2 + 1$$

Differentiating the **above equation** with **respect** to (s).

$$\frac{dA(s)}{ds} = 4s^3 + 2s$$

Placing these coefficients in **row s^3**, we have

s^5	1	1	1
s^4	1	1	1
s^3	$\cancel{4}\ 2$	$\cancel{2}\ 1$	
s^2	$c_1 = \frac{2\times 1 - 1\times 1}{2} = 0.5$	$c_2 = \frac{2\times 1 - 0\times 1}{2} = 1$	
s^1	$d_1 = \frac{0.5\times 1 - 1\times 2}{0.5} = \frac{-1.5}{0.5} = -3$		
s^0	1		

Step 3: **Verifying** the **sufficient condition** for the **Routh-Hurwitz stability**.

There are **two sign changes** in the **first column** of **Routh table**. Hence, the **control system** is **unstable**.

In the **Routh-Hurwitz stability criterion**, we can **know whether** the **closed loop poles** are in **on left half** of the **‘s’ plane** **or** on the **right half** of the **‘s’ plane** **or** on an **imaginary axis**. So, **we can’t find** the **nature** of the **control system**. To **overcome** this **limitation**, there is a **technique known** as the **root locus**

Root Locus

The **Root locus** is the **locus** of the **roots** of the **characteristic equation** on the **complex plane** of **closed-loop poles** as the **feedback** by **varying system gain *K*** from **0** to ∞. In the **diagram** of the **root locus**, the **path** of the **closed loop poles** can be **observed** and then **identify** the **nature** of the **control system**. Using an **open loop transfer function** to **determine** the **stability** of the **closed loop control system** of which the **characteristic equation** is:

$$\mathbf{1 + G(s)H(s) = 0}$$

We can represent $G(s)H(s)$ as:

$$G(s)H(s) = K\frac{N(s)}{D(s)}$$

Where,

- ***K*** represents the **multiplying factor**
- ***N*(s)** represents the **numerator term** having (**factored**) **nth order polynomial** of ‘**s**’.
- ***D*(s)** represents the **denominator term** having **(factored) mth order polynomial** of ‘**s**’.

Substitute, $G(s)H(s)$value in the characteristic equation, we have

$$1 + K\frac{N(s)}{D(s)} = 0 \quad \Rightarrow \quad D(s) + KN(s) = 0$$

❶ **<u>Case</u> 1: (K = 0)**

If $\boldsymbol{K} = \mathbf{0}$, then $\boldsymbol{D}(\mathbf{s}) = \mathbf{0}$.

That means, the **closed loop poles** are **equal** to **open loop poles** when $\boldsymbol{K}$ is **zero**.

❷ **<u>Case 2:</u> (K = ∞)**

The above characteristic equation can be re-written as

$$\boldsymbol{K}\left[\frac{\mathbf{1}}{\boldsymbol{K}}+\frac{\boldsymbol{N}(\boldsymbol{s})}{\boldsymbol{D}(\boldsymbol{s})}\right]=\mathbf{0} \quad \Rightarrow \quad \frac{\mathbf{1}}{\boldsymbol{K}}+\frac{\boldsymbol{N}(\boldsymbol{s})}{\boldsymbol{D}(\boldsymbol{s})}=\mathbf{0}$$

Substituting, $\boldsymbol{K} = \infty$, yields

$$\overset{=\mathbf{0}}{\cancel{\frac{\mathbf{1}}{\infty}}}+\frac{\boldsymbol{N}(\boldsymbol{s})}{\boldsymbol{D}(\boldsymbol{s})}=\mathbf{0} \quad \Rightarrow \quad \frac{\boldsymbol{N}(\boldsymbol{s})}{\boldsymbol{D}(\boldsymbol{s})}=\mathbf{0} \quad \Rightarrow \quad \boldsymbol{N}(\boldsymbol{s})=\mathbf{0}$$

If $\boldsymbol{K} = \infty$, then $\boldsymbol{N}(\boldsymbol{s}) = \mathbf{0}$. It means the **closed loop poles** are **equal** to the **open loop zeros** when $\boldsymbol{K}$ is **infinity**.

From the **above two cases**, we can **conclude** that the **root locus branches <u>start</u>** at **open loop poles** and **<u>end</u>** at **open loop zeros**.

Angle Condition and Magnitude Condition

The **points** on the **root locus branches** satisfy the **angle condition**. So, the **angle condition** is used **to know whether** the **point exist** on **root locus branch** or **not**. The value of K for the points on the root locus branches can be found by using **magnitude condition**. So, the magnitude condition can be used for the points, and this **satisfies** the **angle condition**.

