وزارة التعليم العالي والبحث العلمي
الجامعة التقنية الشمالية
الكلية التقنية الهندسية

الحقيبة التعليمية

القسم العلمي:	هندسة تقنيات الصناعات الكيمياوية والنفطية
اسم المقرر:	Engineering Mechanics & Strength of Materials
رمز المقرر:	TEC0102
المرحلة / المستوى:	1st level
الفصل الدراسي:	Second Semester
السنة الدراسية:	2025 - 2024

معلومات عامة

اسم المقرر:	**الميكانيك الهندسي ومقاومة مواد**				
القسم:	**هندسة تقنيات الصناعات الكيمياوية والنفطية**				
الكلية:	الكلية التقنية الهندسية				
المرحلة / المستوى	الاول				
الفصل الدراسي:	الثاني				
عدد الساعات الاسبوعية:	نظري	3	عملي	3	
عدد الوحدات الدراسية:	6				
الرمز:	**TECO102**				
نوع المادة	نظري		عملي		كلهما √
هل يتوفر نظير للمقرر في الاقسام الاخرى	لا يوجد				
اسم المقرر النظير					
القسم					
رمز المقرر النظير					
معلومات تدريسي المادة					
اسم مدرس (مدرسي) المقرر:	**عماد توما بني**				
اللقب العلمي:	**أستاذ**				
سنة الحصول على اللقب	2023				
الشهادة :	**دكتوراه**				
سنة الحصول على الشهادة	**2012**				
عدد سنوات الخبرة (تدريس)	**38**				

الوصف العام للمقرر

يفهم الطالب مبادئ الميكانيك الهندسي ومقاومة المواد وكيفية أجراء الحسابات التصميمية الخاصة بكل جزء عند حصول الانهيار بسبب القوى الخارجية أو البنية من خلال الاجهادات التي تتكون في ذلك الجزء.

الاهداف المعرفية

1. يتعرف على المفاهيم لمبادئ الميكانيك الهندسي ومقاومة المواد
2. توسيع مدارك الطلبة وتعزيز مفهوم التصميم من خلال إعطائهم مبادئ وحسابات التصميمية.
3. مساعدتهم في تطوير المهارات الوثيقة الصلة في عملية التصميم التي تتضمن صياغة المشاكل، التفكير المبدع، الاتصال الفعال وتحليل المعلومات والعمل الجماعي.

الأهداف المهارتية

1. دراسة تفصيلية للتصميم الهندسي لمبادئ الميكانيك الهندسي ومقاومة المواد
2. دراسة التفاصيل الرياضية التي يحتاجها الطالب خلال دراسة مادة الميكانيك الهندسي ومقاومة المواد
3. إعداد المهندس التقني ليكون مهندسا ناجحا من خلال تعلم المبادئ الصحيحة لتخصص الصناعات الكيمياوية والنفطية والمواد المساعدة لها.

الأهداف الوجدانية والقيمية

1. تعليم الطالب النظام والنظافة
2. تعليم الصبر والمطاولة
3. اكتساب صفة حسن الخلق والتعامل الجيد مع اقرانه.

الأهداف السلوكية او نواتج التعلم

- تنمية مهارة الدقة في مبادئ الميكانيك الهندسي ومقاومة المواد
- تنمية مهارة التعاون ونظام البديل
- تمكين الطلبة من مادة الميكانيك الهندسي ومقاومة المواد في جوانبها التطبيقية و المعرفية
- تطوير قدرة الطالب في تحليل المعلومات و تفسير البيانات التي حصل عليها من خلال أجراء الحسابات المختلفة في مادة الميكانيك الهندسي ومقاومة المواد.
- تمكين الطالب من أجراء المسح الميداني لتحديد المشاكل وحلها على ارض الواقع .

المتطلبات السابقة

- دراسة مادة الرياضيات
- دراسة مادة خواص المواد
- دراسة مادة الرسم الهندسي

الأهداف السلوكية او مخرجات التعليم الأساسية		
ت	تفصيل الهدف السلوكي او مخرج التعليم	آلية التقييم
1	المجال المعرفي (المجال العقلي أو الادراكي)	تذكر المتعلم المعلومات او المعارف او الحقائق او المفاهيم او النظريات او القوانين التي تعلمها سابقاً
2	المجال الوجداني (المجال العاطفي أو الانفعالي)	الانتباه و التقبل مستوى الاستجابة أهمية المادة في مرحلته الدراسية
3	المجال النفس حركي (المهاري أو الحركي)	التقليد (المحاكاة) الاداء الحركي للمهارة الأداء الذي يتطلب التناسق أداء المهارات الحركية المركبة الأداء الطبيعي للمهارة (البسيطة أو المركبة

أساليب التدريس (حدد مجموعة متنوعة من أساليب التدريس لتناسب احتياجات الطلاب ومحتوى المقرر)

ت	الاسلوب أو الطريقة	مبررات الاختيار
1.	المحاضرة بالأساليب الحديثة	شرح المحاضرة بالتفصيل
.	عرض فيديوهات	استخدام التكنولوجيا مثل مشاهدة فيديوهات توضيحية تبين المغزى والمقصود من الفكرة
3.	التعليم القائم	وجود مشكلة حيث يتم منح الطالب مشكلة فيبحث عن حلها وفي نفس الوقت يتعلم وهذه من أفضل الطرق غير المباشرة
4.	التعلم النشط	إجراء بعض الأنشطة والتجارب المفيدة حتى تصل المعلومة بشكل أسرع
5.	طريقة العصف الذهني	لتنمية القدرات الذهنية والابداعية لدى الطلبة
6.	طريقة المجموعات	وذلك من خلال تشكيل مجموعات من الطلبة لحل المسائل المتعلقة بالمقرر والمناقشة بين الطلبة في الطرق المختلفة لحل تلك المسائل

<table>
<tr><td colspan="7">الفصل الاول من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد</td></tr>
<tr><td>عنوان الفصل</td><td colspan="2">الوقت</td><td colspan="4">Introduction, Composition and Resolution of forces, and Resultants of Force Systems</td></tr>
<tr><td>التوزيع الزمني</td><td>النظري</td><td>العملي</td><td>العنوان الفرعي</td><td>طريقة التدريس</td><td>التقنيات</td><td>طرق القياس</td></tr>
<tr><td rowspan="2">الأسبوع الأول</td><td>3</td><td></td><td>مقدمة ، تحليل القوى ، المحصلة ، أنظمة القوى</td><td>محاضرة</td><td>عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة</td><td>التقييم الذاتي
تقييم الاقران</td></tr>
<tr><td></td><td>3</td><td>حل تمارين الفصل</td><td>حل تمارين واسئلة الفصل</td><td>عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل</td><td>الامتحانات</td></tr>
</table>

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<tr><td colspan="7">الفصل الثاني من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد</td></tr>
<tr><td>عنوان الفصل</td><td colspan="2">الوقت</td><td colspan="4">Moment of a Forces and Moment of a Couples</td></tr>
<tr><td>التوزيع الزمني</td><td>النظري</td><td>العملي</td><td>العنوان الفرعي</td><td>طريقة التدريس</td><td>التقنيات</td><td>طرق القياس</td></tr>
<tr><td rowspan="2">الأسبوع الثاني</td><td>3</td><td></td><td>عزم القوى ، عزم المزدوجات</td><td>محاضرة</td><td>عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة</td><td>التقييم الذاتي
تقييم الاقران</td></tr>
<tr><td></td><td>3</td><td>حل تمارين الفصل</td><td>حل تمارين واسئلة الفصل</td><td>عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل</td><td>الامتحانات</td></tr>
</table>

الفصل الثالث من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد						
عنوان الفصل	الوقت		Equilibrium			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع الثالث	3		التوازن	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
		3	حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

الفصل الرابع من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد						
عنوان الفصل	الوقت		Centroids and Centers of Gravity			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع الرابع	3		ايجاد مركز الثقل	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
		3	حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

الفصل الخامس من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد

عنوان الفصل	الوقت		Moments of Inertia			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع الخامس	3	3	عزم القصور	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
			حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

الفصل السادس من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد

عنوان الفصل	الوقت		Friction			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع السادس والسابع	3		الاحتكاك	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
		3	حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

الفصل السابع من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد

عنوان الفصل	الوقت		Truss Structural analysis			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع الثامن والتاسع	3		تحليل القوى على الكمرات	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
		3	حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

الفصل الثامن من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد

عنوان الفصل	الوقت		Strength of Materials			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع العاشر والحادي عشر	3		مقدمة ، الاجهاد ، الانفعال ، انواع القوى ، انواع العوارض ، رسم بياني لقوة وعزم القص	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
		3	حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

الفصل التاسع من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد						
عنوان الفصل	الوقت		Deflection			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع الثاني عشر والثالث عشر	3		الانحناء	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
		3	حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

الفصل العاشر من المحتوى العلمي لمادة الميكانيك الهندسي ومقاومة المواد						
عنوان الفصل	الوقت		Theories of failure			
التوزيع الزمني	النظري	العملي	العنوان الفرعي	طريقة التدريس	التقنيات	طرق القياس
الأسبوع الرابع عشر والخامس عشر	3		نظريات الانهيار	محاضرة	عرض تقديمي، شرح، أسئلة وأجوبة، مناقشة	التقييم الذاتي تقييم الاقران
		3	حل تمارين الفصل	حل تمارين واسئلة الفصل	عرض الاسئلة، شرح طريقة الحل، محاولة الطلبة في حل التمارين، حل ومناقشة تمارين الفصل	الامتحانات

المحتوى العلمي

خارطة القياس المعتمدة

المحتوى التعليمي	عناوين الفصول	الأهمية النسبية	الأهداف السلوكية					عدد الفقرات
			النسبة					
			المعرفة	الفهم	التطبيق	التحليل	التقييم	
الفصل الاول	Introduction, Composition and Resolution of forces, and Resultants of Force Systems	10 %	10 %	25 %	30 %	10 %	25 %	7
الفصل الثاني	Moment of a Forces and Moment of a Couples	10 %	10 %	25 %	30 %	10 %	25 %	7
الفصل الثالث	Equilibrium	10 %	10 %	25 %	30 %	10 %	25 %	6
الفصل الرابع	Centroids and Centers of Gravity	10 %	10 %	25 %	30 %	10 %	25 %	8
الفصل الخامس	Moments of Inertia	10 %	10 %	25 %	30 %	10 %	25 %	10
الفصل السادس	Friction	10 %	10 %	25 %	30 %	10 %	25 %	4
الفصل السابع	Truss Structural analysis	10 %	10 %	25 %	30 %	10 %	25 %	10
الفصل الثامن	Strength of Materials	10 %	10 %	25 %	30 %	10 %	25 %	12
الفصل التاسع	Deflection	10 %	10 %	25 %	30 %	10 %	25 %	8
الفصل العاشر	Theories of failure	10 %	10 %	25 %	30 %	10 %	25 %	7
المجموع	8	100 %	10 %	25 %	30 %	10 %	25 %	**79**

Chapter 1

Introduction, Composition ,

Resolution of forces, and Resultants

of Force Systems

1.1 Introduction

Mechanics: is a branch of the physical science that is concerned with the state of rest or motion of bodies that are subjected to the action of force. Objects of interest in sport biomechanics are human body and sport equipment. According to the nature of studied objects mechanics is divided into several branches (Figure 1).

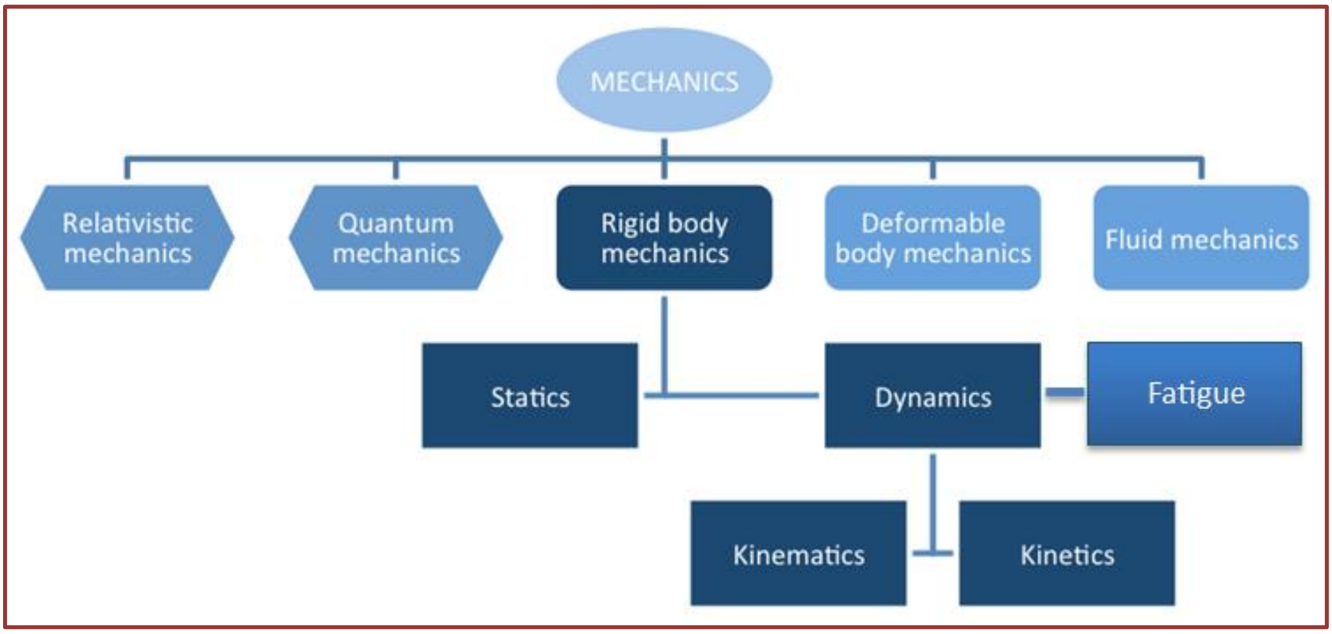


Figure 1. Branches of mechanics divided according to the nature of studied objects, and the division of rigid body mechanics

Static Mechanics

Statics is a branch of engineering mechanics that deals with the analysis of forces and interactions of bodies in equilibrium.

Dynamic Mechanics

Dynamic is concerned with the accelerated motion of bodies under effects of external forces.

Fatigue Mechanics

Fatigue is a failure mechanism that involves the cracking of materials and structural components due to cyclic (or fluctuating) stress.

1.2. Composition of Forces

The process of finding out the resultant force of a number of given forces is called the composition/compounding of forces.

Parallelogram Law

If two forces acting simultaneously on a particle is represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant may be represented in magnitude and direction by the diagonal of the parallelogram which passes through the point of intersection, (figure 2).

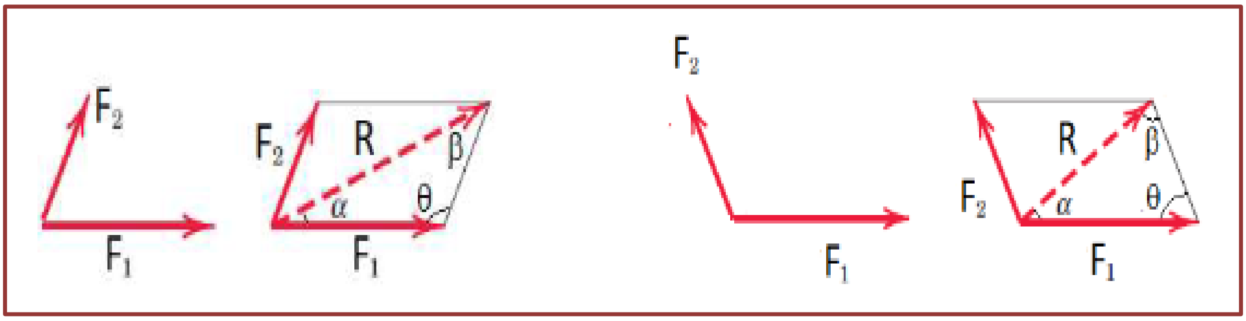


Figure 2. Parallelogram Law

The resultant of a pair of concurrent forces can be determined by the following equation:

$$\textit{Resultant}, R = \sqrt{F_1^2 + F_2^2 - 2F_1 . F_2 \cos\theta}$$

$$\alpha = \tan^{-1}\left(\frac{F_2^2 \sin\theta}{F_1^2 + F_2^2 \cos\theta}\right)$$

2. Triangle Law

Additionally, this equation can be used to determine the direction of the resultant or the unknown forces, (figure 3):

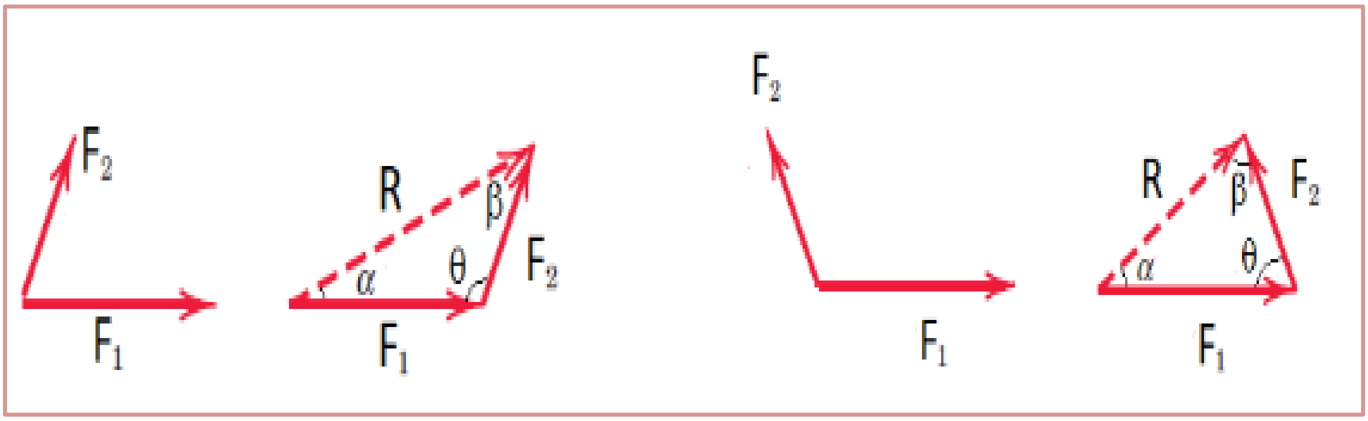


Figure 3. Triangle Law

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$

1.3. Resolution of a Force

The process of substituting a force by its components so that the net effect on the body remains the same is known as resolution of a force.

For each force, there exists an infinite number of possible sets of components.

Suppose a force is to be resolved into two components.

Then:

1. When one of the components is known, the second component can be obtained by applying the triangle rule.

2. When the line of action of each component is known, the magnitude and the sense of the components are obtained by parallelogram law.

1.4. Principle of Resolution

The algebraic sum of the resolved parts of a number of forces in the given direction is equal to the resolved part of their resultant in the same direction.

Replace a single force with its components through the process of resolution.

If a force (F) lies in the plane (x-y). The force (F) may be resolved into two rectangular components. The component of a force parallel to the x-axis is called the Horizontal component (F_x), and parallel to y-axis the is called Vertical component (F_y).

As an illustration of the following force analysis on two axes , (figure 4):

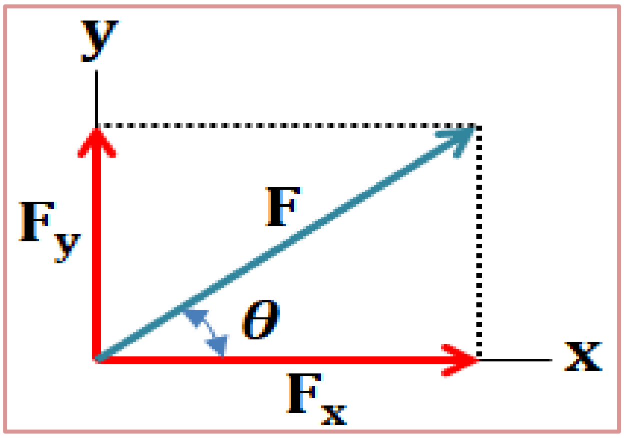


Figure 4. Force analysis on two axes

$$sin\theta = \frac{F_y}{F} \quad \Leftrightarrow \quad \boldsymbol{F_y = F\ sin\theta} \ \ \&\ \ cos\theta = \frac{F_x}{F} \quad \Leftrightarrow \quad \boldsymbol{F_x = F\ cos\theta}$$

$$R = \sqrt{F_x^2 + F_Y^2}$$

$$tan\theta = \frac{F_y}{F_x} \quad \Leftrightarrow \quad \boldsymbol{\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)}$$

1.5. Resultant of Force Systems

Resultant: Simplest force system which have same external effect of the original system.

