



*Lectures in*

# **MATHEMATICS**

*For 2<sup>nd</sup> class students*

## Chapter One

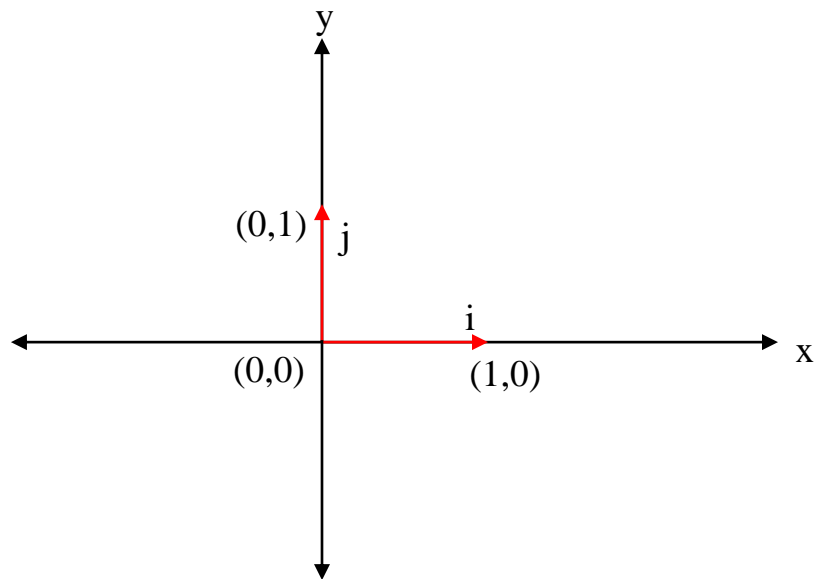
### Vector

#### Vector components:

Quantities can be divided into:

- 1) scalar: which is defined by magnitude only, for example mass, time...
- 2) vector: the quantity which needs magnitude and direction, for example force, velocity, acceleration....

We shall denote the vector from  $(0,0)$  to  $(1,0)$  by  $(i)$  and the vector from  $(0,0)$  to  $(0,1)$  by  $(j)$  as shown in figure below:



Then any vector in the  $x - y$  plane can be divided in the terms of  $i$  and  $j$ .

**Ex. Sketch the following vectors:**

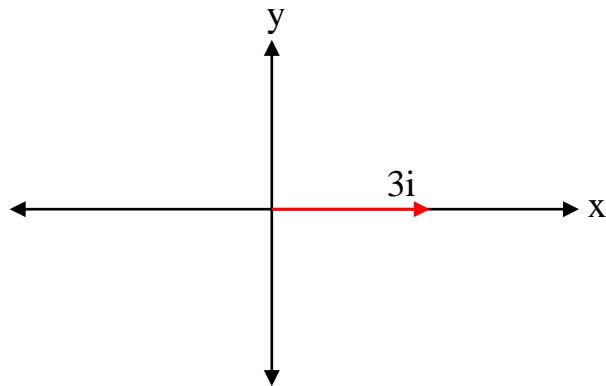
a)  $3i$

b)  $-4i$

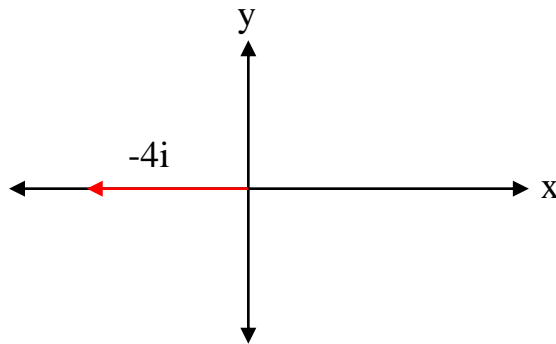
c)  $-2i + 3j$

**Sol.:**

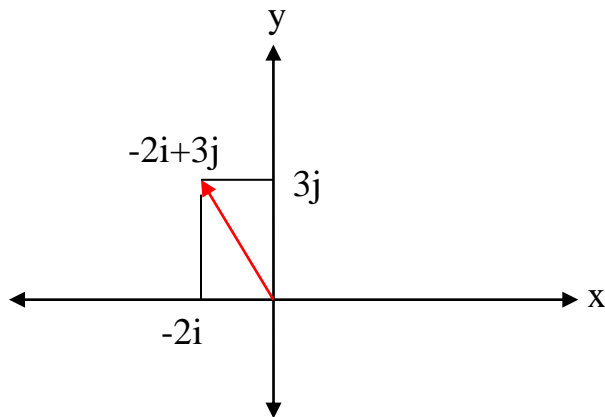
a)



b)



c)



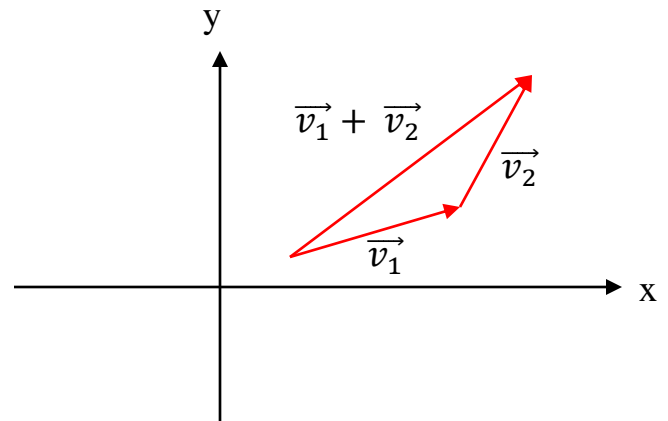
**Note:** when we say the vector  $\overrightarrow{AB}$  means: This vector directed from **point A** to **point B**.

### **Arithmetic operation on vector:**

#### **1) Addition:**

$$\text{Let } \vec{v}_1 = a_1i + b_1j \text{ and } \vec{v}_2 = a_2i + b_2j$$

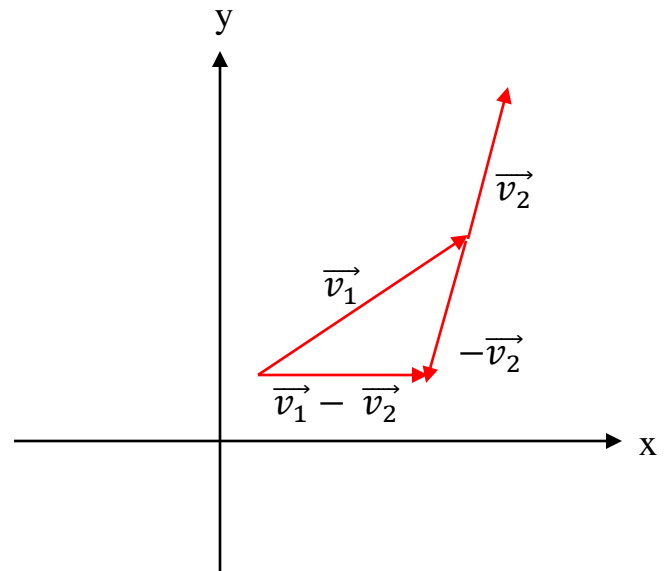
$$\begin{aligned} \text{Then } \vec{v}_1 + \vec{v}_2 &= a_1i + b_1j + a_2i + b_2j \\ &= (a_1 + a_2)i + (b_1 + b_2)j \end{aligned}$$



#### **2) Subtraction:**

$$\text{Let } \vec{v}_1 = a_1i + b_1j \text{ and } \vec{v}_2 = a_2i + b_2j$$

$$\begin{aligned} \text{Then } \vec{v}_1 - \vec{v}_2 &= (a_1i + b_1j) - (a_2i + b_2j) \\ &= (a_1 - a_2)i + (b_1 - b_2)j \end{aligned}$$



#### **3) Multiplication of vector by scalar:**

Let  $\vec{v} = ai + bj$  and  $c$  is scalar.

$$\text{then } c\vec{v} = c(ai + bj) = cai + cbj$$

## **Definitions:**

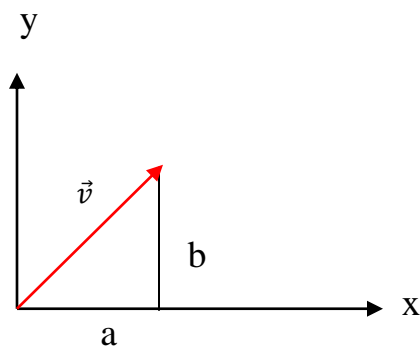
### **1) Length of vector:**

The length of vector  $\vec{v} = ai + bj$  is usually denoted by  $|\vec{v}|$  which may be read as "The magnitude of  $\vec{v}$ ".

$|\vec{v}|$  is found using Pythagoras theorem and as follows:

$$\vec{v} = ai + bj$$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$



### **2) Zero vector:**

Any vector of length zero is called zero vector  $\vec{0}$ ,

$$ai + bj = 0$$

$ai + bj = 0i + 0j$  if and only if  $a = b = 0$ .

### **3) Unit vector:**

It is part from any vector. This part has length equal to unity and it is used to describe the direction of the vector.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} \quad \text{where } \vec{u} \text{ is unit vector of } \vec{v}.$$

**Ex. Find the unit vector of  $\vec{v} = ai + bj$  and prove that unit vector depends on the angle between the vector and the x - axis?**

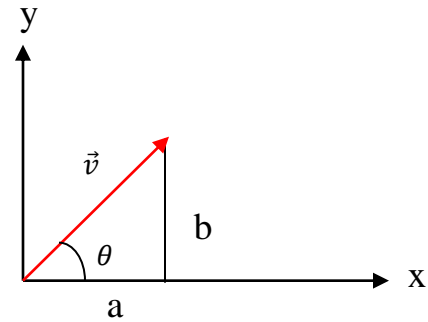
**Sol.:**

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{u} = \frac{ai + bj}{\sqrt{a^2 + b^2}}$$

$$\vec{u} = \frac{a}{\sqrt{a^2 + b^2}}i + \frac{b}{\sqrt{a^2 + b^2}}j$$

$$\vec{u} = \cos\theta i + \sin\theta j$$



**Ex. Find the unit vector of  $\vec{A} = 3i + 4j$ .**

**Sol.:**

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|}$$

$$|\vec{A}| = \sqrt{3^2 + 4^2} = 5$$

$$\vec{u} = \frac{3i+4j}{5} = 0.6i + 0.8j$$

since

$$\vec{u} = \cos\theta i + \sin\theta j$$

$$\cos\theta = 0.6$$

$$\therefore \theta = 53^\circ$$

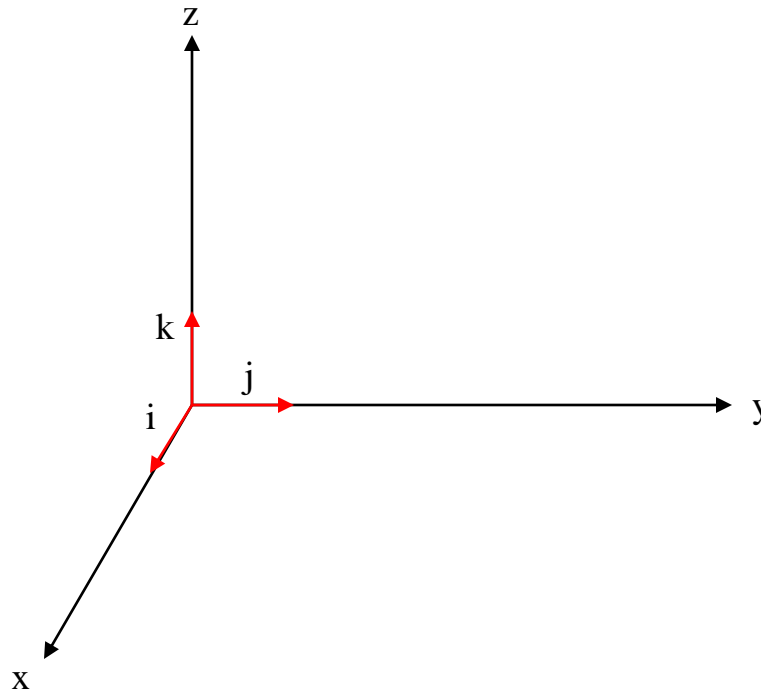
## Vector in Space:

Now we shall consider the vector in three dimensional space as follows:

(i) as a vector pointing from (0,0,0) to (1,0,0)

(j) as a vector pointing from (0,0,0) to (0,1,0)

(k) as a vector pointing from (0,0,0) to (0,0,1)



Any vector  $\vec{A}$  for example may be represented as:

$$\vec{A} = ai + bj + ck$$

and

$$|\vec{A}| = \sqrt{a^2 + b^2 + c^2}$$

**Ex.: Find a unit vector in the direction of vector from  $p_1(1,0,1)$  and  $p_2(3,2,0)$ .**

**Sol.:**

$$\overrightarrow{p_1p_2} = (3 - 1)i + (2 - 0)j + (0 - 1)k$$

$$\overrightarrow{p_1p_2} = 2i + 2j - k$$

$$|\overrightarrow{p_1p_2}| = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\vec{u} = \frac{\overrightarrow{p_1p_2}}{|\overrightarrow{p_1p_2}|} = \frac{2i+2j-k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

**Scalar product (dot product):**

The scalar or dot product of two vectors  $\vec{A}$  and  $\vec{B}$ , denoted by  $\vec{A} \cdot \vec{B}$  (read  $\vec{A}$  dot  $\vec{B}$ ), is defined as the product of the magnitudes of  $\vec{A}$  and  $\vec{B}$  and the cosine of the angle  $\theta$  between them.

$$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\theta, \quad 0 \leq \theta \leq \pi$$

The following laws was valid:

1.  $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
2.  $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
3.  $i \cdot i = j \cdot j = k \cdot k = 1, i \cdot j = j \cdot k = k \cdot i = 0$

The dot product can be used to find:

- 1) The angle between two vectors.
- 2) The projection of vector  $\vec{B}$  on  $\vec{A} = |\vec{B}| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|}$
- 3) The projection of  $\vec{A}$  on  $\vec{B} = |\vec{A}| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$