The characteristic equation of closed loop control system is:

$$\mathbf{1}+\boldsymbol{G}(\boldsymbol{s})\boldsymbol{H}(\boldsymbol{s})=\mathbf{0} \quad \Rightarrow \quad \boldsymbol{G}(\boldsymbol{s})\boldsymbol{H}(\boldsymbol{s})=-\mathbf{1}+\boldsymbol{j}\,\mathbf{0}$$

The **phase angle** of $\boldsymbol{G}(\mathbf{s})\boldsymbol{H}(\mathbf{s})$ is:

$$\angle\,\boldsymbol{G}(\boldsymbol{s})\boldsymbol{H}(\boldsymbol{s})=\boldsymbol{tan}^{-\mathbf{1}}\left(\frac{\mathbf{0}}{-\mathbf{1}}\right)=(\mathbf{2}\boldsymbol{n}+\mathbf{1})\boldsymbol{\pi}$$

The **angle condition** is the point at which the angle of the open loop transfer function is an **odd multiple** of **180°.**

Magnitude of $\boldsymbol{G}$**(s)**$\boldsymbol{H}$**(s)** is:

$$|\boldsymbol{G}(\boldsymbol{s})\boldsymbol{H}(\boldsymbol{s})| = \sqrt{(-\mathbf{1})^{\mathbf{2}} + \mathbf{0}^{\mathbf{2}}} = \mathbf{1}$$

The **magnitude condition** is that the **point** (which **satisfied** the **angle condition**) at which the **magnitude** of the **open loop transfer function** is **<u>one</u>**.

s – plane

In **mathematics** and **engineering**, the **s-plane** is the **complex plane** on which **Laplace transforms** are **graphed**. It is a **mathematical domain** where, **instead of** viewing **processes** in the **time domain modeled** with **time-based functions**, they are **viewed** as **equations** in the **frequency domain**. It is **used** as a **graphical analysis tool** in **engineering** and **physics**.

In mathematics, the **complex plane** or **z-plane** is a **geometric representation** of the **complex numbers** established by the **real axis** and the **perpendicular imaginary axis**. It can be thought of as a **modified Cartesian plane**, with the **real part** of a **complex number represented** by a **displacement** along the ***x*-axis**, and the **imaginary part** by a **displacement** along the **y-axis.**

A **complex number** is a **number** that can be **expressed** in the form:

$$\boldsymbol{a} + \boldsymbol{bi}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are **real numbers**, and $\boldsymbol{i}$ is a **solution** of the equation $\boldsymbol{x}^{\mathbf{2}} = -\mathbf{1}$, because **no real number satisfies** this **equation**, and $\boldsymbol{i}$ is **called** an **imaginary number**. For the **complex number** $\boldsymbol{a} + \boldsymbol{bi}$, $\boldsymbol{a}$ is **called** the **real**

part, and ***b*** is **called** the **imaginary part** and represent as shown in **Figure 1**.

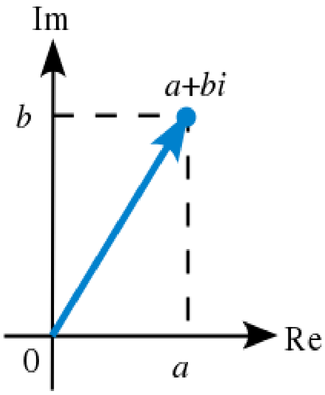


Figure 1.

In **mathematics**, the **Laplace transform** is an **integral transform** named **after** its **inventor Pierre-Simon Laplace**. It **transforms a function** of a **real variable *t*** (**often time**) to a **function** of a **complex variable *s*** (**complex frequency**). The **transform** has **many applications** in **science** and **engineering**.

A **real function *f*** in **time *t*** is **translated into** the **s-plane** by taking the **integral** of the **function multiplied** by $\boldsymbol{e^{-st}}$ **from 0** to ∞, where **s** is a **complex number** with the **form** $\boldsymbol{s = \sigma + \omega i}$.

$$\boldsymbol{F(s) = \int_0^\infty f(t)\ e^{-st} dt|\ s \in \mathbb{C}}$$

This **transformation from** the ***t*-domain** into the **s-domain** is known as a **Laplace transform** and the **function *F*(*s*)** is called the **Laplace transform** of ***f***.