1.5.1. Resultant of Coplanar Concurrent Force System

In x-y plane, the resultant of coplanar concurrent force system where the lines of action of all forces pass through a common point can be found by the following formulas:

$$R_x = \sum F_x \quad \rightarrow^+$$

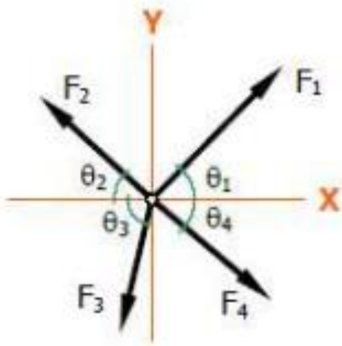
$$R_x = F_{1x} - F_{2x} - F_{3x} + F_{4x}$$

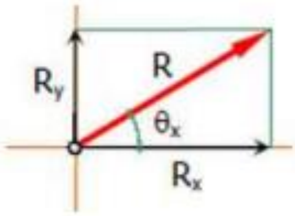
$$R_y = \sum F_y \quad \uparrow^+$$

$$R_y = F_{1y} + F_{2y} - F_{3y} - F_{4y}$$

$$R = \sqrt{{R_x}^2 + {R_y}^2}$$

$$\theta_x = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

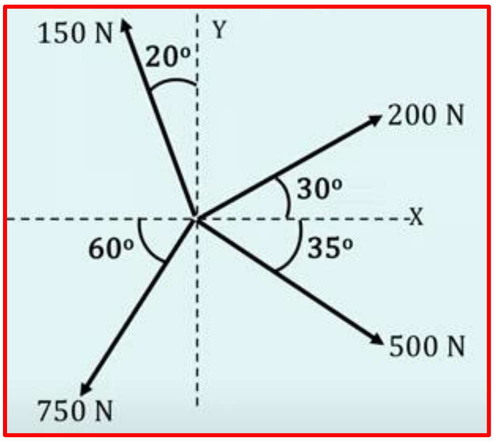


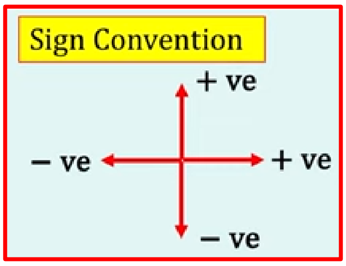


1.6. Solve examples

Example - 1

Calculate the magnitude and direction of resultant vector that is formed when taking the sum of the six forces shown below?





Solution:

First Method

$$\mathbf{\Sigma F_x} = F_1.(\cos 30^\circ) - F_2.(\sin 20^\circ) - F_3.(\cos 60^\circ) + F_4.(\cos 35^\circ)$$

$$\mathbf{\Sigma F_x} = 200 \times 0.866 - 150 \times 0.342 - 750 \times 0.5 + 500 \times 0.819$$
$$= 173.2 - 51.3 - 375 + 409.5 = \mathbf{156.4\ N}$$

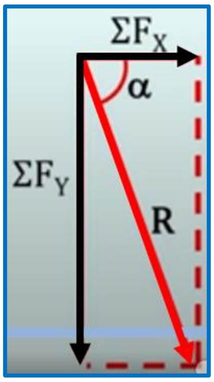
$$\mathbf{\Sigma F_y} = F_1.(\sin 30^\circ) + F_2.(\cos 20^\circ) - F_3.(\sin 60^\circ) - F_4.(\sin 35^\circ)$$
$$\mathbf{\Sigma F_y} = 200 \times 0.5 + 150 \times 0.94 - 750 \times 0.866 + 500 \times 0.574$$
$$= 100 + 141 - 649.5 - 287 = \mathbf{-695.5\ N}$$

Second Method

NO.	*Description*	ΣF_x *(N)*	ΣF_y *(N)*
1.	$200 \llcorner 30^o$	$200 \cos 30^o = 173.21$	$200 \sin 30^o = 100$
2.	$150 \llcorner 110^o$	$150 \cos 110^o = -51.3$	$150 \sin 110^o = 140.95$
3.	$750 \llcorner 240^o$	$750 \cos 240^o = -375$	$750 \sin 240^o = -649.52$
4.	$500 \llcorner 325^o$	$500 \cos 325^o = 409.58$	$500 \sin 325^o = -286.79$
Sum		**156.49**	**-695.34**

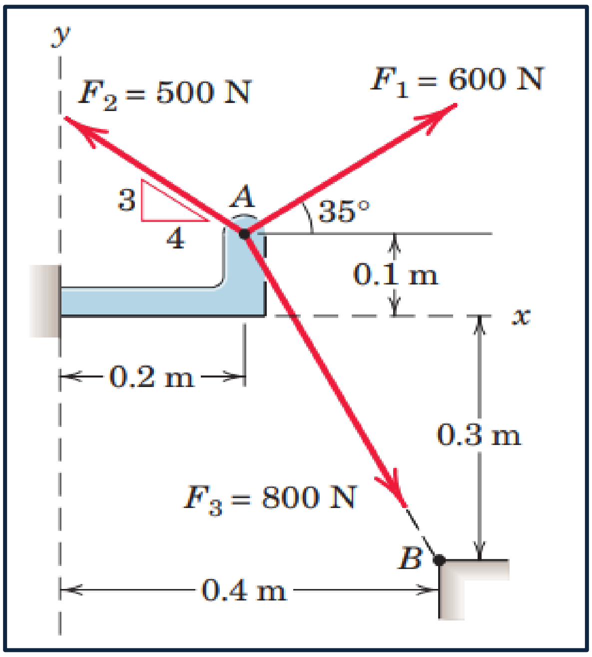
$$R = \sqrt{\Sigma F_x^2 + \Sigma F_y^2} = \sqrt{156.4^2 + 695.5^2} = \mathbf{712.87\ N}$$

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{695.5}{156.4}\right) = \mathbf{77.33^\circ}$$



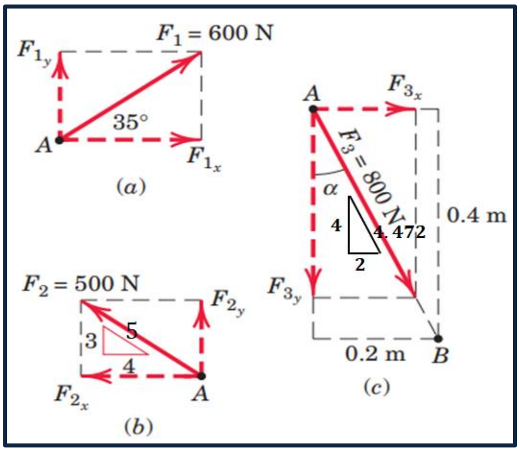
Example - 2

Calculate the magnitude and direction of resultant vector that is formed when taking the sum of the three forces act on point A, shown below?



Solution:

Draw free body diagram all forces, as the following.



First Method

From the figure:

$$\alpha = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{0.2}{0.4}\right) = 26.565^{\circ}$$

$$\boldsymbol{\Sigma F_x} = F_1.(cos\,35^{\circ}) - F_2.(\frac{4}{5}) + F_3.\sin 26.565^{\circ}$$

$$\boldsymbol{\Sigma F_x} = 600 \times 0.819 - 500 \times 0.8 + 800 \times 0.447$$

$$= 491.4 - 400 + 375.6 = \boldsymbol{449\ N}$$

$$\boldsymbol{\Sigma F_y} = F_1.(sin\,35^{\circ}) + F_2.(\frac{3}{5}) - F_3.\cos 26.565^{\circ}$$

$$\boldsymbol{\Sigma F_y} = 600 \times 0.574 + 500 \times 0.6 - 800 \times 0.894$$

$$= 344.4 + 300 - 715.2 = \boldsymbol{-70.8\ N}$$

Second Method

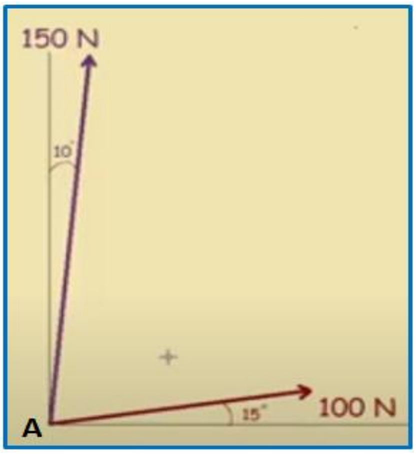
NO.	Description	ΣF_x (N)	ΣF_y (N)
1.	600 ∟35^o	$600\cos 35^o = 491.49$	$600\sin 35^o = 344.4$
2.	500 ∟143.13^o	$500\cos 143.13^o = -400$	$500\sin 143.13^o = 300$
3.	800 ∟296.565^o	$800\cos 296.565^o = 357.77$	$800\sin 296.565^o = -715.2$
Sum		**449.26**	**-70.8**

$$\boldsymbol{R} = \sqrt{\boldsymbol{\Sigma F_x^2 + \Sigma F_y^2}} = \sqrt{449^2 + (-74.8)^2} = \boldsymbol{455.188\ N}$$

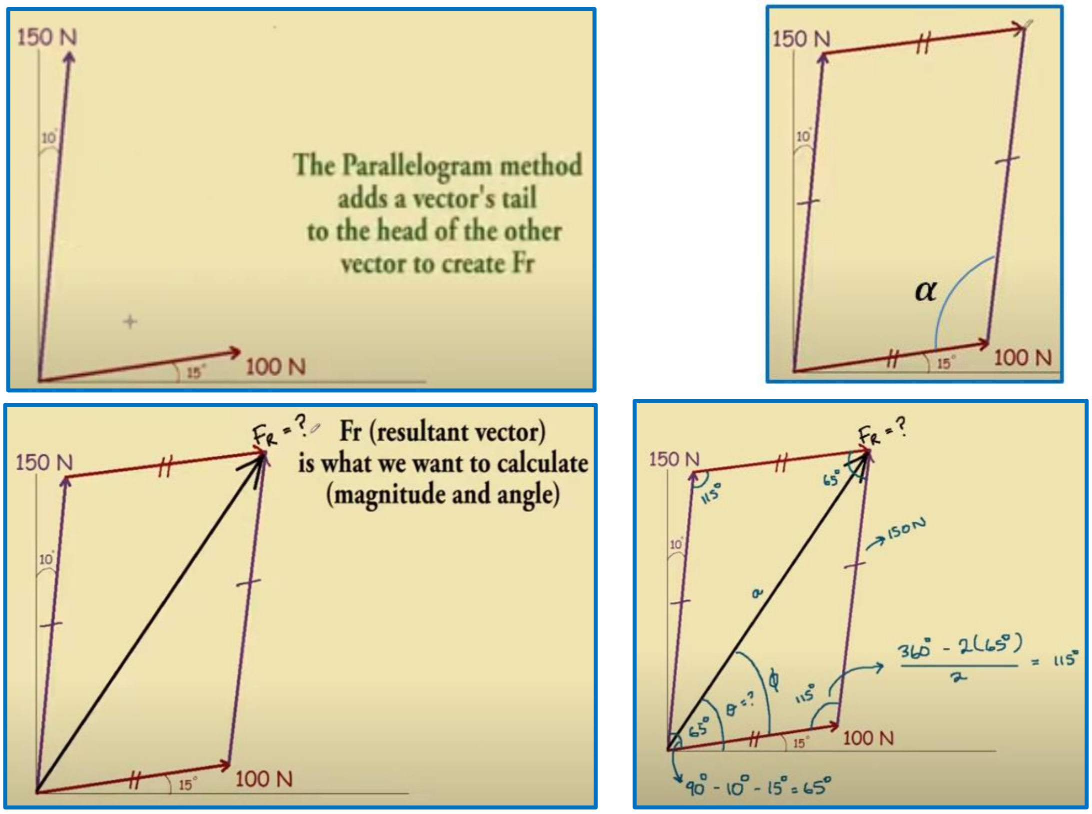
$$\boldsymbol{\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)} = \tan^{-1}\left(\frac{-70.8}{449}\right) = \boldsymbol{8.96^{\circ}}$$

Example - 3

Calculate the magnitude and direction of resultant vector that is formed when taking the sum of the two forces act on point A, shown below?



Solution:



$$Resultant, R = \sqrt{F_1^2 + F_2^2 - 2F_1F_2cos\alpha}$$

$$R = \sqrt{100^2 + 150^2 - 2 \times 100 \times 150 \times \cos 115^{\circ}} = 212.55\ N$$

From Sine Rule

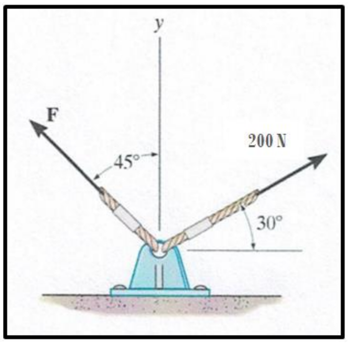
$$\frac{R}{\sin\theta} = \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha}$$

$$\frac{212.55}{\sin 115} = \frac{150}{\sin\emptyset}$$

$$\emptyset = \sin^{-1}\frac{150\sin 115}{212.55} = \sin^{-1}(0.64) = 39.79^{o}$$

Example - 4

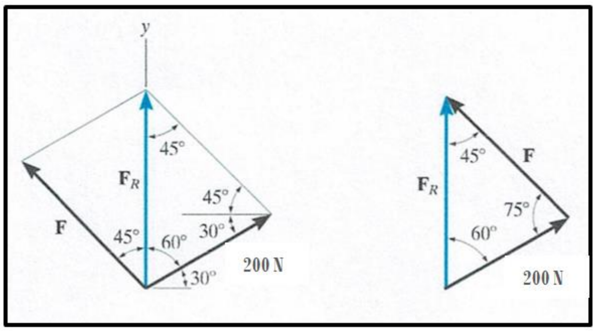
Calculate the magnitude of a force (F). also magnitude and direction of resultant vector that is formed when taking the sum of the two shown below?



__**Solution:**__

From Sine Rule

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$



$$\frac{F}{\sin 60°} = \frac{200}{\sin 45°}$$

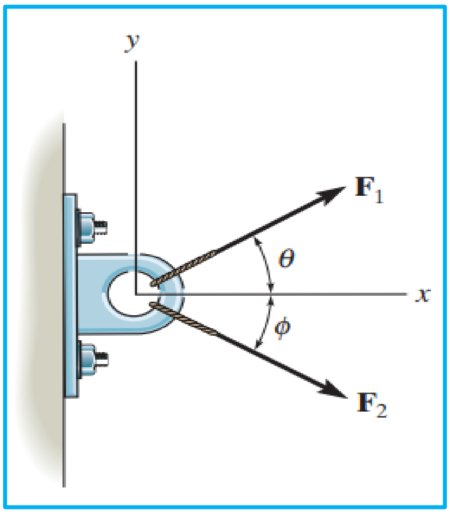
$$F = 245 \text{ N}$$

$$\frac{F_R}{\sin 75°} = \frac{200}{\sin 45°}$$

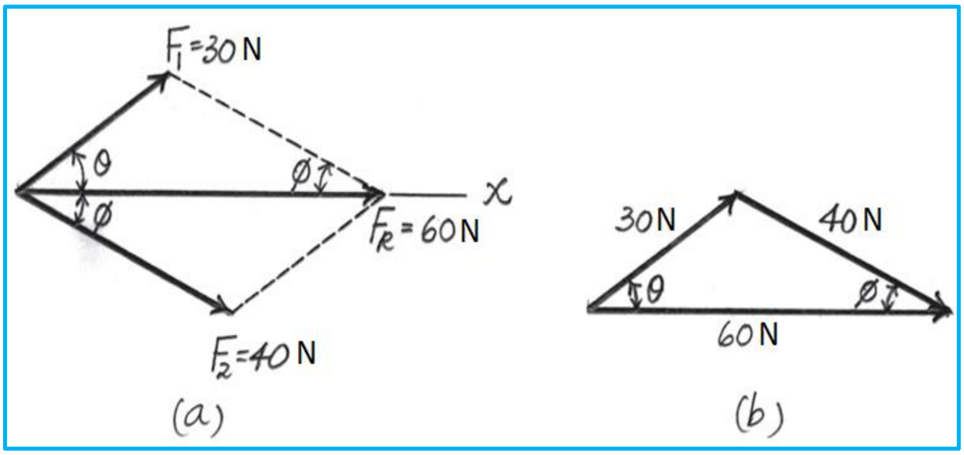
$$F_R = 273 \text{ N}$$

Example - 5

If F_1 = 30 N and F2 = 40 N, determine the angles u and f so that the resultant force is directed along the positive x axis and has a magnitude of F_R = 60 N.



Solution:



Parallelogram Law. The parallelogram law of addition is shown in Fig. *a*.
Trigonometry. Applying the law of cosine by referring to Fig. *b*,

$$40^2 = 30^2 + 60^2 - 2(30)(60)\cos\theta$$

$$\theta = 36.34° = 36.3°$$

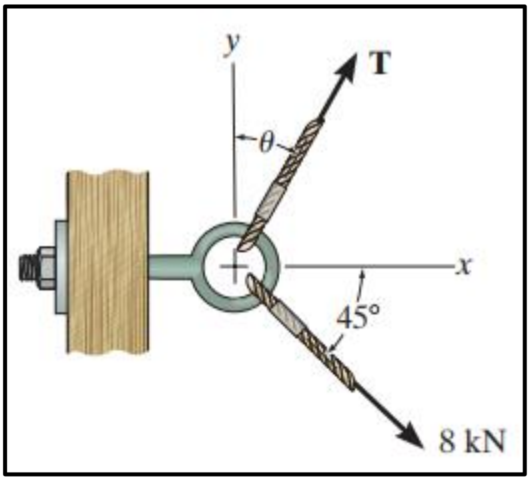
And

$$30^2 = 40^2 + 60^2 - 2(40)(60)\cos\phi$$

$$\phi = 26.38° = 26.4°$$

1.7. Chapter Questions

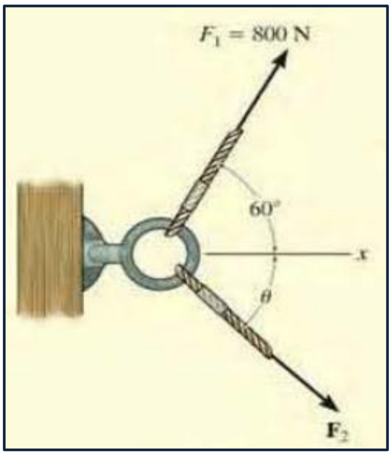
1. If the magnitude of the resultant force is to be (9 KN) directed along the positive x - axis, determine the magnitude of force (T) acting on the eyebolt and its angle.



{Results: $T = 6.57\ KN$; $\theta = 30.6^{\circ}$; $\emptyset = 75.6^{\circ}$}

---

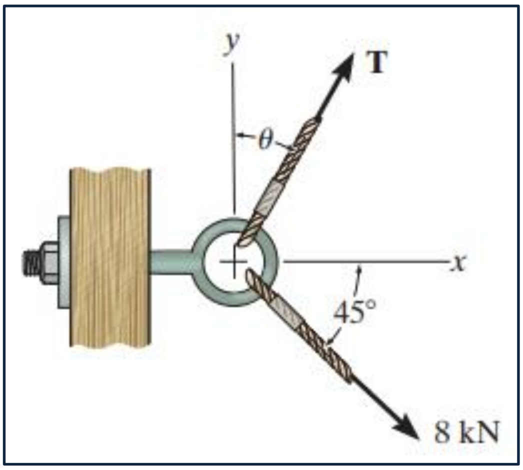
2. It is required that the resultant force acting on the eyebolt in Figure be directed along the positive axis and that (F2) have a minimum magnitude. Determine this magnitude. the angle (θ), and the corresponding resultant force.



{Results: $T = 6.57\ KN$; $\theta = 90^{\circ}$ }

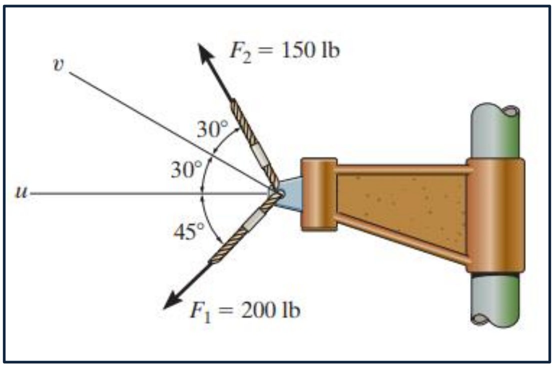
---

3. If ($\theta = 30^{\circ}$) and ($T = 6$ KN) , determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis.



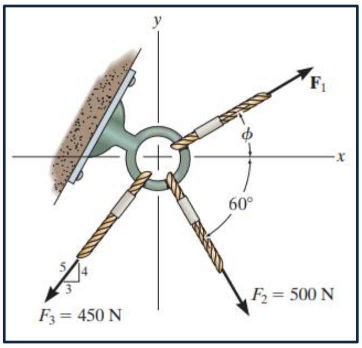
{Results: $F_R = 8.67\ KN$; $\alpha = 63.05^{\circ}$; $\emptyset = 3.05^{\circ}$}

4. Determine the magnitude of the resultant force acting on the bracket and its direction measured counterclockwise from the positive u axis.



