Ministry of Higher Education & Scientific Research

Northern Technical University

Engineering Technical College / Mosul

Dept. of Power Mechanics Engineering Technology

Engineering Mechanics

Dynamics

Prepared by

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The Aim of This Class

The object of this class is to develop the students' abilities in understanding and solving dynamic problems related particles and rigid bodies.

Course Contents

Week no.	Materials to be covered
1	**Introduction to Engineering Mechanics/Dynamics** • Overview of Engineering Mechanics/Dynamics • Fundamental concepts and principles • Unit conversions • Newton's Laws
2	**Kinematics of Particles** • Rectilinear motion
3	• Curvilinear motion
4	• Tangential and normal components
5	• Projectile motion
6	**Review for previous lectures**
7-8	**Kinetics of Particles** • Force, mass, and acceleration • Newton's second law of motion • The equation of motion
9	**Mid-term Exam**
10	• Applications of particle kinetics
11-12	• The work of a force • Work and Energy • Power
12-13	**Vibrations** • Free and forced vibrations • Single degree of freedom systems
14-15	• Damping and damping ratios • Natural frequency and resonance • Vibration isolation and control

Text Book:

- **Engineering Mechanics: Dynamics,** By R. C. Hibbeler 14th edition.

References:

- **Engineering Mechanics: Dynamics,** by J. L. Meriam and L. D. Kraige 9th edition
- **Theory and Problems of Engineering Mechanics: Statics and Dynamics**, Fifth Edition, Shaum's Outline Series.
- **Engineering Mechanics: Dynamics**, by Andrew Pytel and Jaan Kiusalaas, 3rd edition.

Introduction to Engineering Mechanics/Dynamics

Overview of Engineering Mechanics/Dynamics

Engineering Mechanics: The study of how rigid bodies react to forces acting on them. It can be divided into two parts: **Statics** and **Dynamics**.

History of Dynamics

The beginning of a rational understanding of dynamics is credited to Galileo (1564-1642) who made observations on free fall, motion on an inclined plane, and motion of the pendulum.
Newton (1642-1727), guided by Galileo's work, formulated the laws of motion and the law of universal gravitation. His famous work was published in the first edition of Principia. Other scientists such as Euler, D'Alembert, Lagrange made important contributions to mechanics.

Dynamics: is that branch of mechanics which deals with the motion of bodies under the action of forces. Dynamics is divided in to two parts: **kinematics and kinetics**

Kinematics: is the study of motion without reference to the forces which cause motion.

Kinetics: is the study of action of forces on bodies and their resulting motion.

Applications of Dynamics

1. Analysis and design of moving structures.
2. Fixed structures subject to shock loads
3. Robotic devices.
4. Machining of turbines and pumps.
5. Automatic control system
6. Rockets.
7. Missiles and spacecrafts.
8. Ground and air transportation vehicles. And others.

Fundamental concepts and principles

Space: Is the geometric region occupied by bodies

Time: Is a measure of the succession of events (considered absolute in Newtonian Mechanics)

Mass: Is the quantitative measure of inertia or resistance to change in motion of the body.

Force: Is the vector action of one body on another.

A particle: Is a body of negligible dimensions.
The body may be treated as a particle when its dimensions are irrelevant to the description of its motion or the action of forces on it.

A rigid body: is a body whose changes in shape are negligible compared with the overall dimensions of the body or with the changes in position of the body as a whole.

Vector and **Scalar:** We use two kinds of quantities in mechanics—scalars and vectors. **Scalar quantities** are those with which only a magnitude is associated. Examples of scalar quantities are time, volume, density, speed, energy, and mass. **Vector quantities**, on the other hand, possess direction as well as magnitude, examples of vector quantities are displacement, velocity, acceleration, force, moment, and momentum

Units

The units and symbols of the four fundamental quantities of mechanics for the International System of metric units SI And U.S. customary system are summarized in the following table:

Quantity	Dimensional Symbol	SI Units		U.S. Customary Units (FPS)	
		Unit	Symbol	Unit	Symbol
Mass	M	kilogram	kg	slug	-
Length	L	meter	m	foot	ft
Time	T	second	s	second	sec
Force	F	newton	N	pound	lb

The SI system is termed an absolute system because the standard for the base unit kilogram (a platinum - iridium cylinder kept at the International Bureau of standards near Paris, France) is independent of the gravitational attraction of the earth. On the other hand, the U.S. customary system is termed a gravitational system because the standard for the base unit pound (the weight of a standard mass located at sea level and at latitude of 45°) requires the presence of the gravitational field of the earth.

In SI units, by definition, one Newton is that force which will give a one- kilogram mass an acceleration of one meter per second squared. In U.S. customary system a 32.1740 pound mass (1 slug) will have an acceleration of one foot per second squared when acted on by a force of one pound.

SI Prefixes

Multiple	***Exponential Form***	***Prefix***	***SI Symbol***
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k
Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

Conversion Factors (FPS) to (SI)

Quantity	Unit of Measurement (FPS)	Equals	Unit of Measurement (SI)
Force	1 lbf	=	4.448 N
Mass	1 slug	=	14.59 kg
Length	1 ft	=	0.3048 m

Conversion Factors (SI) to (FPS)

Quantity	Unit of Measurement (SI)	Equals	Unit of Measurement (FPS)
Force	1N	=	0.22481 lbf
Mass	1 kg	=	0.0685218 Slug
Length	1 m	=	3.2808ft

Newton's Laws:

Isaac Newton was the first to state correctly the basic laws governing the motion of a particle and to demonstrate their validity. Slightly reworded with modern terminology, these laws are:

- **Law I:** A particle remains at rest or continues to move with uniform velocity (in a straight line with a constant speed) if there is no unbalanced force acting on it.
- **Law II:** The acceleration of a particle is proportional to the vector sum of forces acting on it, and is in the direction of this vector sum.
  Newton's second law forms the basis for most of the analysis in dynamics. As applied to a particle of mass *m*, it may be stated as
  $$F = m\,a \;\ldots\ldots\; (1)$$
  where **F** is the vector sum of forces acting on the particle and ***a*** is the resulting acceleration. This equation is a vector equation because the direction of **F** must agree with the direction of ***a***, and the magnitudes of **F** and (***m a)*** must be equal.
- **Law III:** The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear (they lie on the same line).

Kinematics of Particles

Introduction

Kinematics is the branch of dynamics which describes the motion of bodies without reference to the forces which either cause the motion or are generated as a result of the motion. Kinematics is often described as the “geometry of motion.” Some engineering applications of kinematics include the design of cams, gears, linkages, and other machine elements to control or produce certain desired motions, and the calculation of flight trajectories for aircraft, rockets, and spacecraft. A thorough working knowledge of kinematics is a prerequisite to kinetics, which is the study of the relationships between motion and the corresponding forces which cause or accompany the motion.

Particle Motion

Ways of describing the motion: Depends on the experience and how the data are given. The fig. 1. shows the ways covered in this chapter.

Constrained motion: If the particle is confined to specified path

Unconstrained motion: There are no physical guide

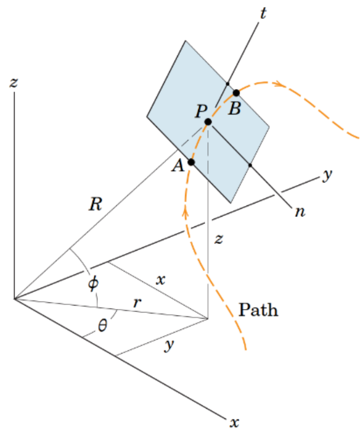


Figure1

Choice of Coordinates

The position of particle P at any time t can be described by:

- Rectangular Coordinates x, y, z
- Cylindrical Coordinates R, θ, z
- Spherical Coordinates R, θ, ϕ

The motion of particle can be described by using: Absolute motion analysis: coordinates measured from fixed reference axis.

Relative-motion analysis: coordinates measured from moving reference axis.

Plane motion: all movements can be represented as occurring in a single plane.

Three-dimensional motion: motion in space.

Rectilinear Motion

Consider a particle P moving along a straight line, Fig. 2. The position of P at any instant of time t can be specified by its distance s measured from reference point O fixed on the line. At time t + Δt the particle has moved to P′ and its coordinate becomes s + Δs. The change in the position coordinate during the interval Δt is called the displacement Δs of the particle. The displacement would be negative if the particle moved in the negative s-direction.

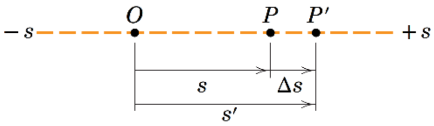


Figure. 2

Displacement: The displacement of the particle is defined as the change in its position. if the particle moves from one point to another, Fig. 2, the displacement is

$$\Delta s = s' - s \text{ ……. (2)}$$

In this case Δs is positive since the particle's final position is to the right of its initial position, i.e., $s' > s$. Likewise, if the final position were to the left of its initial position, Δs would be negative. The displacement of a particle is also a vector quantity, and it should be distinguished from the distance the particle travels. Specifically, the distance traveled is a positive scalar that represents the total length of path over which the particle travels.

Velocity: If the particle moves through a displacement Δs during the time interval Δt, the ***average velocity*** of the particle during this time interval is

$$v_{avg} = \frac{\Delta s}{\Delta t} \text{ …… (3)}$$

If we take smaller and smaller values of Δt, the magnitude of Δs becomes smaller and smaller. Consequently, the ***instantaneous velocity*** is a vector defined as $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ or

$$v = \frac{ds}{dt} = \dot{s} \text{ …… (4)}$$

Thus, the velocity is the time rate of change of the position coordinate s. The velocity is positive or negative depending on whether the corresponding displacement is positive or negative.

Acceleration: The average acceleration of the particle during the interval Δt is the change in its velocity divided by the time interval

$$a_{avg} = \frac{\Delta v}{\Delta t} \text{ …… (5)}$$

As Δt becomes smaller and approaches zero in the limit, the average acceleration approaches the instantaneous acceleration of the particle, which is $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$ or

$$a = \frac{dv}{dt} = \dot{v} \text{ …… (6) or}$$

$$a = \frac{d^2 s}{dt^2} = \ddot{s} \text{ …… (7)}$$

The acceleration is positive or negative depending on whether the velocity is increasing or decreasing. Note that the acceleration would be positive if the particle had a negative velocity which was becoming less negative. If the particle is slowing down, the particle is said to be decelerating.

Finally, an important differential relation involving the displacement, velocity, and acceleration along the path may be obtained by eliminating the time differential dt between Eqs. 4 and 6 or Eqs. 5 and 7. We have

$$v\,dv = a\,ds \text{ …… (8) or}$$

$$s\,d\dot{s} = \ddot{s}\,ds \text{ …… (9)}$$

Equations 6 through 9 are the differential equations for the rectilinear motion of a particle. Problems in rectilinear motion involving finite changes in the motion variables are solved by integration of these basic differential relations. The position coordinate ***s***, the velocity ***v***, and the acceleration ***a*** are algebraic quantities, so that their signs, positive or negative, must be carefully observed. Note that the positive directions for ***v*** and ***a*** are the same as the positive direction for ***s***.

Constant Acceleration, *a = a$_c$*. When the acceleration is constant, each of the three kinematic equations $a_c = dv/dt$, $v = ds/dt$, and $a_c\, ds = v\, dv$ can be integrated to obtain formulas that relate a_c, v, s, and t.

Velocity as a Function of Time: Integrate $a_c = dv/dt$, assuming that initially $v = v_o$ when $t = 0$.

$$\int_{v_o}^{v} dv = \int_{o}^{t} a_c dt$$

$$v = v_o + a_c t \;\ldots\ldots (10)$$

Position as a Function of Time: Integrate $v = ds/dt = v_o + a_c t$, assuming that initially $s = s_o$ when $t = 0$.

$$\int_{s_o}^{s} ds = \int_{o}^{t} v\, dt$$

$$\int_{s_o}^{s} ds = \int_{o}^{t} (v_o + a_c t)\, dt$$

$$s = s_o + v_o t + \frac{1}{2} a_c t^2 \;\ldots\ldots (11)$$

Velocity as a Function of Position: Either solve for ***t*** in Eq. 10 and substitute into Eq. 11, or integrate $v dv = a_c ds$, assuming that initially $v = v_o$ at $s = s_o$.

$$\int_{v_o}^{v} v\, dv = \int_{s_o}^{s} a_c\, ds$$

$$v^2 = v_o^2 + 2a_c(s - s_o) \;\ldots\ldots (12)$$

The algebraic signs of s_o, v_o, and a_c, used in the above three equations, are determined from the positive direction of the s axis as indicated by the arrow written at the left of each equation. Remember that these equations are useful only when the acceleration is constant and when $t = 0$, $s = s_o$, $v = v_o$. A typical example of constant accelerated motion occurs when a body falls freely toward the earth. If air resistance is neglected and the distance of fall is short, then the downward acceleration of the body when it is close to the earth is constant and approximately 9.81 m/s^2 or 32.2 ft/s^2.