**Ex. Find the angle between  $\vec{A} = i - 2j - 2k$  and  $\vec{B} = 6i + 3j + 2k$  also find the projection of  $\vec{A}$  on  $\vec{B}$  and  $\vec{B}$  on  $\vec{A}$ .**

**Sol.:**

$$1) \vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cos\theta$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| \cdot |\vec{B}|}$$

$$\vec{A} \cdot \vec{B} = 1*6 + (-2)*3 + (-2)*2 = -4$$

$$|\vec{A}| = \sqrt{1 + 4 + 4} = 3$$

$$|\vec{B}| = \sqrt{36 + 9 + 4} = 7$$

$$\cos\theta = \frac{-4}{3*7} = \frac{-4}{21}$$

$$\theta = 101^\circ$$

$$2) \text{ The projection of } \vec{A} \text{ on } \vec{B} = |\vec{A}| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|} = \frac{-4}{7}$$

$$3) \text{ The projection of vector } \vec{B} \text{ on } \vec{A} = |\vec{B}| \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|} = \frac{-4}{3}$$

### **Orthogonal vectors:**

The two vectors  $\vec{A}$  and  $\vec{B}$  are orthogonal if and only if:

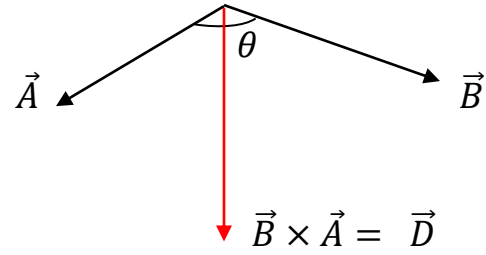
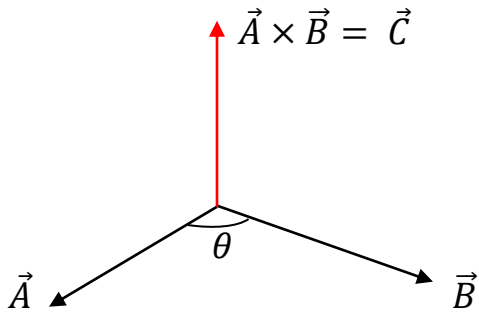
$$\vec{A} \cdot \vec{B} = 0$$

### **The cross product (vector product):**

The vector product or cross product of vector  $\vec{A}$  and  $\vec{B}$  is defined as:

$$\vec{A} \times \vec{B} = \vec{n} |\vec{A}| |\vec{B}| \sin\theta$$

The result of  $\vec{A} \times \vec{B}$  is vector perpendicular to the plane containing  $\vec{A}$  and  $\vec{B}$ , and the direction determined by right hand rule.



The following laws are valid:

1.  $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
2.  $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
3.  $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$
4.  $\vec{i} \times \vec{k} = -\vec{j}, \vec{k} \times \vec{i} = \vec{j}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{j} = -\vec{i}, \vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{i} = -\vec{k}$

**Ex.:**  $\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$  and  $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ .

**Find  $\vec{A} \times \vec{B}$**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

**Ex.:** Find the area of the triangle whose vertices are  $A(1,-1,0)$ ,  $B(2,1,-1)$  and  $C(-1, 1,2)$ .

**Sol.:**

$$|\vec{AB} \times \vec{AC}| = \text{area of parallelogram}$$

$$\frac{1}{2}|\vec{AB} \times \vec{AC}| = \text{area of triangle}$$

$$\overrightarrow{AB} = (2 - 1)\mathbf{i} + (-1 - 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{AC} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = 6\mathbf{i} + 6\mathbf{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{36 + 36} = 6\sqrt{2}$$

$$\text{Area of triangle} = \frac{1}{2} * 6\sqrt{2} = 3\sqrt{2}$$

## Chapter Two

### Complex numbers

Complex numbers are some of the most general numbers used in algebra. Any number that can be expressed in the form  $\mathbf{a} + \mathbf{bi}$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are real numbers and  $\mathbf{i} = \sqrt{-1}$  is a complex number.

complex number can be thought of as a two dimensional vector  $(\mathbf{a}, \mathbf{b})$ , where  $\mathbf{a}$  is the **real part** and  $\mathbf{b}$  is the **imaginary part**.

#### Complex number algebra:

A number is real when the coefficient of  $i$  is zero and is imaginary when real part is zero.

For example  $3 + 4i$

$3 + 0i = 3$  is real,  $0 + 4i = 4i$  is imaginary

#### Addition and subtraction:

Addition of complex numbers is defined by separately adding real and imaginary parts; so if

$$\mathbf{z} = \mathbf{a} + \mathbf{bi} \ , \ \mathbf{w} = \mathbf{c} + \mathbf{di}$$

then

$$\mathbf{z} + \mathbf{w} = (\mathbf{a} + \mathbf{bi}) + (\mathbf{c} + \mathbf{di}) = (\mathbf{a} + \mathbf{c}) + (\mathbf{b} + \mathbf{d})\mathbf{i}$$

Similarly for subtraction:  $\mathbf{z} - \mathbf{w} = (\mathbf{a} + \mathbf{bi}) - (\mathbf{c} + \mathbf{di}) = (\mathbf{a} - \mathbf{c}) + (\mathbf{b} - \mathbf{d})\mathbf{i}$

**Ex.1: Express each of the following in term of  $x + yi$ .**

**a.  $(3 + 5i) + (2 - 3i)$**

**b.  $(3 + 5i) + 6$**

**c.  $7i - (4 + 5i)$**

**Sol.:**

a.  $(3 + 5i) + (2 - 3i) = (3 + 2) + (5 - 3)i = 5 + 2i$

b.  $(3 + 5i) + 6 = (3 + 6) + 5i = 9 + 5i$

c.  $7i - (4 + 5i) = -4 + (7 - 5)i = -4 + 2i$

**Multiplication:**

Multiplication is straightforward provided remember that  $i^2 = -1$ .

So if:  $z = (a + bi)$  and  $w = (c + di)$

then:

$$z*w = (a + bi)(c + di) = (ac - bd) + (ad + cb)i$$

**Ex.2: Simply the following in the form  $x + yi$ .**

**$(2 - 7i)(3 + 4i)$**

**Sol.:**

$$(2 - 7i)(3 + 4i) = 6 + 8i - 21i + 28 = 34 - 13i$$

### **Division:**

The complex conjugate of a complex number is obtained by changing the sign of the imaginary part . So if  $z = a + bi$  , its complex conjugate,  $\bar{z}$  , is defined by:  $\bar{z} = a - bi$

Any complex number  $a + bi$  has a complex conjugate  $a - bi$ .

**Ex.3: Simplify the following expression in the form  $x + yi$ .**

a.  $\frac{3}{1+i}$

b.  $\frac{4+7i}{2+5i}$

c.  $\frac{5}{-3+4i}$

**Sol.:**

$$a. \frac{3}{1+i} = \frac{3}{1+i} * \frac{1-i}{1-i} = \frac{3-3i}{1+1} = \frac{3-3i}{2} = \frac{3}{2} - \frac{3}{2}i$$

$$b. \frac{4+7i}{2+5i} = \frac{4+7i}{2+5i} * \frac{2-5i}{2-5i} = \frac{43-6i}{4+25} = \frac{43-6i}{29} = \frac{43}{29} - \frac{6}{29}i$$

$$c. \frac{5}{-3+4i} = \frac{5}{-3+4i} * \frac{-3-4i}{-3-4i} = \frac{5(-3-4i)}{(-3+4i)(-3-4i)} = \frac{-15-20i}{9+16} =$$

$$\frac{-15-20i}{25} = -\frac{3}{5} - \frac{4}{5}i$$

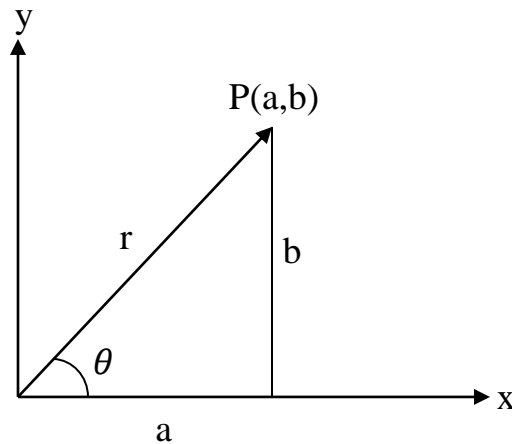
### Argand Diagrams:

There are two geometric representation of the complex number  $z = x + yi$  .

a) as the point  $P(a,b)$  in the  $xy$  - plane , and

b) as the vector  $\overrightarrow{OP}$  from the origin to  $P$ .

In each representation, the  $x$ -axis is called the real axis and the  $y$ - axis is the imaginary axis, as following figure.



In terms of the polar coordinates of  $x$  and  $y$  , we have:

$$a = r\cos\theta, \quad b = r\sin\theta, \quad \tan\theta = \frac{b}{a}$$

$$z = r(\cos\theta + i\sin\theta) \text{ (polar representation)}$$

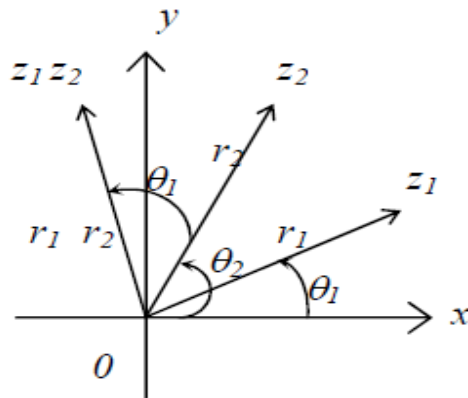
**The length  $r$  of a vector  $\overrightarrow{OP}$  from the origin to  $P(a,b)$  is:**

$$r = |a + bi| = \sqrt{a^2 + b^2}$$

The polar angle  $\theta$  is called the argument of  $z$  and is written  $\theta = \operatorname{arg}z$

The identity  $e^{i\theta} = \cos\theta + i\sin\theta$  is used for calculating products, quotients, powers, and roots of complex numbers. Then  $z = re^{i\theta}$  (**exponential representation**) (**Euler's Formula**).

**a) Product: To multiply two complex numbers (figure below):**



$$z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2} \text{ so that } |z_1| = r_1, \text{ arg}z_1 = \theta_1$$

$$|z_2| = r_2, \text{ arg}z_2 = \theta_2$$

$$\text{Then } z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\text{arg}(z_1 z_2) = \theta_1 + \theta_2 = \text{arg}z_1 + \text{arg}z_2$$

**b. Quotients:**  $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

$$\text{hence } \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \text{ and } \text{arg} \left( \frac{z_1}{z_2} \right) = \theta_1 - \theta_2 = \text{arg}z_1 - \text{arg}z_2$$

**c) Powers: If n is a positive integer, then:**

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \text{ hence } |z^n| = r^n \text{ and } \text{arg}z^n = n\theta$$

**De Moivre's Theorem** :  $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

**d) Roots:** If  $z = re^{i\theta}$  is a complex number different from zero and  $n$  is a positive integer, then there are precisely  $n$  different complex numbers  $w_0, w_1, w_2, \dots, w_{n-1}$ , that are  $n$ th roots of  $z$  given by:

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r} e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)}, k = 0, 1, 2, 3, \dots, n-1$$

**Ex.4:** Let  $z_1 = 1 + i$  and  $z_2 = \sqrt{3} - i$  find:

- 1) the exponential representation for  $z_1$  and  $z_2$ .
- 2) the values of  $z_1 z_2$  and  $\frac{z_1}{z_2}$  in exponential and polar representations.

**Sol.:**

$$\begin{aligned} 1) z_1 = 1 + i, r_1 &= \sqrt{1+1} = \sqrt{2} \\ \tan\theta_1 &= \frac{y_1}{x_1} \implies \theta_1 = \tan^{-1} \frac{y_1}{x_1} = \tan^{-1} 1 \\ \therefore z_1 &= r_1 e^{i\theta_1} = \sqrt{2} e^{i\frac{\pi}{4}} \end{aligned}$$

$$\begin{aligned} z_2 = \sqrt{3} - i, r_2 &= \sqrt{3+1} = \sqrt{4} = 2 \\ \tan\theta_2 &= \frac{y_2}{x_2} \implies \theta_2 = \tan^{-1} \frac{y_2}{x_2} = \tan^{-1} \frac{-1}{\sqrt{3}} = -\frac{\pi}{6} \\ \therefore z_2 &= r_2 e^{i\theta_2} = 2e^{-i\frac{\pi}{6}} \end{aligned}$$

$$2) z_1 z_2 = \sqrt{2} e^{i\frac{\pi}{4}} \cdot 2e^{-i\frac{\pi}{6}} = 2\sqrt{2} e^{i\frac{\pi}{12}} \quad \text{(exponential representation)}$$

$$\begin{aligned} r &= 2\sqrt{2}, \quad \theta = \frac{\pi}{12} \\ z_1 z_2 &= 2\sqrt{2} \left( \cos \frac{\pi}{12} + i\sin \frac{\pi}{12} \right) \quad \text{(Polar representation)} \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}} e^{i\frac{5}{12}\pi} \quad (\text{exponential representation})$$

$$r = \frac{1}{\sqrt{2}}, \theta = \frac{5}{12}\pi$$

$$\frac{z_1}{z_2} = \frac{1}{\sqrt{2}} \left( \cos \frac{5}{12}\pi + i \sin \frac{5}{12}\pi \right) \quad (\text{Polar representation})$$

**Ex.5: Find:  $(\sqrt{3} - i)^{10}$**

**Sol.:**

$$\sqrt{3} - i \longrightarrow r = \sqrt{3+1} = \sqrt{4} = 2 \text{ and } \theta = \tan^{-1} \frac{y}{x}$$

$$\therefore \tan^{-1} \frac{-1}{\sqrt{3}} = -\frac{\pi}{6}$$

$$\sqrt{3} - i = 2 \left( \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right)$$

$$(\sqrt{3} - i)^{10} = 2^{10} \left( \cos 10 \frac{\pi}{6} - i \sin 10 \frac{\pi}{6} \right) = 512 + 512\sqrt{3}i$$

**Ex.6: Find the four fourth roots of (-16).**

**Sol.:**

$$z = -16 \longrightarrow r = \sqrt{(-16)^2 + 0^2} = 16$$

$$\tan \theta = \frac{y}{x} \longrightarrow \theta = \tan^{-1} \frac{0}{-16} = \pi$$

$$\sqrt[4]{-16} = \sqrt[4]{16} e^{i\left(\frac{\pi}{4} + k\frac{2\pi}{4}\right)} = 2e^{i\left(\frac{\pi}{4} + k\frac{\pi}{2}\right)}, k = 0, 1, 2, 3$$

$$\text{at } k = 0 \longrightarrow \text{first root } w_0 = 2e^{i\frac{\pi}{4}} = 2 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = \sqrt{2} + \sqrt{2}i$$

$$\text{at } k = 1 \longrightarrow \text{second root } w_1 = 2e^{i\frac{3\pi}{4}} = -\sqrt{2} + \sqrt{2}i$$

at  $k = 2$   $\longrightarrow$  third root  $w_2 = 2e^{i\frac{5\pi}{4}} = -\sqrt{2} - \sqrt{2}i$

at  $k = 3$   $\longrightarrow$  fourth root  $w_3 = 2e^{i\frac{7\pi}{4}} = \sqrt{2} - \sqrt{2}i$

## Chapter Three

### Partial Derivatives

#### Functions of Two or More Variables:

Functions which have more than one variable arise very commonly. Simple example:

- Formula for the area of a triangle  $A = 1/2(bh)$  is a function of the two variables, base  $b$  and height  $h$ .