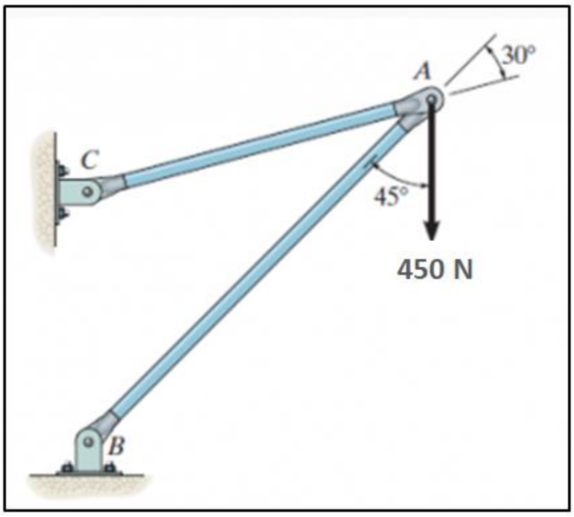
{Results: $F_R = 217\ N$; $\alpha = 63.05^{\circ}$; $\emptyset = 3.05^{\circ}$}

5. If ($F_1 = 600\ N$) and ($\emptyset = 30^{\circ}$) , determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis



{Results: $F_R = 701.91\ N\ ; \theta = 44.06^{\circ}$ }

6. The force (F = 450 N) acts on the frame. Resolve this force into components acting along members AB and AC , and determine the magnitude of each component.



{Results: $F_{AB} = 86\ N\ ; F_{AC} = 636\ N$ }

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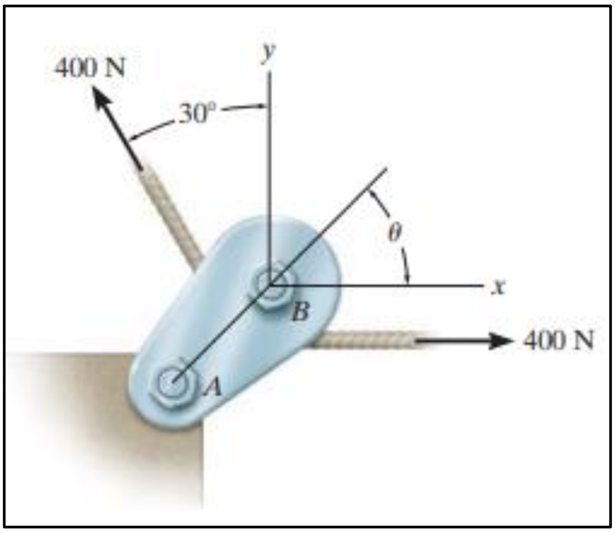
7. If the tension in the cable is 400 N, determine the magnitude and direction of the resultant force acting on the pulley. This angle is the same angle of line AB on the tailboard block.

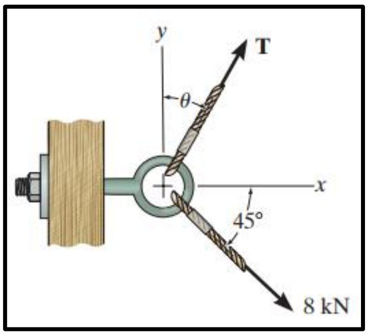


{Results: $R = 400\ N\ ; \theta = 60^{o}$}

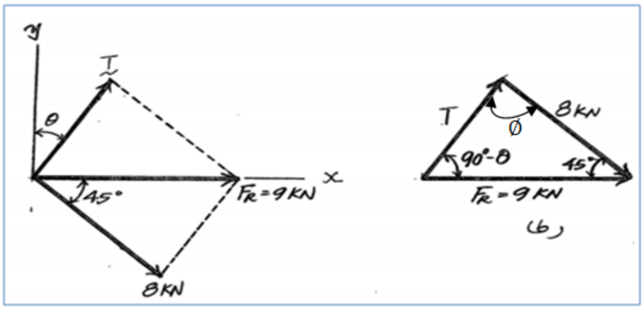
Solve Question
Home Work - 1

1.7. Chapter Questions

8. If the magnitude of the resultant force is to be (9 KN) directed along the positive x - axis, determine the magnitude of force (T) acting on the eyebolt and its angle.



Solution



The parallelogram law of addition and the triangular rule are shown in figures (a & b), respectively.

$$R = \sqrt{F_1^2 + F_2^2 - 2F_1 . F_2 \cos\theta}$$

$$T = \sqrt{8^2 + 9^2 - 2 \times 8 \times 9 \times \cos 45} = 6.57\ KN$$

Appling the law of sine's to figure b. and using this result yield:

$$\frac{R}{\sin\theta} = \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha}$$

$$\frac{6.57}{\sin 45^o} = \frac{8}{\sin(90^o - \theta)} = \frac{9}{\sin\emptyset}$$

$$\frac{6.57}{\sin 45^o} = \frac{9}{\sin\emptyset}$$

$$\sin\emptyset = \frac{9 \sin 45^o}{6.57} = 0.968$$

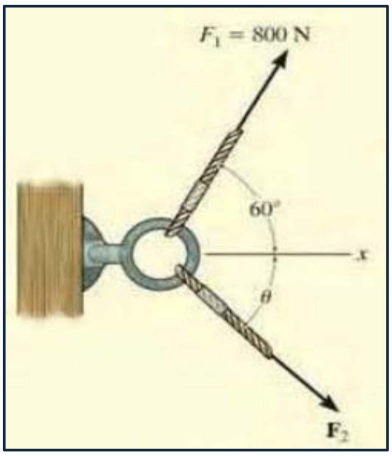
$$\emptyset = \sin^{-1}(0.968) = 75.47^o$$

$$90^o - \theta = 180 - 75.47 - 45$$

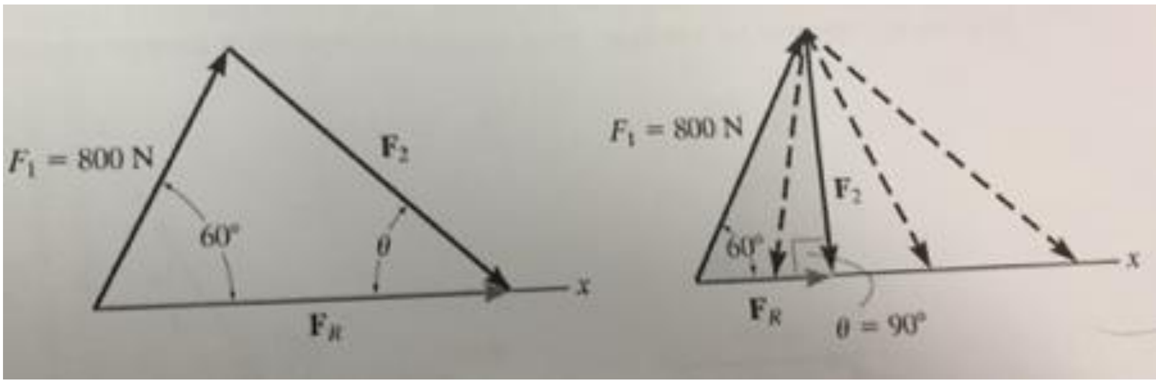
$$90^o - \theta = 119.5$$

$$\theta = 119.5 - 90 = 29.5^o$$

9. It is required that the resultant force acting on the eyebolt in Figure be directed along the positive axis and that (F2) have a minimum magnitude. Determine this magnitude. the angle (θ), and the corresponding resultant force.



Solution



F_2 is a minimum or the shortest length when its line of action is perpendicular to the line of action of F_R, that is, when: $\theta = 90^{\circ}$

$$\frac{800}{sin90^o} = \frac{F_R}{sin30^o} = \frac{F_2}{sin60^o}$$

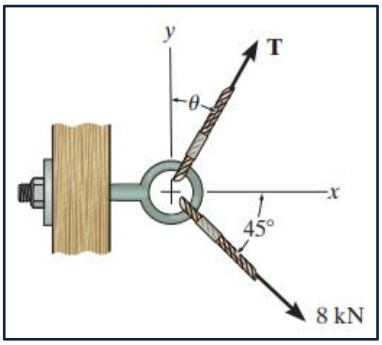
$$\frac{800}{1} = \frac{F_R}{0.5} = \frac{F_2}{0.866}$$

$$F_R = 800 \times 0.5 = 400\ N$$

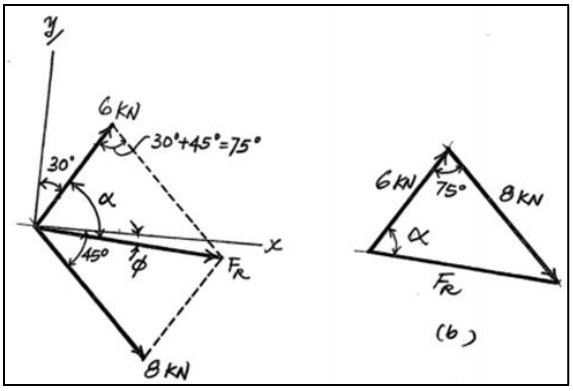
$$F_2 = 800 \times 0.866 = 692.8\ N$$

{Results: $\theta = 90^{\circ}$; $F_R = 400\ N$; $F_2 = 693\ N$}

10. If ($\theta = 30^{\circ}$) and ($T = 6$ KN) , determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis (∅).



Solution



The parallelogram law of addition and the triangular rule are shown in figures (a & b), respectively.

$$R = \sqrt{F_1^2 + F_2^2 - 2F_1 . F_2 \cos\theta}$$

$$R = \sqrt{6^2 + 8^2 - 2 \times 6 \times 8 \times \cos 75^o} = 8.67\ KN$$

Appling the law of sine's to figure b. and using this result yield:

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$

$$\frac{8.67}{sin75^o} = \frac{8}{sin\alpha} = \frac{6}{sin\beta}$$

$$sin\alpha = \frac{8\ sin75^o}{8.67} = 0.891$$

$$\alpha = \sin^{-1}(0.891) = 63^o$$

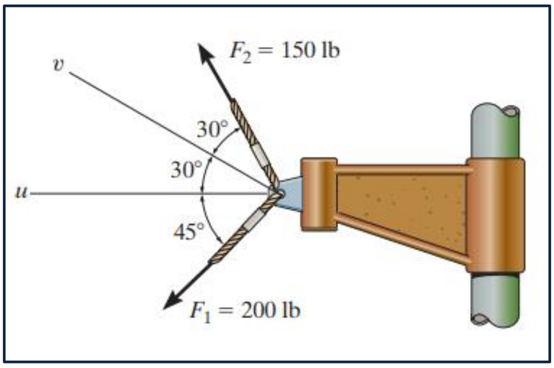
$$sin\beta = \frac{6\ sin75^o}{8.67} = 0.668$$

$$\beta = \sin^{-1}(0.668) = 42^o$$

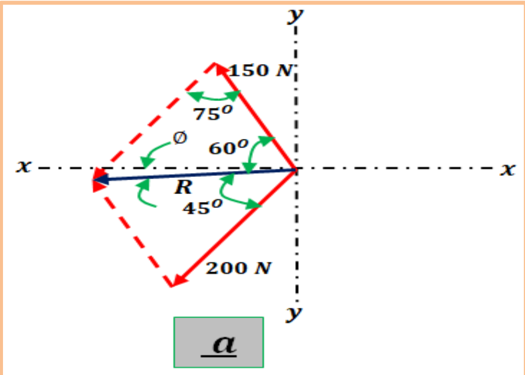
$$\emptyset = \alpha - 60^o = 63^o - 60^o = 3^o$$

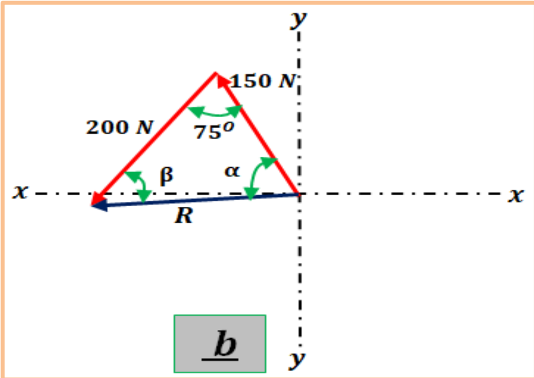
{*Results:* $F_R = 8.67\ KN$; $\alpha = 63.05^{\circ}$; $\emptyset = 3.05^{\circ}$}

11. Determine the magnitude of the resultant force acting on the bracket and its direction measured counterclockwise from the positive u axis.



Solution





The parallelogram law of addition and the triangular rule are shown in figures (a & b), respectively.

$$R = \sqrt{F_1^2 + F_2^2 - 2F_1 . F_2 \cos\theta}$$

$$R = \sqrt{200^2 + 150^2 - 2 \times 200 \times 150 \times \cos 75^o} = 216.72\ N$$

Appling the law of sine's to figure b. and using this result yield:

$$\frac{R}{sin\theta} = \frac{F_1}{sin\beta} = \frac{F_2}{sin\alpha}$$

$$\frac{216.72}{sin75^o} = \frac{200}{sin\alpha} = \frac{150}{sin\beta}$$

$$sin\alpha = \frac{200\ sin75^o}{216.72} = 0.891$$

$$\alpha = \sin^{-1}(0.891) = 63^o$$

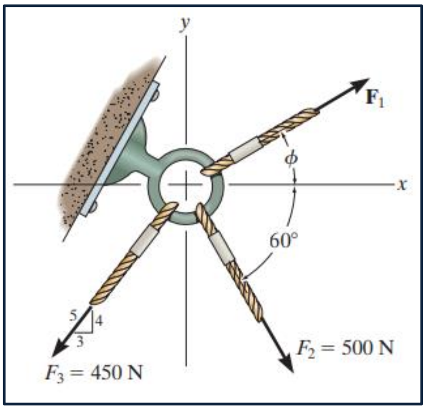
$$sin\beta = \frac{150\ sin75^o}{216.72} = 0.668$$

$$\beta = \sin^{-1}(0.668) = 42^o$$

$$\emptyset = \alpha - 60^o = 63^o - 60^o = 3^o$$

{Results: $F_R = 217\ N$; $\alpha = 63.05^{\circ}$; $\emptyset = 3.05^{\circ}$}

12. If ($F_1 = 600\ N$) and ($\emptyset = 30^{\circ}$) , determine the magnitude of the resultant force acting on the eyebolt and its direction measured clockwise from the positive x axis



Solution

$$\theta = \tan^{-1}\frac{3}{4} = 36.87^{o}$$

NO.	***Description***	$\boldsymbol{\Sigma\, F_x}$ ***(N)***	$\boldsymbol{\Sigma\, F_y}$ ***(N)***
1.	$600 \llcorner 30^{o}$	$600\cos 30^{o} = 519.6$	$600\sin 30^{o} = 300$
2.	$450 \llcorner 233.13^{o}$	$450\cos 233.13^{o} = -270$	$450\sin 233.13^{o} = -353.83$
3.	$500 \llcorner 300^{o}$	$500\cos 300^{o} = 250$	$500\sin 300^{o} = -433.01$
Sum		**499.6**	**-486.84**

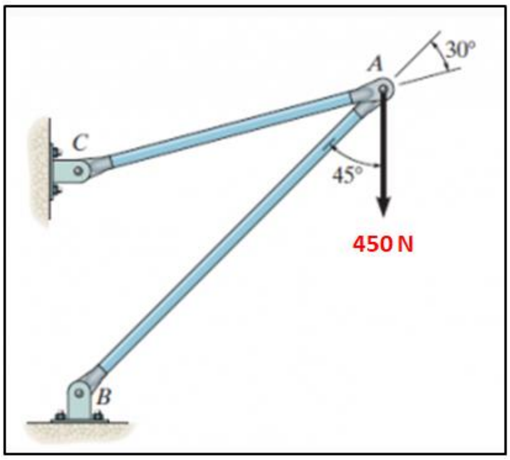
$$\boldsymbol{R} = \sqrt{\boldsymbol{\Sigma F_x^2 + \Sigma F_y^2}} = \sqrt{(499.6)^2 + (-486.84)^2} = \mathbf{697.58\ N}$$

$$\boldsymbol{\theta} = \mathbf{tan^{-1}}\left(\frac{\boldsymbol{F_y}}{\boldsymbol{F_x}}\right) = \tan^{-1}\left(\frac{486.84}{499.6}\right) = \mathbf{44.26^{\circ}}$$

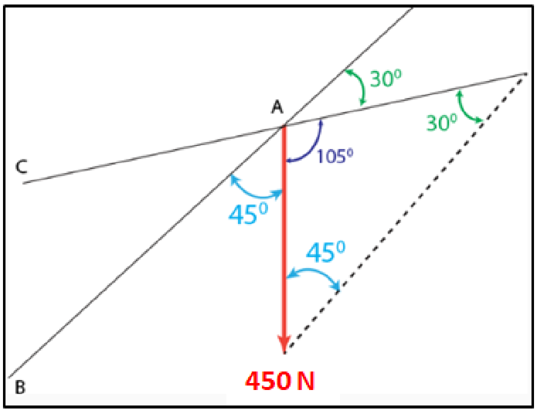
{Results: $F_R = 701.91\ N\ ; \theta = 44.06^{\circ}$ *}*

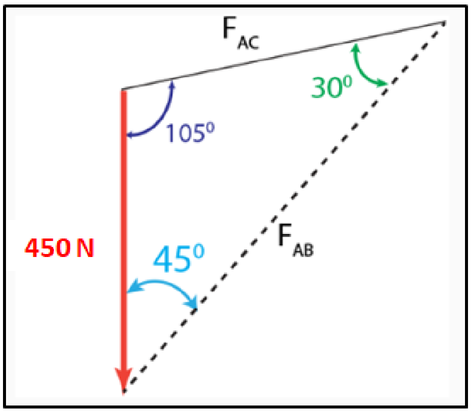
---

13. The force (F = 450 N) acts on the frame. Resolve this force into components acting along members AB and AC , and determine the magnitude of each component.



Solution





Appling the law of sine's to find (F_{AB} & F_{AC}):

$$\frac{R}{sin\theta} = \frac{F_{AB}}{sin\alpha} = \frac{F_{AC}}{sin\beta}$$

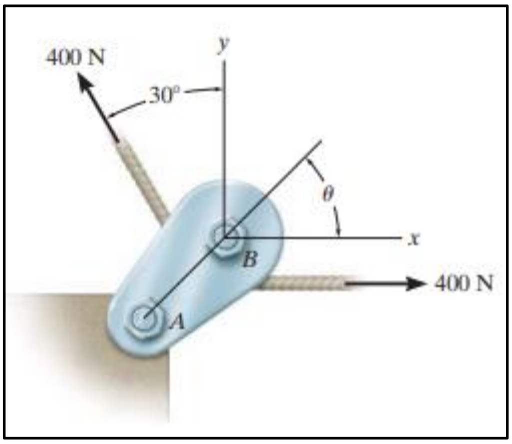
$$\frac{450}{sin30^o} = \frac{F_{AB}}{sin105^o} = \frac{F_{AC}}{sin45^o}$$

$$F_{AB} = \frac{450\ sin105^o}{sin30^o} = 869.33\ N$$

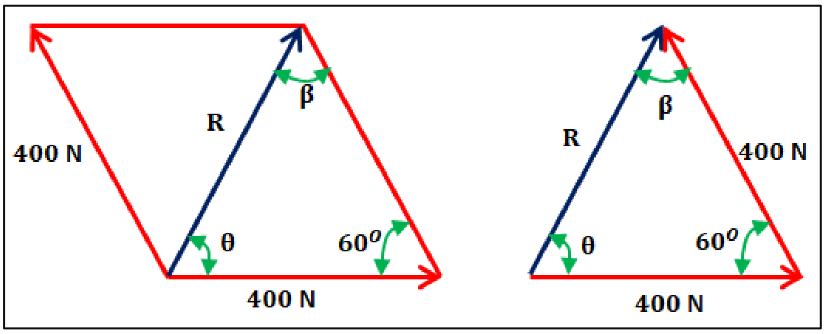
$$F_{AC} = \frac{450\ sin45^o}{sin30^o} = 636.4\ N$$

{Results: $F_{AB} = 869\ N\ ;\ F_{AC} = 636\ N$ }

14. If the tension in the cable is 400 N, determine the magnitude and direction of the resultant force acting on the pulley. This angle is the same angle of line AB on the tailboard block.