Important Points

- **Dynamics is concerned with bodies that have accelerated motion.**
- **Kinematics is a study of the geometry of the motion.**
- **Kinetics is a study of the forces that cause the motion.**
- **Rectilinear kinematics refers to straight-line motion.**
- **Speed refers to the magnitude of velocity.**
- **Average speed is the total distance traveled divided by the total time. This is different from the average velocity, which is the displacement divided by the time.**
- **A particle that is slowing down is decelerating.**
- **A particle can have an acceleration and yet have zero velocity.**
- **The relationship *a ds = v dv* is derived from *a = dv/dt* and *v = ds/dt*, by eliminating *dt*.**

Example 1

The position coordinate of a particle which is confined to move along a straight line is given by $\boldsymbol{s}=2\boldsymbol{t}^3-24\boldsymbol{t}+6$, where $\boldsymbol{s}$ is measured in meters from a convenient origin and $\boldsymbol{t}$ is in seconds. Determine

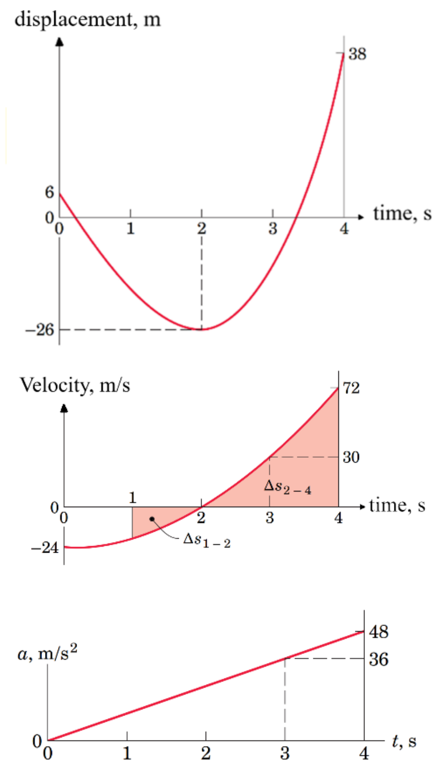
a. The time required for the particle to reach a velocity of 72 m/s from its initial condition at $\boldsymbol{t} = 0$
b. The acceleration of the particle when $\boldsymbol{v} = 30$ m/s, and
c. The net displacement of the particle during the interval from $\boldsymbol{t} = 1$ s to $\boldsymbol{t} = 4$ s.
d. Plot the displacement, velocity, and acceleration as functions of time for the first 4 seconds of motion

Solution: The velocity and acceleration are obtained by successive differentiation of $\boldsymbol{s}$ with respect to the time. Thus,

$$[\, v = \dot{s}\,] \rightarrow \; \mathbf{v = 6\, t^2 - 24 \text{ m/s}}$$

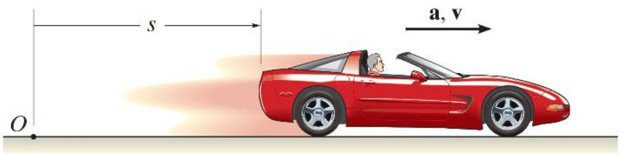
$$[\, a = \dot{v}\,] \rightarrow \; \mathbf{a = 12\, t \text{ m/s}^2}$$

a. Substituting $\boldsymbol{v}$ = 72 m/s into the expression for $\boldsymbol{v}$ gives us $72 = 6t^2 - 24$, from which $t = \pm 4$ s.
The negative root describes a mathematical solution for $\boldsymbol{t}$ before the initiation of motion, so this root is of no physical interest. 1 Thus, the desired result is **t = 4** s.
b. Substituting $\boldsymbol{v}$ = 30 m/s into the expression for $\boldsymbol{v}$ gives $30=6t^2 - 24$, from which the positive root is t = 3 s, and the corresponding acceleration is
$\boldsymbol{a} = 12(3) = 36 \text{ m/s}^2$
c. The net displacement during the specified interval is
$\Delta s = s4 - s1$ or
$\Delta s = [2(4^3) - 24(4) + 6] - [2(1^3) - 24(1) + 6] = 54$ m



Example 2

The car as shown in below figure moves in a straight line such that for a short time its velocity is defined by $v = \frac{ds}{dt} = t^2 + \frac{2}{3}t$ where ***v*** in m/s and ***t*** in seconds. Determine its position and acceleration when ***t*** =3 s. When ***t*** = 0, ***s*** = 0.



SOLUTION:

Coordinate System. The position coordinate extends from the fixed origin O to the car, positive to the right.

Position. Since ***v*** = f(t), the car's position can be determined from ***v = ds/dt***, since this

$$v = \frac{ds}{dt} = t^2 + \frac{2}{3}t$$

$$\int_{s_o}^{s} ds = \int_{o}^{t} v\, dt$$

$$\int_{s_o}^{s} ds = \int_{o}^{t} (t^2 + \frac{2}{3}t)\, dt$$

$$s\Big|_{s_o}^{s} = \frac{t^3}{3} + \frac{t^2}{3}\Big|_{o}^{t}$$

$$s = \frac{t^3}{3} + \frac{t^2}{3}$$

When t = 3 s

$$s = \frac{3^3}{3} + \frac{3^2}{3} = 12 \text{ m}$$

Acceleration. Since ***v*** = f(t), the acceleration is determined from ***a = dv/dt***, since this equation relates ***a, v,*** and ***t***.

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(t^2 + \frac{2}{3}t\right) = 2t + \frac{2}{3}$$

When t = 3 s

$$a = 2t + \frac{2}{3} = 2(3) + \frac{2}{3} = 6.67 \text{ m/s}^2$$

NOTE: The formulas for ***constant acceleration*** cannot be used to solve this problem, because the acceleration is a function of time.

Example 3

A particle moves along the x-axis with an initial velocity v_o = 15 m/s at the origin when t = 0. For the first 4 seconds it has no acceleration, and thereafter it is acted on by a retarding force which gives it a constant acceleration $a_x = -3\ m/s^2$. Calculate the velocity and the x-coordinate of the particle for the conditions of t=8 s and t = 12 s and find the maximum positive x-coordinate reached by the particle.

Solution: The velocity of the particle after t = 4 sec is computed from

$$[\int dv = \int a\, dt] \rightarrow \int_{15}^{v_x} dv = \int_4^t a_x dt \rightarrow \int_{15}^{v_x} dv_x = \int_4^t (-3)dt \rightarrow v|_{15}^{v_x} = -3t|_4^t$$

$$v_x - 15 = -3(t-4) \rightarrow v_x = -3t + 27$$

and is plotted as shown. At the specified times, the velocities are

t = 8 s, v_x = − 3(8)+27 = 3 m/s.

t = 12 s , v_x = − 3(12)+27 = −9 m/s.

The x-coordinate of the particle at any time greater than 4 seconds is the distance traveled during the first 4 seconds plus the distance traveled after the discontinuity in acceleration occurred. Thus,

$$s_t = s_{1-4} + s_{4-t} = v.t + \int_4^t ds = v.t + \int_4^t (-3t + 27)dt$$

$$s_x = 15(4) + \int_4^t (-3t + 27)dt = \left(\frac{-3}{2}t^2 + 27t\right) - 24$$

For the two specified times,

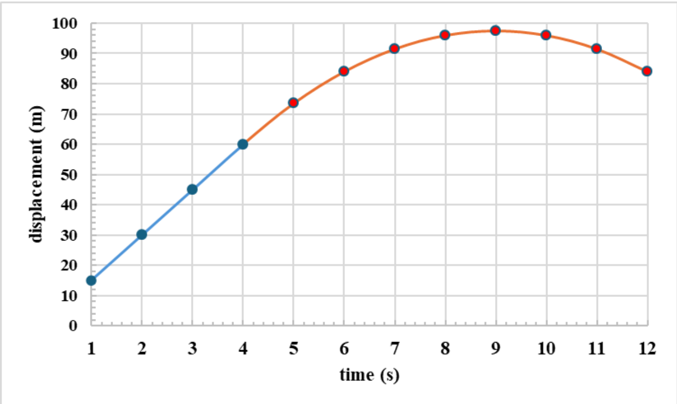
t = 8 s, $s_8 = \left(\frac{-3}{2}t^2 + 27t\right) - 24 = \left(\frac{-3}{2}8^2 + 27.8\right) - 24 = 96\ \ m$

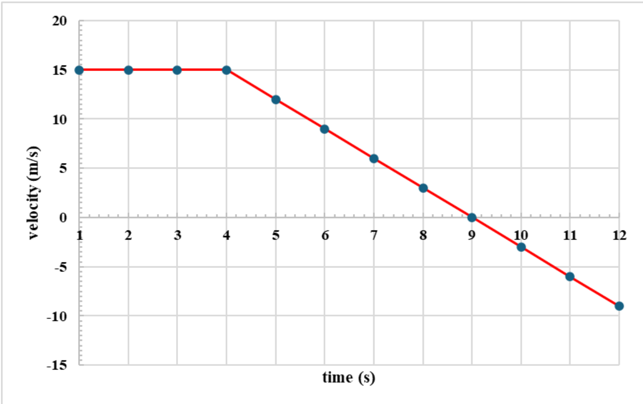
t = 12 s, $s_{12} = \left(\frac{-3}{2}t^2 + 27t\right) - 24 = \left(\frac{-3}{2}12^2 + 27.12\right) - 24 = 84\ \ m$

The x-coordinate for t = 12 s is less than that for t = 8 s since the motion is in the negative x-direction after t = 9 s. The maximum positive x-coordinate is, then, the value of x for t = 9 s which is

$$x_{max} = \left(\frac{-3}{2}t^2 + 27t\right) - 24 = \left(\frac{-3}{2}9^2 + 27.9\right) - 96 = 97.5\, m$$

These displacements are seen to be the net positive areas under the v-t graph up to the values of t in question.





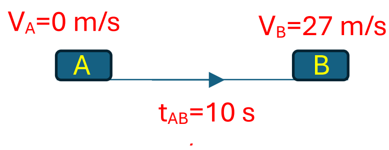
Homework no. 1

1. **Initially, the car travels along a straight road with a speed of 35 m/s. If the brakes are applied and the speed of the car is reduced to 10 m/s in 15 s, determine the constant deceleration of the car.**

Ans: ($-1.67\ m/s^2$)

Solution:

$$v_B = v_A + a_c t_{AB}$$
$$10 = 35 + a_c \times 15$$
$$10 - 35 = a_c \times 15$$
$$-25 = a_c \times 15$$
$$a_c = \frac{-25}{15} = -1.67\ \mathrm{m/s^2}$$



2. A particle travels along a straight line with a velocity of $\boldsymbol{v}$ = ($4t - 3t^2$) m/s, where $\boldsymbol{t}$ is in seconds. Determine the position of the particle when t = 4 s, s_o = 0 when t_o = 0. **Ans: (-32 m)**

Solution:

Velocity is the derivative of position with respect to time:

$$v = \frac{ds}{dt} = 4t - 3t^2$$

Velocity is the derivative of position with respect to time:

$$ds = (4t - 3t^2)\ dt$$

$$\int_{s_o}^{s} ds = \int_{t_o}^{t} (4t - 3t^2)\ dt$$

$$\int_{0}^{s} ds = \int_{0}^{t} (4t - 3t^2)\ dt$$

$$s\Big|_0^s = 4 \times \frac{t^2}{2} - 3 \times \frac{t^3}{3}\Big|_0^t$$

$$s\Big|_0^s = (2t^2 - t^3)\Big|_0^t$$

$$(s - 0) = (2t^2 - t^3) - [2(0)^2 - (0)^3]$$

$$s = 2t^2 - t^3$$

At $t = 4$ s :

$$s(4) = 2(4)^2 - (4)^3 = -32\ \mathrm{m}$$

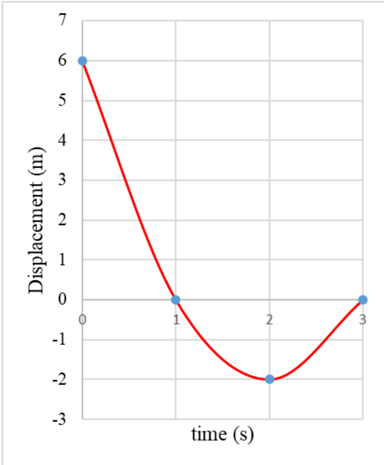
3. The position of the particle is given by s = (2t^2 - 8t + 6) m, where *t* is in seconds. Determine the time when the velocity of the particle is zero, and the total distance traveled by the particle when *t* = 3 s. **Ans: (2 s, 10 m)**

Solution

First, find velocity ***v*** by differentiating position ***s***

$$v = \frac{ds}{dt} = 4t - 8$$

To find the time when the velocity equals zero, set v=0 in the above equation:



$$0 = 4t - 8 \ \rightarrow t = 2\,\text{s}$$

Now, find the total **distance** traveled up to t=3 s.