You should be used to the notation  $y = f(x)$  for a function of one variable, and that the graph of  $y = f(x)$  is a curve.

For functions of two variables the notation simply becomes:  $z = f(x,y)$

Where the two **independent** variables are  $x$  and  $y$ , while  $z$  is the **dependent** variable.

Since  $z = f(x, y)$  is a function of two variables, if we want to differentiate we have to decide whether we are differentiating with respect to  $x$  or with respect to  $y$  (the answers are different). A special notation is used. We use the symbol  $\partial$  instead of  $d$  and introduce the partial derivatives of  $z$ , which are:

- $\frac{\partial z}{\partial x}$  is read as partial derivative of  $z$  with respect to  $x$ , holding  $y$  constant.
- $\frac{\partial z}{\partial y}$  means differentiate with respect to  $y$  holding  $x$  constant.

**Ex.: Calculate  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  of the following functions:**

**a.  $z = x^2 + 3xy + y - 1$**

**b.  $z = \ln(x^2 - y)$**

**c.  $z = x\cos(y) + ye^x$**

**d.  $z = y\sin(xy)$**

**Sol.:**

**a.** To find  $\frac{\partial z}{\partial x}$  treat  $y$  as a constant and differentiate with respect to  $x$ . We

have  $z = x^2 + 3xy + y - 1$  so:

$$\frac{\partial z}{\partial x} = 2x + 3y$$

Similarly

$$\frac{\partial z}{\partial y} = 3x + 1$$

**b.**  $z = \ln(x^2 - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{(x^2 - y)} 2x = \frac{2x}{(x^2 - y)}, \quad \frac{\partial z}{\partial y} = \frac{-1}{(x^2 - y)}$$

**c.**  $z = x \cos(y) + ye^x$

$$\frac{\partial z}{\partial x} = \cos(y) + ye^x, \quad \frac{\partial z}{\partial y} = -x \sin(y) + e^x$$

**d.**  $z = y \sin(xy)$

$$\frac{\partial z}{\partial x} = y \cos(xy) y = y^2 \cos(xy)$$

$$\frac{\partial z}{\partial y} = x y \cos(xy) + \sin(xy)$$

### **Functions of Three or More Variables:**

The general notation would be something like:

$$w = f(x, y, z)$$

where  $x$ ,  $y$  and  $z$  are the independent variables.

**Ex.:** Calculate  $\frac{\partial}{\partial x}$ ,  $\frac{\partial}{\partial y}$  and  $\frac{\partial}{\partial z}$  of the following functions:

**a.**  $w = x \sin(y + 3z)$

**b.**  $u(x, y, t) = 75 + 300e^{-0.2t} \cos(x) \cosh(y)$

**c.**  $w = \tan(x^2 y^4 + e^{z^3})$

**Sol.:**

**a.**  $w = x\sin(y + 3z)$

$$\frac{\partial w}{\partial x} = \sin(y + 3z), \quad \frac{\partial w}{\partial y} = x\cos(y + 3z), \quad \frac{\partial w}{\partial z} = 3x\cos(y + 3z)$$

**b.**  $u(x, y, t) = 75 + 300e^{-0.2t} \cos(x) \cosh(y)$

$$\frac{\partial u}{\partial x} = -300e^{-0.2t} \sin(x) \cosh(y)$$

$$\frac{\partial u}{\partial y} = 300e^{-0.2t} \cos(x) \sinh(y)$$

$$\frac{\partial u}{\partial t} = -60e^{-0.2t} \cos(x) \cosh(y)$$

**c.**  $w = \tan(x^2y^4 + e^{z^3})$

$$\frac{\partial w}{\partial x} = 2xy^4 \sec^2(x^2y^4 + e^{z^3})$$

$$\frac{\partial w}{\partial y} = 4x^2y^3 \sec^2(x^2y^4 + e^{z^3})$$

$$\frac{\partial w}{\partial z} = 3z^2 e^{z^3} \sec^2(x^2y^4 + e^{z^3})$$

### **Second order partial derivatives:**

Again, let  $z = f(x, y)$  be a function of  $x$  and  $y$ .

- $\frac{\partial^2 z}{\partial x^2}$  means the second derivative with respect to  $x$  holding  $y$  constant.
- $\frac{\partial^2 z}{\partial y^2}$  means the second derivative with respect to  $y$  holding  $x$  constant.
- $\frac{\partial^2 z}{\partial x \partial y}$  means differentiate first with respect to  $y$  and then with respect to  $x$ .

The "mixed" partial derivative  $\frac{\partial^2 z}{\partial x \partial y}$  is as important in applications as the others.

It is a general result that:  $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

i.e. you get the same answer whichever order the differentiation is done.

**Ex.: a. Let  $z = 4x^2 - 8xy^4 + 7y^5 - 3$ . Find all the first and second order partial derivatives of  $z$ .**

**b. Find all the first and second order partial derivatives of the function  $z = \sin(xy)$ .**

**Sol.:**

**a.**  $z = 4x^2 - 8xy^4 + 7y^5 - 3$

First partial derivative:  $\frac{\partial z}{\partial x} = 8x - 8y^4$ ,  $\frac{\partial z}{\partial y} = -32xy^3 + 35y^4$

Second partial derivative:  $\frac{\partial^2 z}{\partial x^2} = 8$ ,  $\frac{\partial^2 z}{\partial y^2} = -96xy^2 + 140y^3$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (-32xy^3 + 35y^4) = -32y^3$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (8x - 8y^4) = -32y^3$$

**b.**  $z = \sin(xy)$

First partial derivative:  $\frac{\partial z}{\partial x} = y \cos(xy)$ ,  $\frac{\partial z}{\partial y} = x \cos(xy)$

Second partial derivative:  $\frac{\partial^2 z}{\partial x^2} = -y^2 \sin(xy)$ ,  $\frac{\partial^2 z}{\partial y^2} = -x^2 \sin(xy)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (x \cos(xy)) = -xy \sin(xy) + \cos(xy)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (y \cos(xy)) = y(-x \sin(xy)) + \cos(xy) = -xy \sin(xy) + \cos(xy)$$

**Subscript notation for second order partial derivatives:**

If  $z = f(x, y)$  then:

- $z_{xx}$  means  $\frac{\partial^2 z}{\partial x^2}$
- $z_{yy}$  means  $\frac{\partial^2 z}{\partial y^2}$
- $z_{xy}$  means  $\frac{\partial^2 z}{\partial x \partial y}$  or  $\frac{\partial^2 z}{\partial y \partial x}$

### Compute partial derivatives with Chain Rule:

**Formula:** If  $w = f(x,y)$  and  $x = x(t)$  and  $y = y(t)$  such that  $f, x, y$  are all differentiable. Then:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Where:

$$f_x = \frac{\partial w}{\partial x}$$

$$f_y = \frac{\partial w}{\partial y}$$

Also If  $w = f(x,y,z)$  and  $x = x(t)$ ,  $y = y(t)$  and  $z = z(t)$  such that  $f, x, y, z$  are all differentiable. Then:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

Where:

$$f_x = \frac{\partial w}{\partial x}$$

$$f_y = \frac{\partial w}{\partial y}$$

$$f_z = \frac{\partial w}{\partial z}$$

**Ex.: a.** If  $w = xy + z$  and  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$ . Find  $\frac{dw}{dt}$  by using chain rule.

**b.** Given  $w = \ln(u + v + z)$ , with  $u = \cos^2 t$ ,  $v = \sin^2 t$  and  $z = t^2$ , find  $\frac{dw}{dt}$  both by using the chain rule and by expressing  $w$  explicitly as a function of  $t$  before differentiating.

**Sol.:**

$$\text{a. } \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$\frac{\partial w}{\partial x} = y, \quad \frac{\partial w}{\partial y} = x, \quad \frac{\partial w}{\partial z} = 1$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t, \quad \frac{dz}{dt} = 1$$

$$\frac{dw}{dt} = -y \sin t + x \cos t + 1$$

$$\frac{dw}{dt} = -\sin t \sin t + \cos t \cos t + 1$$

$$\mathbf{b.} \quad \frac{dw}{dt} = \frac{\partial w}{\partial u} \frac{du}{dt} + \frac{\partial w}{\partial v} \frac{dv}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$\frac{\partial w}{\partial u} = \frac{1}{(u+v+z)}, \quad \frac{du}{dt} = -2 \cos t \sin t$$

$$\frac{\partial w}{\partial v} = \frac{1}{(u+v+z)}, \quad \frac{dv}{dt} = 2 \cos t \sin t$$

$$\frac{\partial w}{\partial z} = \frac{1}{(u+v+z)}, \quad \frac{dz}{dt} = 2t$$

$$\frac{dw}{dt} = \frac{1}{(u+v+z)} (-2 \cos t \sin t + 2 \cos t \sin t + 2t)$$

$$\frac{dw}{dt} = \frac{2t}{(u+v+z)}$$

### Gradient derivative:

The **position vector** of a point  $P(x, y)$  in two dimensions is  $x\mathbf{i} + y\mathbf{j}$ . We will often denote this important vector by  $\mathbf{r}$ . (In three dimensions the position vector is  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ).

The **vector differential operator**  $\nabla$ , called “del” or “nabla”, is defined in two and three dimensions to be:

- $\text{grad}f(x,y) = \nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$
- $\text{grad}f(x,y,z) = \nabla f(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

The **gradient** of a function is a **vector field**. It is obtained by applying the vector operator  $\nabla$  to the scalar function  $f(x,y)$ . Such a vector field is called a **gradient (or conservative) vector field**.

**Ex.:** The **gradient** of the function  $f(x, y) = x+y^2$  is given by:

$$\begin{aligned}\nabla f(x,y) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \\ &= \frac{\partial}{\partial x}(x + y^2)\mathbf{i} + \frac{\partial}{\partial y}(x + y^2)\mathbf{j} \\ &= (1 + 0)\mathbf{i} + (0 + 2y)\mathbf{j} \\ &= \mathbf{i} + 2y\mathbf{j}\end{aligned}$$

**Ex. Calculate the gradient of the following functions.**

**a.**  $f(x, y) = \sqrt{x^2 + y^2}$

**b.**  $f(x, y, z) = 3x^2\sqrt{y} + \cos(3z)$

**Sol.:**

**a.**  $f(x, y) = \sqrt{x^2 + y^2}$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$\begin{aligned}
&= \frac{\partial}{\partial x} (\sqrt{x^2 + y^2}) \mathbf{i} + \frac{\partial}{\partial y} (\sqrt{x^2 + y^2}) \mathbf{j} \\
&= \left( \frac{2x}{2\sqrt{x^2 + y^2}} \right) \mathbf{i} + \left( \frac{2y}{2\sqrt{x^2 + y^2}} \right) \mathbf{j}
\end{aligned}$$

**b.  $f(x, y, z) = 3x^2\sqrt{y} + \cos(3z)$**

$$\begin{aligned}
\nabla f(x,y,z) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \\
&= \frac{\partial}{\partial x} (3x^2\sqrt{y} + \cos(3z)) \mathbf{i} + \frac{\partial}{\partial y} (3x^2\sqrt{y} + \cos(3z)) \mathbf{j} \\
&\quad + \frac{\partial}{\partial z} (3x^2\sqrt{y} + \cos(3z)) \mathbf{k} \\
&= (6x\sqrt{y}) \mathbf{i} + \left( \frac{3x^2}{2\sqrt{y}} \right) \mathbf{j} - (3\sin(3z)) \mathbf{k}
\end{aligned}$$

**Ex.) The surface is defined by the equation  $x^2yz = 0$ . Find the normal to the surface at point (1,1,1). (note: use gradient method).**

**Sol.:  $f(x,y,z) = x^2yz$**

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

$$\nabla f(x,y,z) = 2xyzi + x^2zj + x^2yk$$

**At the point (1, 1, 1) this is:**

$$\nabla f(x,y,z) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

### Directional Derivatives:

**Definition:** if  $\vec{u}$  is a unit vector, then  $\vec{u} \cdot \nabla f$  is called the **directional derivative** of  $f$  in the direction  $\vec{u}$ . The directional derivative is the rate of change of  $f$  in the direction  $\vec{u}$ .