The **s-plane** is the **complex plane** on which **Laplace transforms** are **graphed**. Where in **mathematics**, the **complex plane** or **z-plane** is a

geometric representation of the **complex numbers** established by the **real axis** and the **perpendicular imaginary axis**. It can be thought of as a **modified Cartesian plane**, with the **real part** of a **complex number represented** by a **displacement along** the **x-axis**, and the **imaginary part** by a **displacement along** the **y-axis**.

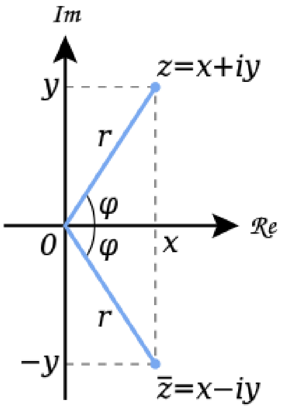


Figure 2.

Figure 2, shows the Geometric representation of **z** and its **conjugate $\bar{z}$** in the **complex plane**. The distance along the **light blue line from** the **origin** to the **point z** is the **modulus** or **absolute value** of **z**. The **angle φ** is the **argument** of **z**.

In **complex analysis**, the **complex numbers** are **customarily represented** by the **symbol z**, which can be **separated** into its **real (x)** and **imaginary (y) parts**:

$$\boldsymbol{z = x + iy}.$$

for example: $\boldsymbol{z = 4 + 5i}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are **real numbers**, and $\boldsymbol{i}$ is the **imaginary unit**. In this **customary notation** the **complex number z corresponds to** the **point (x, y)** in the **Cartesian plane**.

In the **Cartesian plane** the **point (*x, y*) can** also **be represented** in **polar coordinates** as:

$$(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{r\,cos\theta},\boldsymbol{r\,sin\theta}) \quad (\boldsymbol{r},\boldsymbol{\theta}) = \left(\sqrt{\boldsymbol{x^2+y^2}}\,, \qquad \boldsymbol{arctan\,\frac{y}{x}}\right)$$

In the **Cartesian plane** it may be assumed that the **arctangent** takes **values from − π/2 to π/2** (in **radians**), and some **care must be taken** to **define** the **more complete arctangent function** for **points (*x, y*)** when $\boldsymbol{x \le 0}$.

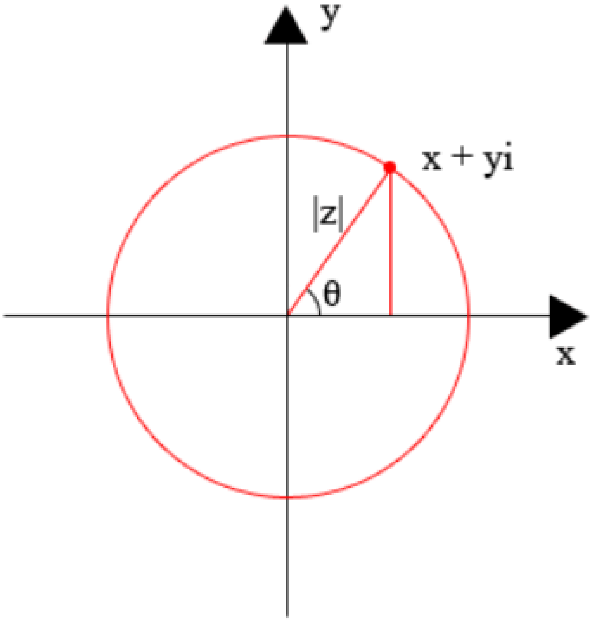
In **the complex plane** these **polar coordinates** take the **form:**

$$\boldsymbol{z = x + iy = |z|(cos\theta + i\,sin\theta) = |z|e^{i\theta}}$$

Where,

$$\boldsymbol{|z| = \sqrt{x^2+y^2}\,, \theta = arg(z) = \frac{1}{i}\,ln\frac{z}{|z|} = -i\,ln\frac{z}{|z|}}$$

Argand diagram refers to a **geometric plot** of **complex numbers** using the ***x*-axis** as the **real axis** and **y-axis** as the **imaginary axis**. It is **frequently used** to **plot** the **positions** of the **zeros** and **poles** of a **function** in the **complex plane**.



In **complex analysis** (a branch of mathematics), **zeros** of **holomorphic functions** which are **points z** where ***f*(z) = 0**, play an important role.

For **meromorphic functions**, particularly, there is a **duality between zeros** and **poles**. A **function *f*** of a **complex variable z** is **meromorphic** in the **neighborhood** of a **point z_0** if either *f* or its **reciprocal function 1/*f*** is **holomorphic** in **some neighborhood** of **z_0** (that is, if ***f*** or **1/*f*** is **differentiable** in a **neighborhoo**d of **z_0**). If **z_0** is a **zero** of 1/*f*, then **it is** a **pole** of ***f***.