At t = 0 s: $s(0) = 2(0)^2 - 8(0) + 6 = 6\,\text{m}$

At t = 1 s: $s(1) = 2(1)^2 - 8(1) + 6 = 0\,\text{m}$

At t = 2 s: $s(2) = 2(2)^2 - 8(2) + 6 = -2\,\text{m}$

At t = 3 s: $s(3) = 2(3)^2 - 8(3) + 6 = 0\,\text{m}$

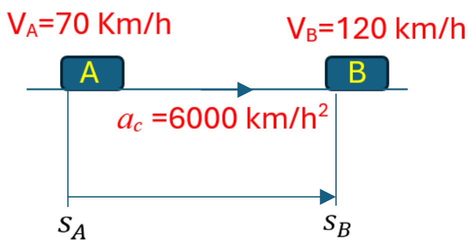
The particle moves from s=6m to s=−2 m (distance 8m), then back to s=0 m (distance 2 m). Total distance:

$$s_t = |s_{0-1}| + |s_{1-2}| + |s_{2-3}|$$

$$s_t = |6| + |-2| + |2| = 10\,\text{m}$$

4. Traveling with an initial speed of 70 km/h, a car accelerates at 6000 km/h^2 along a straight road. How long will it take to reach a speed of 120 km/h? Also, through what distance does the car travel during this time? **Ans: (30 s, 0.791 km)**

Solution:



$$v_A = 70\,\text{km/h} = \frac{70 \times 1000}{3600} = 19.44\,\text{m/s}$$

$$v_B = 120\,\text{km/h} = \frac{120 \times 1000}{3600} = 33.33\,\text{m/s}$$

$$a_c = 6000\,\text{km/h}^2 = \frac{6000 \times 1000}{(3600)^2} = 0.463\,\text{m/s}^2$$

$$v_B = v_A + a_c t$$
$$33.33 = 19.44 + 0.463 \times t \ \rightarrow \ t = 30\,\text{s}$$

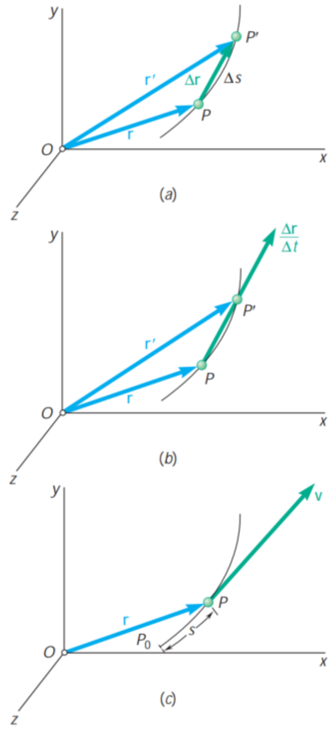
Distance traveled during this time is given as

$$s_B = s_A + v_A t + \frac{1}{2} a_c t^2$$

$$s_B = 0 + 19.44 \times 30 + \frac{1}{2} \times 0.463 \times (30)^2 = 791.5\,\text{m}$$

Curvilinear Motion

When a particle moves along a curve other than a straight line, we say that the particle is in curvilinear motion.



The instantaneous velocity

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} \ \ldots\ldots (13)$$

The vector v obtained in the limit must therefore be tangent to the path of the particle

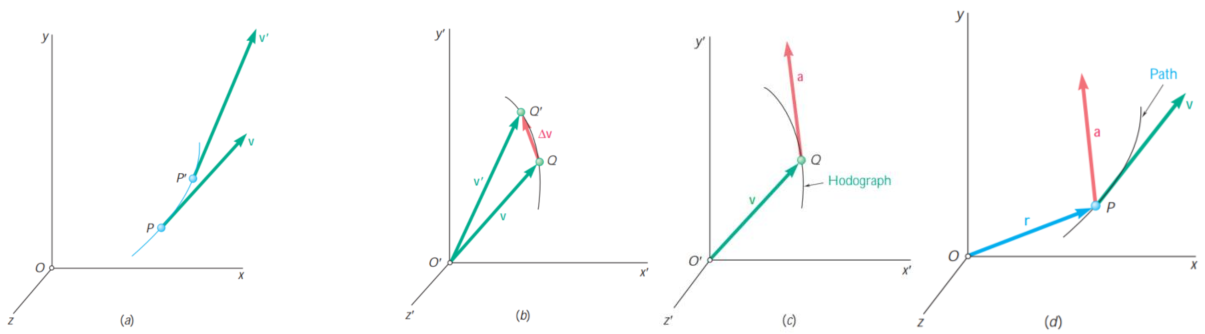
Since the position vector **r** depends upon the time *t*, we can refer to it as a *vector function* of the scalar variable *t* and denote it by **r**(*t*). Extending the concept of derivative of a scalar function introduced in elementary calculus, we will refer to the limit of the quotient Δ**r**/Δ*t* as the *derivative* of the vector function **r**(*t*). We write

$$v = \frac{dr}{dt} \ldots\ldots (14)$$

The instantaneous acceleration

$$a = \frac{dv}{dt} \ldots\ldots (15)$$

The curve described by the tip of ***v*** and shown in Figure below is called the hodograph of the motion

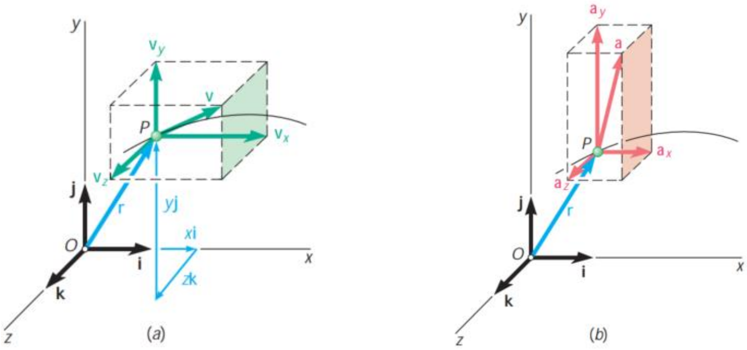


Three different coordinate systems are used:

1. Rectangular coordinates.
2. Normal and Tangential coordinates.
3. Polar coordinates.

Rectangular Components of Velocity and Acceleration

When the position of a particle *P* is defined at any instant by its rectangular coordinates *x*, *y*, and *z*, it is convenient to resolve the velocity ***v*** and the acceleration ***a*** of the particle into rectangular components.



Resolving the position vector **r** of the particle into rectangular components, we write

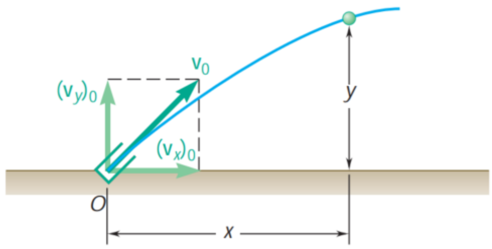
$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad \ldots\ldots (16)$$

where the coordinates *x*, *y*, *z* are functions of *t*. Differentiating twice, we obtain

$$\boldsymbol{v} = \frac{dr}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \quad \ldots\ldots (17)$$

$$\boldsymbol{a} = \frac{dv}{dt} = \ddot{x}\,\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \quad \ldots\ldots (18)$$

In the case of the ***motion of a projectile***



(*a*) Motion of a projectile

the acceleration components are

$a_x = 0$ …… (19)

$a_y = -g$ …… (20)

$v_x = v_{ox}$ …… (21)

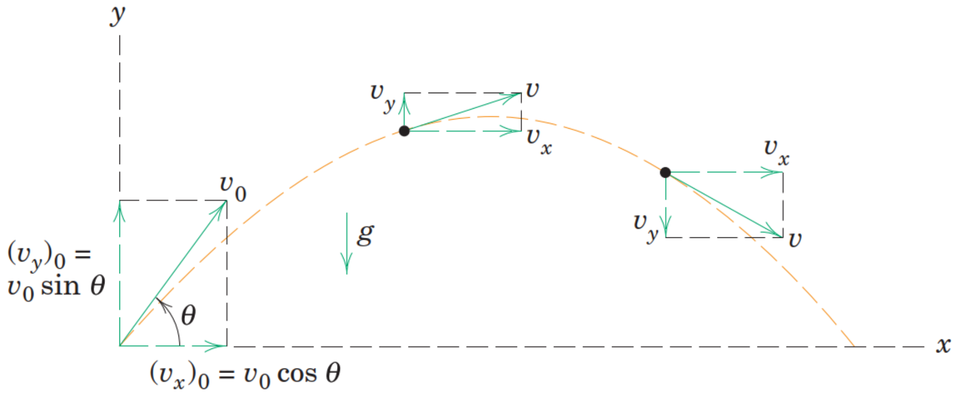
$v_y = v_{oy} - gt$ …… (22)

$x = v_{ox}\, t$ …… (23)

$y = y_o + v_{oy}\, t - \frac{1}{2} g t^2$ …… (24)

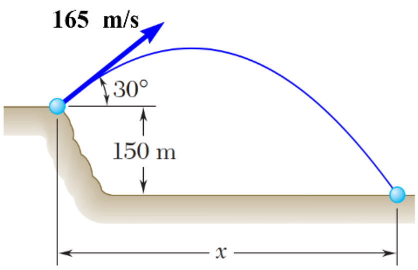
$v_y^2 = v_{oy}^2 - 2g(y - y_o)$ …… (25)

In all these expressions, the subscript zero denotes initial conditions, frequently taken as those at launch where, for the case illustrated.



Example 5

A projectile is fired from the edge of a 150-m cliff with an initial velocity of **165 m/s** at an angle of 30° with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.

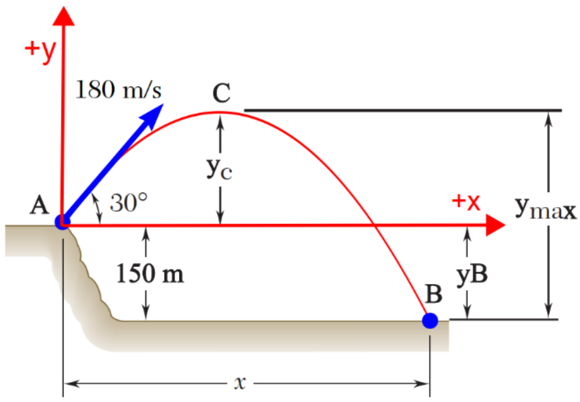


Solution:

$v_{Ay} = \boldsymbol{v} \sin 30° \rightarrow v_{Ay} = 180 \sin 30° = 90 \text{ m/s}$

$v_{Ax} = \boldsymbol{v} \cos 30° \rightarrow v_{Ax} = 180 \cos 30° = 155.9 \text{ m/s}$

$a_y = -9.81 \text{ m/s}^2$



$$y_B = y_A + v_{Ay}\, t - \frac{1}{2} g t^2$$

$$-150 = 0 + 90t - 0.5 \times 9.81 t^2$$

$$4.9t^2 - 90t - 150 = 0$$

$$t = \frac{-(-90) \pm \sqrt{(-90)^2 - 4 \times 4.9 \times (-150)}}{2 \times 4.9} = \frac{90 \pm 105.07}{9.8} = 19.9 \text{ s}$$

$$x = v_{Ax}\, t = 155.9 \times 19.9 = 3102\, m$$

When the projectile reaches its greatest elevation, we have $v_{cy} = 0$

$$v_{Cy}^2 = v_{Ay}^2 - 2g(y_C - y_A)$$

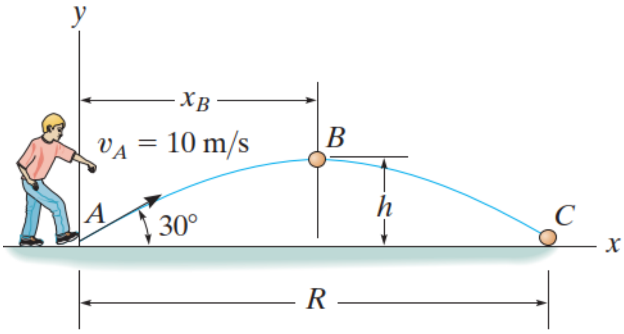
$$0 = (90)^2 - 2 \times 9.81 \times (y_C - 0)$$

$$y_C = 412.84\, m$$

$$y_{max} = y_C + 150 = 412.84 + 150 = 562.84\, m$$

Example 6

The ball is kicked from point A with the initial velocity $\mathbf{v}_A$ = 10 m/s at an angle of 30° with the horizontal. Determine the range R.