Solution



$$R = \sqrt{400^2 + 400^2 - 2 \times 400 \times 400 \times \cos(60^\circ)} = 400\ N$$

From sine law:

$$\frac{R}{sin\alpha} = \frac{F_{AB}}{sin\theta} = \frac{F_{AC}}{sin\beta}$$

$$\frac{400}{\sin 60^o} = \frac{400}{\sin\theta} = \frac{400}{\sin\beta}$$

$$\sin\theta^o = \frac{400 sin\ 60^o}{400} = 0.866$$

$$\theta^o = 60^o$$

$$\sin\beta = \frac{400 sin\ 60^o}{400} = 0.866$$

$$\beta = 60^o$$

Chapter 2

Moment of a Forces

and

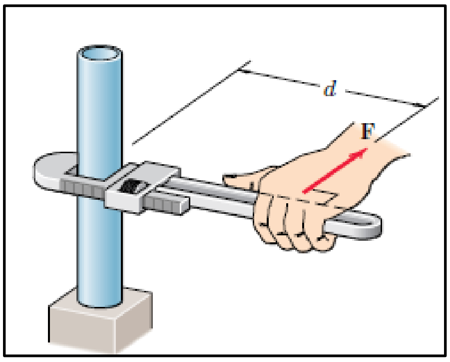
Moment of a couples

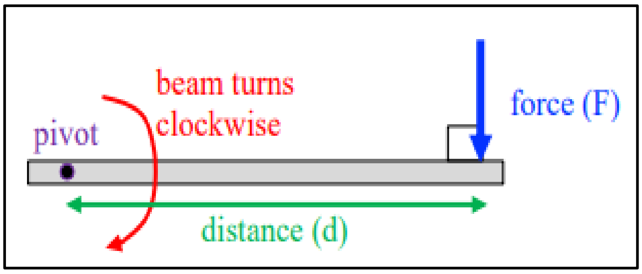
2.1. Introduction

In the previous chapter, we have been discussing the effects of forces, acting on a body, through their lines of action or at the point of their intersection. But in this chapter, we shall discuss the effects of these forces, at some other point, away from the point of intersection or their lines of action

2.2. Moment of a Force

Moment is ability of the force to produce twisting or turning a body about an axis.





$$M = F \cdot d$$

Where:

$\boldsymbol{M}: The\ moment\ of\ the\ force\ (N.m).$

$\boldsymbol{F}: Applied\ force\ (N).$

$\boldsymbol{d}: is\ the\ perpendicular\ distance\ from\ the\ axis\ moment$

$\quad to\ the\ line\ of\ action\ of\ the\ force\ (m).$

$$Units: kN.m, N.m, N.mm$$

Sign Convention:

<u>Note:</u> Always taking clockwise as positive moment.

2.3. Principle of moments

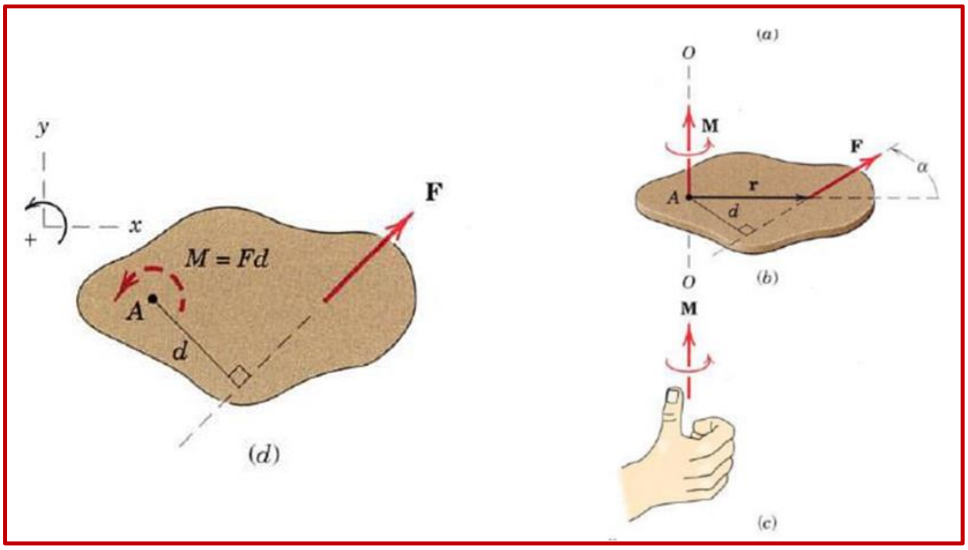
The moment of a force with respect to any axis (or point) is equal to the algebraic sum of the moments of its components with respect to the same axis.

$$M = \sum F \cdot d$$

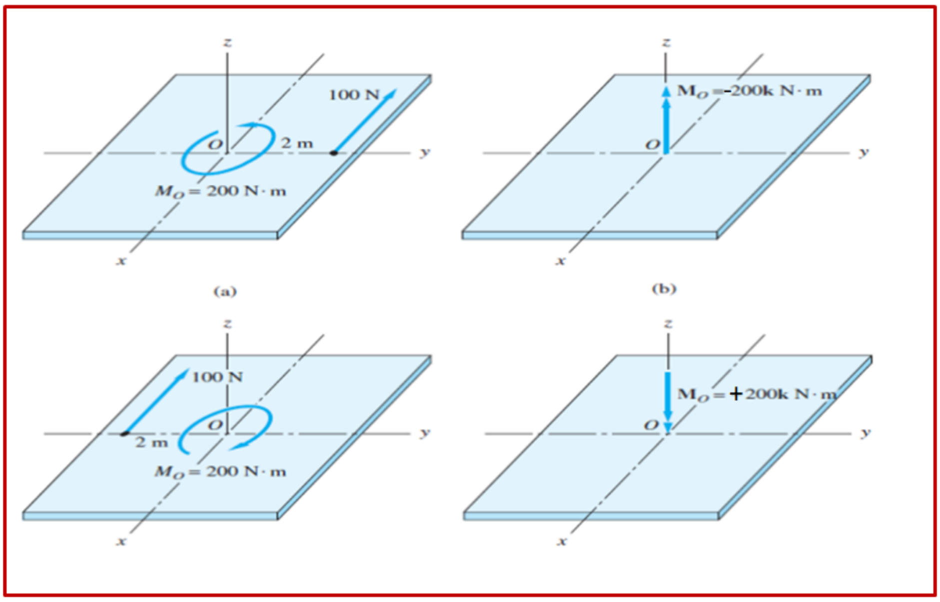
Where:

Is the moment arm, which is the perpendicular distance from the axis of rotation to the line of action of the force.

$$M = F.r.\sin \propto$$



2.3.1. Moment's Direction

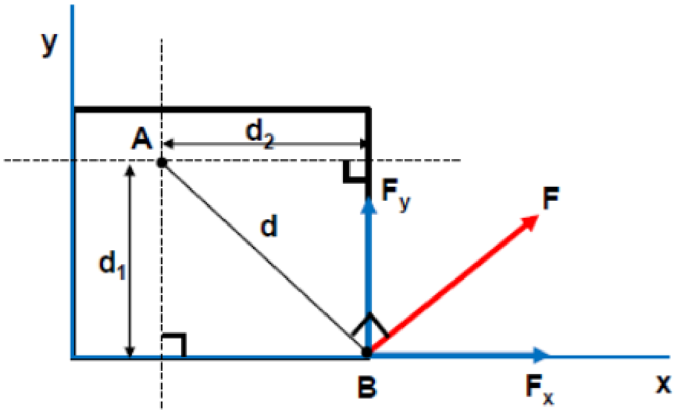


2.3.2. Varignon's Theorem

One of the most useful principles of mechanics is Varignon's theorem, which states that the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point .

$$M_A = F.d$$

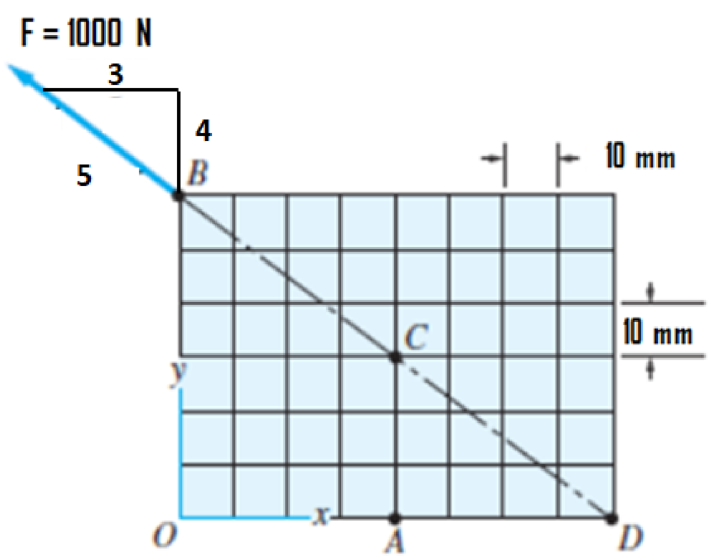
$$M_A = F_x.d_1 + F_Y.d_2$$



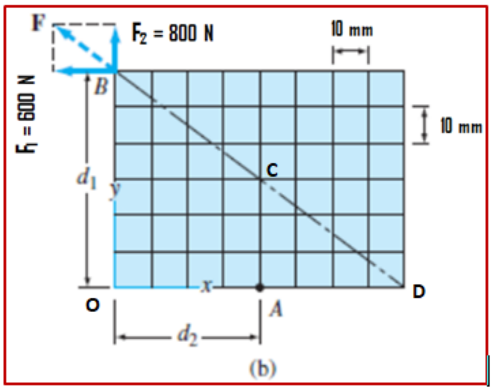
2.4. Solve examples

Example - 1

Determine the moment of the force (F) in Figure below about points (A), (C) , (D) and (O) ?



Solution



$$\circlearrowright +M_A = F_1.d_1 - F_2.d_2$$

$$\circlearrowright +M_A = -600 \times 60 + 800 \times 40 = -4000\ N.mm$$

$$= 4000\ N.mm \circlearrowleft$$

$$\circlearrowright +M_C = F_1.d_1 - F_2.d_2$$

$$\circlearrowright +M_C = -600 \times 30 + 800 \times 4000 = 18000\ N.mm \circlearrowright$$

$$\circlearrowright +M_D = F_1.d_1 - F_2.d_2$$

$$\circlearrowright +M_D = -600 \times 60 + 800 \times 80 = 28000\ N.mm \circlearrowright$$

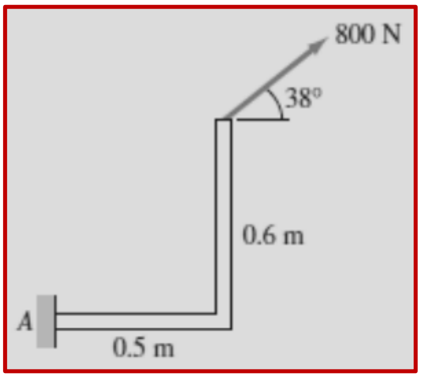
$$\circlearrowright +M_O = F_1.d_1 - F_2.d_2$$

$$\circlearrowright +M_O = -600 \times 60 + 0 = -36000\ N.mm$$

$$= 36000\ N.mm \circlearrowleft$$

Example - 2

Determine the magnitude and sense of the moment of the (F = 800 N) force about point (A)?



Solution

$$\circlearrowright +M_A = F_x \times 0.6 - F_y \times 0.5$$

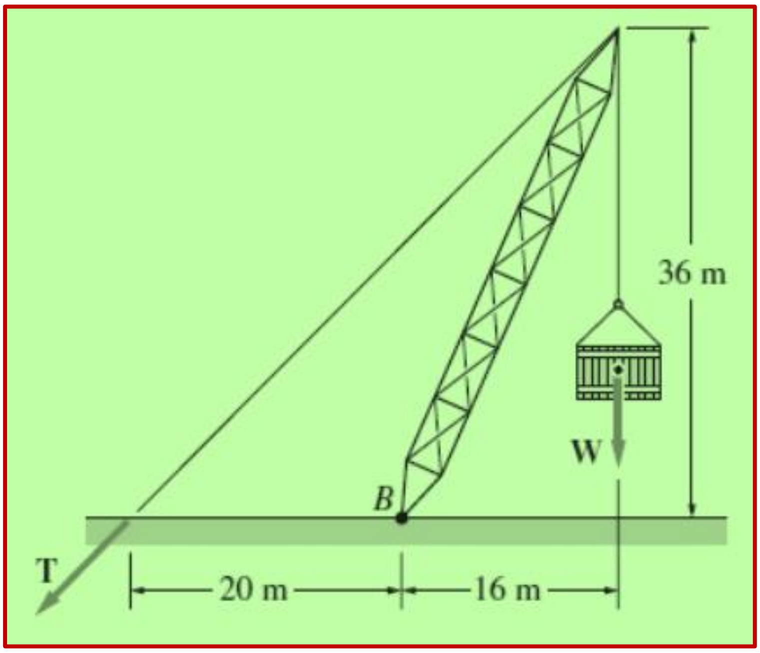
$$= (800 \cos 38^o) \times 0.5 - (800 \sin 38^o) \times 0.6$$

$$M_A = 378.25 - 246.26 = 131.99\ N.m$$

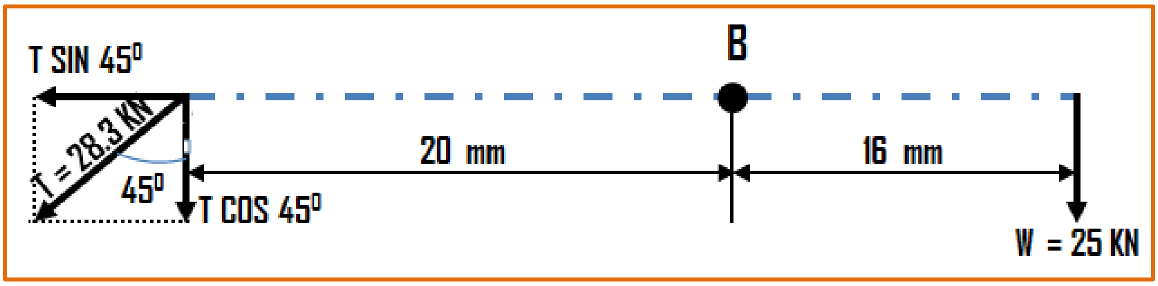
$$\therefore\ M_A = 131.99\ N.m\ \circlearrowright$$

Example – 3

Given that (T =28.3 KN) and (W =25 KN), determine the magnitude and sense of the moments about point (B) of the following: (a) the force T; (b) the force W; and (c) forces T and W combined?



Solution



(a). For T:

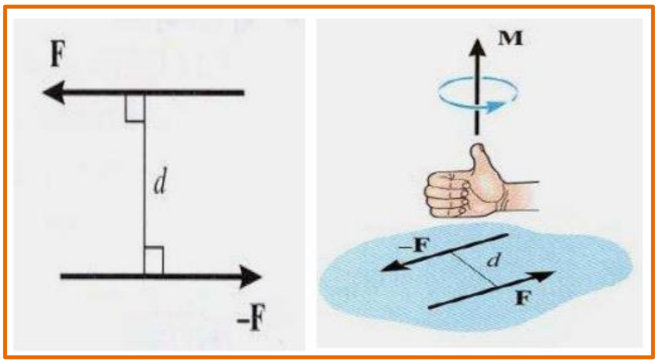
$\circlearrowright +M_B = -20\ (28.3\ \cos 45^O) = -400\ = 400\ KN.mm \circlearrowleft$

(b). For W: $\circlearrowright +M_B = -25\ (16) = -400 = 400\ KN.mm \circlearrowleft$

(c). For T & W: $\circlearrowright \sum M_B = 400 - 400 = 0$

2.5. Moment of a Couple

A couple is defined as two parallel forces with the same magnitude but opposite in direction separated by a perpendicular distance (d).



The moment of a couple is defined as:

$$M_O = F \cdot d$$

(Using a scalar analysis) or as.

$$M_O = r \cdot F$$

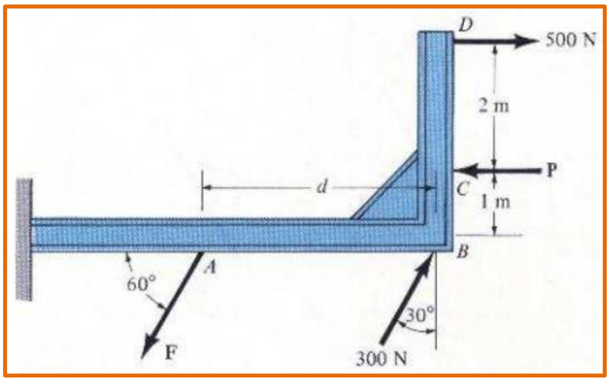
(Using a vector analysis). Here r is any position vector from the line of action of −F to the line of action of F.

The net external effect of a couple is that the net force equals zero and the magnitude of the net moment equals (F . d) Since the moment of a couple depends only on the distance between the forces, the moment of a couple is a free vector. It can be moved anywhere on the body and have the same external effect on the body. Moments due to couples can be added using the same rules as adding any vectors.

Two couples act on the beam. One couple is formed by the forces at A and B, and other by the forces at C and D. If the resultant couple is zero, determine the magnitudes of P and F, and the distance d between A and B.

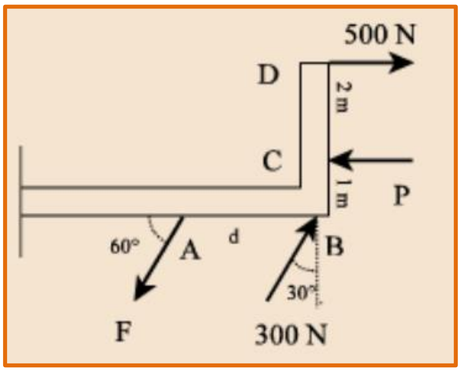
Example – 4

Two couples act on the beam. One couple is formed by the forces at A and B, and other by the forces at C and D. If the resultant couple is zero, determine the magnitudes of P and F, and the distance d between A and B.



Solution

Free Body Diagram (F. B. D) of the figure.



Since these are couples we must have:

$F = 300$ N

$P = 500$ N

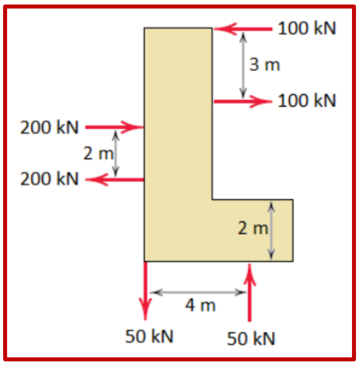
The resultant coupe is:

$M = +500 * 2 - 300 * d \cos 30^0 = 0$

Thus d = 3.85 m

Examples - 5

Determine the resultant moment of the three couples acting on the plate?



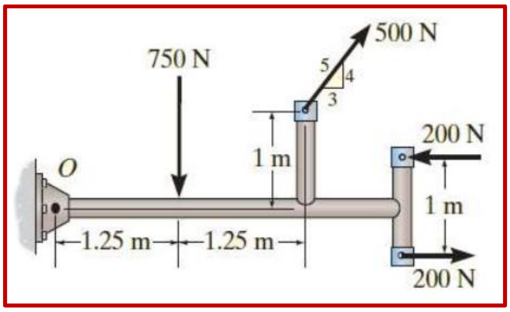
Solution

$$\circlearrowright +M = \Sigma\, F.d = 200 * 2 - 100 * 3 - 50 * 4 = -100\ KN.m$$

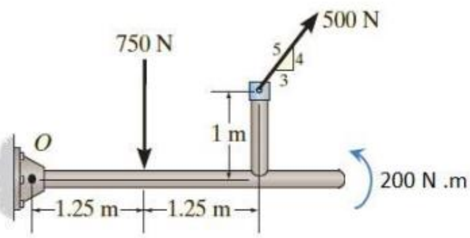
$$= 100\ KN.m\ \circlearrowleft\ Anticlockwise$$

Example - 6

Determine the resultant moment with respect to point (O)?



Solution



$$M_{couple} = F \cdot d = 200 \times 1 = 200\ N.m \circlearrowright$$

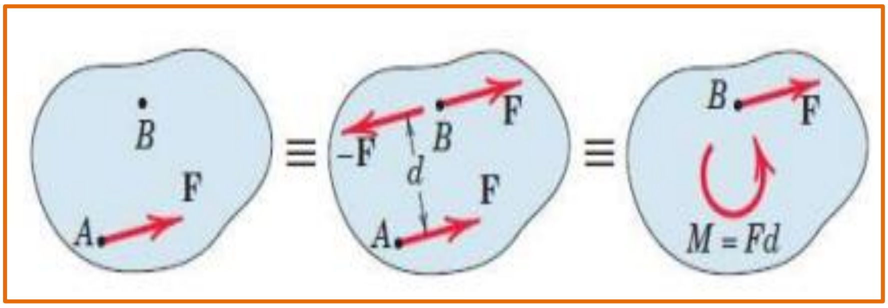
$$\circlearrowleft + M_o = \sum F \cdot d$$

$$M_o = 750 \times 1.25 + 500 \times \frac{3}{5} \times 1 - 500 \times \frac{4}{5} \times 2.5 - 200$$

$$M_o = 37.5\ N.m \circlearrowright$$

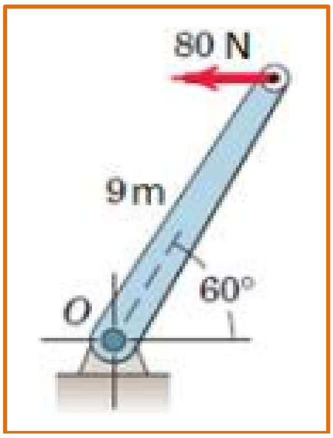
2.6. Force – Couple Systems

According to the principle of transmissibility, the force can be moved to any point along its line of action, as it produces the same effect on the body. However, if we want to move the force to a point not lying on its line of action, it must generate a couple such that it produces the same effect as the force. This is known as forcecouple system. The replacement of a force into a force and a couple is explain in Figure below, where the given force F acting at point A is replaced by an equal force F at point B and the counterclockwise couple $M = Fd$.



Example – 7

Replace the horizontal (80 N) force acting on the lever by an equivalent system consisting of a force at (O) and a couple.