**Ex.: Find the directional derivative of the function  $f(x, y) = x^3y^4 + x^4y^3$  at the point (1,1) in the direction of the vector (4,3).**

**Sol.:**

$$\nabla f(x,y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} = \frac{\partial}{\partial x} (x^3y^4 + x^4y^3) \mathbf{i} + \frac{\partial}{\partial y} (x^3y^4 + x^4y^3) \mathbf{j}$$

$$= (3x^2y^4 + 4x^3y^3) \mathbf{i} + (4x^3y^3 + 3x^4y^2) \mathbf{j}$$

$$= (3 + 4) \mathbf{i} + (4 + 3) \mathbf{j} = 7\mathbf{i} + 7\mathbf{j}$$

$$\vec{v} = 4\mathbf{i} + 3\mathbf{j}$$

$$|\vec{v}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j}$$

**Directional derivative:**

$$\vec{u} \cdot \nabla f = \left(\frac{4}{5} \mathbf{i} + \frac{3}{5} \mathbf{j}\right) \cdot (7\mathbf{i} + 7\mathbf{j}) = \frac{28}{5} + \frac{21}{5} = \frac{49}{5}$$

**Ex.: Find the directional derivative of  $f(x, y, z) = x \cos y \sin z$  at  $(1, \pi, \frac{\pi}{4})$  in the direction of  $\vec{u} = (2, -1, 4)$ .**

**Sol.:**

$$\begin{aligned} \nabla f(x,y,z) &= \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = \frac{\partial}{\partial x} (x \cos y \sin z) \mathbf{i} + \frac{\partial}{\partial y} (x \cos y \sin z) \mathbf{j} + \\ &+ \frac{\partial}{\partial z} (x \cos y \sin z) \mathbf{k} \end{aligned}$$

$$= (\cos y \sin z) \mathbf{i} + (-x \sin y \sin z) \mathbf{j} + (x \cos y \cos z) \mathbf{k}$$

$$= \cos(\pi) \sin(\pi/4) \mathbf{i} - \sin(\pi) \sin(\pi/4) \mathbf{j} + \cos(\pi) \cos(\pi/4) \mathbf{k}$$

$$= -0.7071 \mathbf{i} - 0 - 0.7071 \mathbf{k} = -0.7071 \mathbf{i} - 0.7071 \mathbf{k}$$

$$\vec{v} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

$$|\vec{v}| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$$

$$\begin{aligned}\vec{u} &= \frac{\vec{v}}{|\vec{v}|} = \frac{2i-j+4k}{\sqrt{21}} \\ \vec{u} \cdot \nabla f &= \left( \frac{2i-j+4k}{\sqrt{21}} \right) \cdot (-0.7071i - 0.7071k) \\ &= \frac{-1.4142}{\sqrt{21}} - \frac{2.8284}{\sqrt{21}} = \frac{-4.2426}{\sqrt{21}} = -0.9258\end{aligned}$$

**Ex.: Find the directional derivative of  $f(x,y) = x^2 - y^2$  at the point  $(4, 2)$  in the direction of a unit vector that makes an angle of  $\frac{\pi}{3}$  with the positive  $x$ -axis.**

**Sol.:**

$$\nabla f(x,y) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j = \frac{\partial}{\partial x}(x^2 - y^2)i + \frac{\partial}{\partial y}(x^2 - y^2)j = 2xi - 2yj$$

$$\nabla f(4,2) = 8i - 4j$$

$$\vec{v} = \cos\left(\frac{\pi}{3}\right)i + \sin\left(\frac{\pi}{3}\right)j$$

$$|\vec{v}| = \sqrt{\left(\cos\left(\frac{\pi}{3}\right)\right)^2 + \left(\sin\left(\frac{\pi}{3}\right)\right)^2} = \sqrt{0.25 + 0.75} = 1$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\cos\left(\frac{\pi}{3}\right)i + \sin\left(\frac{\pi}{3}\right)j}{1}$$

$$\vec{u} \cdot \nabla f = \left( \frac{\cos\left(\frac{\pi}{3}\right)i + \sin\left(\frac{\pi}{3}\right)j}{1} \right) \cdot (8i - 4j) = 4 - 3.4641 = 0.54$$

## Chapter Four

### Double Integral

#### Introduction:

The integral  $\iint f(x,y)dA$  is used to denote the double integral of the function  $f(x,y)$  over the region of integration  $A$ .

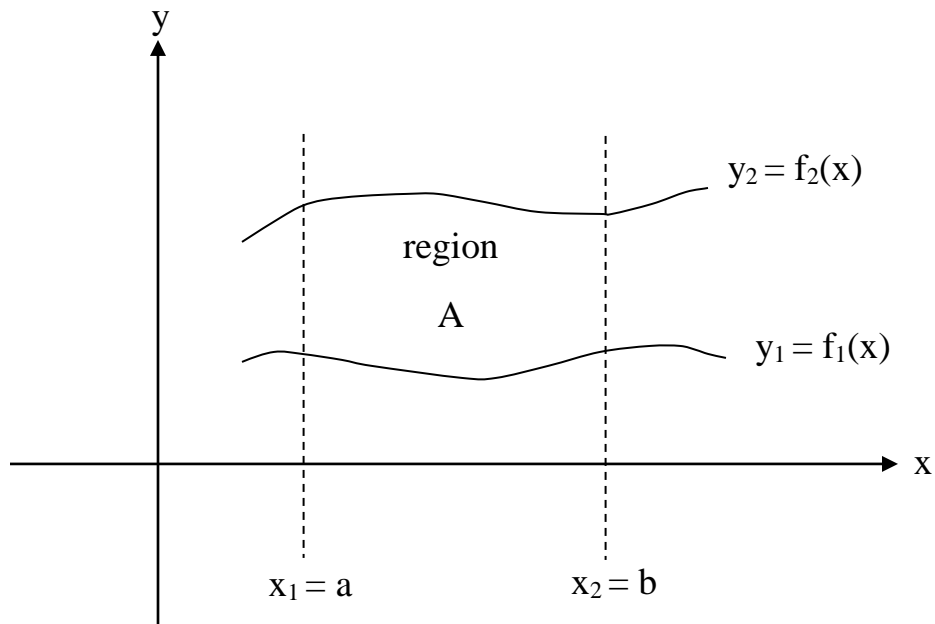
$$dA = dx dy \text{ or } dA = dy dx$$

The integral  $\iint f(x,y)dA$  is evaluated either by  $\iint f(x,y)dx dy$  or  $\iint f(x,y)dy dx$ .

There exists a theory which state that both integrals are equal.

Let us consider the figure where region of integration is given by:

$$y_1 = f_1(x), y_2 = f_2(x), x_1 = a \text{ and } x_2 = b$$



The integral  $\iint f(x,y)dy dx$  is determined by:

1. Integrating  $\int f(x, y)dy$  with respect to  $y$  (with  $x$  is treated as a constant)

and evaluating the resulting integral between  $y_1 = f_1(x)$  to  $y_2 = f_2(x)$ .

2. Integrating the result in (1) with respect to  $x$  and evaluating between

$x_1 = a$  to  $x_2 = b$ .

$$\iint f(x, y)dA = \int_{x_1=a}^{x_2=b} \left[ \int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x, y)dy \right] dx$$

**Ex.: Find I, where :**

$$I = \int_0^{\pi} \int_0^{\sin x} y dy dx$$

**sol.:**

$$I = \int_0^{\pi} \left[ \int_0^{\sin x} y dy \right] dx$$

$$\int_0^{\sin x} y dy = \left[ \frac{y^2}{2} \right]_0^{\sin x} = \frac{1}{2} \sin^2 x$$

$$I = \int_0^{\pi} \frac{1}{2} \sin^2 x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$I = \frac{1}{2} \int_0^{\pi} \frac{1}{2} (1 - \cos 2x) dx = \frac{1}{4} \int_0^{\pi} (1 - \cos 2x) dx$$

$$I = \frac{1}{4} \left[ x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{4} \left[ \pi - \frac{\sin 2\pi}{2} - \left( 0 - \frac{\sin 0}{2} \right) \right]$$

$$I = \frac{\pi}{4}$$

### Volume:

The geometrical meaning of the integral  $\int \int_A f(x,y) dA$  is a volume of a solid having a base (region of integration) and altitude  $f(x,y)$ .

Region of integration = A

Altitude,  $z = f(x,y)$

**Ex.: Find the volume of the solid whose base is the region in the xy - plane which is bounded by the parabola  $y = 4 - x^2$  and the line  $y = 3x$  which the low top of the solid is bounded by the plane  $z = 4 + x$ .**

### Sol.:

$$V = \int \int_A F(x,y) dA$$

$$y_1 = f_1(x) = 3x$$

$$y_2 = f_2(x) = 4 - x^2$$

$$4 - x^2 = 3x$$

$$x^2 + 3x - 4 = 0$$

$$(x + 4)(x - 1) = 0$$

$$x = 1, a = 1$$

$$x = -4, b = -4$$

$$V = \int_{-4}^1 \int_{3x}^{4-x^2} F(x, y) dy dx$$

$$z = 4 + x, F(x, y) = 4 + x$$

$$V = \int_{-4}^1 \int_{3x}^{4-x^2} (4 + x) dy dx = \int_{-4}^1 [\int_{3x}^{4-x^2} (4 + x) dy] dx$$

$$= \int_{-4}^1 [4y + xy]_{3x}^{4-x^2} = \int_{-4}^1 [(16 - 4x^2) - (12x + 3x^2)] dx$$

$$= \int_{-4}^1 [(16 + 4x - 4x^2 - x^3) - (12x + 3x^2)] dx$$

$$= \int_{-4}^1 [16 - 8x - 7x^2 - x^3] dx$$

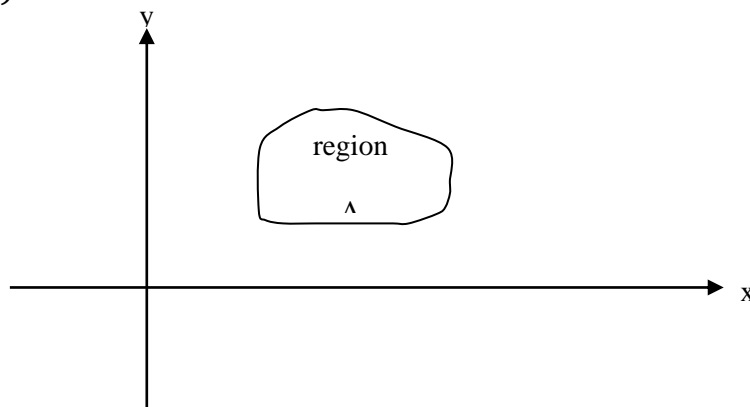
$$= \left[ 16x - 4x^2 - \frac{7x^3}{3} - \frac{x^4}{4} \right]_{-4}^1 = \left[ 16 - 4 - \frac{7}{3} - \frac{1}{4} \right] - \left[ -64 - 64 + \frac{7}{3} 64 - 64 \right]$$

$$V = \frac{52}{12}$$

### Physical application:

We shall study some of the physical application of the double integration in the mechanical engineering for example the mass, centroid, first moment and moment of inertia.

Let us consider in the figure below a thin plate in the xy - plane this plate has density  $\delta(x, y)$ .



1.  $M = \iint_A \delta(x, y) dA$

$dM = \text{mass of } dA$

$$dM = \delta(x, y) dA$$

$$dA = dx dy$$

$$dA = dy dx$$

2. First moment of the mass with respect to x - axis.

$$M_x = \iint y \delta(x, y) dA$$

3. First moment of the mass with respect to y - axis.

$$M_y = \iint x \delta(x, y) dA$$

4. Centroide  $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{M}$$

$$\bar{y} = \frac{M_x}{M}$$

5. Second moment of the mass with respect to x - axis.

$$I_x = \iint y^2 \delta(x, y) dA$$

It is also called the moment of inertia with respect to x - axis.

6. Second moment of the mass with respect to y - axis.

$$I_y = \iint x^2 \delta(x, y) dA$$

It is also called the moment of inertia with respect to y - axis.

7. Moment of inertia with respect to origin point.

$$I_o = \iint (x^2 + y^2) \delta(x, y) dA$$

$$I_o = I_x + I_y$$

**Ex.: Find the mass and centroide of the plane lamina Triangle bounded by  $x = 0$ ,  $y = 0$  and  $x + y = 1$ ,  $\delta(x, y) = xy$ .**

**Sol.:**

$$\begin{aligned} M &= \iint \delta(x, y) dA = \int_0^1 \int_0^{1-x} xy dy dx = \int_0^1 \left[ \frac{xy^2}{2} \right]_0^{1-x} dx \\ &= \int_0^1 \left[ \frac{x(1-x)^2}{2} \right]_0^{1-x} dx = \int_0^1 \left[ \frac{x-2x^2+x^3}{2} \right] dx = \left[ \frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \right]_0^1 = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} \\ &= \frac{1}{24} \end{aligned}$$

$$\bar{x} = \frac{M_y}{M}$$

$$\begin{aligned} M_y &= \iint x \delta(x, y) dA = \int_0^1 \int_0^{1-x} x^2 y dy dx = \int_0^1 \left[ \frac{x^2 y^2}{2} \right]_0^{1-x} dx \\ &= \frac{1}{2} \int_0^1 x^2 [y^2]_0^{1-x} dx = \frac{1}{2} \int_0^1 x^2 [1 - 2x + x^2] dx = \frac{1}{2} \int_0^1 [x^2 - 2x^3 + x^4] dx \\ &= \frac{1}{2} \left[ \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right]_0^1 = \frac{1}{2} \left[ \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right] = \frac{1}{60} \end{aligned}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{60}}{\frac{1}{24}} = \frac{2}{5}$$

$$\bar{y} = \frac{M_x}{M}$$

$$\begin{aligned}
M_x &= \iint y \delta(x, y) dA = \int_0^1 \int_0^{1-y} xy^2 dx dy = \int_0^1 y^2 \left[ \frac{x^2}{2} \right]_0^{1-y} dy \\
&= \frac{1}{2} \int_0^1 y^2 [(1-y)^2] dy = \frac{1}{2} \int_0^1 (y^2 - 2y^3 + y^4) dy = \frac{1}{2} \left[ \frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5} \right]_0^1 \\
&= \frac{1}{2} \left[ \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right] = \frac{1}{30}
\end{aligned}$$

$$\bar{y} = \frac{M_x}{M} = \frac{2}{5}$$

The mass of region =  $\frac{1}{24}$  and its centroid is  $(\frac{2}{5}, \frac{2}{5})$