Thus a **pole** is a **certain type** of **singularity** of a **function**, nearby which the **function behaves relatively regularly**, in contrast to essential singularities, such as 0 for the logarithm function, and branch points, such as 0 for the complex square root function.

Construction of Root Locus

The **root locus** is a **graphical representation** in **s-domain** and it is **symmetrical about** the **real axis**. Because the **open loop poles** and **zeros exist** in the **s-domain** having the **values** either as **real** or as **complex conjugate pairs**.

Rules for Construction of Root Locus

The **root locus** is a **graphical representation** in **s – domain** and it is **symmetrical** about the **real axis**. Because the **open loop poles** and **zeros exist** in the **s – domain** having the values either as **real** or as **complex conjugate pairs**.

For constructing a root locus, it has to follow these rules:

Rule 1 − Locate the **open loop poles** and **zeros** in the ‘**s**’ **plane**.

Rule 2 − Find the **number** of **root locus branches**.

Since the **root locus branches start** at the **open loop poles** and **end** at **open loop zeros**, so the **number** of **root locus branches** ***N*** is **equal** to the **number** of **finite open loop poles** $\boldsymbol{P}$ or the **number** of **finite open loop zeros** $\boldsymbol{Z}$, **whichever** is **greater**.

Mathematically, the **number** of **root locus branches** ***N*** we can be written as:

$$\boldsymbol{N = P \quad if \quad P \geq Z}$$

$$\boldsymbol{N = Z \quad if \quad P < Z}$$

Rule 3 − Identify and **draw** the **real axis root locus branches**.

- If the **angle** of the **open loop transfer function** at a **point** is an **odd multiple** of **180^0**, **then** that **point** is **on** the **root locus**.
- If **odd number** of the **open loop poles** and **zeros exist** to the **left side** of a **point** on the **real axis**, **then** that **point** is **on** the **root locus branch**.

Therefore, the **branch** of **points** which **satisfies this condition** is the **real axis** of the **root locus branch**.

Rule 4 − Find the **centroid** and the **angle of asymptotes**.

- If ***P* = *Z***, then **all the root locus branches start** at **finite open loop poles** and **end** at **finite open loop zeros**.
- If ***P* > *Z***, then ***Z*** **number** of **root locus branches start** at **finite open loop poles** and **end** at **finite open loop zeros** and ***P−Z*** **number** of **root locus branches start** at **finite open loop poles** and **end** at **infinite open loop zeros**.
- If ***P* < *Z***, then ***P*** **number** of **root locus branches start** at **finite open loop poles** and **end** at **finite open loop zeros** and ***Z−P*** **number** of **root locus branches start** at **infinite open loop poles** and **end** at **finite open loop zeros**.

So, **some** of the **root locus branches approach infinity**, when $P \neq Z$. **Asymptotes give** the **direction** of these **root locus branches**. The intersection point of **asymptotes** on the **real axis** is known as **centroid**.

The **centroid** α can be calculated by using the formula,

$$\alpha = \frac{\sum \boldsymbol{Real\ Part\ of\ Finite\ open\ loop\ poles} - \sum \boldsymbol{Real\ Part\ of\ Finite\ open\ loop\ zeros}}{P - Z}$$

The formula for the **angle** of **asymptotes θ** is:

$$\theta = \frac{(2q + 1)180°}{P - Z}$$

Where, $q = 0, 1, 2, 3, \ldots, (P - Z) - 1,$

Rule 5 − Find the **intersection points** of **root locus branches with an imaginary axis.**

The **point** at which the **root locus branch intersects** the **imaginary axis** and the **value** of K at that point can be **calculated** by using the **Routh array method** and **special case (ii)**.

- If **all elements** of **any row** of the **Routh array** are **zero**, **then** the **root locus branch intersects** the **imaginary axis** and **vice-versa**.
- **Identify** the **row** in such a way that if we make the **first element as zero**, **then** the **elements** of the **entire row** are **zero**. **Find** the **value** of K for this combination.
- **Substitute** this K **value** in the **auxiliary equation**, **then** the **intersection point** of the **root locus branch** with an **imaginary axis** will be gotten.

Rule 6 − Find Break-away and **Break-in points**.

- If there **exists** a **real axis root locus branch between two open loop poles**, **then** a **break-away point** will be in **between** these **two open loop poles**.

- If there **exists** a **real axis root locus branch between two open loop zeros**, **then** a **break-in point** will be in between these **two open loop zeros**.