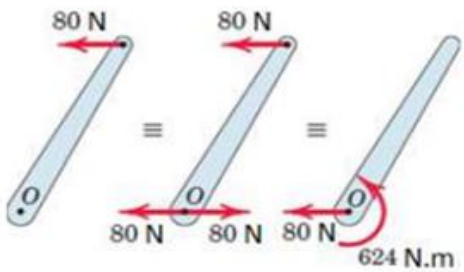


Solution

$$M_o = F \cdot d$$

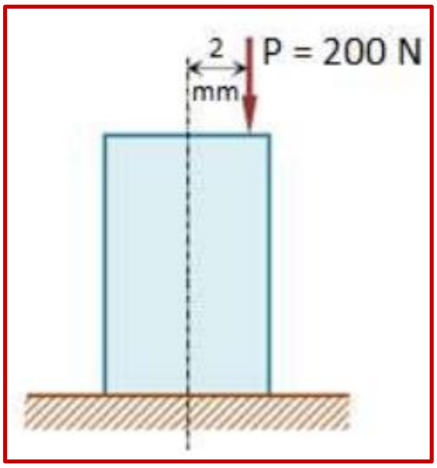
$$M_o = 80 \times (9 \ \sin 60)$$

$$M_o = 624 \ N.m$$



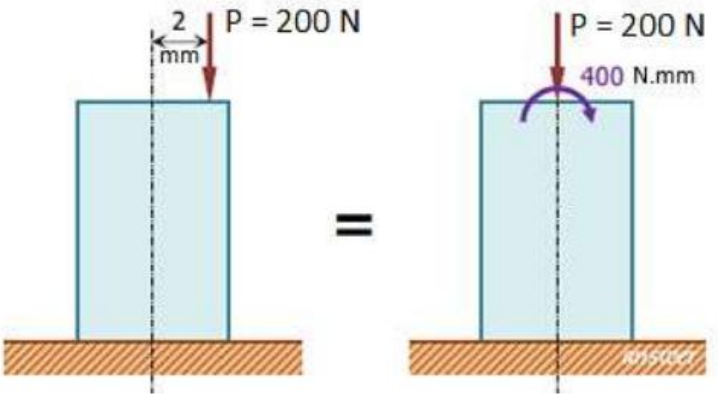
Example - 8

For the compression member shown in the figure, replace the force (P = 200 N) by an equivalent axial load and a couple.



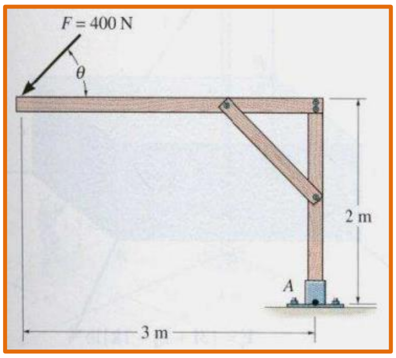
Solution

$$M_{couple} = F \cdot d = 200 \times 2 = 400\ N.mm \circlearrowright$$



2.7. Chapter Questions

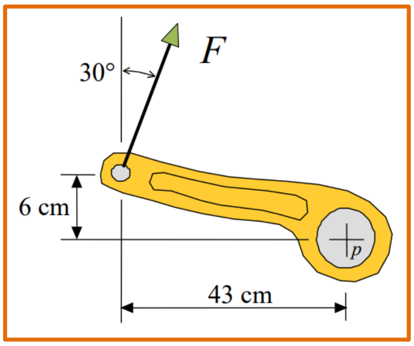
1. A 400 N force is applied to the frame and ϴ = 20º, as in the following figure. Find the moment of the force at A?



{Answer: $M_B = 1160\ N.m$}

---

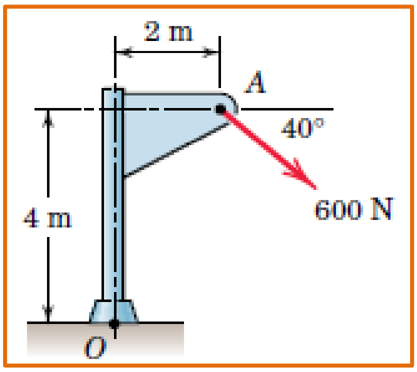
1. The wrench shown is used to turn drilling pipe. If a torque (moment) of (800 N.m) about point (p) is needed to turn the pipe, determine the required force (F).



{Answer: $F = 239\ N$}

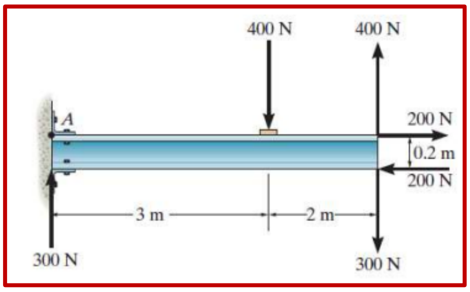
---

2. Calculate the moment about the base point (O) of the (600 N)?



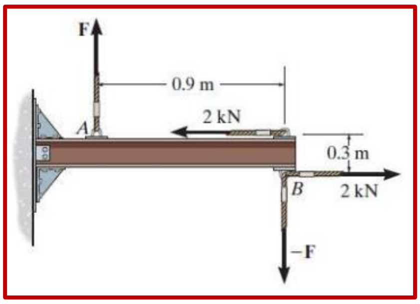
{Answer: $M_o = 2610\ N.m$}

3. Determine the resultant moment acting on the beam?



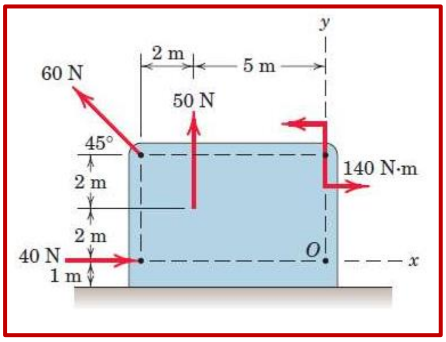
{Answer: $M_{Couple} = 740\ N.m$ ↻}

4. Determine the magnitude of (F), so that the resultant moment acting on the beam is (1.5 kN.m) clockwise.?



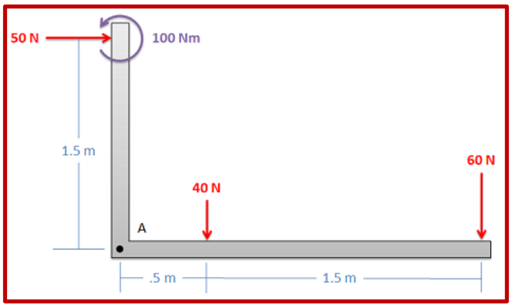
{Answer: $F = 2.333\ KN$}

5. Determine the resultant moment of the three forces and one couple which act on the plate shown about point (O)?



{Answer:: $M_O = 237279\ N.m$ ↻}

6. Find the equivalent force couple system about point A for the set of forces shown below?



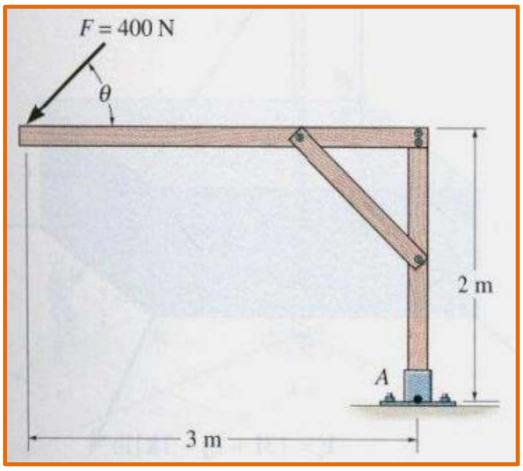
{Answer:: $M_A = 115\ N.m$ ↺}

Solve Question

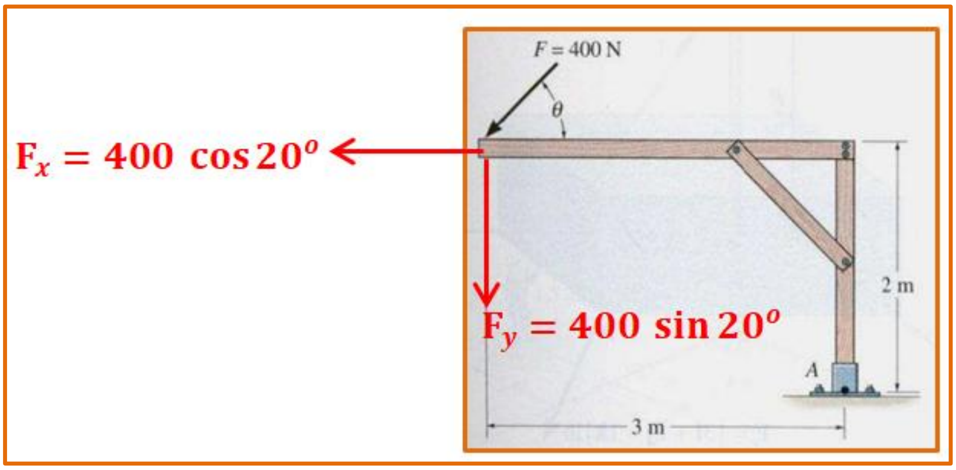
Home Work - 2

2. 7. Chapter Questions

1. A 400 N force is applied to the frame and ϴ = 20º, as in the following figure. Find the moment of the force at A?



Solution:



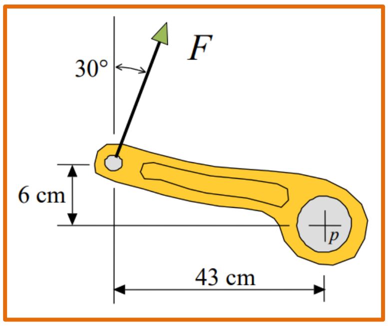
$$\circlearrowright +\mathbf{M_B} = \ \mathbf{\Sigma\, F\,.\,d}$$

$$\circlearrowright +\mathrm{M_B} = \ -400\cos 20^o \times 2 - 400\sin 20^o \times 3 = -751.75 - 410.42$$

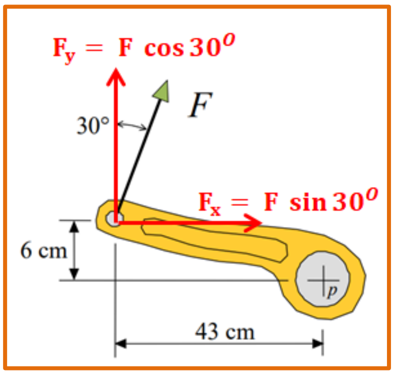
$$= -1162.17\ N\,.\,m = 1162.17\ N\,.\,m\ \circlearrowleft$$

{Answer: $M_B = 1160\ N.m$*}*

2. The wrench shown is used to turn drilling pipe. If a torque (moment) of (800 N.m) about point (p) is needed to turn the pipe, determine the required force (F).



Solution:



$$\circlearrowright +\mathbf{M_p} = \mathbf{\Sigma\, F\,.\,d}$$

$$\circlearrowright + M_p = F\,\sin 30^o \times 0.06 + F\,\cos 30^o \times 0.43$$

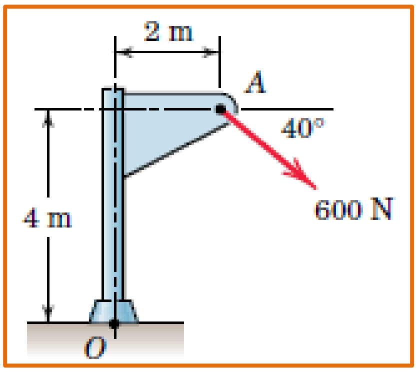
$$800 = 0.03\ F\ + 0.372\ F$$

$$F = \frac{800}{0.402} = \ 1990\ N$$

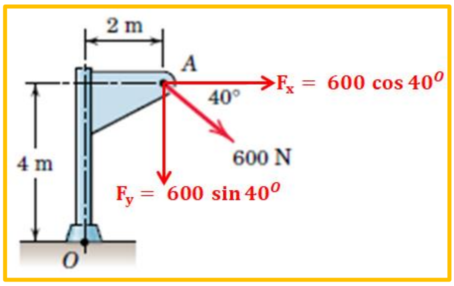
{Answer: $\boldsymbol{F = 1990\ N}$*}*

---

3. Calculate the moment about the base point (O) of the (600 N)?



Solution:



$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\,d}$$

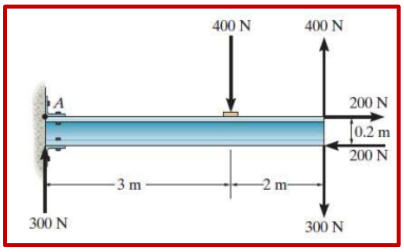
$$\circlearrowright + M_O = 600\,.\cos 40^o \times 4 + 600.\sin 40^o \times 2$$

$$\circlearrowright + M_O = 1838.51 + 771.35 = 2609.86\ N.m \circlearrowright$$

{*Answer:* $M_o = 2610\ N.m$}

---

4. Determine the resultant moment acting on the beam?



Solution:

$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\,d}$$

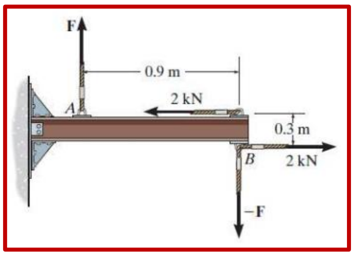
$$\circlearrowright + M_A = -400 \times 2 + 300 \times 5 + 200 \times 0.2$$

$$\circlearrowright + M_A = 800 + 1500 + 40 = 740\ N.m \circlearrowright$$

{*Answer:* $M_{Couple} = 740\ N.m \circlearrowright$}

---

5. Determine the magnitude of (F), so that the resultant moment acting on the beam is (1.5 kN.m) clockwise.?



Solution:

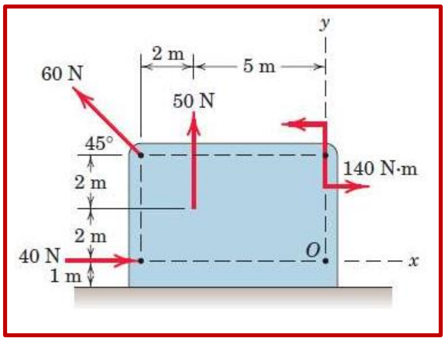
$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\,d}$$

$$1.5 = F \times 0.9 - 2 \times 0.3$$

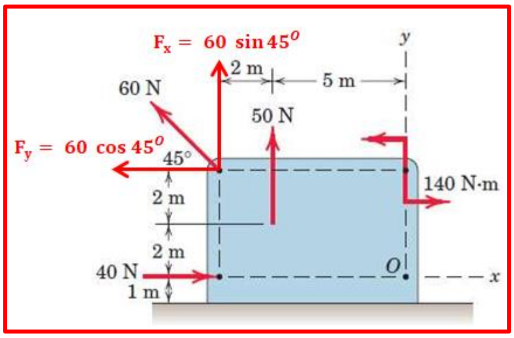
$$F = \frac{1.5 + 0.6}{0.9} = 2.333\ KN$$

{*Answer:* $F = 2.333\ KN$}

6. Determine the resultant moment of the three forces and one couple which act on the plate shown about point (O)?



Solution:



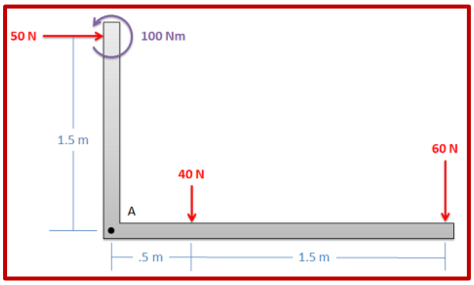
$$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\, d}$$

$$\circlearrowright + M_O = -140 + 50 \times 5 - 60\,.\cos45^o \times 4 + 60.\sin45^o \times 7 - 40 \times 0$$

$$\circlearrowright + M_O = -140 + 250 - 169.7 + 296.98 - 0 = 237.28\ N.m \circlearrowright$$

{Answer:: $M_O = 237.28\ N.m\ \circlearrowright$}

7. Find the equivalent force couple system about point A for the set of forces shown below?



Solution:

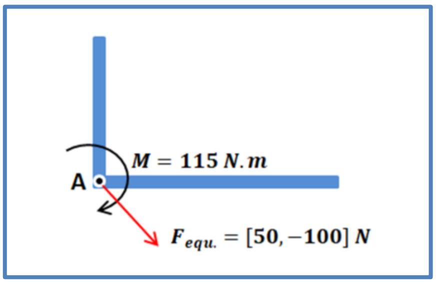
$\mathbf{\Sigma\, F_x} = 50\ N$

$\mathbf{\Sigma\, F_y} = -40 - 60 = -100\ N$

$\circlearrowright +\mathbf{M_A} = \mathbf{\Sigma\, F\,.\, d}$

$\circlearrowright + M_A = 500 \times 1.5 - 100 + 40 \times 0.5 + 60 \times 2$

$\circlearrowright + M_A = 75 - 100 + 20 + 120 = 115\ N.m \circlearrowright$



{Answer:: $M_A = 115\ N.m\ \circlearrowleft$}

Chapter 3

Equilibrium

3.1. Equilibrium

When all the sums of the forces of the system in certain directions and the sum of moments of the forces with respect to certain axes are zero for any particular force system, its resultant is zero, and the body on which the system acts is in equilibrium. The conditions assuring equilibrium of a body with a particular type of force system can therefore be expressed as a set of algebraic equations which must be satisfied. By means of these conditions, it is possible to determine one or more unknown forces or reactions acting on a body which is in equilibrium.

3.2. Free body Diagrams

Is a sketch of a body, a portion of a body, or two or more bodies completely isolated or free from all other bodies, showing the forces exerted by all other bodies on the one being considered.

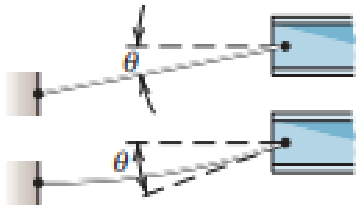
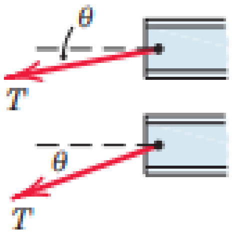
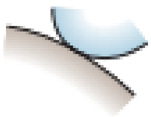
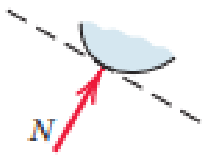
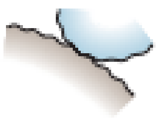
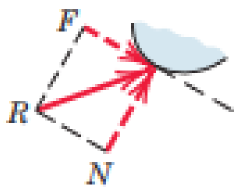
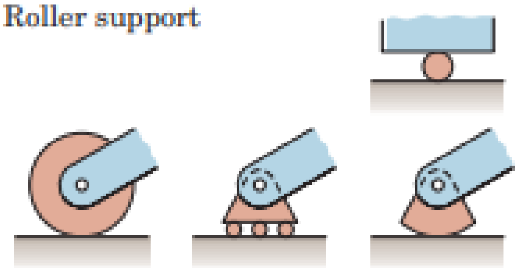
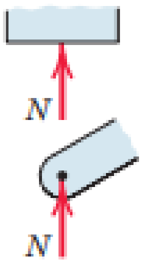
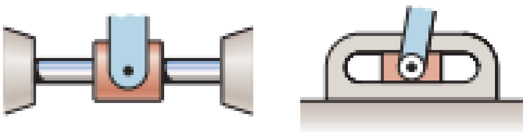
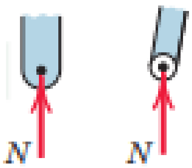
Procedure for drawing a free body diagram

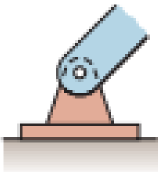
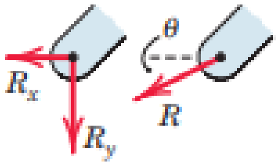
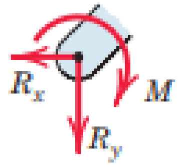
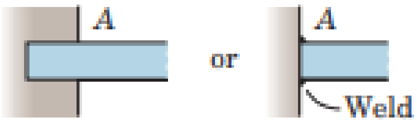
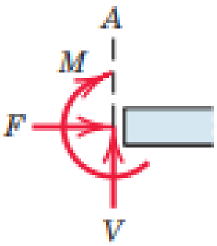
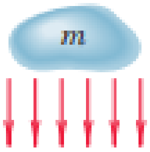
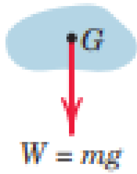
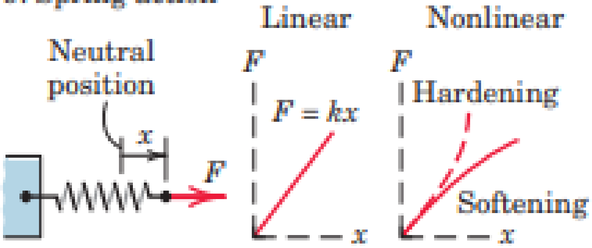
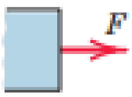
1. **Draw outlined shape**
   Imagine the particle to be isolated or cut (free) from its surroundings by drawing its outlined shape.
2. **Show all forces**
   Indicate on this sketch all the forces that acts on the particle. These forces can be active forces, which tend to set the particle in motion. Or they can be reactive forces which are the result of the constraints or support that tend to prevent motion.