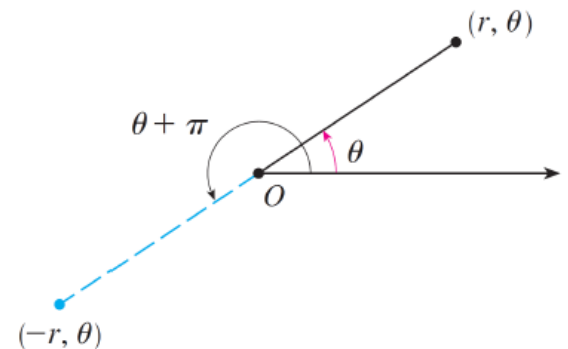
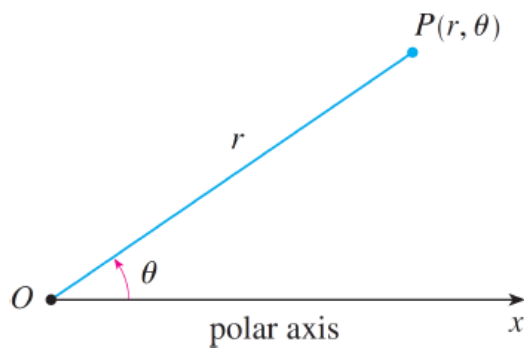
$$\begin{aligned}
I_x &= \iint y^2 \delta(x, y) dA = \int_0^1 \int_0^{1-y} xy^3 dx dy = \int_0^1 y^3 \left[ \frac{x^2}{2} \right]_0^{1-y} dy \\
&= \frac{1}{2} \int_0^1 y^3 [(1-y)^2] dy = \frac{1}{2} \int_0^1 y^3 (1 - 2y + y^2) dy = \frac{1}{2} \int_0^1 (y^3 - 2y^4 + \\
&y^5) dy \\
&= \frac{1}{2} \left[ \frac{y^4}{4} - \frac{2y^5}{5} + \frac{y^6}{6} \right]_0^1 = \frac{1}{2} \left[ \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \right] = \frac{13}{120}
\end{aligned}$$

$$\begin{aligned}
I_y &= \iint x^2 \delta(x, y) dA = \int_0^1 \int_0^{1-x} yx^3 dy dx = \int_0^1 x^3 \left[ \frac{y^2}{2} \right]_0^{1-x} dx \\
&= \frac{1}{2} \int_0^1 x^3 [(1-x)^2] dx = \frac{1}{2} \int_0^1 x^3 (1 - 2x + x^2) dx = \frac{1}{2} \int_0^1 (x^3 - 2x^4 + \\
&x^5) dx \\
&= \frac{1}{2} \left[ \frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6} \right]_0^1 = \frac{1}{2} \left[ \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \right] = \frac{13}{120}
\end{aligned}$$

## Polar Coordinates

**DEFINITION:** The **polar coordinate system** is a two-dimensional coordinate system in which each point **P** on a plane is determined by a distance **r** from a fixed point **O** that is called the **pole** (or origin) and an angle  **$\theta$**  from a fixed direction. The point **P** is represented by the ordered pair  **$(r, \theta)$**  and **r,  $\theta$**  are called **polar coordinates**.

We extend the meaning of polar coordinates  $(r, \theta)$  to the case in which  $r$  is negative by agreeing that the points  $(-r, \theta)$  and  $(r, \theta)$  lie in the same line through  $O$  and at the same distance  $|r|$  from  $O$ ; but on opposite sides of  $O$ ; If  $r > 0$ ; the point  $(r, \theta)$

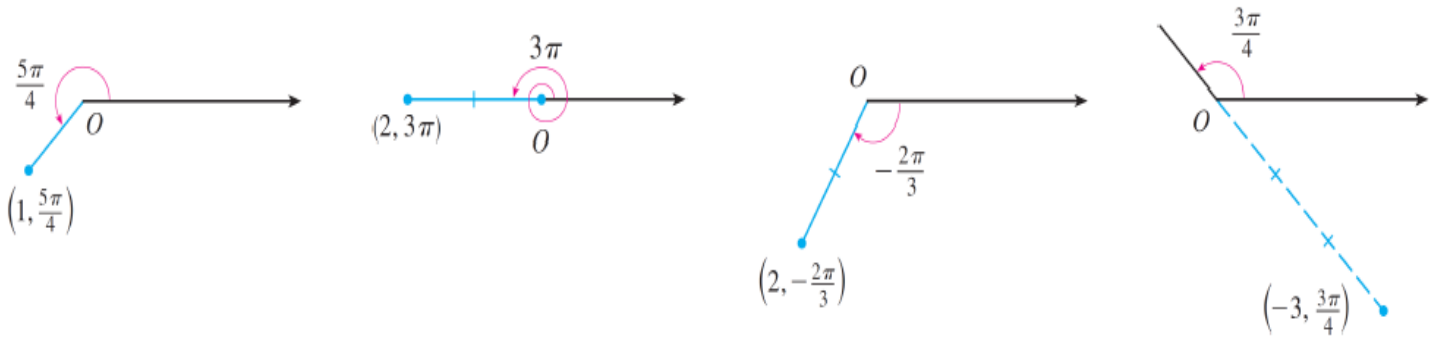


lies in the same quadrant as  $\theta$ ; if  $r < 0$ ; it lies in the quadrant on the opposite side of the pole.

**Ex.: Plot the points whose polar coordinates are given:**

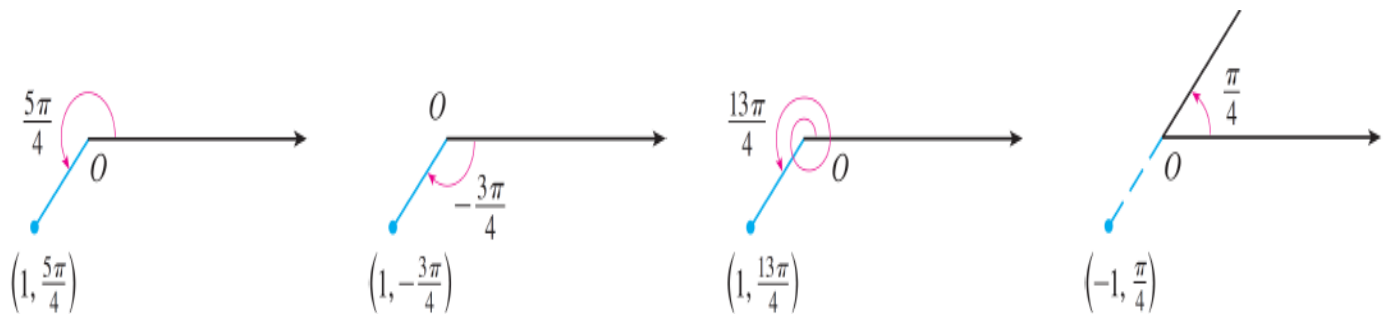
- a.  $(1, 5\pi/4)$ , b.  $(2, 3\pi)$ , c.  $(2, -2\pi/3)$  d.  $(-3, 3\pi/4)$

**Sol.:**

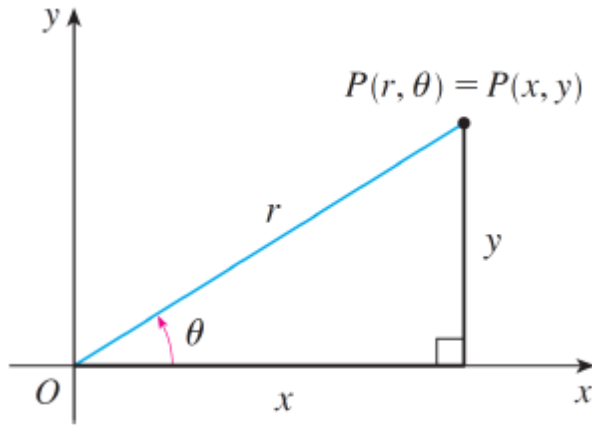


**Note:** In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For example, the point

$(1, 5\pi/4)$  in the example above could be written as  $(1, -3\pi/4)$  or  $(1, 13\pi/4)$  or  $(-1, \pi/4)$ :



**The connection between polar and Cartesian coordinates can be seen from the figure below and described by the following formulas:**



$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2, \quad r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

**Ex.:**

**a. Convert the point  $(2, \pi/3)$  from polar to Cartesian coordinates.**

**b. Represent the point with Cartesian coordinates  $(1, -1)$  in terms of polar coordinates.**

**Sol.:**

a.  $x = r\cos\theta$

$$x = 2\cos(\pi/3) = 2 \cdot 1/2 = 1$$

$$y = r\sin\theta = 2\sin(\pi/3) = 2 \cdot \sqrt{3}/2 = \sqrt{3}$$

Therefore, the point is  $(1, \sqrt{3})$  in Cartesian coordinates.

b.  $r = \sqrt{x^2 + y^2}, \quad r = \sqrt{1 + 1} = \sqrt{2}$

$$\tan\theta = \frac{y}{x} = -1, \quad \theta = -\pi/4 \text{ or } \theta = 7\pi/4$$

point  $(1, -1)$  can be represented in polar coordinates as  $(\sqrt{2}, -\pi/4)$  or  $(\sqrt{2}, 7\pi/4)$

**Ex.: Express the equation  $x^2 = 4y$  in polar coordinates.**

**Sol.:**

$$x = r\cos\theta$$

$$y = r\sin\theta$$

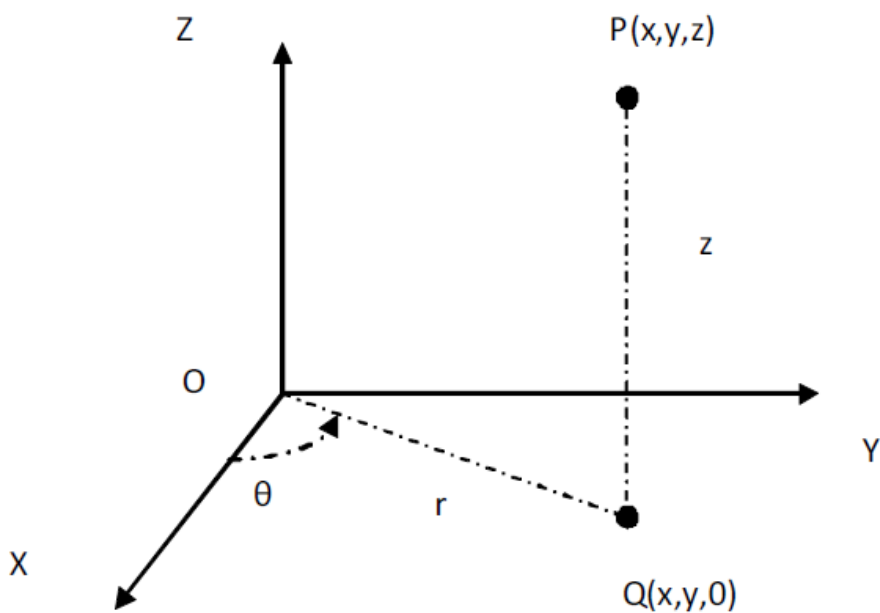
$$(r\cos\theta)^2 = 4r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

$$r = \frac{4\sin\theta}{\cos^2\theta} = 4\sec\theta \tan\theta$$

**Cylindrical Coordinate System:**

point P whose Cartesian coordinate is  $(x, y, z)$ , is assigned by the ordered triple  $(r, \theta, z)$  where  $(r, \theta)$  is the polar coordinate of  $(x, y)$ , the vertical projection along z - axis of P onto xy - plane.



Thus, we readily have the conversion formula:

$$x = r \cos\theta$$

$$y = r \sin\theta$$

$$z = z$$

$$r^2 = x^2 + y^2, \quad r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

**Ex.: Convert the point  $(2, 4\pi/3, 8)$  from cylindrical to rectangular coordinates.**

**Sol.:**

$$x = r\cos\theta = 2\cos(4\pi/3) = -1$$

$$y = r\sin\theta = 2\sin(4\pi/3) = -\sqrt{3}$$

$$z = 8$$

Thus, the point is  $(-1, -\sqrt{3}, 8)$  in rectangular coordinates.

**Ex.: Convert the point  $(\sqrt{3}, 1, 4)$  from rectangular to cylindrical coordinates.**

**Sol.:**

$$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$$

$$\theta = \tan^{-1}(y/x) = \tan^{-1}(1/\sqrt{3}) = \pi/6$$

$$z = 4$$

Thus, the point is  $(2, \pi/6, 4)$  in cylindrical coordinates.

**Ex.: Find an equation in cylindrical coordinates for the ellipsoid**

$$4x^2 + 4y^2 + z^2 = 1.$$

**Sol.:**

$$\text{Since } r^2 = x^2 + y^2$$

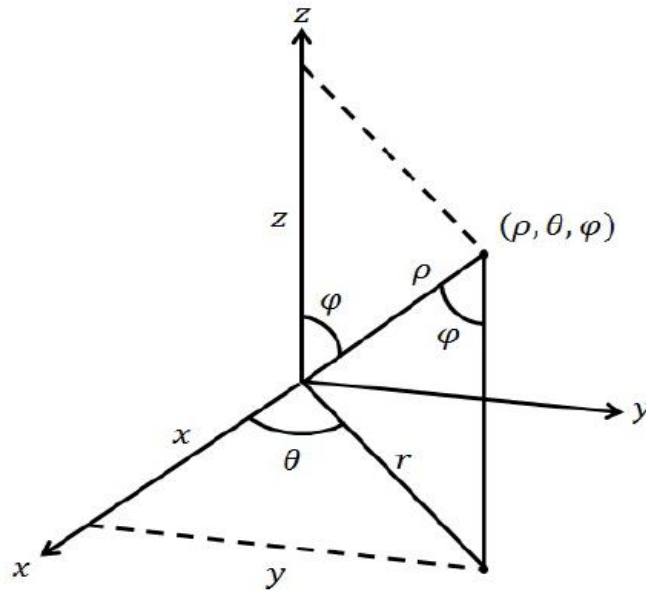
$$4x^2 + 4y^2 + z^2 = 1$$

$$4r^2 + z^2 = 1$$

$$z^2 = 1 - 4r^2$$

### Spherical Coordinate System:

Point P in space is represented by the ordered triple  $(\rho, \theta, \varphi)$ , where  $\rho > 0$  is the distance from the origin to P,  $\theta$  is the same angle as in cylindrical coordinates, and  $0 \leq \varphi \leq \pi$  is the angle between the positive z-axis and the line segment OP, as shown in figure below.