Note: Break-away and **break-in points exist** only **on the real axis root locus branches**.

To find the break-away and break-in points, the steps below should be followed:

- **Write K** in **terms** of **s** from the **characteristic equation**

$$\mathbf{1 + G(s)H(s) = 0}$$

Differentiate K with **respect to s** and **make it equal** to **zero**. **Substitute** these **values** of s in the above **equation**.

- The **values** of **s** for which the **K value** is **positive** are the **break points**.

Rule 7 − Find the **angle of departure** and the **angle of arrival**.

The **Angle of departure** and the **angle of arrival** can be calculated at **complex conjugate open loop poles** and **complex conjugate open loop zeros** respectively.

The **formula** for the **angle of departure ϕ_d** is:

$$\phi_d = 180^\circ - \phi$$

The **formula** for the **angle of arrival ϕ_a is**

$$\phi_a = 180^\circ + \phi$$

Where, $$\phi = \sum \phi_P - \sum \phi_Z$$

Example: To draw the root locus of the control system having open loop transfer function, $G(s)H(s) = \frac{K}{s(s+1)(s+5)}$

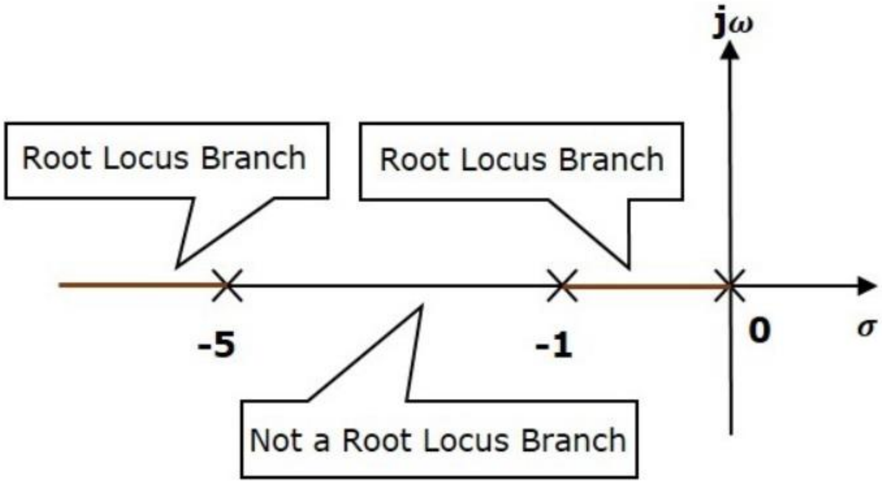
Step 1 − The **given open loop transfer function** has **three poles** at

$$s = 0, \qquad s = -1, \qquad \text{and } s = -5$$

It **doesn't have** any **zero**. Therefore, the **number of root locus branches** is **equal** to the **number of poles** of the **open loop transfer function**.

$$N = P = 3$$

The **three poles** are located as shown in the figure below



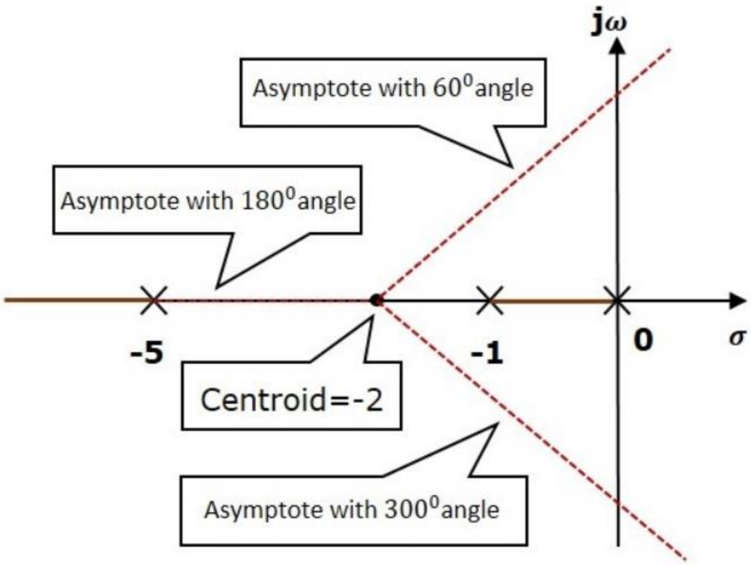
The line segment between $s = -1$ and $s = 0$ is **one branch** of **root locus on real axis**. And the **other branch** of the **root locus** on the **real axis** is the **line segment** to the **left** of $s = -5$.