Solution

From A to B

$v_{By} = v_{Ay} - gt$

$v_{By} = v_A \sin 30 - g \times t_{AB}$

$0 = 10 \sin 30 - 9.81 \times t_{AB}$

$t_{AB} = 0.5096 \text{ s}$

The total time of flight

$t = 2 \times t_{AB}$

$t = 2 \times 0.5096 = 1.019 \text{ s}$

the range for projectile motion

$x = v_{ox}\, t$

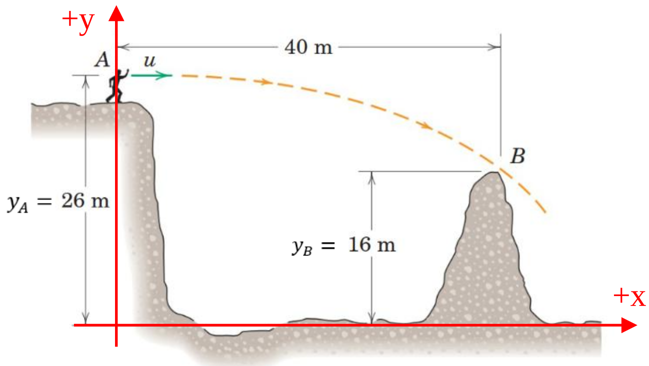
$R = v_A \times \cos 30 \times t$

$R = 10 \times \cos 30 \times 1.019$

$R = 8.824 \text{ m}$

Homework

1. **With what minimum horizontal velocity *u* can a boy throw a rock at A and have it just clear the obstruction at B?**



Solution

$v_{Ay} = 0$

$v_{Ax} = u$

$y_B = y_A + v_{Ay}\, t - \frac{1}{2} g t^2$

$16 = 26 + 0 \times t - 0.5 \times 9.81 \times t^2$

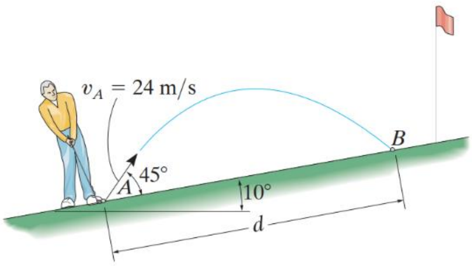
$t = 1.43 \text{ s}$

$x = v_{Ax}\, t$

$40 = u \times 1.43$

$u = 28 \text{ m/s}$

2. A golf ball is struck with a velocity of 24 m/s as shown. Determine (a) the time of flight from A to B, (b) the distance *d* to where the ball will land, and (c) the speed at which it strikes the ground at B



Solution:

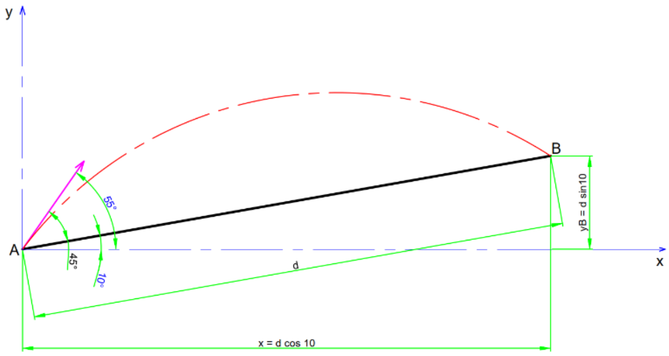
$v_{Ay} = v_A \sin 55 = 24 \sin 55 = 19.66\ m/s$

$v_{Ax} = v_A \cos 55 = 24 \cos 55 = 13.76\ m/s$

$x = v_{Ax}\, t$

$d \times \cos 10 = 13.76 \times t$

$d = \frac{13.76 \times t}{\cos 10} = 13.97\, t \ldots\ldots\ldots (1)$



$y_B = y_A + v_{Ay}\, t - \frac{1}{2} g t^2$

$d \times \sin 10 = 0 + 19.66 \times t - 0.5 \times 9.81 \times t^2 \quad \ldots\ldots (2)$

Substitute equation (1) into equation (2) we obtain :

$13.97 \times t \times \sin 10 = 0 + 19.66 \times t - 0.5 \times 9.81 \times t^2$

$2.42 \times t = 19.66 \times t - 4.9 \times t^2 \quad \rightarrow divide\ by\ t\ as\ t \neq 0$

$2.42 = 19.66 - 4.9 \times t \quad \rightarrow t = 3.52\ s$

To get the distance ***d*** to where the ball will land, substitute value of ***t*** in equation (1)

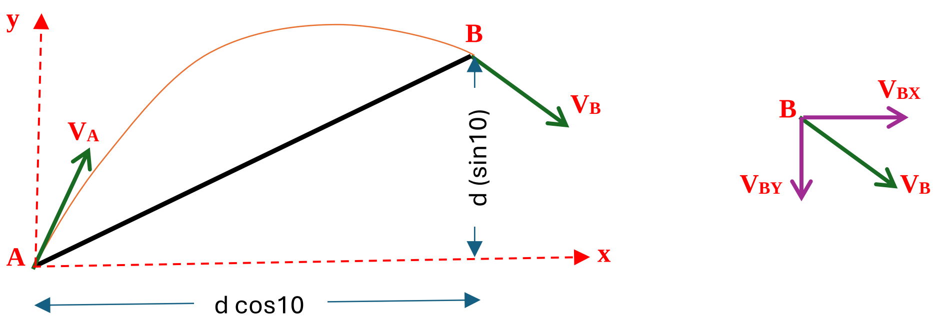
$d = 13.97 \times t = 13.97 \times 3.52 = 49.2\ m$

the speed at which it strikes the ground at **B**

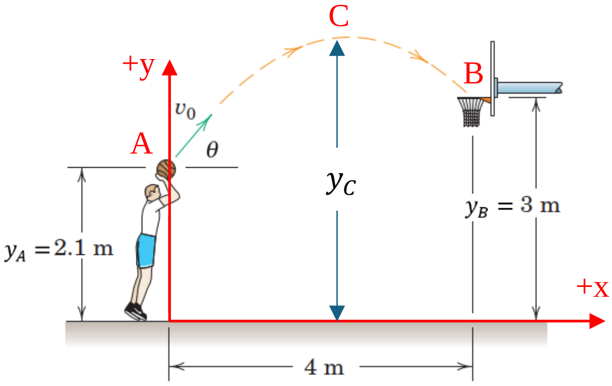
$v_{By} = v_{Ay} - gt$

$v_{By} = 19.66 - 9.81 \times 3.52 = -14.67$

$v_B = \sqrt{(v_{Bx})^2 + (v_{By})^2} = \sqrt{(13.76)^2 + (-14.67)^2} = 20.11\ m/s$



3. The basketball player likes to release his foul shots at an angle θ = 50° to the horizontal as shown. What initial speed v_0 will cause the ball to pass through the center of the rim? Determine the greatest elevation above the ground reached by the ball.



Solution:

Use x-y coordinate with origin at the ground

$v_{Ay} = v_o \sin 50° = 0.766\, v_o$ ... ... (1)

$v_{Ax} = v_o \cos 50° = 0.642\, v_o$ ... ... (2)

$x = v_{Ax}\, t$

$4 = 0.642\, v_o \times t$

$t = \frac{4}{0.642\, v_o} = \frac{6.23}{v_o}$ ... ... (3)

$y_B = y_A + v_{Ay}\, t - \frac{1}{2} g t^2 \rightarrow 3 = 2.1 + v_{Ay}\, t - \frac{1}{2} g t^2$ ... ... (4)

Substitute equation (1) *and* (3) *into equation* (4) *we obtain* :

$$3 = 2.1 + 0.766\, v_o \times \left(\frac{6.23}{v_o}\right) - \frac{1}{2} \times 9.81 \times \left(\frac{6.23}{v_o}\right)^2$$

$$3 - 2.1 = 0.766 \times 6.23 - 4.9 \times \frac{(6.23)^2}{(v_o)^2}$$

$$0.9 = 4.772 - \frac{190.18}{(v_o)^2}$$

$$0.9 - 4.772 = -\frac{190.18}{(v_o)^2}$$

$$-3.872 = -\frac{190.18}{(v_o)^2}$$

$$(v_o)^2 = \frac{190.18}{3.872}$$

$(v_o)^2 = 49.116 \rightarrow v_o = \sqrt{49.116} = 7.008$ m/s

$v_{By}^2 = v_{oy}^2 - 2g(y_B - y_o)$

$(0)^2 = (0.766\, v_o)^2 - 2 \times 9.81(y_B - 2.1)$

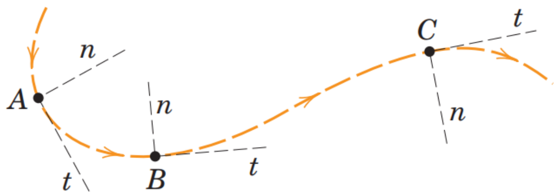
$(0)^2 = (0.766 \times 7.008)^2 - 2 \times 9.81(y_B - 2.1)$

$y_B = 3.568\, m$

Normal and Tangential Coordinates of Velocity and Acceleration

It is one of the very natural descriptions of the curvilinear motion along the tangent **t** and normal **n** to the path of the particle.

When the particle advances from A to B to C, the positive n is always taken toward the center of the curvature of the path.



Velocity and Acceleration

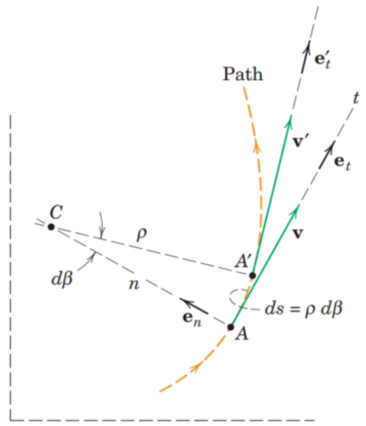
Since the particle moves, *s* is a function of time. the particle's velocity ***v*** has a *direction* that is *always tangent to the path and a magnitude that is determined by taking the time derivative of the path function s*

We introduce:

$\vec{e}_t$ a unit vector in t direction

$\vec{e}_n$ a unit vector in n direction

During dt, the particle moves a differential distance ds along the curve from A to $\acute{A}$ where ρ the radius of curvature of the path.



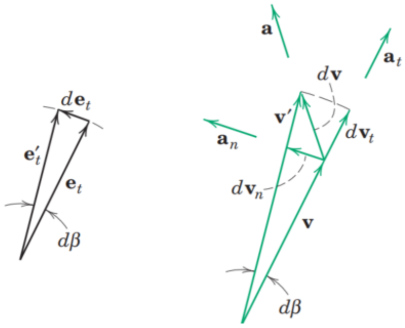
$ds = \rho\, d\beta$ divide both sides by dt

$$\frac{ds}{dt} = \rho\,\frac{d\beta}{dt}$$

$$v = \rho\,\frac{d\beta}{dt} = \rho\dot{\beta}$$

we can write the velocity as the vector

$$\vec{v} = v\,\vec{e}_t = \rho\,\dot{\beta}\,\vec{e}_t$$



The acceleration of the particle is the time rate of change of the velocity. Thus,

$$a = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\,\vec{e}_t) = v\,\dot{\vec{e}_t} + \dot{v}\,\vec{e}_t$$

acceleration becomes

$$a = \frac{v^2}{\rho}\,\vec{e}_n + \dot{v}\,\vec{e}_t$$

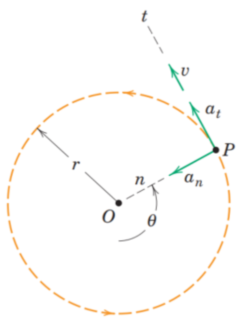
$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\,\dot{\beta} \quad \ldots\ldots (26)$$

$$a_t = \dot{v} \quad \ldots\ldots (27)$$

$$a = \sqrt{a_n{}^2 + a_t{}^2} \quad \ldots\ldots (28)$$

Circular Motion

It is a special case of plane curvilinear motion.



$$v = r\dot{\theta} \quad \ldots\ldots (29)$$

$$a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\,\dot{\theta} \quad \ldots\ldots (30)$$

$$a_t = \dot{v} = r\ddot{\theta} \quad \ldots\ldots (31)$$

$$a = \sqrt{a_n{}^2 + a_t{}^2} \quad \ldots\ldots (32)$$

Example 7

An automobile is traveling on a curve having a radius of 250 m. If the acceleration of the automobile is 2 m/s^2, determine the constant speed at which the automobile is traveling.

Solution

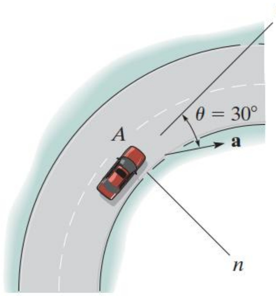
Since the automobile is traveling at a constant speed, $a_t = 0$

$$\therefore\ a_n = 2\ \text{m/s}^2$$

$$a_n = \frac{v^2}{r} \rightarrow 2 = \frac{v^2}{250} \rightarrow v = \sqrt{2 \times 250} = 22.36\ \text{m/s}$$

Example 8

The automobile has a speed of 25 m/s at point A and an acceleration having a magnitude of *a* = 3 m/s^2, acting in the direction shown. Determine the radius of curvature of the path at point A and the tangential component of acceleration.



The tangential acceleration is

$$a_t = a\cos 30 = 3 \times \cos 30 = 2.6\ \text{m/s}^2$$

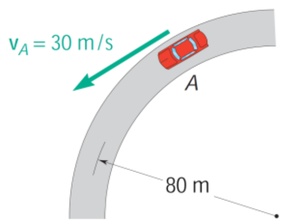
and the normal acceleration is

$$a_n = a\sin 30 = 3 \times \sin 30 = 1.5\ \text{m/s}^2$$

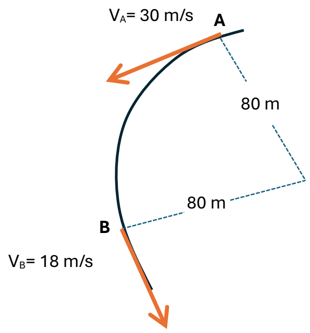
$$a_n = \frac{v^2}{r} \rightarrow 1.5 = \frac{(25)^2}{r} \rightarrow r = \frac{625}{1.5} = 416.6\ \text{m}$$

Homework

1. **A motorist is traveling on a curved section of highway of radius 80 m at the speed of 30 m/s. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to 18 m/s, determine the acceleration of the automobile after the brakes have been applied. (Ans: 4.318 m/s^2)**



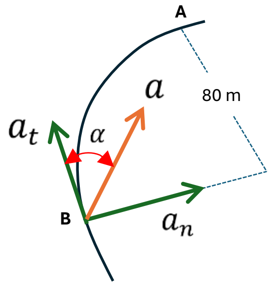
Solution:



$$a_t = \frac{\Delta v}{\Delta t} = \frac{V_B - V_A}{8} = \frac{18 - 30}{8} = -1.5\ \text{m/s}^2$$

The normal component of acceleration of the automobile after applying brakes when the speed reduced to 18 m/s is given by.

$$a_n = \frac{v^2}{r} = \frac{(18)^2}{80} = 4.05\ m/s^2$$



The magnitude and direction of the resultant a of the components a_n and a_t are

$$a = \sqrt{a_n{}^2 + a_t{}^2} = \sqrt{(4.05)^2 + (-1.5)^2} = 4.318\ \text{m/s}^2$$

$$\tan\alpha = \frac{a_n}{a_t} \rightarrow \alpha = \tan^{-1}\frac{a_n}{a_t} \rightarrow \alpha = \tan^{-1}\frac{4.05}{1.5} = 69.67°$$

Kinetics of Particles

Introduction

According to Newton's second law, a particle will accelerate when it is subjected to unbalanced forces. Kinetics is the study of the relations between unbalanced forces and the resulting changes in motion. Kinetics of particles can be solved by three general approaches:

a. Direct application of Newton's second law (Force- mass – acceleration method).
b. Work and energy principles.
c. Impulse and momentum methods.