The forces that are known should be labeled with their proper magnitudes and directions. Letters are used to represent the magnitudes and directions of forces that are unknown.

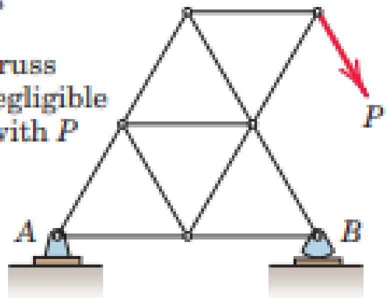
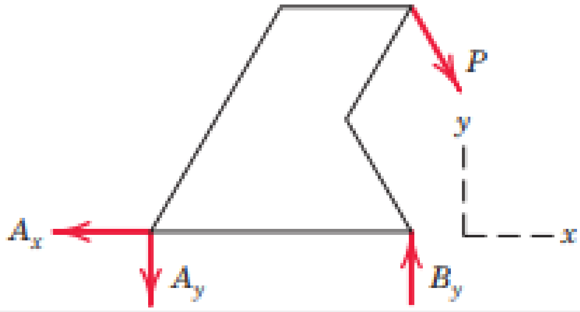
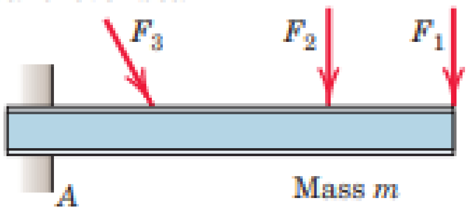
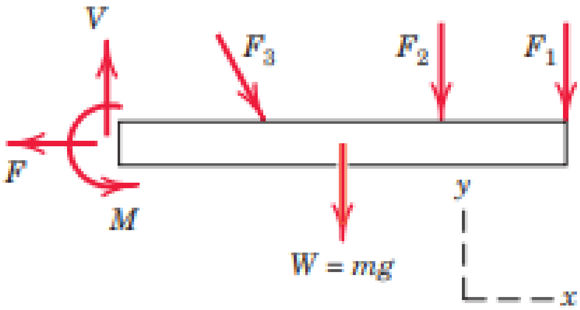
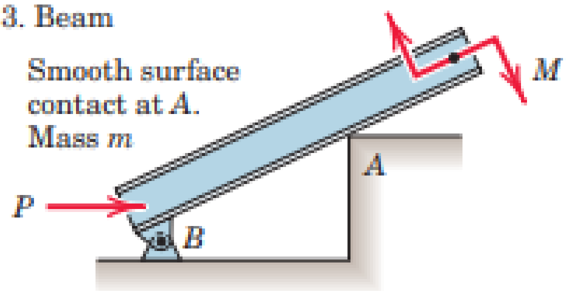
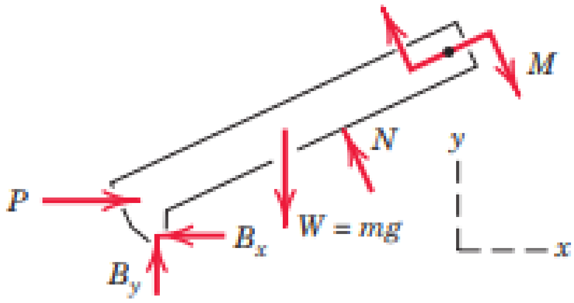
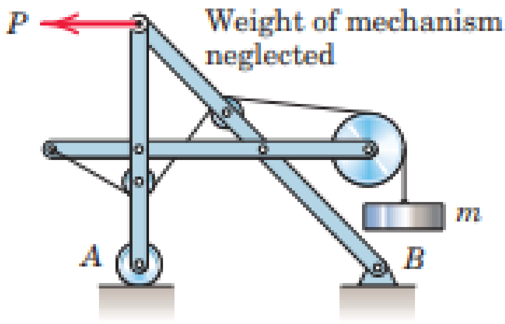
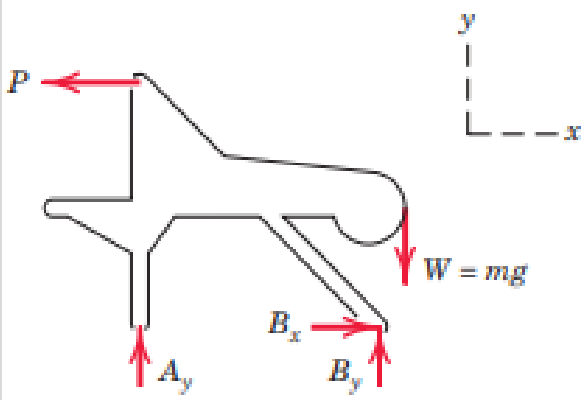
Note

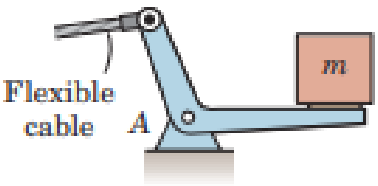
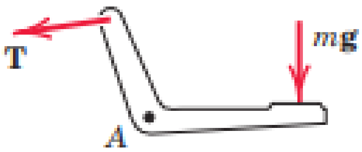
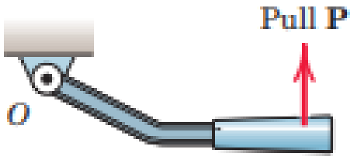
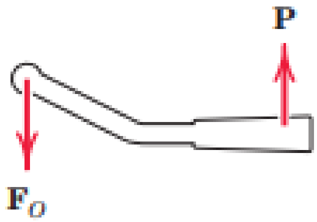
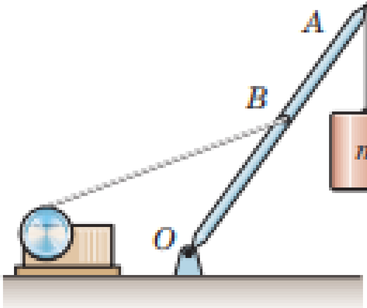
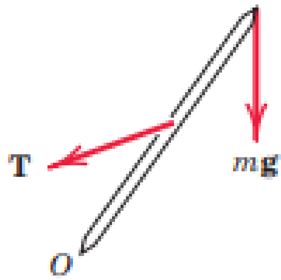
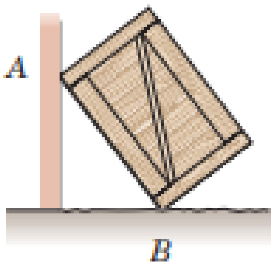
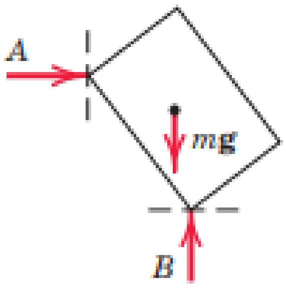
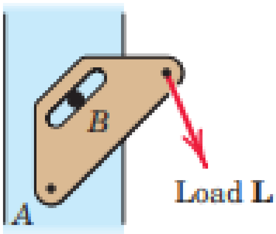
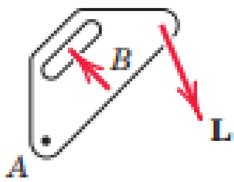
All cables will be assumed to have negligible weight and they cannot stretch. Also, a cable can support only a tension or pulling force and this force always acts in the direction of the cable.

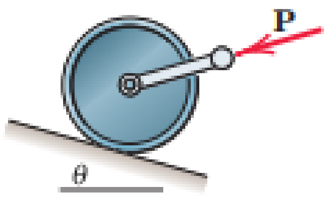
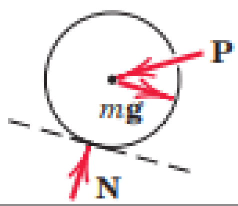
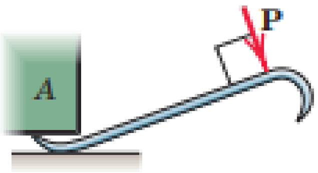
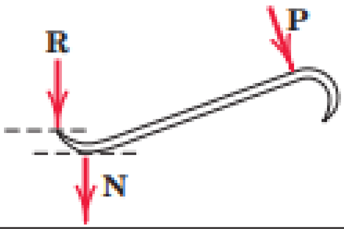
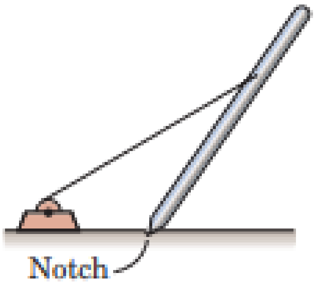
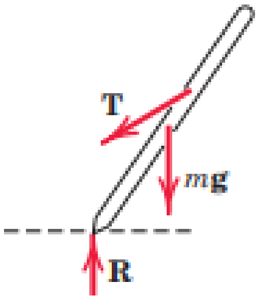
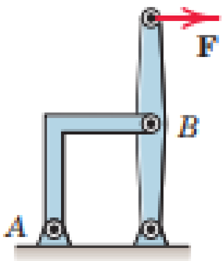
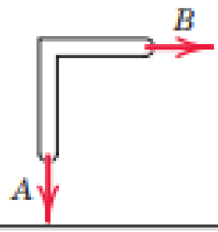
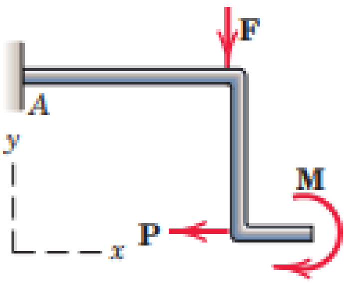
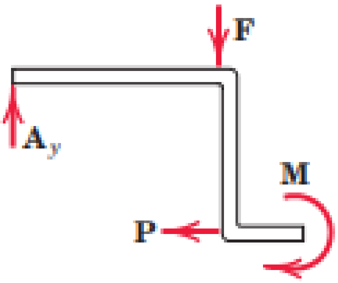
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	Action on Body to Be Isolated
6. Pin connection 	Pin free to turn  Pin not free to turn  A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Examples of Free Body Diagrams

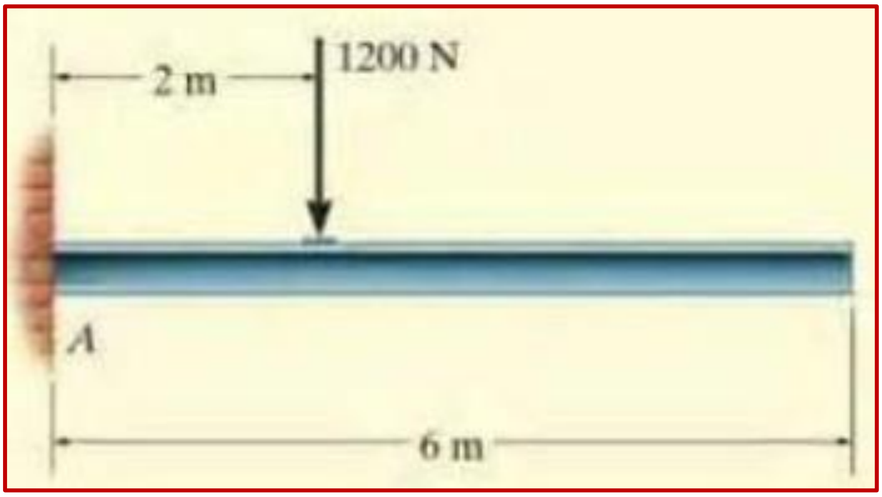
SAMPLE FREE-BODY DIAGRAMS	
Mechanical System	**Free-Body Diagram of Isolated Body**
1. Plane truss Weight of truss assumed negligible compared with P 	
2. Cantilever beam 	
3. Beam Smooth surface contact at A. Mass m 	
4. Rigid system of interconnected bodies analyzed as a single unit Weight of mechanism neglected 	

	Body	Incomplete FBD
1. Bell crank supporting mass m with pin support at A.		
2. Control lever applying torque to shaft at O.		
3. Boom OA, of negligible mass compared with mass m. Boom hinged at O and supported by hoisting cable at B.		
4. Uniform crate of mass m leaning against smooth vertical wall and supported on a rough horizontal surface.		
5. Loaded bracket supported by pin connection at A and fixed pin in smooth slot at B.		

	Body	Wrong or Incomplete FBD
1. Lawn roller of mass m being pushed up incline θ.		
2. Prybar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.		
3. Uniform pole of mass m being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.		
4. Supporting angle bracket for frame; pin joints.		
5. Bent rod welded to support at A and subjected to two forces and couple.		

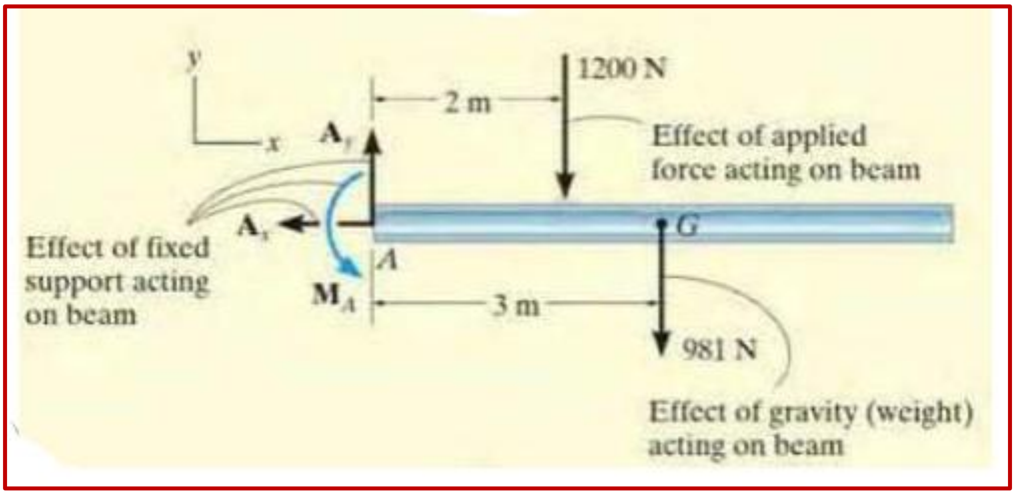
Example - 1

Draw the free body diagram of the uniform beam shown in figure. The beam has a mass of (100 kg)?



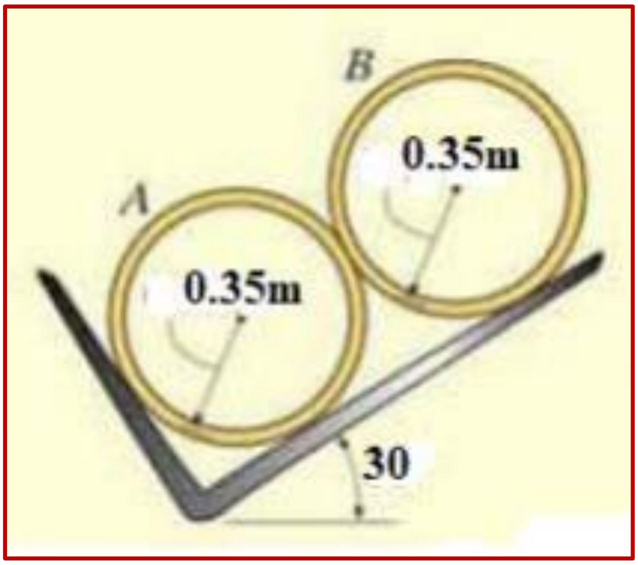
Solution:

Free Body Diagram (F. B. D)



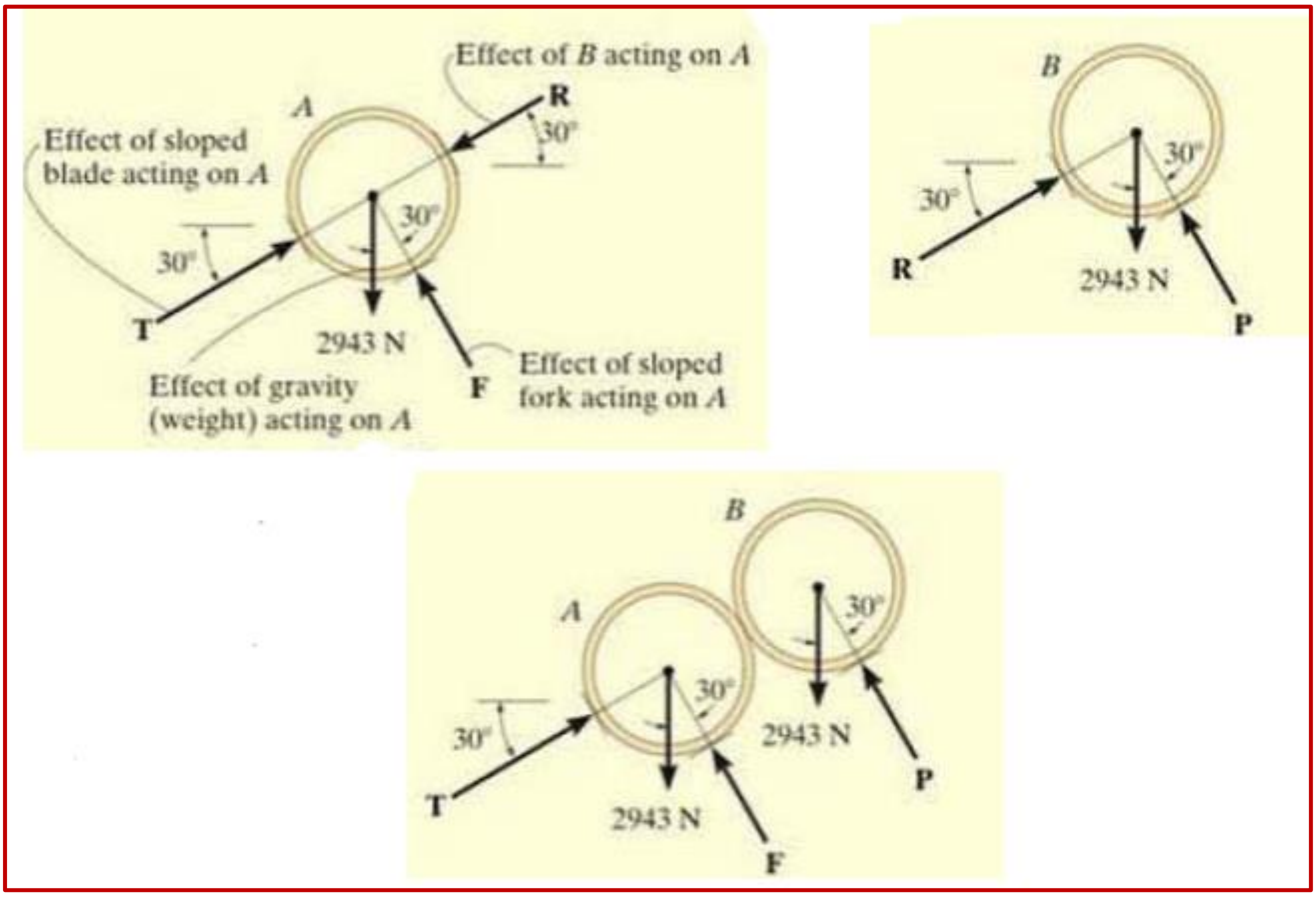
Example - 2

Two smooth pipes, each having a mass of (300 kg), are supported by the forked tines of the tractor. Draw the F,B.D for each pipe and both pipes together.



Solution:

Free Body Diagram (F. B. D)



3.3. General Procedure for the Solution of Problems in Equilibrium

1. Determine the given data and the unknown.
2. Draw the F.B.D for the member on which the unknown forces are acting.
3. Determine the type of force system acting on the F.B.D and the number of independent equations of equilibrium.
4. Compare the number of unknown on the F.B.D with the number of independent equations of equilibrium.
    A. If the number of equations=the number of unknowns, then start the solution.
    B. If the number of unknown > the number of independent equations, then draw F.B.D. for another body and repeat step 3 and 4.
5. If the number of unknowns in the second F.B.D = the number of equations then solve the problem. If it is not repeat step 4-6
6. If there are still too many unknowns after drawing F.B.D for all bodies, then the problem is statically indeterminate.