Step 2 − The values of the **centroid** and the **angle of asymptotes** can be obtained by using the formulae for **Centroid α**.

$$\alpha = -2$$

The angles of asymptotes are: $\theta = 60°, 180°,$ and $300°$.

The **centroid** and **three asymptotes** are shown in the following figure.

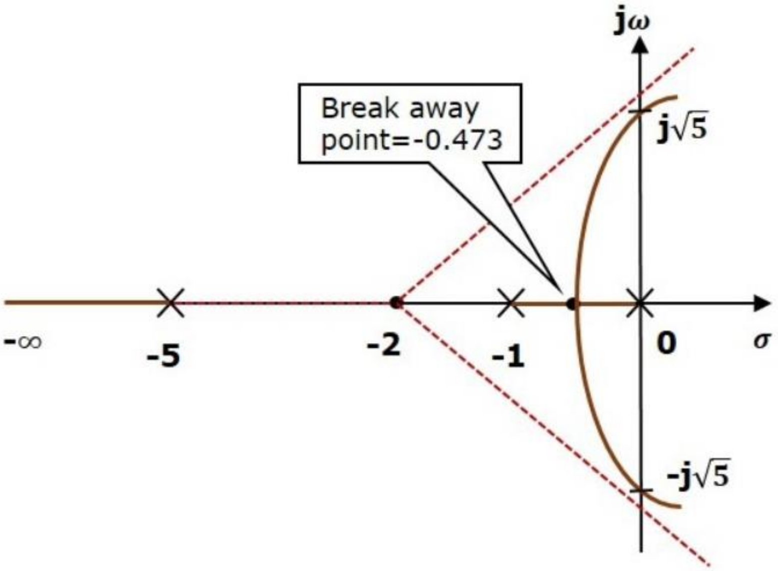


Step 3 − Since two asymptotes have the angles of $\boldsymbol{\theta = 60°}$ and $\boldsymbol{\theta = 300°}$, **two root locus branches <u>intersect</u>** the **imaginary axis**. By using the **Routh array method** and **special case (ii),** the **root locus branches <u>intersect</u>** the **imaginary axis** at ($\boldsymbol{j\sqrt{5}}$) and ($\boldsymbol{-j\sqrt{5}}$).

There will be **one break-away point** on the **real axis root locus branch between** the **poles** ($\boldsymbol{s = -1}$) and ($\boldsymbol{s = 0}$). By **following** the **procedure** given for the **calculation** of **break-away point**, we will get

$$\boldsymbol{s = -0.473}$$

The root locus diagram for the given control system is shown in the followin



In this way, we can draw the root locus diagram of any control system and observe the movement of poles of the closed loop transfer function. From the **root locus diagrams,** we can **know** the **range** of ***K*** **values** for **different types of damping**.

Effects of Adding Open Loop Poles and Zeros on Root Locus

The **root locus** can be shifted in **'s' plane** by **adding** the **open loop poles** and **open loop zeros**.

- If we **include** a **pole** in the **open loop transfer function**, then some of **root locus branches** will **move towards** **right half** of **'s' plane**. Because of this, the **damping ratio** δ **decreases**. Which implies, **damped frequency** ω_d **increases** and the **time domain specifications** like **delay time** t_d, **rise time** t_r and **peak time** t_p **decrease**. But, it effects the system stability.
- If we **include** a **zero** in the **open loop transfer function**, then some of **root locus branches** will **move towards** **left half** of **'s' plane**. So, it will **increase** the **control system stability**. In this case, the **damping ratio** δ **increases**. Which implies, **damped frequency** ω_d **decreases** and the **time domain specifications** like **delay time** t_d, **rise time** t_r and **peak time** t_p **increase**.

So, based on the requirement, we **can include** (add) the **open loop poles** or **zeros** to the **transfer function**.

Frequency Response Analysis

In the previous section, the **time response analysis** of the control systems and the **time domain specifications** of the **second order control systems** has been discussed. In the next section, the **frequency response analysis** of the control systems and the **frequency domain specifications** of the second order control systems will be discuss.

What is Frequency Response?

The **response of a system** can be **partitioned into** both the **transient response** and **steady state response**. The **transient response** can be **found** by using **Fourier integrals**. The **steady state response** of a system for an **input sinusoidal signal** is known as the **frequency response**. In this section, we will focus only on the **steady state response**.

If a **sinusoidal signal** is **applied as** an **input** to a **Linear Time-Invariant** (**LTI**) **system**, then it **produces** the **steady state output**, which is **also a sinusoidal signal**. The **input** and **output sinusoidal signals** have the **same frequency**, but **different amplitudes** and **phase angles**.