The connection between rectangular and spherical coordinates can be seen in figure in above figure. If the point P has rectangular coordinates  $(x, y, z)$  and spherical coordinates  $(\rho, \theta, \varphi)$ , then:

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

**Ex.: Convert the point  $(4, \frac{\pi}{4}, \frac{\pi}{6})$  from spherical to rectangular coordinates.**

**Sol.:**

$$\rho = 4$$

$$\theta = \frac{\pi}{4}$$

$$\varphi = \frac{\pi}{6}$$

$$x = \rho \sin \varphi \cos \theta = \sqrt{2}$$

$$y = \rho \sin \varphi \sin \theta = \sqrt{2}$$

$$z = \rho \cos \varphi = 2\sqrt{3}$$

Thus, the point is  $(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$  in rectangular coordinates.

**Ex.: Convert the point  $(1, -1, -\sqrt{2})$  from rectangular to spherical coordinates.**

**Sol.:**

$$x = 1$$

$$y = -1$$

$$z = -\sqrt{2}$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 2} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{7\pi}{4}$$

$$\varphi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

Thus, the point is  $(2, \frac{7\pi}{4}, \frac{3\pi}{4})$  in spherical coordinates.

**Ex.: Find an equation in spherical coordinates for the surface  $3x^2 - x + 3y^2 + 3z^2 = 0$ .**

**Sol.:**

Since  $\rho^2 = x^2 + y^2 + z^2$  and  $x = \rho \sin \varphi \cos \theta$ , it follows that:

$$3x^2 + 3y^2 + 3z^2 = x$$

$$3\rho^2 = \rho \sin \varphi \cos \theta ; \rho = \frac{1}{3} \sin \varphi \cos \theta$$

## Matrices

### Concept:

Let us now introduce the concept of a *matrix*. Consider a set of scalar quantities arranged in a rectangular array containing  $m$  rows and  $n$  columns:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

Matrices are conventionally identified by **bold uppercase** letters such as **A**, **B**, etc. The entries of matrix **A** may be denoted as  $A_{ij}$  or  $a_{ij}$ , according to the intended use. Occasionally we shall use the short-hand component notation:

$$A = [a_{ij}]$$

**Ex.: The following is a  $2 \times 3$  numerical matrix:**

$$B = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 9 & 1 \end{bmatrix}$$

This matrix has 2 rows and 3 columns. The first row is (2, 6, 3), the second row is (4, 9, 1), the first column is (2, 4), and so on.

### The determinant of matrix:

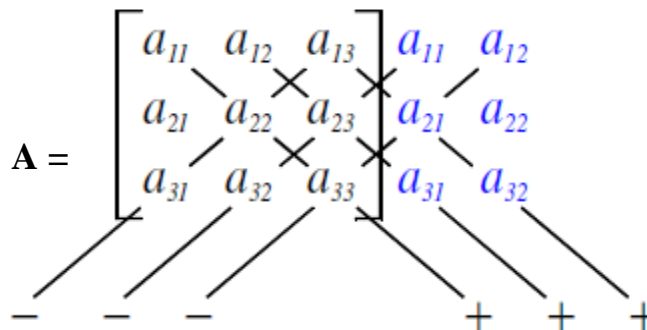
The determinant of a matrix is a scalar value that is used in many matrix operations. The matrix must be **square** (equal number of columns and rows) to have a determinant. The notation for absolute value is used to indicate "the determinant

of", e.g.  $|A|$  means "the determinant of matrix A" and  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$  means to take the determinant of the enclosed matrix. Methods for finding the determinant vary depending on the size of the matrix.

The **determinant of a 2×2 matrix** is simply:

where  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,  $\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

The **determinant of a 3×3 matrix** can be calculated by repeating the first two columns as shown below.



$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

The **determinant of a 3×3 matrix** can be calculated by the following process:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

**Ex.: Find the determinant of the following matrix:**

$$A = \begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 3 \\ -1 & 0 & 5 \end{bmatrix}$$

**Solution:**

**First method:**

$$|A| = \begin{vmatrix} 2 & 1 & 6 & 2 & 1 \\ 0 & 2 & 3 & 0 & 2 \\ -1 & 0 & 5 & -1 & 0 \end{vmatrix}$$

-   -   -   +   +   +

$$|A| = (2 \cdot 2 \cdot 5) + (1 \cdot 3 \cdot -1) + (6 \cdot 0 \cdot 0) - (6 \cdot 2 \cdot -1) - (2 \cdot 3 \cdot 0) - (1 \cdot 0 \cdot 5)$$

$$|A| = 20 - 3 + 12 = 29$$

**Second method:**

$$|A| = 2(2 \cdot 5 - 3 \cdot 0) - 1(0 \cdot 5 - 3 \cdot -1) + 6(0 - 2 \cdot -1)$$

$$= 20 - 3 + 12 = 29$$

**Matrices addition and subtraction:**

For example purposes, let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$\text{then } A \pm B = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

**Ex.: Find addition and subtraction between A and B of the following matrices:**

$$A = \begin{bmatrix} 5 & -2 \\ -1 & 3 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -2 \\ 0 & 1 \\ 4 & -1 \end{bmatrix}$$

**Solution:**

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 5 & -2 \\ -1 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 0 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -4 \\ -1 & 4 \\ 5 & -1 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 5 & -2 \\ -1 & 3 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 0 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ -3 & 1 \end{bmatrix}$$

**Matrix by Matrix Product:**

We now pass to the most general matrix-by-matrix product, and consider the operations involved in computing the product  $\mathbf{C}$  of two matrices  $\mathbf{A}$  and  $\mathbf{B}$ :

$$\mathbf{C} = \mathbf{AB}$$

Let  $\mathbf{A}$  be a  $1 \times 2$  row matrix and  $\mathbf{B}$  be a  $2 \times 1$  column matrix:

$$\mathbf{A} = [\mathbf{a} \quad \mathbf{b}] \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

The product of these two matrices is written  $\mathbf{AB}$  and is the  $1 \times 1$  matrix defined by:

$$\mathbf{AB} = [\mathbf{ac} + \mathbf{bd}]$$

For example: 1.  $\mathbf{A} = [2 \quad -3]$  and  $\mathbf{B} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$

$$\mathbf{AB} = [12 - 15] = [-3]$$

2.  $\mathbf{A} = [2 \quad -4 \quad 3 \quad 2]$  and  $\mathbf{B} = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix}$

$$\mathbf{AB} = [6 - 12 - 6 + 10] = [-2]$$

### Multiplying two 2×2 matrices:

If A and B are two matrices then the product AB is obtained by multiplying the rows of A with the columns of B in the manner described above.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$
$$A \quad B \quad = \quad C$$

**Ex.: Find the product AB where:**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

**Solution:**

$$C = A \times B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -5 & 1 \end{bmatrix}$$

### Multiplying two 3×3 matrices:

The definition of the product  $C = AB$  where A and B are two  $3 \times 3$  matrices is as follows:

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} ar + bu + cx & as + bv + cy & at + bw + cz \\ dr + eu + fx & ds + ev + fy & dt + ew + fz \\ gr + hu + ix & gs + hv + iy & gt + hw + iz \end{bmatrix}$$

**Ex.: Find the product AB where:**

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 1 & 5 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix}$$

**Solution:**

$$C = A \times B = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 1 & 5 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -8 & 7 \\ 10 & -11 & 13 \\ 7 & -17 & 12 \end{bmatrix}$$

**Inverse of Matrix:**

We must follow the following steps to calculate inverse matrix:

1. Calculate determinant of matrix.
2. Minors of matrix.
3. Cofactor of matrix  $(-1)^{i+j}a_{ij}$
4. Adjoint matrix (adj) (Transpose of cofactor)
5.  $A^{-1} = \frac{adjA}{|A|}$

**Ex.: Calculate inverse matrix of the following matrix:**

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

**Solution:**

$$|A| = -1(-4 - 3) - (12 + 1) + 2(9 - 1) = 7 - 13 + 16 = 10$$

$$\text{Minors of Matrix A} = \begin{bmatrix} -7 & 13 & 8 \\ -2 & -2 & -2 \\ 3 & -7 & -2 \end{bmatrix}$$

$$\text{Cofactor matrix} = \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}$$

$$\text{Adjoint matrix} = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adjoint matrix}}{|A|} = \frac{\begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}}{10} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

### Solving simultaneous equations using the inverse matrix:

Matrix algebra allows us to write the solution of the system using the inverse matrix of the coefficients. In practice the method is suitable only for small systems. Its main use is the theoretical insight into such problems which it provides.

$$AX = B$$

$$X = A^{-1}B$$

**X** = variables

**A** = matrix of coefficient variables

**B** = matrix of constant

**Ex.: Solve the following equations using matrix methods:**

$$x_1 - 2x_2 + x_3 = 3$$

$$2x_1 + x_2 - x_3 = 5$$

$$3x_1 - x_2 + 2x_3 = 12$$

**Solution:**

$$AX = B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ -0.7 & -0.1 & 0.3 \\ -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 & 0.1 \\ -0.7 & -0.1 & 0.3 \\ -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix}$$

$$\mathbf{x}_1 = 3$$

$$\mathbf{x}_2 = 1$$

$$\mathbf{x}_3 = 2$$

## Line Integration

and

## Green's Theorem

### Line Integration:

If  $f$  is any continuous function (not just a positive one), defined on a smooth curve  $C$  given in equation (a), then the line integral of  $f$  along  $C$  can be computed by the formula (b):

$$\vec{r}(t) = \langle x(t), y(t) \rangle \text{ for } a \leq t \leq b. \quad (\text{a})$$

$$\begin{aligned} \int_C f(x, y) ds &= \int_a^b f(x(t), y(t)) |\vec{r}'(t)| dt \\ &= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \end{aligned} \quad (\text{b})$$

**Ex.: Evaluate  $\int_C (2 + x^2y)ds$  where  $C$  is the upper half of the unit circle  $x^2 + y^2 = 1$ .**

**Solution:**

$$\begin{aligned}\int_C f(x, y) ds &= \int_a^b f(x(t), y(t)) |\vec{r}'(t)| dt \\ &= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt\end{aligned}$$

$$x(t) = \cos t, y(t) = \sin t, \quad 0 \leq t \leq \pi$$

$$\begin{aligned}\int_C f(x, y) ds &= \int_a^b f(x(t), y(t)) |\vec{r}'(t)| dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) (\sqrt{\sin^2 t + \cos^2 t}) dt \\ &= \int_0^\pi (2 + \cos^2 t \sin t) dt \\ &= \int_0^\pi 2 dt + \int_0^\pi (\cos^2 t \sin t) dt \\ &= \left[ 2t - \frac{\cos^3 t}{3} \right]_0^\pi \\ &= 2\pi + \frac{1}{3} + \frac{1}{3} = 2\left(\pi + \frac{1}{3}\right)\end{aligned}$$

**Ex.: Evaluate  $\int_C 2x ds$  where  $C = C_1 + C_2$ ,  $C_1$  being the arc of the parabola  $y = x^2$  between  $(0,0)$  and  $(1,1)$  and  $C_2$  being vertical line from  $(1,1)$  to  $(1,2)$ .**

$$\int_C 2x ds = \int_{C_1} 2x ds + \int_{C_2} 2x ds$$

**Solution:**

1. To evaluate  $\int_{C_1} 2x ds$ , we need to parametrize  $C_1$ .  $y = x^2$  can be parametrized by  $x = t$ ,  $y = t^2$ ,  $0 \leq t \leq 1$ . Thus

$$\begin{aligned} \int_{C_1} 2x ds &= \int_0^1 2t \sqrt{1 + 4t^2} dt \\ &= \frac{5\sqrt{5} - 1}{6} \end{aligned}$$

2. To evaluate  $\int_{C_2} 2x ds$ , we need to parametrize  $C_2$ . A vertical line between the given points can be parametrized by  $x = 1$ ,  $y = t$ ,  $1 \leq t \leq 2$ . Thus

$$\begin{aligned} \int_{C_2} 2x ds &= \int_1^2 2\sqrt{1} dt \\ &= 2 \end{aligned}$$

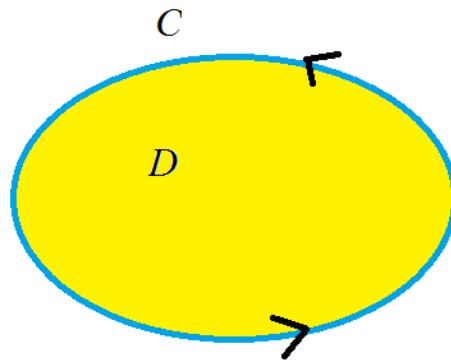
3. Therefore

$$\int_C 2x ds = \frac{5\sqrt{5} - 1}{6} + 2$$

### Green's Theorem:

Suppose  $C$  is a positively oriented, piecewise-smooth, simple closed curve in the plane that bounds a region  $D$ . If  $P$  and  $Q$  have continuous derivatives (in an open set containing the region  $D$ ), then:

$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$



Sometimes the line integral is written  $\int_C P dx + Q dy$  to emphasize that the curve is closed. For each of the following regions  $D$ , associated boundary curves  $C$ , and line integrals. . .

- Compute the given line integral directly by parameterizing the path  $C$ .
- Compute the given line integral by applying Green's theorem and computing a double integral.

**Ex.: Evaluate the line integral of the following by using Green's theorem:**

$$\int_C y^2 dx + x^2 dy$$

**where  $C$  is the closed curve which is the boundary of the triangle with vertices  $(0; 0)$ ,  $(1; 1)$  and  $(1; 0)$ , with the counterclockwise orientation.**

**Solution:**

$$\oint_C P(x, y)dx + Q(x, y)dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where  $P(x, y) = y^2$ ,  $Q(x, y) = x^2$  and  $D$  is the inside of the triangle. Clearly

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y$$

We can describe the triangle  $D$  by

$$0 \leq x \leq 1, \quad 0 \leq y \leq x,$$

$$\int_0^1 \int_0^x (2x - 2y) dy dx = \int_0^1 \left[ -y^2 + 2xy \right]_{y=0}^{y=x} dx = \int_0^1 x^2 dx = \frac{1}{3}$$

**Examples:**

1. Let  $P = xy$ ,  $Q = y^2$  and  $D = \{(x, y); 0 \leq x \leq 1, x^2 \leq y \leq x\}$ . Let  $C$  be the positively oriented boundary of  $D$ . Calculate both sides of Green's theorem.