Section A: Force, Mass, and Acceleration

Newton's Second Law

$$\vec{F} = m\,\vec{a} \text{ ...... } (33)$$

The verification of this equation is entirely experimental. The experiment involves subjecting a mass particle to the action of a single force $\vec{F}_1$, and the acceleration $\vec{a}_1$ is measured in the primary inertial system. By repeating the procedure with different forces, the followings were noticed:

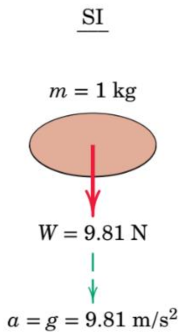
$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \cdots = \frac{F}{a} = C, \qquad \text{a constant}$$

The acceleration is always in the direction of the applied force.

Force And Mass Units

Consider the free -fall experiment as shown in the figure below:

In SI units, for a mass m = 1 kg, the weight is W = 9.81 N and the corresponding *downward a*cceleration a is g = 9.81 m / s^2



Rectilinear Motion

If force and acceleration vectors can be resolved into rectangular components along x and y directions for plane motion then Newton’s second law can also be expressed in scalar forms as:

$$\sum F_x = m\,a_x \text{ ... ... } (34)$$

$$\sum F_y = m\,a_y \text{ ... ... } (35)$$

where F_x and F_y are respectively x and y components of the individual forces in a system of forces.

Curvilinear Motion

Equation (33) is re-written in these ways:

- Rectangular coordinates.
  $\sum F_x = m\,a_x$ …… (36)
  $\sum F_y = m\,a_y$ …… (37)
  Where $a_x = \ddot{x}$ and $a_y = \ddot{y}$
- Normal and tangential coordinates.
  $\sum F_n = m\,a_n$ …… (38)
  $\sum F_t = m\,a_t$ …… (39)
  Where

  $v = r\dot{\theta}$ …… (40)

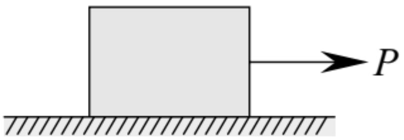
  $a_n = \frac{v^2}{r} = r\dot{\theta}^2 = v\,\dot{\theta}$ …… (41)

  $a_t = \dot{v} = r\ddot{\theta}$ …… (42)

  $a = \sqrt{{a_n}^2 + {a_t}^2}$ …… (43)

Example 9

A block of 15 kg weight is resting on a rough horizontal table. What horizontal force P is required to move the block with an acceleration of 1.5 m/s^2? The coefficient of kinetic friction between the contact surfaces is 0.2.



Solution

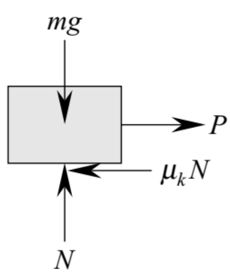
as shown on the FBD

[Note that as the block moves horizontally, its acceleration along the Y-direction, i.e., a_y is zero]

$\sum F_y = m\,a_y \rightarrow N - mg = 0 \rightarrow N = mg = 15 \times 9.81 = 147.15\,N$

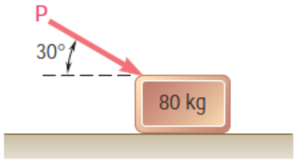
$\sum F_x = m\,a_x \rightarrow P - \mu_k N = m\,a_x \rightarrow P - 0.2 \times 147.15 = 15 \times 1.5$

$P = 51.93\,N$



Example 10

An 80-kg block rests on a horizontal plane. Find the magnitude of the force P required to give the block an acceleration of 2.5 m/s² to the right. The coefficient of kinetic friction between the block and the plane is 0.25.



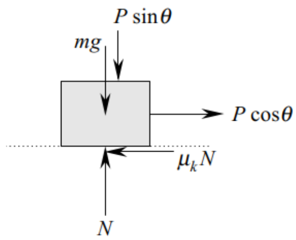
Solution:

as shown on the FBD

$\sum F_y = m\, a_y \rightarrow N - mg - P\sin 30 = 0 \rightarrow N = mg + P\sin 30 = 80 \times 9.81 + P\sin 30$ ….. (1)

[Note that as the block moves horizontally, its acceleration along the Y-direction, i.e., a_y is zero]

$\sum F_x = m\, a_x \rightarrow P\cos 30 - \mu_k N = m\, a_x \rightarrow P\cos 30 - 0.25 \times N = 80 \times 2.5$ ….. (2)



Substitute eq (1) in eq (2) we obtain

$$P\cos 30 - 0.25 \times (80 \times 9.81 + P\sin 30) = 80 \times 2.5$$

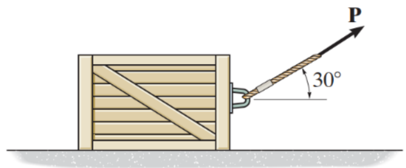
$$P\cos 30 - 0.25 \times 80 \times 9.81 - 0.25 \times P\sin 30) = 200$$

$$P\,(\cos 30 - 0.25 \times \sin 30) - 0.25 \times 80 \times 9.81 = 200$$

$$P = \frac{200 + 0.25 \times 80 \times 9.81}{(\cos 30 - 0.25 \times \sin 30)} = 534.664\ \text{N}$$

Homework

1. **The 50-kg crate rests on a horizontal surface for which the coefficient of kinetic friction is 0.3. If the crate is subjected to the 400-N towing force as shown, determine the velocity of the crate in 3 s starting from rest.**



Solution:

as shown on the FBD

[Note that as the block moves horizontally, its acceleration along the Y-direction, i.e., a_y is zero]

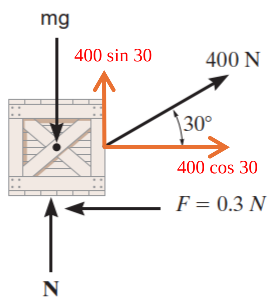
$\sum F_y = m\, a_y \rightarrow N - mg + P\sin 30 = 0$

$N = mg - P\sin 30 = 50 \times 9.81 - 400\sin 30 \rightarrow N = 290.5\ \text{N}$

$\sum F_x = m\, a_x \rightarrow P\cos 30 - \mu_k N = m\, a_x$

$400\cos 30 - 0.3 \times 290.5 = 50 \times a_x$

$a_x = 5.185\ \text{m/s}^2$

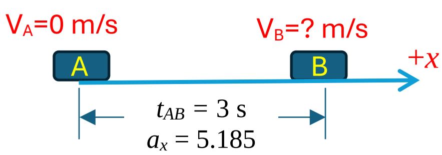


The acceleration is *constant*, since the applied force **P** is constant. the initial velocity is zero, the velocity of the crate after 3 s is

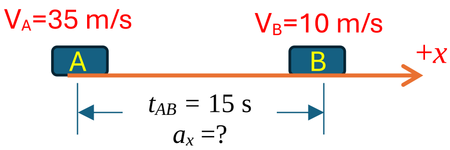
$v_B = v_A + a_x t_{AB}$

$v_B = 0 + 5.185 \times 3$

$v_B = 15.6\ \text{m/s}$



2. **A car of 2 tons mass travels along a straight road with a speed of 35 m/s. If the brakes are applied and the speed of the car is reduced to 10 m/s in 15 s, determine the constant deceleration of the car, and what should be the braking force applied assuming it to be uniform?**



Solution:

2 tons = 2000 kg

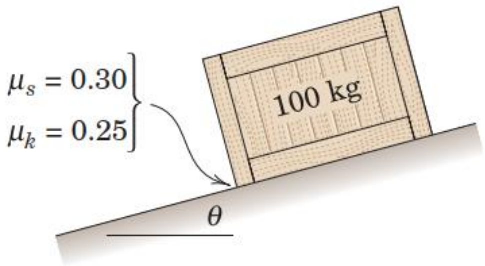
$v_B = v_A + a_x t$

$10 = 35 + a_x \times 15$

$a_x = \dfrac{10 - 35}{15} = -1.67\ \mathrm{m/s^2}$

$\sum F_x = m\, a_x = \rightarrow\ F_x = 2000 \times (-1.67) = -3340\ \mathrm{N}$

3. **The 100-kg crate is carefully placed with zero velocity on the incline. Describe what happens if (*a*) θ =15° and (*b*) θ =20°.**

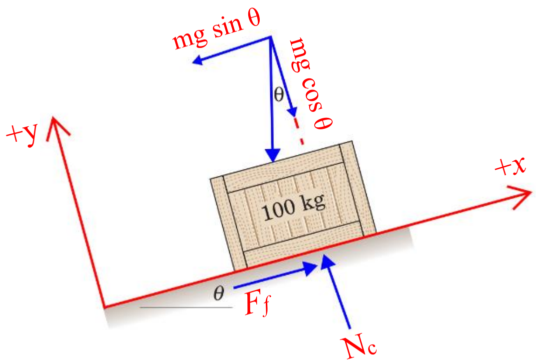


Solution:

$\theta_{max} = \tan^{-1} \mu_s = \tan^{-1} 0.3 = 16.7°$

Which is the minimum inclination angle to have motion.

The free body diagram shown below



Case (a) when θ = 15°

$since\ \theta < \theta_{max}$, there is no motion (both $\boldsymbol{v}$ and $\boldsymbol{a} = 0$)

Case (b) when θ = 20 °

$\sum F_y = 0 \rightarrow \quad N_c - mg \cos 20° = 0$

$N_c = mg \cos 20° = 100 \times 9.81 \times \cos 20° = 921.838\ \mathrm{N}$

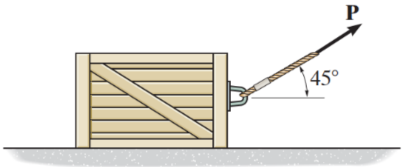
$\sum F_x = m\, a_x \quad \rightarrow\ -mg \sin 20° + \mu_k N_c = m\, a_x$

$-100 \times 9.81 \sin 20° + 0.25 \times 921.838 = 100 \times\ a_x$

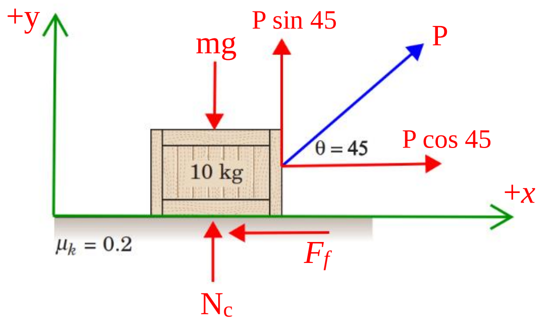
$a_x = -1.0506\ \mathrm{m/s^2}$

$the\ block\ will\ accelerates\ down\ plane\ with\ a\ rate\ of\ 1.0506\ \mathrm{m/s^2}$

4. A block of 10 kg is resting on a rough horizontal table. What force P inclined at 45° to the horizontal is required to move the block horizontally with an acceleration of 2 m/s²? The coefficient of kinetic friction between the contact surfaces is 0.2.



Solution
as shown on the FBD



$\sum F_y = m\, a_y$

[Note that as the block moves horizontally, it has no acceleration along the Y-direction, i.e., a_y is zero]

$\sum F_y = m \times 0$

$\sum F_y = 0$

$N_c - mg + P \sin 45 = 0$

$N_c = mg - P \sin 45$ …….. (1)

$\sum F_x = m\, a_x$

$P \cos 45 - F_f = m\, a_x$

$P \cos 45 - \mu_k N_c = m\, a_x$ …… (2)

Substitute eq (1) in eq (2) we obtain

$P \cos 45 - \mu_k (mg - P \sin 45) = m\, a_x$

$P \cos 45 - \mu_k \times mg + \mu_k \times P \sin 45 = m\, a_x$

$P\,(\cos 45 + \mu_k \sin 45) - \mu_k \times mg = m\, a_x$

$P\,(\cos 45 + \mu_k \sin 45) = m\, a_x + \mu_k \times mg$

$$P = \frac{m\, a_x + \mu_k \times mg}{\cos 45 + \mu_k \sin 45} = \frac{10 \times 2 + 0.2 \times 10 \times 9.81}{\cos 45 + 0.2 \times \sin 45} = 46.7\ \text{N}$$

Work and Energy

Introduction

In the preceding chapter, we solved kinetic problems using Newton's laws of motion. From the second law of motion, we determined the acceleration of a body or a system of bodies. Once acceleration is known, we could describe the kinematics of the system, i.e., its displacement and velocity as functions of time. In this chapter, we will introduce an alternative approach, called work-energy method to solve the same type of kinetic problems.