Let the **input signal** be:

$$\boldsymbol{r(t) = Asin(\omega_0 t)}$$

The **open loop transfer function** will be:

$$\boldsymbol{G(s) = G(j\omega)}$$

We can represent $\boldsymbol{G(j\omega)}$ in terms of **magnitude** and **phase** as shown below.

$$\boldsymbol{G(j\omega) = |G(j\omega_0)| \angle G(j\omega)}$$

Substitute, $\boldsymbol{\omega = \omega_0}$ in the equation.

$$\boldsymbol{G(j\omega_0) = |G(j\omega_0)| \angle G(j\omega_0)}$$

The **output signal** is:

$$\boldsymbol{c(t) = A|G(j\omega_0)|sin[\omega_0 t + \angle G(j\omega_0)]}$$

- The **amplitude** of the **output sinusoidal signal** is obtained by **multiplying** the **amplitude** of the **input sinusoidal signal** and the **magnitude** of $\boldsymbol{G(j\omega)}$ at $\boldsymbol{\omega = \omega_0}$.
- The **phase** of the **output sinusoidal signal** is **obtained** by **adding** the **phase** of the **input sinusoidal signal** and **phase** of $\boldsymbol{G(j\omega)}$ at $\boldsymbol{\omega = \omega_0}$.

Where,

$\boldsymbol{A}$**:** is the **amplitude** of the **input sinusoidal signal**.

$\boldsymbol{\omega_0}$: is **angular frequency** of the **input sinusoidal signal**.

We can write, **angular frequency** $\boldsymbol{\omega_0}$ as:

$$\boldsymbol{\omega_0 = 2\,\pi\, f_0}$$

Here, $\boldsymbol{f_0}$ is the **frequency** of the **input sinusoidal signal**. Similarly, you can follow the **same procedure** for **closed loop control system**.

Frequency Domain Specifications

The **frequency domain specifications** are **resonant peak**, **resonant frequency** and **bandwidth**. Consider the **transfer function** of the **second order closed loop control system** as,

$$\boldsymbol{T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}}$$

Substitute, $s = \boldsymbol{j\omega}$, we have

$$\boldsymbol{T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\delta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + 2j\delta\omega\omega_n + \omega_n^2}}$$

$$\Longrightarrow \quad \boldsymbol{T(j\omega) = \frac{\omega_n^2}{\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}}}$$

$$\Longrightarrow \quad \boldsymbol{T(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(2\delta\frac{\omega}{\omega_n}\right)}}$$

Let, $u = \boldsymbol{\frac{\omega}{\omega_n}}$, and substitute in the above equation.

$$\Longrightarrow \quad \boldsymbol{T(j\omega) = \frac{1}{(1 - u^2) + j(2\delta u)}}$$

Magnitude of $\boldsymbol{T(j\omega)}$ is:

$$\boldsymbol{M = |T(j\omega)| = \frac{1}{\sqrt{(1 - u^2)^2 + (2\delta u)^2}}}$$

Phase of $\boldsymbol{T(j\omega)}$ is:

$$\boldsymbol{\angle T(j\omega) = -tan^{-1}\left(\frac{2\delta u}{1 - u^2}\right)}$$

Resonant Frequency

It is the **frequency** at which the **magnitude** of the **frequency response** has **peak value** for the **first time**. It is **denoted** by $\boldsymbol{\omega_r}$. At $\boldsymbol{\omega = \omega_r}$, the **first derivate** of the magnitude of $\boldsymbol{T(j\omega)}$ is **zero**.

Differentiate *M* with respect to $\boldsymbol{u}$.

$$\boldsymbol{\frac{dM}{du} = -\frac{1}{2}[(1 - u^2)^2 + (2\delta u)^2]^{\frac{3}{2}}[2(1 - u^2)(-2u) + 2(2\delta u)(2\delta)]}$$

$$\Longrightarrow \quad \boldsymbol{\frac{dM}{du} = -\frac{1}{2}[(1 - u^2)^2 + (2\delta u)^2]^{\frac{3}{2}}\,[4u(u^2 - 1 + 2\delta^2)]}$$