**Solution:**

$$\begin{aligned} \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy &= \iint_D -x dx dy = \int_0^1 \int_{x^2}^x -x dy dx = \int_0^1 -x(x - x^2) dx \\ &= -\frac{x^3}{3} + \frac{x^4}{4} \Big|_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}. \end{aligned}$$

2. Evaluate the line integral by using Green's theorem:

$$\oint_C (3x - y) dx + (x + 5y) dy, \quad \text{where } C := x^2 + y^2 = 1$$

**Solution:**

$$\begin{aligned} P &= 3x - y, \quad \frac{\partial P(x, y)}{\partial y} = -1 \\ Q &= x + 5y, \quad \frac{\partial Q(x, y)}{\partial x} = 1 \end{aligned}$$

Hence

$$\begin{aligned} \oint_C (3x - y) dx + (x + 5y) dy &= \iint_D [1 - (-1)] d(xy) \\ &= 2 \iint_D d(x, y) \\ &= 2 \times \text{area of unit disk} = 2\pi \end{aligned}$$

## The Maclaurin Series

and

## Taylor Series

### The Maclaurin Series:

Consider a function  $f(x)$  which can be differentiated at  $x = 0$ . For example  $e^x$ ,  $\cos x$ ,  $\sin x$  would fit into this category but  $|x|$  would not. Let us assume that  $f(x)$  can

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots = \sum_{p=0}^{\infty} \frac{1}{p!}f^{(p)}(0)x^p$$

be represented by a power series in  $x$ :

This is called Maclaurin expansion of  $f(x)$ .

**Ex.: Find the Maclaurin expansion of  $\cos x$ .**

**Solution:**

Here  $f(x) = \cos x$  and, differentiating a number of times:

$$f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x \quad \text{etc.}$$

Evaluating each of these at  $x = 0$ :

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0 \quad \text{etc.}$$

Substituting into  $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$ , gives:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

**Ex.: Find the Maclaurin expansion of  $e^x \ln(1 + x)$ .**

**Solution:**

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots = \sum_{p=0}^{\infty} \frac{x^p}{p!}f^{(p)}(0)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

and

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

$$\begin{aligned} e^x \ln(1 + x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right) \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ &\quad + x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \dots \\ &\quad + \frac{x^3}{2} - \frac{x^4}{4} \dots \\ &\quad + \frac{x^4}{6} \dots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^4}{40} + \dots \end{aligned}$$

## The Taylor series:

The Taylor series is a generalization of the Maclaurin series being a power series developed in powers of  $(x - x_0)$  rather than in powers of  $x$ .

If the function  $f(x)$  can be differentiated as often as required at  $x = x_0$  then:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots$$

This is called the Taylor series of  $f(x)$  about the point  $x_0$ .

**Ex.: Find the Taylor series of degrees one and two for  $f(x) = e^x$ , centered at  $x = 0$ .**

**Solution:**

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots$$

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 1$$

$$f(x) = 1 + (x - 0) + \frac{(x-0)^2}{2!}$$

$$f(x) = 1 + x + \frac{x^2}{2}$$

**Ex.: Find the Taylor series of degree three for  $f(x) = \sin x$ , centered at**

$$x = 5\pi/6.$$

**Solution:**

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots$$

$$f(x) = \sin x \qquad f\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

$$f'(x) = \cos x \qquad f'\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f''(x) = -\sin x \qquad f''\left(\frac{5\pi}{6}\right) = -\frac{1}{2}$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{5\pi}{6}\right) - \frac{1}{4}\left(x - \frac{5\pi}{6}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{5\pi}{6}\right)^3$$

## Solution of Differential Equations

### Differential Equation:

A differential equation is an equation involving an unknown function and its derivatives. The following are differential equations involving the unknown function  $y$ .

$$y' = \frac{dy}{dx} = 5x + 3 \quad (1)$$

$$y'' = \frac{d^2y}{dx^2}, \quad e^y y'' - 2y' = 1 \quad (2)$$

$$y''' = \frac{d^3y}{dx^3}, \quad 4y''' + \sin(x)y'' + 5xy = 0 \quad (3)$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x \quad (4)$$

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0 \quad (5)$$

A differential equation is an Ordinary Differential Equation (ODE) if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a Partial Differential Equation (PDE) .

Equations (1) through (4) are examples of ordinary differential equations, since the unknown function  $y$  depends solely on the variable  $x$ . Equation (5) is a partial differential equation since the unknown function  $y$  depends on both the independent variables  $t$  and  $x$ .

The order of a differential equation is the order of the highest derivative appearing in the equation. So the equation (1) is a first - order differential equation, (2) and (4) are second - order differential equation. (Note in Equation (4) that the order of the highest derivative appearing in the equation is two). Equation (3) is a third – order differential equation.

### **Notation**

The expressions  $y'$  ,  $y''$  ,  $y'''$  ,  $y^{(4)}$  , ... ,  $y^{(n)}$  are often used to represent, respectively, the first, second, third, fourth, ... ,  $n$ th derivatives of  $y$  with respect to the independent variable under consideration . Thus ,  $y''$  represents  $\frac{d^2 y}{d x^2}$  if the

independent variable is  $x$  , but represents  $\frac{d^2 y}{d p^2}$  if the independent variable is  $p$  .

Observe that parentheses are used in  $y^{(n)}$  to distinguish it from the  $n$ th power ,  $y^n$  .

If the independent variable is time , usually denoted by  $t$  , primes are often replaced by dots . Thus  $y'$ ,  $y''$  and  $y'''$  represent  $\frac{d y}{d t}$  ,  $\frac{d^2 y}{d t^2}$  , and  $\frac{d^3 y}{d t^3}$  , respectively .

### **First – Order Differential Equations:**

Standard form for a first – order differential equation in the unknown function  $y = f(x)$  is :

$$y' = f(x, y)$$

where the derivative  $y'$  appears only on the left side of the equation . Many, but not all , first –order differential equations can be written in standard form by

algebraically solving for  $y'$  and then setting  $f(x, y)$  equal to the right side of the resulting equation .

The right side can always be written as a quotient of two other functions  $M(x, y)$  and  $-N(x, y)$  . Then becomes

$$\frac{dy}{dx} = \frac{M(x, y)}{-N(x, y)}$$

which is equivalent to the differential form

$$M(x, y)dx + N(x, y)dy = 0 \quad \dots\dots (*)$$

### Classifications of First-Order Differential Equations

- **Separable First – Order Differential Equation**

Consider the differential form of the first – order differential equations

$$M(x, y)dx + N(x, y)dy = 0$$

If  $M(x, y) = A(x)$ , is a function only of  $x$  , and  $N(x, y) = B(y)$ , is a function only of  $y$  , then the differential equation is Separable , or has its variables separated .

$$A(x)dx + B(y)dy = 0 \tag{1}$$

The general solution to the equation (1) is :

$$\int A(x)dx + \int B(y)dy = c \tag{2}$$

Where  $c$  represents an arbitrary constant .

The integrals obtained in equation (2) may be , for all practical purposes , impossible to evaluate . In such cases , numerical techniques are used to obtain an

approximate solution . Even if the indicated integrations in equation (2) can be performed , it may not be algebraically possible to solve for  $y$  explicitly in terms of  $x$  . In that case , the solution is left in implicit form .

### **Solution to the Initial – Value Problem**

The solution to the initial – value problem

$$A(x)dx + B(y)dy = 0 ; y(x_0) = y_0 \quad (3)$$

can be obtained , as usual , by first using equation (2) to solve the differential equation and then applying the initial condition directly to evaluate  $c$  .

Alternatively, the solution to equation (3) can be obtained from

$$\int_{x_0}^x A(x)dx + \int_{y_0}^y B(y)dy = c \quad (4)$$

Equation (4) , however, may not determine the solution of (3) uniquely ; that is , (4) may have many solutions , of which only one will satisfy the initial – value problem.

**Example (1) : Solve the differential equation**  $y' = \frac{x^2 + 2}{y}$  .

### **Solution :**

We first rewrite equation in the differential form

$$\frac{dy}{dx} = \frac{x^2 + 2}{y} \Rightarrow (x^2 + 2)dx - ydy = 0$$

By integrating both sides we'll obtain:

$$\int (x^2 + 2)dx - \int ydy = C \Rightarrow \frac{x^3}{3} + 2x - \frac{y^2}{2} = C$$

Solving for  $y$  we obtain :

$$y^2 = \frac{2}{3}x^3 + 4x - 2C \Rightarrow y = \pm \sqrt{\frac{2}{3}x^3 + 4x - 2C}$$

**Example (2) : Solve the differential equation**  $e^x dx - y dy = 0$  ;  $y(0) = 1$

**Solution :**

This differential equation is a separable , so the solution is given by :

$$\int A(x)dx + \int B(y)dy = c$$

where ;  $A(x) = e^x$  and  $B(y) = -y$

$$\text{Thus , } \int e^x dx + \int (-y)dy = c \Rightarrow e^x - \frac{y^2}{2} = c$$

$$\Rightarrow y^2 = 2e^x - 2c$$

Applying the initial condition, we obtain

$$1^2 = 2e^0 - 2c \Rightarrow c = \frac{1}{2}$$

Thus, the solution to the initial – value problem is :

$$y^2 = 2e^x - 2c \Rightarrow y^2 = 2e^x - 1 \Rightarrow y = \sqrt{2e^x - 1}$$

Note that we cannot choose the negative square root , since then

$y(0) = -1$  , which violates the initial condition .

To ensure that  $y$  remains real , we must restrict  $x$  so that  $2e^x - 1 \geq 0$

.To guarantee that  $y'$  exists [ note that  $y'(x) = dy/dx = e^x/y$  ] , we must

restrict  $x$  so that  $2e^x - 1 \neq 0$  . Together these conditions imply

that  $2e^x - 1 > 0$  , or  $x > \ln 1/2$  .

## Applications of First – Order Differential Equations:

### Temperature Problems

Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. Let  $T$  denote the temperature of the body and let  $T_m$  denote the temperature of the surrounding medium. Then the time rate of change of the temperature of the body is:  $\frac{dT}{dt}$

and Newton's law of cooling can be formulated as:

$$\frac{dT}{dt} = -k(T - T_m) \quad \text{or as} \quad \frac{dT}{dt} + kT = kT_m$$

where  $k$  is a positive constant of proportionality. Once  $k$  is chosen positive, the minus sign is required in Newton's law to make  $\frac{dT}{dt}$  negative in a cooling process, when  $T$  is greater than  $T_m$ , and positive in a heating process, when  $T$  is less than  $T_m$ .

**Example (3): A metal bar at a temperature of 100 ° F is placed in a room at a constant temperature of 0 ° F. If after 20 minutes the temperature of the bar is 50 ° F, find (a) the time it will take the bar to reach a temperature of 25 ° F and (b) the temperature of the bar after 10 minutes.**

#### **Solution :**

Using equation  $\frac{dT}{dt} + kT = kT_m$  with  $T_m = 0$ ; the medium here is the room

which is being held at a constant temperature of 0 ° F. Thus we have:

$$\frac{dT}{dt} + kT = 0 \quad \text{or} \quad \frac{dT}{T} + k dt = 0$$

whose solution is :  $\ln T = -kt + c \Rightarrow T = ce^{-kt}$

Since  $T = 100$  at  $t = 0$  ( the temperature of the bar is initially  $100^\circ F$  ), it follows that :  $100 = ce^{-k(0)} \Rightarrow c = 100$

Substituting this value into equation , we obtain :  $T = 100 e^{-kt}$

At  $t = 20$  , we are given that  $T = 50$  ; hence ,

$$50 = 100 e^{-20k} \Rightarrow k = \frac{-1}{20} \ln \frac{50}{100} = \frac{-1}{20} (-0.693) = 0.035$$

Substituting this value into equation , we obtain the temperature of the bar at any time  $t$  as :  $T = 100 e^{-0.035t}$

(a) We require  $t$  when  $T = 25$  . Substituting  $T = 25$  into equation ,

$$\text{we have : } 25 = 100 e^{-0.035t} \text{ or } -0.035t = \ln \frac{1}{4} \Rightarrow t = 39.6 \text{ min}$$

(b) We require  $T$  when  $t = 10$  . Substituting  $t = 10$  into equation

and then solving for  $T$  , we find that :

$$T = 100 e^{-0.035t} = 100 e^{-0.035(10)} = 100(0.705) = 70.5^\circ F$$

It should be noted that since Newton's law is valid only for small temperature differences , the above calculations represent only a first approximation to the physical situation .

## ■ Homogeneous First – Order Differential Equation

A first – order differential equation in standard form  $y' = f(x, y)$  is homogeneous if  $f(\lambda x, \lambda y) = f(x, y)$ , for every real number  $\lambda$ .

### Reduction of Homogeneous Equations

The homogeneous differential equation

$$\frac{dy}{dx} = f(x, y) \quad \dots (1)$$

having the property that can be transformed into a separable equation by making the substitution :

$$z = \frac{y}{x} \Rightarrow y = xz \quad \dots (2)$$

along with its corresponding derivative :

$$\frac{dy}{dx} = y' = z + x \frac{dz}{dx} = z + xz'$$

The resulting equation in the variables  $z$  and  $x$  is solved as a separable differential equation ; the required solution to Equation (1) is obtained by back substitution .

**Example (4) :** Determine if the following differential equation is homogeneous and solve it  $y' = \frac{2y^4 + x^4}{xy^3}$  .

#### Solution :

This differential equation is not separable. Instead it has

the form  $y' = f(x, y)$ , with  $y' = f(x, y) = \frac{2y^4 + x^4}{xy^3}$

Since ,  $f(\lambda x, \lambda y) = \frac{2(\lambda y)^4 + (\lambda x)^4}{\lambda x (\lambda y)^3} = \frac{\lambda^4 (2y^4 + x^4)}{\lambda^4 (xy^3)} = f(x, y)$  Thus , the differential equation is homogeneous .