Unlike Newton's second law of motion, which relates force and acceleration, the work-energy method relates force, velocity and displacement. The work-energy method has certain advantages over Newton's method for the following reasons: Firstly, work and energy are scalar quantities and hence, they add up algebraically; thus, avoiding the need to deal with directional aspects of force vectors under Newton's method. Secondly, the kinematics of the problem, i.e., displacement and velocity could be determined directly without knowing the acceleration of the system. Displacement and velocity are more real to understand than acceleration, which seems to be an abstract quantity. Thirdly, motion of inter connected bodies could be solved without drawing separate free-body diagrams for each body in the system.

Work Done by A Force

When a force acting on a particle causes a *displacement* of the particle, the force is then said to have done **work** on the particle. This definition of work is quite different from our daily usage of the word 'work,' which we refer to any activity involving muscular or mental effort. Consider a force $\vec{F}$ acting on a particle at A causing a displacement $\vec{s}$ (from the point A to B) in the direction of the force. We then define work done on the particle as a **product** of magnitudes of **force** and **displacement**. Mathematically, we can write this as

$$U = F\,s \ldots\ldots (44)$$

Calculation of Work

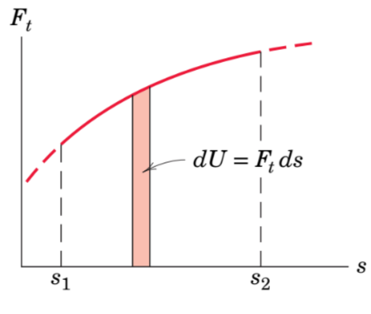
During a finite movement of the point of application of a force, the force does an amount of work equal to

$$U = \int_1^2 F\,d\mathbf{r} = \int_1^2 (F_x\,dx + F_y\,dy + F_z\,dz)$$

Or

$$U = \int_{s_1}^{s_2} F_t\,ds$$

In order to carry out this integration, it is necessary to know the relations between the force components and their respective coordinates or the relation between *Ft* and *s*. If the functional relationship is not known as a mathematical expression which can be integrated but is specified in the form of approximate or experimental data, then we can compute the work by carrying out a numerical or graphical integration as represented by the area under the curve of $\boldsymbol{F_t}$ versus ***s***, as shown in below figure.



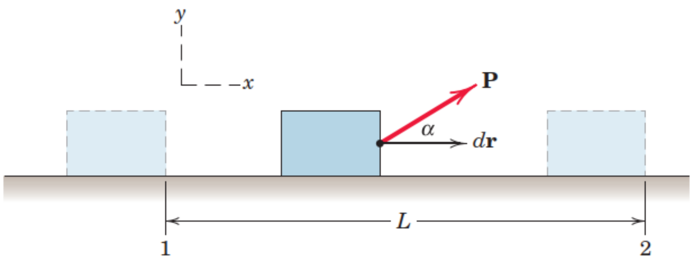
Examples of Work

When work must be calculated, we may always begin with the definition of work, $U = \int \mathbf{F}\cdot d\mathbf{r}$, insert appropriate vector expressions for the force F and the differential displacement vector $d\mathbf{r}$, and carry out the required integration. With some experience, simple work calculations, such as those associated with constant forces, may be performed by inspection. We now formally compute the work associated with three frequently occurring forces: constant forces, spring forces, and weights.

1. Work Associated with a Constant External Force. Consider the constant force **P** applied to the body as it moves from position 1 to position 2, With the force **P** and the differential displacement $d\mathbf{r}$ written as vectors, the work done on the body by the force is

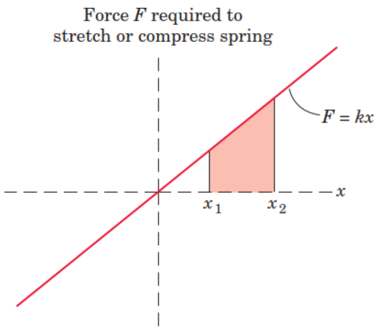
$$U_{1-2} = \int_1^2 [(P\cos\alpha)\mathbf{i} + (P\sin\alpha)\mathbf{j}]\, dx\mathbf{i}$$

$$= \int_{x_1}^{x_2} P\cos\alpha\ dx = P\cos\alpha\,(x_2 - x_1) = PL\cos\alpha \quad \ldots\ldots (45)$$

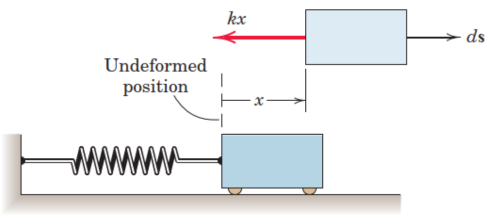


As previously discussed, this work expression may be interpreted as the force component $P\cos\alpha$ times the distance L traveled. Should α be between 90° and 270°, the work would be negative. The force component $P\sin\alpha$ normal to the displacement does no work.

2. Work Associated with a Spring Force. We consider here the linear spring of stiffness k where the force required to stretch or compress the spring is proportional to its deformation x.



(a)



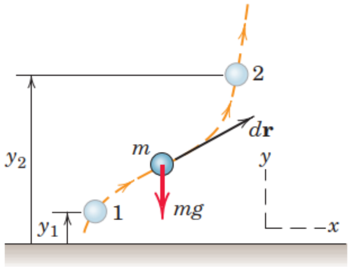
(b)

For both cases, tension or compression, the work done on the body by the spring is negative and is given by:

$$U_{1-2} = -\int_{x_1}^{x_2} F\,dx = -\int_{x_1}^{x_2} kx\,dx = \frac{-1}{2}k(x_2^2 - x_1^2) = \frac{1}{2}\,k\,(x_1^2 - x_2^2) \quad \dots\dots (46)$$

For both cases, when the spring is being released, the force exerted on the body by the spring is in the same sense as the displacement, and therefore, the work is positive.

3. Work Associated with Weight



$$U_{1-2} = \int_1^2 \mathbf{F}\,d\mathbf{r} = \int_1^2 (-mg\mathbf{i}).(dx\mathbf{i} + dy\mathbf{j})$$

$$= -mg\int_{y_1}^{y_2} dy = -mg(y_2 - y_1) \quad \dots\dots (47)$$

Principle of Work and Kinetic Energy

The *kinetic energy* T of the particle is defined as:

$$T = \frac{1}{2}\,m\,v^2 \quad \dots\dots (48)$$

and is the total work which must be done on the particle to bring it from a state of rest to a velocity ***v***. Kinetic energy ***T*** is a scalar quantity with the units of or joules (J) in SI units and ft-lb in U.S. customary units. Kinetic energy is always positive, regardless of the direction of the velocity.

the work-energy relation may be expressed as the initial kinetic energy T_1 plus the work done U_{1-2} equals the final kinetic energy T_2, or

$$T_1 + U_{1-2} = T_2 \quad \dots\dots (49)$$

Advantages of Work- Energy Method

1. No necessity of computing the acceleration.
2. Involves only forces do work.
3. Enables us to analyze a system of particles joined without dismembering the system. (interconnected member).

Power

The capacity of a machine is measured by the time rate at which it can do work or deliver energy. The total work or energy output is not a measure of this capacity since a motor, no matter how small, can deliver a large amount of energy if given sufficient time. On the other hand, a large and powerful machine is required to deliver a large amount of energy in a short period of time. Thus, the capacity of a machine is rated by its ***power***, which is defined as the *time rate of doing work*. Accordingly, the power ***P*** developed by a force ***F*** which does an amount of work ***U*** is ***P*** = dU / dt = **F·ds / dt**. Because **ds / dt** is the velocity ***v*** of the point of application of the force, we have

$$\boldsymbol{P} = \boldsymbol{F}.\boldsymbol{v} \quad \dots\dots (50)$$

Power is clearly a scalar quantity, and in SI it has the units of N.m/s = J/s. The special unit for power is the watt (W), which equals one joule per second (J/s). In U.S. customary units, the unit for mechanical power is the horsepower (hp). These units and their numerical equivalences are

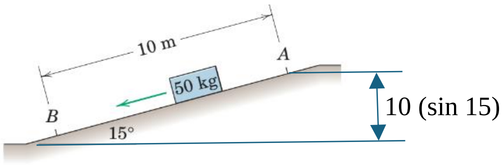
1 W = 1 J/s

1 hp = 550 ft-lb/sec = 33,000 ft-lb/min

1 hp = 746 W = 0.746 kW

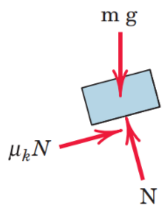
Example 11

Calculate the velocity *v* of the 50-kg crate when it reaches the bottom of the chute at *B* if it is given an initial velocity of 4 m/s down the chute at *A*. The coefficient of kinetic friction is 0.30.



Solution:

As shown on the free-body diagram



$\sum F_y = 0$

$N - mg\cos 15° = 0$

$N = mg\cos 15° = 50 \times 9.81 \times \cos 15°$

$N = 474$ N

The work done by the weight component is positive, whereas that done by the friction force is negative

$U_{A-B} = F \,.\, s = mg \times 10\sin 15° - \mu_k N \times 10$

$U_{A-B} = 50 \times 9.81 \times 10\sin 15° - 0.3 \times 474 \times 10$

$U_{A-B} = -159.1$ J

The work-energy equation gives

$T_A + U_{A-B} = T_B$

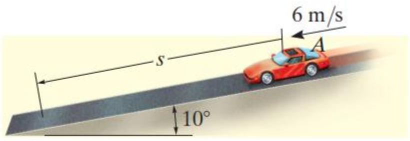
$\frac{1}{2} m v_A^2 + U_{A-B} = \frac{1}{2} m v_B^2$

$\frac{1}{2}(50) \times (4)^2 - 151.9 = \frac{1}{2}(50) v_B^2$

$v_B = 3.15$ m/s

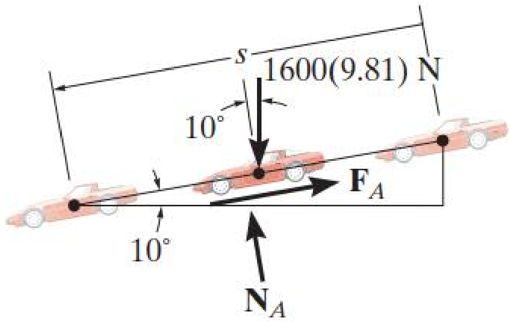
Example 12

The 1600-kg automobile shown in below figure travels down the 10° inclined road at a speed of 6 m/s. If the driver jams on the brakes, causing the wheels to lock, determine how far *s* the tires skid on the road. The coefficient of kinetic friction between the wheels and the road is 0.5.



Solution:

This problem can be solved using the principle of work and energy, since it involves force, velocity, and displacement.



$\sum F_y = 0$

$N - mg\cos 10° = 0$

$N = mg\cos 10° = 1600 \times 9.81 \times \cos 10°$

$N = 15457.7\ \text{N}$

The work done by the weight component is positive, whereas that done by the friction force is negative

$U_{A-B} = F\,.\,s = mg \times s \times \sin 10° - \mu_k\, N \times s$

$U_{A-B} = 1600 \times 9.81 \times s \times \sin 10° - 0.5 \times 15457.7 \times s = -5003.27\, s$

The work-energy equation gives

$T_A + U_{A-B} = T_B$

$\frac{1}{2}mv_A^2 + U_{A-B} = \frac{1}{2}mv_B^2$

$\frac{1}{2}(1600) \times (6)^2 - 5003.27\, s = 0$

$s = 5.76\ \text{m}$

Note: If this problem is solved by using the equation of motion, two steps are involved. First, from the free-body diagram, the equation of motion is applied along the incline to obtain the deceleration. These yields

$\sum F_x = m\, a_x \rightarrow m \times g \times \sin 10 - \mu_k\, N = m\, a_x$

$1600 \times 9.81 \times \sin 10 - 0.5 \times 15457.7 = 1600\, a_x$

$a_x = -3.127\ \text{m/s}^2$

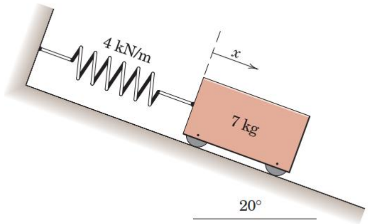
$v_B^2 = v_A^2 + 2a_x(s - s_o)$

$0 = (6)^2 + 2 \times (-3.127) \times (s - 0)$

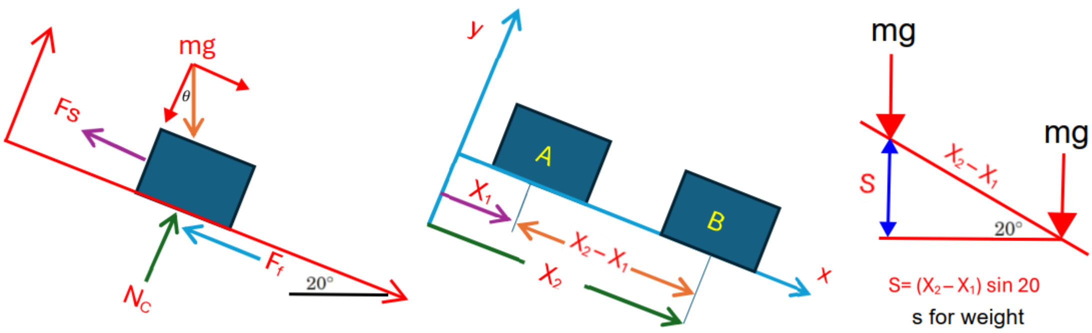
$s = 5.76\ \text{m}$

Example 13

The spring is unstretched when $x = 0$. If the body moves from the initial position $x_1 = 0.1$ m to the final position $x_2 = 0.2$ m, determine (a) the work done by the spring on the body (b) the work done on the body by its weight, and (3) the total work done.