Substitute, $\boldsymbol{u = u_r}$ and $\boldsymbol{\frac{dM}{du} = 0}$, yields

$$\Longrightarrow \quad \boldsymbol{0 = -\frac{1}{2}[(1 - u_r^2)^2 + (2\delta u_r)^2]^{-\frac{3}{2}}\,[4u_r(u_r^2 - 1 + 2\delta^2)]}$$

$$\Longrightarrow \quad \boldsymbol{4u_r(u_r^2 - 1 + 2\delta^2) = 0} \quad \Longrightarrow \quad \boldsymbol{u_r^2 - 1 + 2\delta^2 = 0}$$

$$\Rightarrow \qquad u_r^2 = 1 - 2\delta^2 \quad \Rightarrow \qquad u_r = \sqrt{1 - 2\delta^2}$$

Substituting $u_r = \dfrac{\omega_r}{\omega_n}$, we have

$$\Rightarrow \quad \frac{\omega_r}{\omega_n} = \sqrt{1 - 2\delta^2} \qquad \Rightarrow \qquad \omega_r = \omega_n\sqrt{1 - 2\delta^2}$$

Resonant Peak

It is the peak (maximum) value of the magnitude of $T(j\omega)$. It is denoted by M_r.

At $u = u_r$, the Magnitude of $T(j\omega)$ is:

$$M_r = |T(j\omega)| = \frac{1}{\sqrt{(1 - u_r^2)^2 + (2\delta u_r)^2}}$$

Substitute, $u_r = \sqrt{1 - 2\delta^2} \quad \Rightarrow \quad 2\delta^2 = 1 - u_r^2$, we get

$$M_r = \frac{1}{\sqrt{(2\delta^2)^2 + \left(2\delta\sqrt{1 - 2\delta^2}\right)^2}} = \frac{1}{2\delta\sqrt{1 - 2\delta^2}}$$

Resonant peak in **frequency response corresponds to** the peak overshoot in the **time domain transient response** for **certain values** of **damping ratio** δ. So, the **resonant peak** and **peak overshoot** are **correlated** to **each other**.

Bandwidth

It is the range of **frequencies** over which, the **magnitude** of $T(j\omega)$ **drops** to **70.7%** from its **zero frequency value**.

At $\omega = 0$, the value of u will be **zero**.

Thus, substitute, $u = 0$ in M**,** we have

$$M = T(j\omega) = \frac{1}{\sqrt{(1 - 0^2)^2 + [2\delta(0)]^2}} = 1$$

Therefore, the magnitude of $T(j\omega)$ at $\omega = 0$**,** is **one**.

At **3 – dB frequency**, the **magnitude** of $\boldsymbol{T(j\omega)}$ will be **70.7%** of **magnitude** of $\boldsymbol{T(j\omega)}$ at $\boldsymbol{\omega = 0}$.

$$\text{i.e., at} \quad \boldsymbol{\omega = \omega_b} \quad \Longrightarrow \quad \boldsymbol{M = 0.707(1) = \frac{1}{\sqrt{2}}}$$

$$\Longrightarrow \quad \boldsymbol{M = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{\left(1 - u_b^2\right)^2 + (2\delta u_b)^2}}}$$

$$\Longrightarrow \quad \boldsymbol{2 = \left(1 - u_b^2\right)^2 + (2\delta)^2 u_b^2}$$

Let, $\boldsymbol{u_b^2 = x}$, then

$$\Longrightarrow \quad \boldsymbol{2 = (1 - x^2)^2 + (2\delta)^2 x^2}$$

$$\Rightarrow \quad \boldsymbol{x^2 + (4\delta^2 - 2)x - 1 = 0}$$

$$\Longrightarrow \quad \boldsymbol{x = \frac{-(4\delta^2 - 2) \pm \sqrt{(4\delta^2 - 2)^2 + 4}}{2}}$$

Consider only the **positive value** of ***x***.

$$\boldsymbol{x = 1 - 2\delta^2 + \sqrt{(2\delta^2 - 1)^2 + 1}}$$

$$\Longrightarrow \quad \boldsymbol{x = 1 - 2\delta^2 + \sqrt{2 - 4\delta^2 + 4\delta^4}}$$

Substitute, $\boldsymbol{x = u_b^2 = \frac{\omega_b^2}{\omega_n^2}}$, yields

$$\boldsymbol{\frac{\omega_b^2}{\omega_n^2} = 1 - 2\delta^2 + \sqrt{2 - 4\delta^2 + 4\delta^4}}$$

$$\Longrightarrow \boldsymbol{\omega_b = \omega_n \sqrt{1 - 2\delta^2 + \sqrt{2 - 4\delta^2 + 4\delta^4}}}$$

Bandwidth $\boldsymbol{\omega_b}$ in the **frequency response** is **inversely proportional** to the **rise time *t_r*** in the **time domain transient response**.