3.4. Equilibrium of Force System

The body is said to be in equilibrium if the resultant of all forces acting on it is zero. There are two major types of static equilibrium, namely, translational equilibrium and rotational equilibrium.

a) Formulas Concurrent force system

$$\sum F_x = 0\,, \qquad \sum F_y = 0$$

b) Parallel Force System Non-Concurrent

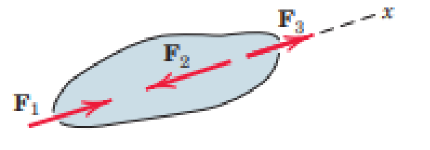
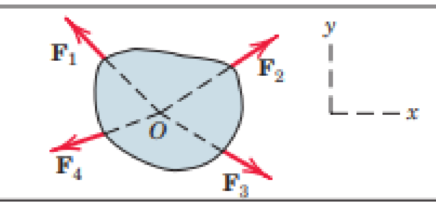
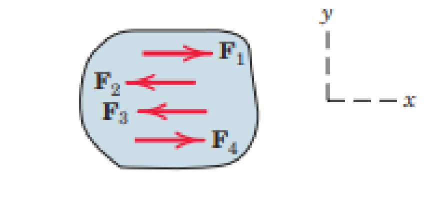
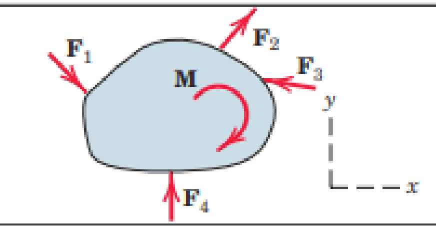
$$\sum F_x = 0\,, \qquad \sum M_O = 0$$

c) Non-Parallel Force System

$$\sum F_x = 0\,, \qquad \sum F_y = 0\,, \qquad \sum M_O = 0$$

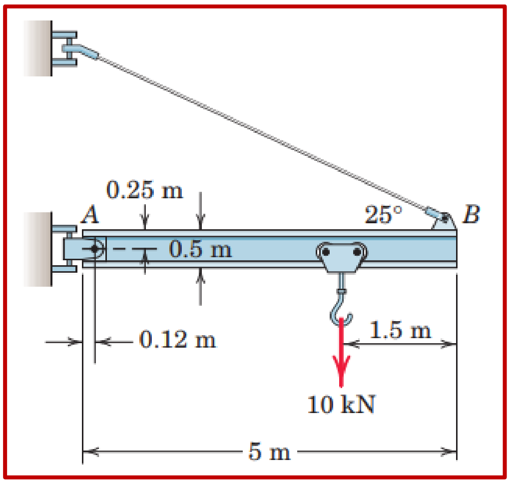
3.5. Important Points for Equilibrium Forces

1. Two forces are in equilibrium if they are equal and oppositely directed.
2. Three coplanar forces in equilibrium are concurrent.
3. Three or more concurrent forces in equilibrium form a close polygon when connected in head to-tail manner.

CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0 \quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0 \quad \Sigma M_z = 0$ $\Sigma F_y = 0$

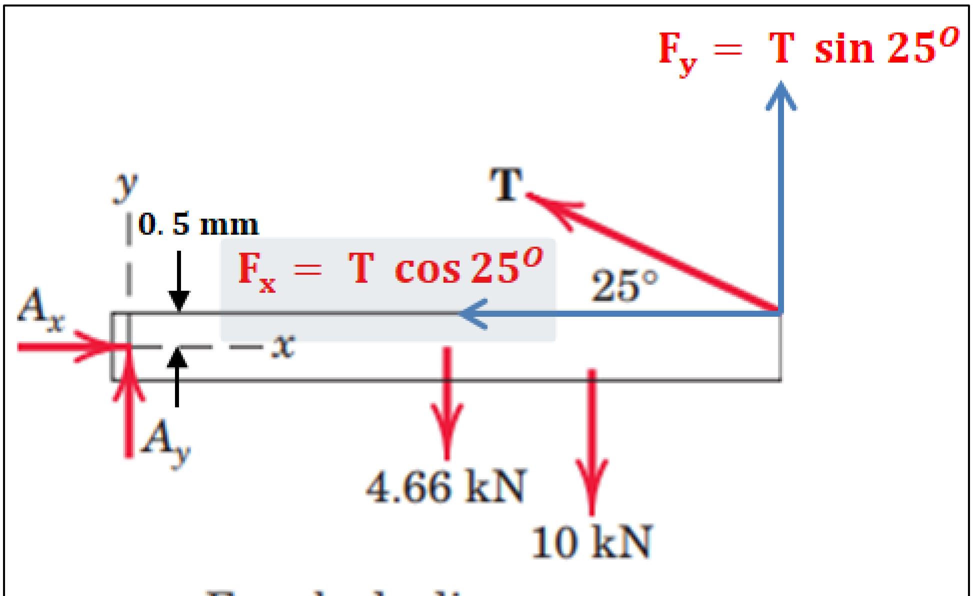
Example – 3

Determine the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane shown. The beam AB is a standard (0.5 m) I-beam with a mass of (95 kg) per meter of length?



Solution:

Draw free body diagram (F. B. D)



Free-body diagram

$$\boldsymbol{\Sigma F_x = 0 \qquad \Sigma F_y = 0\,, \qquad \Sigma M_A = 0}$$

$$\circlearrowright +\Sigma M_A = -(T\cos 25^o) \times 0.25 - (T\sin 25^o)(5 - 0.12) + 10 \times (5 - 1.5 - 0.12) + 4.66(2.5 - 0.12) = 0$$

$$-0.227\,T - 2.062\,T + 33.8 + 11.091 = 0$$

$$-2.289\,T + 44.891 = 0$$

$$T = \frac{44.891}{2.289} = 19.612\ KN$$

$$\Sigma F_x = A_x - T\cos 25^o = 0$$

$$A_x - 19.612 \times \cos 25^o = 0$$

$$A_x = 17.775\ KN$$

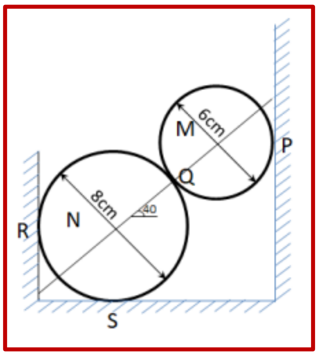
$$\Sigma F_y = A_y + T\sin 25^o - 4.66 - 10 = 0$$

$$A_y = -19.612 \times \sin 25^o + 4.66 + 10 = 6.372\ KN$$

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{17.775^2 + 6.372^2} = 18.883\ KN$$

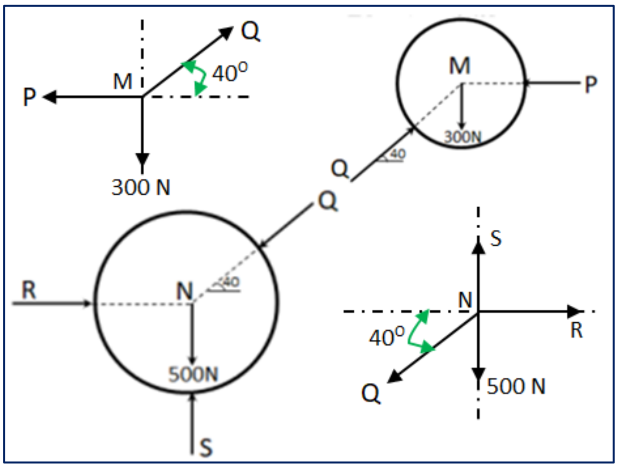
Example – 4

The (300 N) shaft M and the (500 N) shaft N are supported as shown in the figure. Neglecting friction at the contact surfaces P,Q,R and S, determine the reaction at (R and S) on shaft N.?



Solution:

Draw free body diagram (F. B. D)



From F.B.D of M

$$\uparrow \sum F_y = 0$$

$Q \sin 40 - 300 = 0 \qquad \therefore Q = 467N$ on M

From the F.B.D. of N

$$\uparrow \sum F_y = 0$$

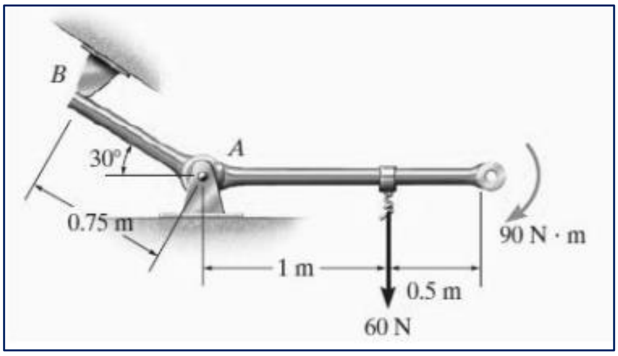
$S - 500 - Q\sin 40 = 0 \qquad \therefore S = 800N \uparrow$ on N

$$\rightarrow \sum F_x = 0$$

$R - Q \cos 40 = 0 \qquad \therefore R = 358N \rightarrow$ on N

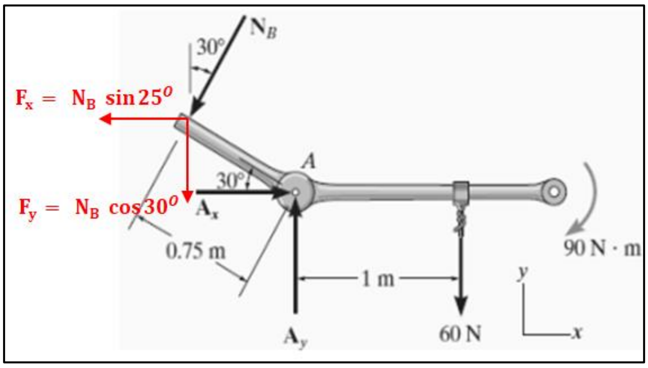
Example – 5

The member shown in Figure is pin-connected at **A** and rests against a smooth support at **B**. Determine the horizontal and vertical components of reaction at the pin **A.**



Solution:

Draw free body diagram (F. B. D)



Equations of Equilibrium: Summing moments about (A), to find direct solution for (N_B).

$$\circlearrowleft +\Sigma M_A = 0$$

$$\circlearrowleft +\Sigma M_A = 90 + 60 \times 1 - N_B \times 0.75$$

$$0 = 90 + 60 - N_B \times 0.75$$

$$N_B = \frac{150}{0.75} = 200\ N$$

$$\Sigma F_x = 0$$

$$\Sigma F_x = A_x - 200 \sin 30^o$$

$$0 = A_x - 200 \sin 30^o$$

$$A_x = 200\, sin\, 30^o = 100\ N$$

$$\Sigma F_y = 0$$

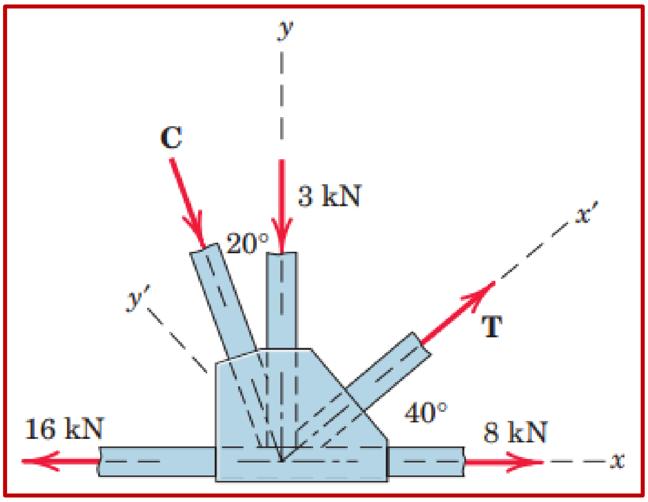
$$\Sigma F_y = A_y - 200 \cos 30^o - 60$$

$$0 = A_y - 200\, cos\, 30^o - 60$$

$$A_y = 200\, cos\, 30^o + 60 = 233\ N$$

3.6. Chapter Questions

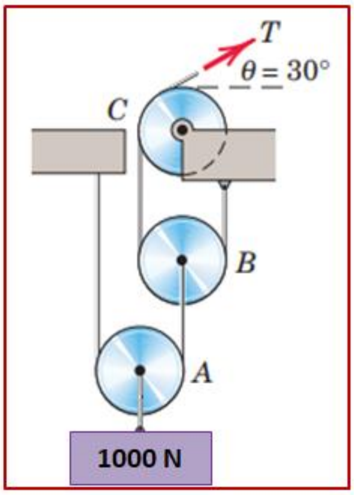
1. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



{Answer: $T = 9.09\ KN\ , C = 3.03\ KN$}

---

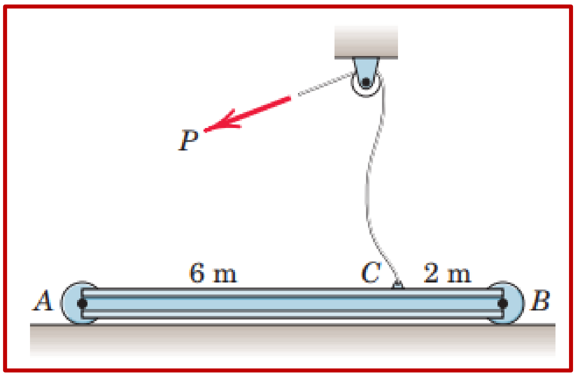
2. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



{Answer: $F = 250\ N$}

---

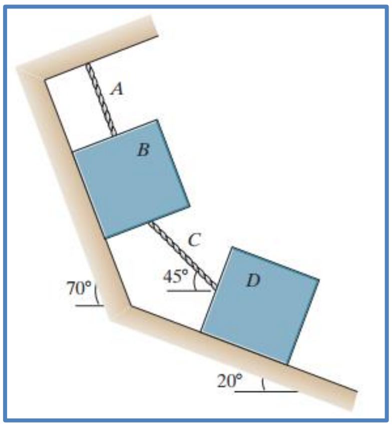
3. The uniform (100 kg) I-beam is supported initially by its end rollers on the horizontal surface at A and B. By means of the cable at C it is desired to elevate end B to a position (3 m) above end A. Determine the required tension P, the reaction at A, and the angle made by the beam with the horizontal in the elevated position.



{Answer: $P = 654\ N\ ,\ \theta = 22^{o}$}

---

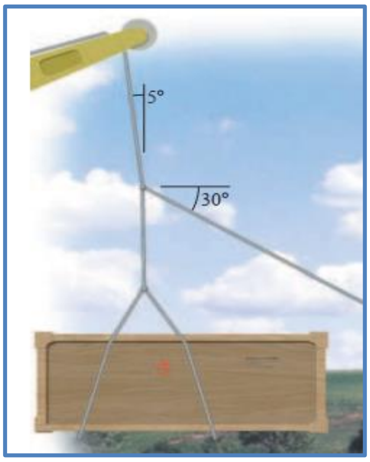
4. Each box weighs 40 N. The angles are measured relative to the horizontal. The surfaces are smooth. Determine the tension in the rope A and the normal force exerted on box B by the inclined surface?



{Answer: $T_A = 51.2\ N\ ,\ N_B = 7.03\ N$}

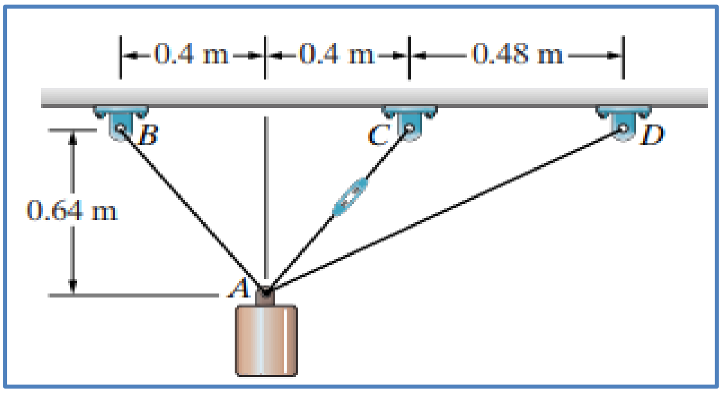
---

5. The construction worker exerts a 90 N force on the rope to hold the crate in equilibrium in the position shown. What is the weight of the crate?



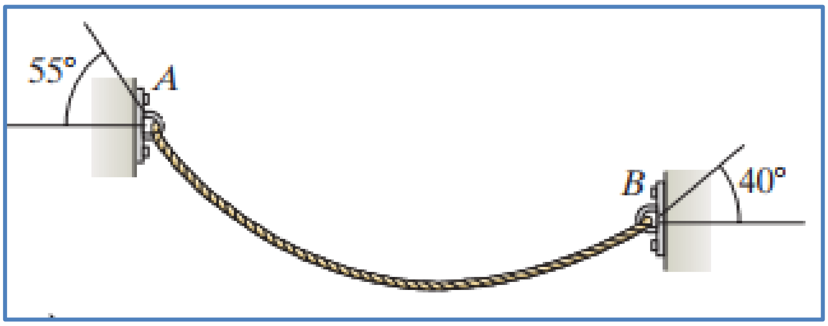
{Answer: $W = 935.9\ N$}

6. The 20-kg mass is suspended from three cables. Cable AC is equipped with a turnbuckle so that
its tension can be adjusted and a strain gauge that allows its tension to be measured. If the tension in cable AC is 40 N, what are the tensions in cables AB and AD?



{Answer: $T_{AB} = 144.1\ N$, $T_{AD} = 68.2\ N$}

7. A heavy rope used as a mooring line for a cruise ship sags as shown. If the mass of the rope is
90 kg, what are the tensions in the rope at A and B?



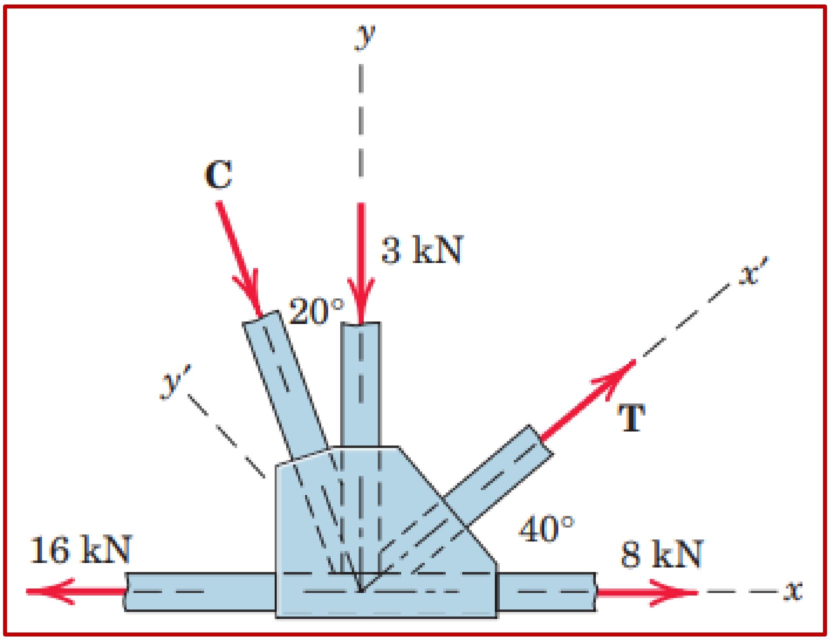
{Answer: $T_A = 679\ N$, $T_B = 508\ N$}

Solve Question

Home Work - 3

3.6. Chapter Questions

1. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



Solution I (scalar algebra). For the x-y axes as shown we have

$[\Sigma F_x = 0]$ $$8 + T\cos 40° + C\sin 20° - 16 = 0$$

$$0.766T + 0.342C = 8$$

$[\Sigma F_y = 0]$ $$T\sin 40° - C\cos 20° - 3 = 0$$

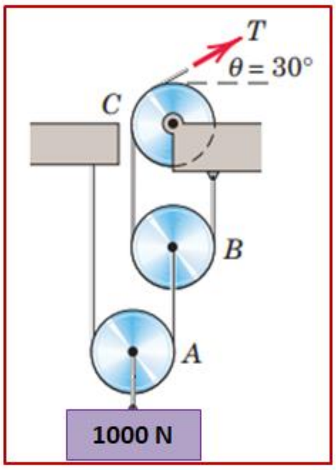
$$0.643T - 0.940C = 3$$

Simultaneous solution of Eqs. (a) and (b) produces

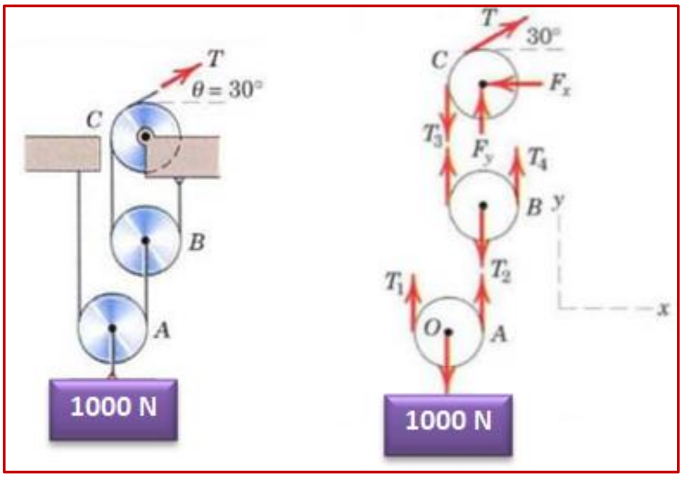
$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN}$$

{*Answer:* $T = 9.09\ KN$, $C = 3.03\ KN$}

2. Determine the magnitudes of the forces C and T, which, along with the other three forces shown, act on the bridge-truss joint?