Solution:



$k = 4000$ N/m , $x_1 = 0.1\,m$, $x_2 = 0.2\,m$

the work done by the spring on the body

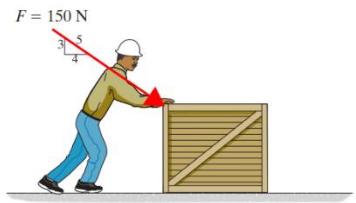
$$U_{1-2} = \frac{1}{2}\,k\,(x_1^2 - x_2^2) = \frac{1}{2}\,(4000)(0.1^2 - 0.2^2) = -60\text{ J}$$

the work done on the body by its weight

$$U_{1-2} = F\,.\,s = m \times g \times (x_2 - x_1)\sin 20 = 7 \times 9.81 \times (0.2 - 0.1)\sin 20 = 2.35\text{ J}$$

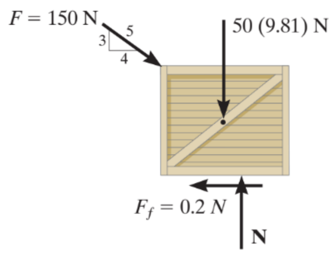
Example 14

The man in below figure pushes on the 50-kg crate with a force of F = 150 N. Determine the power supplied by the man when t = 4 s. The coefficient of kinetic friction between the floor and the crate is 0.2. Initially the create is at rest.



Solution:

To determine the power developed by the man, the velocity of the 150-N force must be obtained first. The free-body diagram of the crate is shown in below figure. Applying the equation of motion.



$$\sum F_y = m\,a_y \;;\rightarrow N - m \times g - 150\sin\theta = 0$$

$$N - 50 \times 9.81 - 150\frac{3}{5} \rightarrow N = 580.5\text{ N}$$

$$\sum F_x = m\,a_x \rightarrow 150 \times \sin\theta - \mu_k\,N = m\,a_x$$

$$150 \times \frac{4}{5} - 0.2 \times 580.5 = 50\, a_x \;\rightarrow\; a_x = 0.078\ \text{m/s}^2$$

The velocity of the crate when t = 4 s is therefore

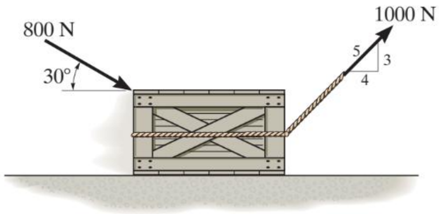
$$v_x = v_{ox} + a_x t \;\rightarrow\; v_x = 0 + 0.078 \times 4 = 0.312\ \text{m/s}$$

The power supplied to the crate by the man when *t* = 4 s is therefore

$$\boldsymbol{P} = \boldsymbol{F}.\boldsymbol{v} \;\rightarrow\; \boldsymbol{P} = \boldsymbol{F_x}.\boldsymbol{v} \;\rightarrow\; \boldsymbol{P} = 150 \times \frac{4}{5} \times 0.312 = 37.4\ W$$

Homework

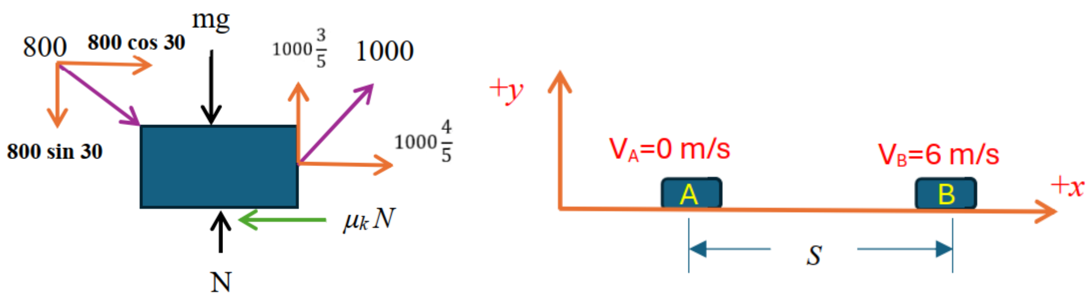
1. The crate, which has a mass of 100 kg, is subjected to the action of the two forces. If it is originally at rest, determine the distance it slides in order to attain a speed of 6 m/s. The coefficient of kinetic friction between the crate and the surface is 0.2.



Solution

$$\sum F_y = m\, a_y\ ;\ \rightarrow N + 1000 \sin\theta - m \times g - 800 \sin 30° = 0$$

$$N + 1000\frac{3}{5} - 100 \times 9.81 - 800 \sin 30° = 0 \;\rightarrow N = 781\ \text{N}$$



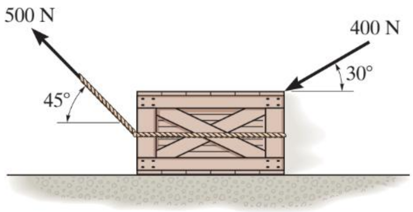
$$T_A + \sum U_{A-B} = T_B$$

$$\frac{1}{2}\, m\, v_A^2 + 1000 \cos\theta \times s + 800 \cos 30° \times s - \mu_k N.s = \frac{1}{2}\, m\, v_B^2$$

$$\frac{1}{2}(100)(0)^2 + 1000\frac{4}{5} \times s + 800 \cos 30° \times s - 0.2 \times 781 \times s = \frac{1}{2}(100)(6)^2$$

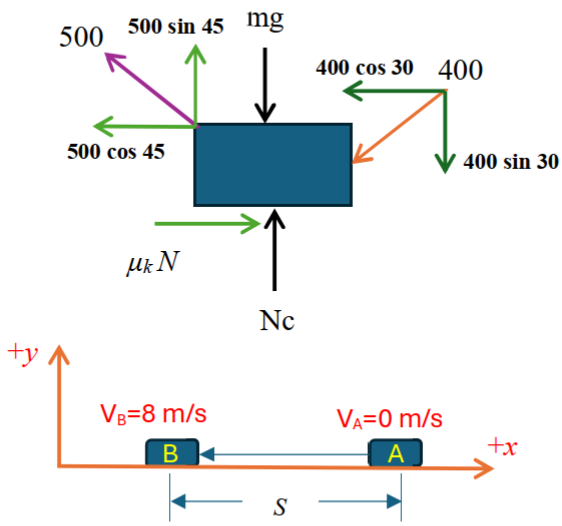
$$s = 1.35\ \text{m}$$

2. The 100-kg crate is subjected to the forces shown. If it is originally at rest, determine the distance it slides in order to attain a speed of *v* = 8 m/s. The coefficient of kinetic friction between the crate and the surface is 0.2.



Solution

Work. Consider the force equilibrium along the *y* axis by referring to the FBD of the crate



$$\sum F_y = m\,a_y \;;\; \rightarrow N + 500\sin 45° - 100 \times 9.81 - 400\sin 30° = 0 \;\rightarrow\; N = 827.45\ \text{N}$$

$$U_{f1} = 400\cos 30° \times s = 346.41\, s$$

$$U_{f2} = 500\cos 45° \times s = 353.55\, s$$

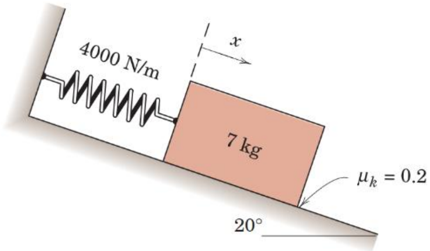
$$U_{ff} = -F_f \times s = -\mu_k N \times s = -0.2 \times 827.45 \times s = -165.49\, s$$

$$T_A + \sum U_{A-B} = T_B$$

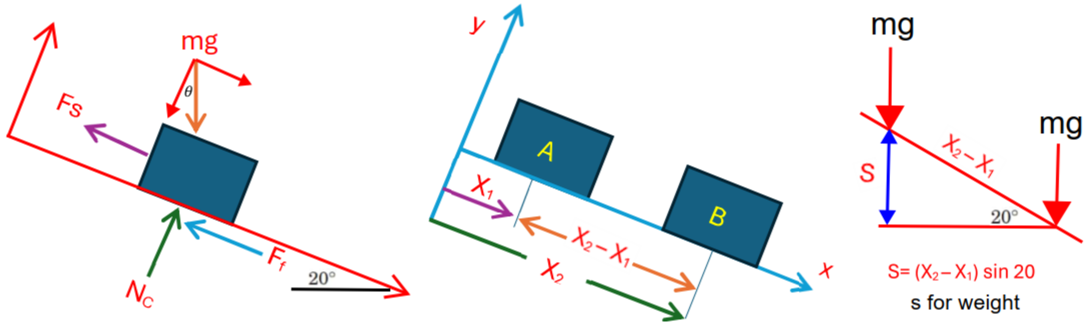
$$\frac{1}{2}\, m\, v_A^2 + U_{f1} + U_{f2} + U_{ff} = \frac{1}{2}\, m\, v_B^2$$

$$\frac{1}{2}(100)(0)^2 + 346.41\, s + 353.55\, s + (-165.49\, s) = \frac{1}{2}(100)\,(8)^2 \quad \rightarrow \quad s = 5.9\, m$$

3. The 7 kg body connected to spring with stiffness of 4000 N/m, spring is unstretched when $\boldsymbol{x} = 0$. If the body moves from the initial position $\boldsymbol{x}_1 = 0.1$ m to the final position $\boldsymbol{x}_2 = 0.3$ m, if the coefficient of kinetic friction is 0.20, determine (a) the work done by the spring on the body (b) the work done on the body by its weight, (c) Work done by friction, and (d) the total work done.



Solution:



$k = 4000 \text{ N/m}, \quad x_1 = 0.1\,m \quad , \quad x_2 = 0.3\,m$

the work done by the spring on the body

$$U_{Spring} = \frac{1}{2}\,k\,(x_1^2 - x_2^2) = \frac{1}{2}\,(4000)(0.1^2 - 0.3^2) = -160 \text{ J}$$

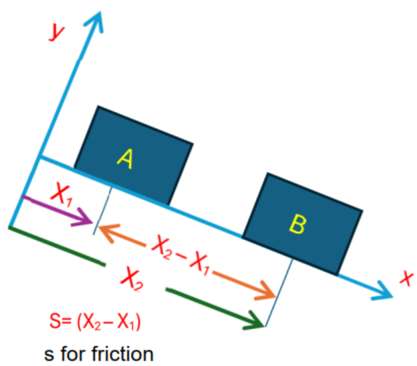
the work done on the body by its weight

$$U_{weight} = F\,.\,s = m \times g \times (x_2 - x_1)\sin 20° = 7 \times 9.81 \times (0.3 - 0.1)\sin 20° = 4.7 \text{ J}$$

Work Done by Friction

Friction opposes motion, and its work is given by:

$U_{friction} = -F_f . s$



$$\sum F_y = m\,a_y \;;\rightarrow N - m \times g \times \cos\theta = 0$$

$$N = m \times g \times \cos\theta \;\rightarrow N = 7 \times 9.81 \times \cos 20° = 64.53 \text{ N}$$

$$F_f = \mu_k\,N = 0.2 \times 64.53 = 13.1 \text{ N}$$

$$U_{friction} = -13.1 \times (0.3 - 0.1) = -2.62 \text{ J}$$

Friction does **negative work** (resisting motion).

Total Work Done.

Summing all work contributions:

$U_{total} = U_{weight} + U_{Spring} + U_{friction}$

$U_{total} = 4.71 - 160 - 2.62 = -157.91 \text{ J}$

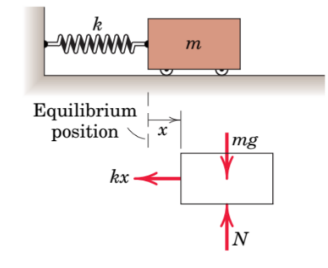
Vibrations

Free Vibration of Particles

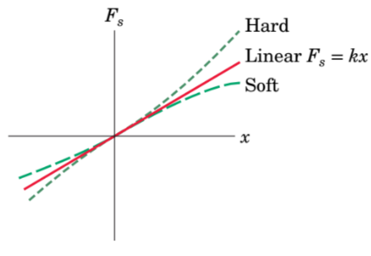
When a spring-mounted body is disturbed from its equilibrium position, its ensuing motion in the absence of any imposed external forces is termed free vibration. In every actual case of free vibration, there exists some retarding or damping force which tends to diminish the motion. Common damping forces are those due to mechanical and fluid friction. In this article we first consider the ideal case where the damping forces are small enough to be neglected. Then we treat the case where the damping is appreciable and must be accounted for.