The solution is given by substituting :  $z = \frac{y}{x} \Rightarrow y = xz$

with its corresponding derivative :  $y' = z + x \frac{dz}{dx}$  into the differential equation as

originally given, we obtain :

$$z + x \frac{dz}{dx} = \frac{2(xz)^4 + x^4}{x(xz)^3}$$

$$z + x \frac{dz}{dx} = \frac{2z^4 + 1}{z^3} = 2z + \frac{1}{z^3} \Rightarrow x \frac{dz}{dx} = z + \frac{1}{z^3}$$

which can be algebraically simplified to

$$x \frac{dz}{dx} = \frac{z^4 + 1}{z^3} \Rightarrow \frac{dx}{x} - \frac{z^3}{z^4 + 1} dz = 0$$

This last equation is separable ; its solution is :

$$\int \frac{dx}{x} - \int \frac{z^3}{z^4 + 1} dz = c \Rightarrow \ln x - \frac{1}{4} \ln (z^4 + 1) = c \Rightarrow$$

$$\Rightarrow \ln \frac{x}{\sqrt[4]{z^4 + 1}} = \ln c \Rightarrow z^4 + 1 = \left(\frac{x}{c}\right)^4 = kx^4 , \text{ where } k = \frac{1}{c^4}$$

Finally , substituting  $z = \frac{y}{x}$  back into equation , we obtain :

$$\begin{aligned} \left(\frac{y}{x}\right)^4 + 1 &= kx^4 \Rightarrow y^4 = x^4(kx^4 - 1) \Rightarrow \\ &\Rightarrow y = x\sqrt[4]{kx^4 - 1} \end{aligned}$$

## Exact First – Order Differential Equation

A first – order differential equation in differential form

$$M(x, y)dx + N(x, y)dy = 0 \dots (1)$$

is exact if there is a function  $u(x, y)$  such that :

$$du(x, y) = M(x, y)dx + N(x, y)dy \dots (2)$$

### Test for exactness

If  $M(x, y)$  and  $N(x, y)$  are continuous functions and have continuous first partial derivatives on some rectangle of the  $xy$  – plane , then the differential equation is exact if and only if :

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \dots (3)$$

### Method of Solution

To solve Equation (1) , assuming that it is exact , first solve the equations :

$$\frac{\partial u(x, y)}{\partial x} = M(x, y) \dots (4)$$

$$\frac{\partial u(x, y)}{\partial y} = N(x, y) \dots (5)$$

for  $u(x, y)$  .

The solution to equation (1) is then given implicitly by :

$$u(x, y) = c \dots (6)$$

where  $c$  represents an arbitrary constant .

Equation (6) is immediate from equations (1) and (2) . If Equation (2) is substituted into Equation (1), we obtain :

$$d u(x, y(x)) = 0$$

Integrating this equation (note that we can write 0 as  $0 dx$ ), we have :

$$\int d u(x, y(x)) = \int 0 dx ,$$

which , in turn , implies Equation (6) .

### **Integrating Factor**

In general , if Equation (1) is not exact , then we can transform it into an exact differential equation by multiplying it with a suitable function  $I(x, y) \neq 0$  . This function is called an Integrating factor .

Hence , the function  $I(x, y) \neq 0$  is an integrating factor for equation (1) if the equation  $I(x, y)[M(x, y)dx + N(x, y)dy] = 0 \dots\dots (7)$

is exact .

A solution to Equation (1) is obtained by solving the exact differential equation defined by (7), and the conditions that follow are :

If  $\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \equiv g(x)$  , a function of x alone , then

the integrating factor is  $I(x, y) = e^{\int g(x)dx}$

If  $\frac{1}{M} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \equiv h(y)$  , a function of y alone, then

the integrating factor is  $I(x, y) = e^{-\int h(y)dy}$

If  $M = y f(x, y)$  and  $N = x g(x, y)$ , then the integrating factor is

$$I(x, y) = \frac{1}{xM - yN}$$

**Note : A separable equation is always exact .**

**Example (5) : Determine whether the differential equation**

**$2xy dx + (1 + x^2) dy = 0$  is exact and solve it .**

**Solution :**

This equation has the form with  $M(x, y) = 2xy$  and  $N(x, y) = 1 + x^2$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$ , the differential equation is exact. We now determine a

function  $u(x, y)$  that satisfies:

$$\frac{\partial u(x, y)}{\partial x} = M(x, y) \dots (1) \quad \text{and} \quad \frac{\partial u(x, y)}{\partial y} = N(x, y) \dots (2)$$

Substituting  $M(x, y) = 2xy$  into (1), we obtain:

$$\frac{\partial u}{\partial x} = 2xy \Rightarrow du = 2xy dx$$

Integrating both sides of this equation with respect to  $x$ , we find :

$$\int du = \int 2xy dx \Rightarrow u(x, y) = x^2 y + h(y) \dots (3)$$

Note that when integrating with respect to  $x$ , the constant of integration can depend on  $y$ .

We now determine  $h(y)$ . Differentiating (3) with respect to  $y$ , we obtain:

$\frac{\partial u}{\partial y} = x^2 + h'(y)$  Substituting this equation along with  $N(x, y) = 1 + x^2$  into (2),

we have :

$$\frac{\partial u}{\partial y} = N(x, y) = x^2 + h'(y) = 1 + x^2 \Rightarrow$$

$$\Rightarrow h'(y) = 1 \Rightarrow \frac{dh}{dy} = 1 \Rightarrow dh = dy$$

Integrating this last equation with respect to  $y$ , we obtain :

$$\int dh = \int dy \Rightarrow h(y) = y + c_1, \quad (c_1 = \text{constant}).$$

Substituting this expression into (3) yields:

$$u(x, y) = x^2 y + y + c_1$$

The solution to the differential equation, which is given implicitly by  $u(x, y) = c$ , is :

$$x^2 y + y = c_2, \quad (c_2 = c - c_1)$$

Solving for  $y$  explicitly, we obtain the solution as:

$$y = \frac{C_2}{x^2 + 1}$$

**Example (6) : Convert the following differential equation into an exact differential equation and solve it .**

$$y' = 2xy - x$$

**Solution :**

Rewriting this equation in differential form, we have

$$(2xy - x)dx - dy = 0$$

Here ,  $M(x, y) = 2xy - x$  and  $N(x, y) = -1$

$$\frac{\partial M}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$  , the differential equation is not exact .

$$\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{-1} (2x - 0) = -2x \equiv g(x)$$

is a function of  $x$  alone. Thus , the integrating factor is :

$$I(x, y) = e^{\int g(x)dx} = e^{\int -2xdx} = e^{-x^2}$$

Multiplying the differential equation by  $I(x, y) = e^{-x^2}$  we obtain :

$$(2xye^{-x^2} - xe^{-x^2})dx - e^{-x^2}dy = 0 ,$$

which is exact .

We now determine a function  $u(x, y)$  that satisfies:

$$\frac{\partial u(x, y)}{\partial x} = M(x, y) = 2xye^{-x^2} - xe^{-x^2} \quad \dots \quad (1)$$

$$\text{and} \quad \frac{\partial u(x, y)}{\partial y} = N(x, y) = -e^{-x^2} \quad \dots \quad (2)$$

First solve Equation (1) , we obtain:

$$du = (2xye^{-x^2} - xe^{-x^2})dx$$

Integrating both sides of this equation with respect to  $x$  , we find :

$$\int du = \int (2xye^{-x^2} - xe^{-x^2})dx = [-x^2 = u \Rightarrow 2xdx = -du] \Rightarrow$$

$$\Rightarrow u(x, y) = -ye^{-x^2} + \frac{1}{2}e^{-x^2} + h(y) \quad \dots \quad (3)$$

Note that when integrating with respect to  $x$ , the constant of integration can depend on  $y$ .

We now determine  $h(y)$ . Differentiating (3) with respect to  $y$ , we obtain:

$$\frac{\partial u}{\partial y} = -e^{-x^2} + h'(y)$$

Substituting this equation along with  $N(x, y) = -e^{-x^2}$  into (2), we have:

$$\frac{\partial u}{\partial y} = -e^{-x^2} + h'(y) = N = -e^{-x^2} \Rightarrow h'(y) = 0 \Rightarrow dh = 0$$

Integrating this last equation with respect to  $y$ , we obtain:

$$\int dh = c_1 \Rightarrow h(y) = c_1, \quad (c_1 = \text{constant}).$$

Substituting this expression into (3) yields:

$$\Rightarrow u(x, y) = -ye^{-x^2} + \frac{1}{2}e^{-x^2} + c_1 = c$$

The solution to the differential equation, which is given implicitly by  $u(x, y) = c$ , is:

$$\Rightarrow u(x, y) = -ye^{-x^2} + \frac{1}{2}e^{-x^2} = c_2, \quad c_2 = c - c_1$$

Solving for  $y$  explicitly, we obtain the solution as:

$$y = \frac{1}{2} - c_2 e^{x^2}$$

**Example (7) : Convert the following differential equation into an exact differential equation and solve it .**

$$y^2 dx + xy dy = 0$$

**Solution :**

Here  $M(x, y) = y^2$  and  $N(x, y) = xy$

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = y$$

Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , the differential equation is not exact.

Equation however, is applicable and provides the integrating factor:

$$\frac{1}{M} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{y^2} (2y - y) = \frac{1}{y} \equiv h(y)$$

is a function of  $y$  alone. Thus, the integrating factor is :

$$I(x, y) = e^{-\int h(y) dy} = e^{-\int \frac{dy}{y}} = e^{-\ln y} = \frac{1}{y}$$

Multiplying the differential equation by  $I(x, y) = \frac{1}{y}$  we obtain:

$$y dx + x dy = 0 \quad \text{which is exact .}$$

We now determine a function  $u(x, y)$  that satisfies:

$$\frac{\partial u(x, y)}{\partial x} = M(x, y) = y \quad \dots \quad (1)$$

$$\text{and} \quad \frac{\partial u(x, y)}{\partial y} = N(x, y) = x \quad \dots \quad (2)$$

First solve Equation (2), we obtain:

$$du(x, y) = x dy$$

Integrating both sides of this equation with respect to  $x$ , we find :

$$\int du = \int x dy \Rightarrow u = xy + k(x) \quad \dots \quad (3)$$

Note that when integrating with respect to  $x$ , the constant of integration can depend on  $y$ .

We now determine  $k(x)$ . Differentiating (3) with respect to  $y$ , we obtain :

$$\frac{\partial u}{\partial x} = y + k'(x)$$

Substituting this equation along with  $M(x, y) = y$  into (2), we have :

$$\frac{\partial u}{\partial x} = y + k'(x) = M = y \Rightarrow k'(x) = 0 \Rightarrow dk = 0$$

Integrating this last equation with respect to  $y$ , we obtain :

$$\int dk = c_1 \Rightarrow k(x) = c_1, \quad (c_1 = \text{constant}).$$

Substituting this expression into (3) yields:

$$\Rightarrow u(x, y) = xy + c_1 = c$$

The solution to the differential equation, which is given implicitly by  $u(x, y) = c$ , is :

$$\Rightarrow u(x, y) = xy = c_2, \quad c_2 = c - c_1$$

Solving for  $y$  explicitly, we obtain the solution as:  $y = \frac{c_2}{x}$

**Example (8): Convert  $y' = \frac{xy^2 - y}{x}$  into an exact differential equation.**

**Solution :**

Rewriting this equation in differential form, we have :

$$(y - xy^2)dx + xdy = 0 \dots\dots (1)$$

Here  $M(x, y) = y - xy^2$  and  $N(x, y) = x$

$\frac{\partial M}{\partial y} = 1 - 2xy$  and  $\frac{\partial N}{\partial x} = 1$ . Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , The differential equation is not

exact. Equation however, is applicable and provides the integrating factor:

$$I(x, y) = \frac{1}{xM - yN}$$

$$I(x, y) = \frac{1}{xM - yN} = \frac{1}{x(y - xy^2) - xy} = \frac{-1}{x^2 y^2}$$

Multiplying the differential equation by  $I(x, y) = \frac{-1}{x^2 y^2}$  we obtain :

$$\frac{xy - 1}{x^2 y} dx - \frac{1}{xy^2} dy = 0 \quad \text{which is exact .}$$

We now determine a function  $u(x, y)$  that satisfies:

$$\frac{\partial u(x, y)}{\partial x} = M(x, y) = \frac{1}{y} - \frac{x^{-2}}{y} \quad \dots \quad (1)$$

$$\text{and} \quad \frac{\partial u(x, y)}{\partial y} = N(x, y) = -\frac{y^{-2}}{x} \quad \dots \quad (2)$$

First solve Equation (2) , we obtain:  $du(x, y) = -\frac{y^{-2}}{x} dy$

Integrating both sides of this equation with respect to  $x$  , we find :

$$\int du = \int -\frac{y^{-2}}{x} dy \Rightarrow u = \frac{x}{y} + k(x) \quad \dots \dots (3)$$

Note that when integrating with respect to  $x$  , the constant of integration can depend on  $y$  . We now determine  $k(x)$  . Differentiating (3) with respect to

$y$  , we obtain:  $\frac{\partial u}{\partial x} = \frac{1}{y} + k'(x)$

Substituting this equation along with  $M(x, y) = \frac{1}{y} - \frac{x^{-2}}{y}$  into (2) , we have :

$$\frac{\partial u}{\partial x} = \frac{1}{y} + k'(x) = M = \frac{1}{y} - \frac{x^{-2}}{y} \quad \Rightarrow \quad k'(x) = -\frac{x^{-2}}{y} \quad \Rightarrow$$

Integrating this last equation with respect to  $y$  , we obtain :

$$\int dk = \int -\frac{x^{-2}}{y} dx \Rightarrow k(x) = \frac{1}{xy} + c_1, \quad (c_1 = \text{constant}).$$

Substituting this expression into (3) yields:  $u = \frac{x}{y} + k(x) = \frac{x}{y} + \frac{1}{xy} + c_1 = c$

The solution to the differential equation, which is given implicitly by  $u(x, y) = c$ , is :

$$\frac{x}{y} + \frac{1}{xy} = c_2$$

Solving for  $y$  explicitly, we obtain the solution as :  $y = \frac{x^2 + 1}{c_2 x}$