Equation of Motion for Undamped Free Vibration

We begin by considering the horizontal vibration of the simple frictionless spring-mass system of Fig. *a*. Note that the variable x denotes the displacement of the mass from the equilibrium position, which, for this system, is also the position of zero spring deflection. Figure *b* shows a plot of the force F_s necessary to deflect the spring versus the corresponding spring deflection for three types of springs. Although nonlinear hard and soft springs are useful in some applications, we will restrict our attention to the linear spring. Such a spring exerts a restoring force - kx on the mass—that is, when the mass is displaced to the right, the spring force is to the left, and vice versa. We must be careful to distinguish between the forces of magnitude F_s which must be applied to both ends of the massless spring to cause tension or compression and the force $\boldsymbol{F = -k\,x}$ of equal magnitude which the spring exerts on the mass. The constant of proportionality k is called the *spring constant*, *modulus*, or *stiffness* and has the units N/m or lb/ft.



(*a*)



(*b*)

The equation of motion for the body of Fig. *a* is obtained by first drawing its free-body diagram. Applying Newton's second law in the form $\Sigma \boldsymbol{F}\boldsymbol{x} = \boldsymbol{m}\,\ddot{\boldsymbol{x}}$ gives

$$-kx = m\ddot{x} \quad \textbf{or} \quad m\ddot{x} + kx = 0 \quad \ldots\ldots (51)$$

The oscillation of a mass subjected to a linear restoring force as described by this equation is called simple harmonic motion and is characterized by acceleration which is proportional to the displacement but of opposite sign. The above Equation is normally written as

$$\ddot{x} + \omega_n^2 x = 0 \quad \ldots\ldots (52)$$

Where

$$\omega_n = \sqrt{k/m} \quad \ldots\ldots (53)$$

is a convenient substitution whose physical significance will be clarified shortly.

Solution for Undamped Free Vibration

Because we anticipate an oscillatory motion, we look for a solution which gives x as a periodic function of time. Thus, a logical choice is

$$x = A\cos\omega_n t + B\sin\omega_n t \quad \ldots\ldots (54)$$

or, alternatively,

$$x = C\sin(\omega_n t + \psi) \quad \ldots\ldots (55)$$

Direct substitution of these expressions into Eq. (52) verifies that each expression is a valid solution to the equation of motion. We determine the constants A and B, or C and, from knowledge of the initial displacement x_o and initial velocity $\dot{x}_o$ of the mass. For example, if we work with the solution form of Eq. (54) and evaluate x and $\dot{x}$ at time $t = 0$, we obtain

$$x_o = A \quad \text{and} \quad \dot{x}_o = B\omega_n$$

Substitution of these values of A and B into Eq. (54) yields

$$x = x_o \cos \omega_n t + \frac{\dot{x}_o}{\omega_n} \sin \omega_n t \qquad \ldots\ldots (56)$$

The constants C and ψ of Eq. (55) can be determined in terms of given initial conditions in a similar manner. Evaluation of Eq. (55) and its first time derivative at $t = 0$ gives

$$x_o = C \sin \psi \quad and \quad \dot{x}_o = C\omega_n \cos \psi$$

Solving for C and ψ yields

$$C = \sqrt{x_o^2 + (\dot{x}_o/\omega_n)^2} \qquad \psi = \tan^{-1}(x_o\omega_n/\dot{x}_o)$$

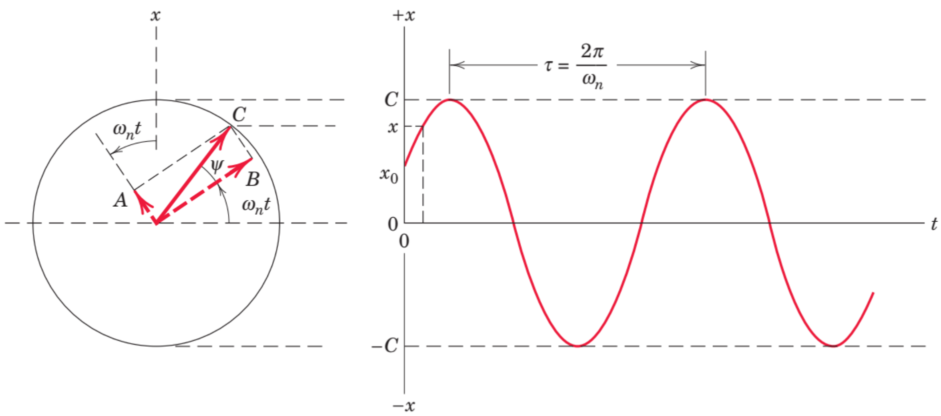
Substitution of these values into Eq. (55) gives

$$x = \sqrt{x_o^2 + (\dot{x}_o/\omega_n)^2} \, \sin[\,\omega_n t + \tan^{-1}(x_o\omega_n/\dot{x}_o)] \quad \ldots\ldots (57)$$

Equations (56) and (57) represent two different mathematical expressions for the same time-dependent motion. We observe that $C = \sqrt{A^2 + B^2}$ and $\psi = \tan^{-1}(A/B)$.

Graphical Representation of Motion

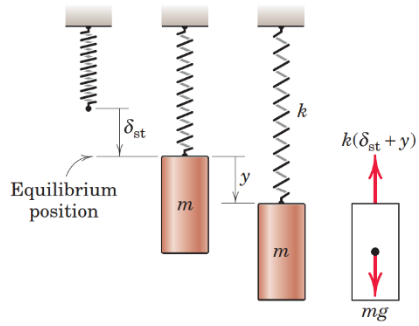
The motion may be represented graphically, as shown in below figure, where x is seen to be the projection onto a vertical axis of the rotating vector of length C. The vector rotates at the constant angular velocity $\omega_n = \sqrt{k/m}$, which is called the *natural circular frequency* and has the units radians per second. The number of complete cycles per unit time is the *natural frequency* $f_n = \omega_n/2\pi$ and is expressed in hertz (1 hertz (Hz) = 1 cycle per second). The time required for one complete motion cycle (one rotation of the reference vector) is the *period* of the motion and is given by $\tau = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$.



We also see from the figure that x is the sum of the projections onto the vertical axis of two perpendicular vectors whose magnitudes are A and B and whose vector sum C is the *amplitude*. Vectors A, B, and C rotate together with the constant angular velocity ω_n. Thus, as we have already seen, $\sqrt{A^2 + B^2}$ and $\psi = \tan^{-1}(A/B)$

Equilibrium Position as Reference

As a further note on the free undamped vibration of particles, we see that, if the system in the below figure, where the motion is vertical direction. The equilibrium position now involves a nonzero spring deflection δ_{st}. From the free-body diagram, Newton's second law gives:



$$-k(\delta_{st} + y) + mg = m\,\ddot{y}$$

At the equilibrium position $x = 0$, the force sum must be zero, so that

$$-k\delta_{st} + mg = 0$$

Thus, we see that the pair of forces $-k\delta_{st}$ and mg on the left side of the motion equation cancel, giving

$$m\,\ddot{y} + ky = 0$$

which is identical to Eq. (51).

The equation of motion will be given as:

$$\ddot{y} + \omega_n^2 y = 0$$

Where ω_n is ***natural circular frequency*** and has the units radians per second and given as:

$$\omega_n = \sqrt{k/m}$$

The solution of the above differential equation is given as:

$$y = y_o \cos \omega_n t + \frac{\dot{y}_o}{\omega_n} \sin \omega_n t$$

Where

y_o is the initial position in (m) and

$\dot{y}_o$ is the initial velocity in (m/s)

The number of complete cycles per unit time is the ***natural frequency*** which expressed in hertz (1 hertz (Hz) = 1 cycle per second)

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega_n}{6.28}$$

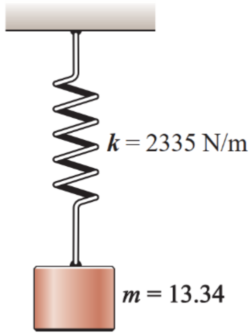
The time required for one complete motion cycle (one rotation of the reference vector) is the ***period*** of the motion and is given by

$$\tau = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = \frac{6.28}{\omega_n}$$

Example 15

A body weighing 11.34 kg is suspended from a spring of constant k = 2335 N/m. At time t = 0, it has a downward velocity of 0.6 m/s as it passes through the position of static equilibrium. Determine

(a) The equation of motion.
(b) The static spring deflection δ_{st}
(c) The natural frequency of the system in both rad/sec (ω_n) and cycles/sec (f_n)
(d) The system period τ
(e) The displacement y, velocity $\dot{y}$ and acceleration $\ddot{y}$ as a function of time, where y is measured from the position of static equilibrium.
(f) The maximum velocity $\dot{y}_{max}$ and acceleration $\ddot{y}_{max}$ attained by the mass.



Solution

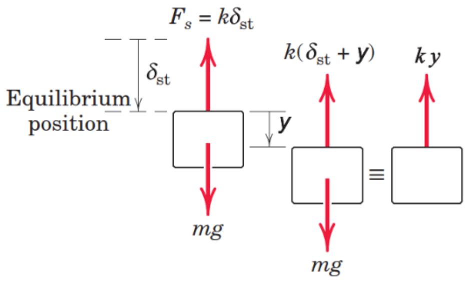
(a) the equation of motion will be given as:

$$\ddot{y} + \omega_n^2 y = 0$$

(b) at equilibrium position the static deflection given as:

$$\delta_{st} = \frac{mg}{k} = \frac{11.34 \times 9.81}{2335} = 0.0476\ m$$

(c) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2335}{11.34}} = 14.35\ \text{rad/s}$



$$f_n = \frac{\omega_n}{2\pi} = \frac{14.35}{2\pi} = 2.28\ Hz\ or\ \text{Cycles/s}$$

(d) $\tau = \frac{1}{f_n} = \frac{1}{2.28} = 0.438\ \text{s}$

(e) $y = y_o \cos \omega_n t + \frac{\dot{y}_o}{\omega_n} \sin \omega_n t$

$$y = 0\cos(14.35t) + \frac{0.6}{14.35}\sin(14.35t)$$
$$y = 0.0418\sin(14.35t)$$
$$\dot{y} = 0.0418 \times 14.35\cos(14.35t)$$
$$\dot{y} = 0.5998\cos(14.35\,t)$$
$$\ddot{y} = -0.5998 \times 14.35\sin(14.35\,t)$$
$$\ddot{y} = -8.607\ \sin(14.35\,t)$$

(f) the maximum velocity

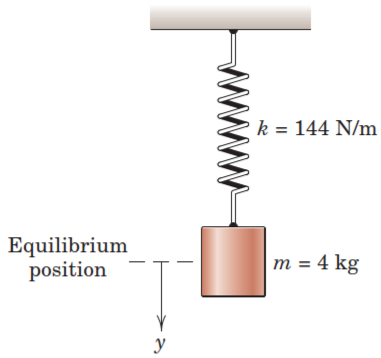
$$\dot{y}_{max} = 0.5998\ \text{m/s}$$

the maximum acceleration

$$\ddot{y}_{max} = -8.607\ \text{m/s}^2$$

Example 16

For the spring-mass system shown, the cylinder is displaced 0.1 m downward from its equilibrium position and is released at time determine (a) the equation of motion, (b) the static deflection δ_{st}, (c) the natural frequency of the system in both rad/sec (ω_n) and cycles/sec (f_n), (d) the system period τ, (e) The displacement y, velocity $\dot{y}$ and acceleration $\ddot{y}$ as a function of time, where y is measured from the position of static equilibrium, and (f) the maximum velocity and acceleration?



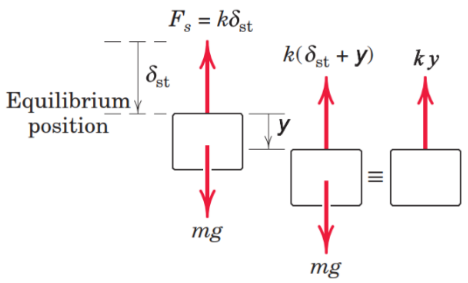
Solution

(a) the equation of motion will be given as:

$$\ddot{y} + \omega_n^2 y = 0$$

(b) at equilibrium position the static deflection given as:

$$\delta_{st} = \frac{mg}{k} = \frac{4 \times 9.81}{144} = 0.2725\, m$$



(c) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{144}{4}} = 6 \text{ rad/s}$

$$f_n = \frac{\omega_n}{2\pi} = \frac{6}{2\pi} = 0.955 \text{ Cycles/s}$$

(d) $\tau = \frac{1}{f_n} = \frac{1}{0.955} = 1.047 \text{ s}$

(e) $y = y_o \cos\omega_n t + \frac{\dot{y}_o}{\omega_n} \sin\omega_n t$

$$y = 0.1\cos 6t + \frac{0}{6}\sin 6t$$

$$y = 0.1\cos(6t)$$

$$\dot{y} = -0.1\sin(6t) \times 6$$

$$\dot{y} = -0.6\sin(6t)$$

$$\ddot{y} = -0.6\cos(6t) \times 6$$

$$\ddot{y} = -3.6\cos(6t)$$

(f) The maximum velocity

$$\dot{y}_{max} = 0.6\, m/s$$

The maximum acceleration

$$\ddot{y}_{max} = 3.6 \text{ m/s}^2$$