<u>**Solution:**</u>



$$M_o = 0$$

$$T_1 \, . \, r - T_2 \, .r = 0$$

$$\Sigma F_y = 0$$

$$T_1 + T_2 - 1000 = 0$$

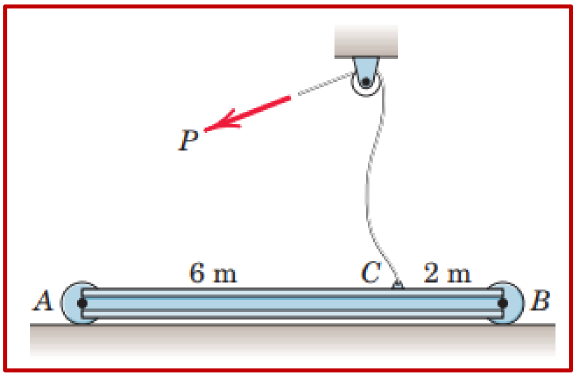
$$T_1 = T_2 = 500 \; N$$

$$T_3 = T_4 = \frac{T_2}{2} = 500 \; N$$

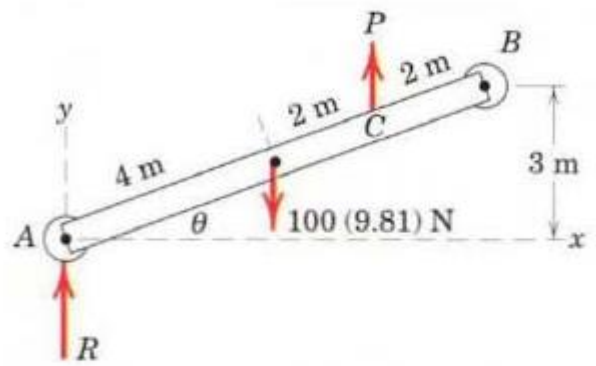
$$T = T_3 = 500 \; N$$

{*Answer:* $\boldsymbol{F} = \mathbf{250}\ \boldsymbol{N}$}

3. The uniform (100 kg) I-beam is supported initially by its end rollers on the horizontal surface at A and B. By means of the cable at C it is desired to elevate end B to a position (3 m) above end A. Determine the required tension P, the reaction at A, and the angle made by the beam with the horizontal in the elevated position.



Solution



Moment equilibrium about A eliminates force *R* and gives

$\Sigma M_o = 0$ $\quad$ $P(6 \cos \theta) - 981(4 \cos \theta) = 0$ $\quad$ $P = 654$ N

Equilibrium of vertical forces requires

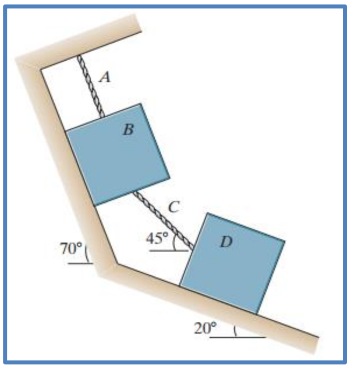
$\Sigma F_y = 0$ $\quad$ $654 + R - 981 = 0$ $\quad$ $R = 327$ N

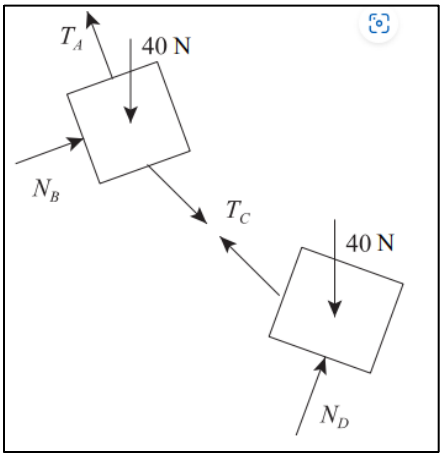
The angle θ depends only on the specified geometry and is

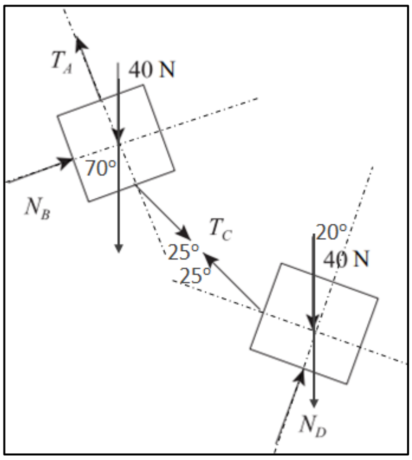
$\sin \theta = 3/8$ $\quad$ $\theta = 22^{\circ}$

{*Answer:* $P = 654\ N\ ,\ \theta = 22^o$}

4. Each box weighs 40 N. The angles are measured relative to the horizontal. The surfaces are smooth. Determine the tension in the rope A and the normal force exerted on box B by the inclined surface?







The free-body diagrams are shown. The equilibrium equations for box D are

$$\sum F_x : (40\ \text{N})\sin 20^\circ - T_C \cos 25^\circ = 0$$

$$\sum F_y : N_D - (40\ \text{N})\cos 20^\circ + T_C \sin 25^\circ = 0$$

The equilibrium equations for box B are

$$\sum F_x : (40\ \text{N})\sin 70^\circ + T_C \cos 25^\circ - T_A = 0$$

$$\sum F_y : N_B - (40\ \text{N})\cos 70^\circ + T_C \sin 25^\circ = 0$$

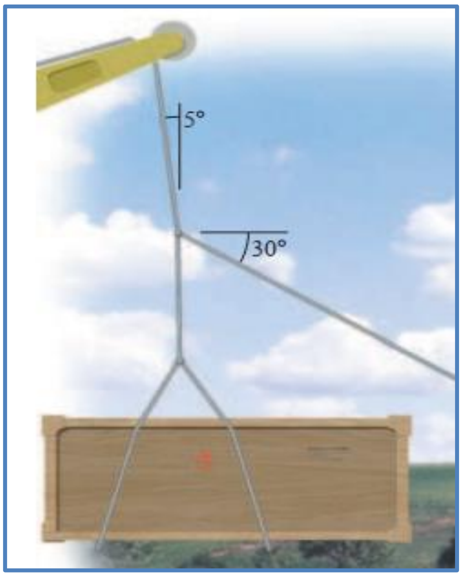
Solving these four equations yields:

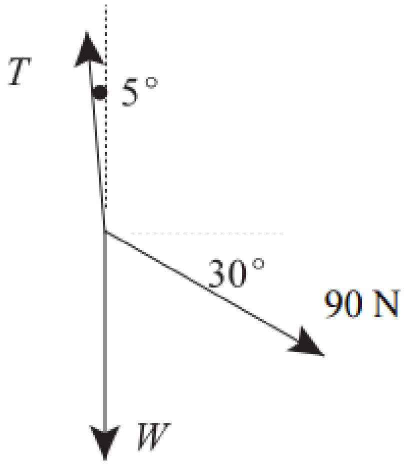
$T_A = 51.2$ N, $T_C = 15.1$ N, $N_B = 7.30$ N, $N_D = 31.2$ N

Thus **T_A = 51.2 N, NB = 7.30 N.**

{Answer: $T_A = 51.2\ N\ ,\ N_B = 7.03\ N$*}*

5. The construction worker exerts a 90 N force on the rope to hold the crate in equilibrium in the
position shown. What is the weight of the crate?





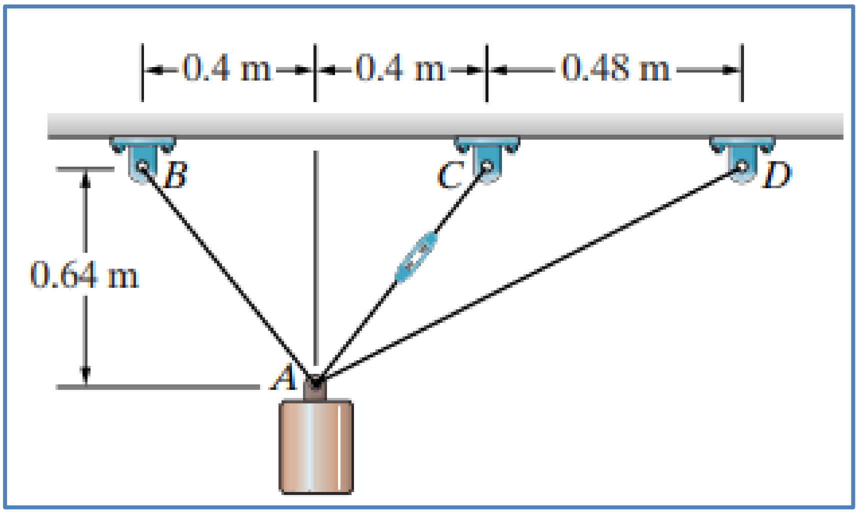
The free-body diagram is shown. The equilibrium equations for the part of the rope system where the three ropes are joined are

$$\sum F_x : (90\text{ N})\cos 30^\circ - T\sin 5^\circ = 0$$
$$\sum F_y : -(90\text{ N})\sin 30^\circ + T\cos 5^\circ - W = 0$$

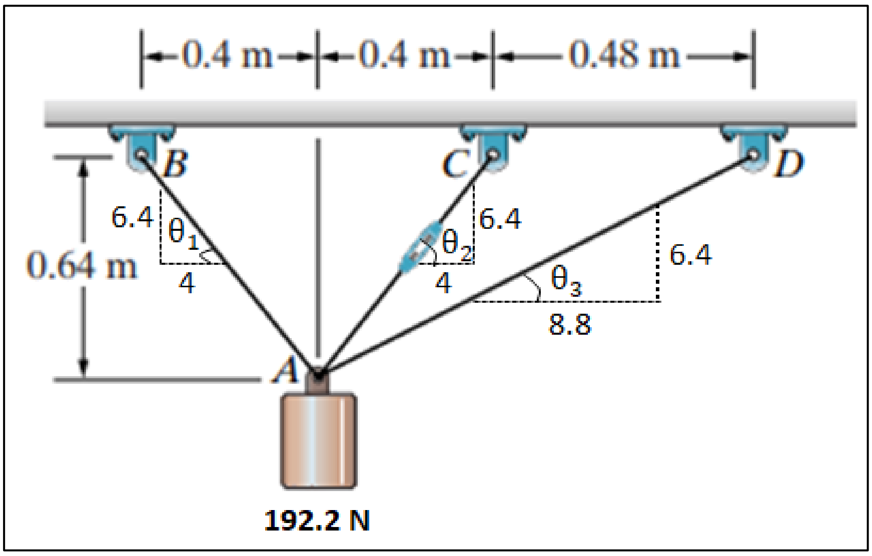
Solving yields **W = 935.9 N.**

{Answer: $W = 935.9\ N$*}*

6. The 100 kg mass is suspended from three cables. Cable AC is equipped with a turnbuckle so that
its tension can be adjusted and a strain gauge that allows its tension to be measured. If the tension in cable AB is 200 N, what are the tensions in cables AB and AD?



Solution:



$$T_{Ac} = 40\ N$$

$$\theta_1 = tan^{-1}\left(\frac{6.4}{4}\right) = 57.99^o$$

$$\theta_2 = tan^{-1}\left(\frac{6.4}{4}\right) = 57.99^o$$

$$\theta_3 = tan^{-1}\left(\frac{6.4}{8.8}\right) = 36.03^o$$

$$\Sigma F_x = 0$$

$$\sin 57.99^o = 0.848 \quad ; \quad cos\, 57.99^o = 0.531$$

$$\sin 36.03^o = 0.588 \quad ; \quad cos\, 36.03^o = 0.809$$

$$-T_{AB}.cos\theta_1 + T_{Ac}.cos\theta_2 + T_{AD}.cos\theta_3 = 0$$

$$-106 + 0.53\, T_{AC} + 0.81\, T_{AD} = 0$$

$$T_{AC} = \frac{106 - 0.81\, T_{AD}}{0.53} \qquad (1)$$

$$\Sigma F_y = 0$$

$$T_{AB}.sin\theta_1 + T_{Ac}.sin\theta_2 + T_{AD}.sin\theta_3 - 981 = 0$$

$$169.6 + 0.848\, T_{AC} + 0.588\, T_{AD} - 981 = 0 \qquad (2)$$

Substituting the first equation with the second equation results:

$$169.6 + 0.848\, (\frac{106 - 0.81\, T_{AD}}{0.53}) + 0.588\, T_{AD} - 981 = 0 \qquad (2)$$

$$169.6 + 1.6\,(106 - 0.81\, T_{AD}) \ + 0.588\, T_{AD} - 981 = 0$$

$$169.6 + 169.6 - 0.81\, T_{AD} \ + 0.588\, T_{AD} - 981 = 0$$

$$1.6\ (21.2 + 0.81\, T_{AD}) + 0.588\, T_{AD} - 196.2 = 0$$

$$33.92 + 1.296\, T_{AD} + 21.2 + 0.81\, T_{AD} - 196.2 = 0$$

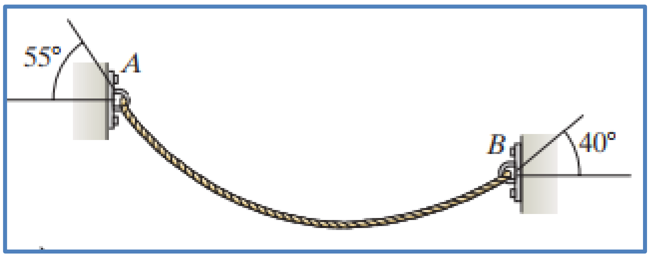
$$T_{AD} = \frac{128.36}{1.884} = 68.13\, N$$

Substituting the value of (T_{AD}) into the equation (I) to obtain the value of (T_{AB}) results in:

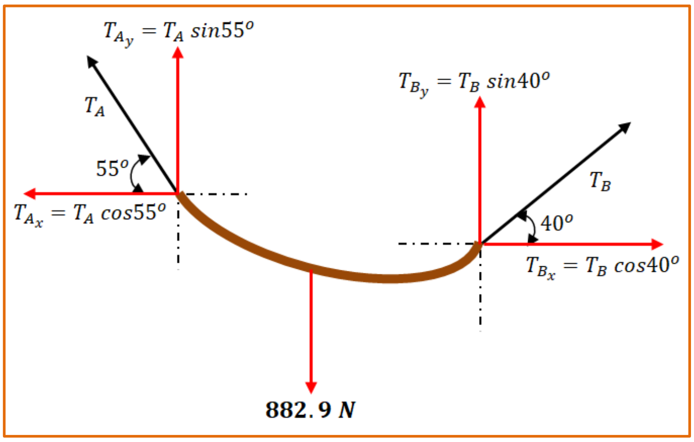
$$T_{AB} = \frac{21.2 + 0.81(68.13)}{0.53} = 144.14\, N$$

{Answer: $\boldsymbol{T_{AB} = 144.1\, N}$, $\boldsymbol{T_{AD} = 68.2\, N}$}

7. A heavy rope used as a mooring line for a cruise ship sags as shown. If the mass of the rope is
90 kg, what are the tensions in the rope at A and B?



Solution:



$$\Sigma F_x = 0$$
$$T_B\ cos40^o - T_A\ cos55^o = 0$$
$$0.766\ T_B - 0.575\ T_A = 0$$
$$T_B = \frac{0.575\ T_A}{0.766}$$
$$T_B = 0.751\ T_A \qquad (1)$$
$$\Sigma F_y = 0$$
$$T_B\ sin40^o + T_A\ sin55^o - 882.9 = 0$$
$$0.643\ T_B + 0.819\ T_A - 882.9 = 0 \qquad (2)$$

Substituting the first equation with the second equation results:

$$0.643\ (0.751\ T_A) + 0.819\ T_A - 882.9 = 0$$
$$T_A = \frac{882.9}{1.302} = 678.11\ N$$

Substituting the value of (T_A) into the equation (1) to obtain the value of (T_B) results in:

$$T_B = 0.751\ (678.11) = 509.26\ N$$

{Answer: $\boldsymbol{T_A = 679\ N\ ,\ \ T_B = 508\ N}$}

Chapter 4

Centroids and Centers of Gravity

4.1. Introduction

A centroid is a weighted average like the center of gravity, but weighted with a geometric property like area or volume, and not a physical property like weight or mass. This means that centroids are properties of pure shapes, not physical objects. They represent the coordinates of the "middle" of the shape.

To design the structure for supporting a water tank, we will need to know the weight of the tank and water as well as the locations where the resultant forces representing these distributed loads act.

4.2. Objectives

1. To discuss the concept of the center of gravity, center of mass, and the centroid.
2. To show how to determine the location of the center of gravity and centroid for a system of discrete particles and a body of arbitrary shape.

4.3. Centroids and Centers of Gravity

A centroid is the geometric center of a geometric object: a one-dimensional curve, a two-dimensional area or a three-dimensional volume. Centroids are useful for many situations in Statics and subsequent courses, including the analysis of distributed forces, beam bending, and shaft torsion.

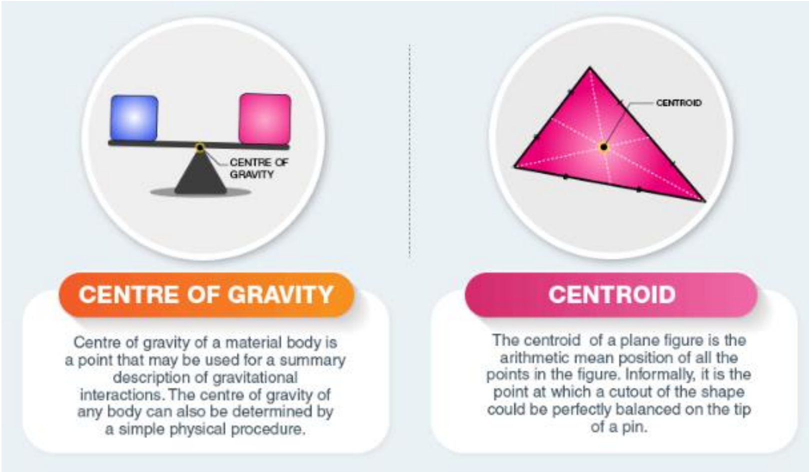
Two related concepts are the **center of gravity**, which is the average location of an object's weight, and the **center of mass** which is the average location of an object's mass. In many engineering situations, the centroid, center of mass, and center of gravity are all coincident. Because of this, these three terms are often used interchangeably without regard to their precise meanings.

Consciously and subconsciously use centroids for many things in life and engineering, including:

1. Keeping your body's balance: Try standing up with your feet together and leaning your head and hips in front of your feet. You have just moved your body's center of gravity out of line with the support of your feet.
2. Computing the stability of objects in motion like cars, airplanes, and boats: By understanding how the center of gravity interacts with the accelerations caused by motion, we can compute safe speeds for sharp curves on a highway.
3. Designing the structural support to balance the structure's own weight and applied loadings on buildings, bridges, and dams: We design most large infrastructure not to move. To keep it from moving, we must understand how the structure's weight, people, vehicles, wind, earth pressure, and water pressure balance with the structural supports.

4.4. Difference between Centre of Gravity and Centroid

1. In an object, a center of mass is referred to as the point where the whole object's mass is focused, which means the point's mass is represented as the whole object's mass. The center of gravity of any object is the point where gravity acts on the body.
2. On the other hand, the centroid is referred to as the geometrical center of a uniform density object. This means the object has its weight distributed equally across all body parts. If the body is homogeneous (having constant density), its center of gravity is equivalent to the centroid.



4.5. Equations for Centroids

The defining equations for centroids are similar to the equations for Centers of Gravity , but with volume used as the weighting factor for three-dimensional shapes.

Centers of Gravity Equations

$$\bar{x} = \frac{\sum \bar{x}_i W_i}{\sum W_i} \quad \bar{y} = \frac{\sum \bar{y}_i W_i}{\sum W_i} \quad \bar{z} = \frac{\sum \bar{z}_i W_i}{\sum W_i}$$

Centroids Equations

$$\bar{x} = \frac{\sum \bar{x}_i V_i}{\sum V_i} \quad \bar{y} = \frac{\sum \bar{y}_i V_i}{\sum V_i} \quad \bar{z} = \frac{\sum \bar{z}_i V_i}{\sum V_i},$$

and area for two-dimensional shapes

$$\bar{x} = \frac{\sum \bar{x}_i A_i}{\sum A_i} \quad \bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i}.$$

If the shape has an axis of symmetry, every point on one side of the axis is mirrored by another point equidistant on the other side. One has a positive distance from the axis, and the other is the same distance away in the negative direction. These two points will add to zero the numerator, as will every other point making up the shape, and the first moment will be zero. This means that the centroid must lie along the line of symmetry if there is one. If a shape has multiple symmetry lines, then the centroid must exist at their intersection.

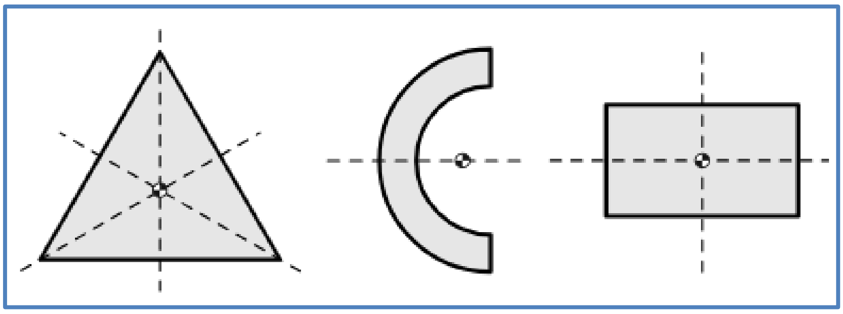


Figure. Centroids lie upon axes of symmetry

Since rectangles, circles, cubes, spheres, etc. have multiple lines of symmetry, their centroids must be exactly in the center as we would expect.

Table. Centroids of Common Shapes

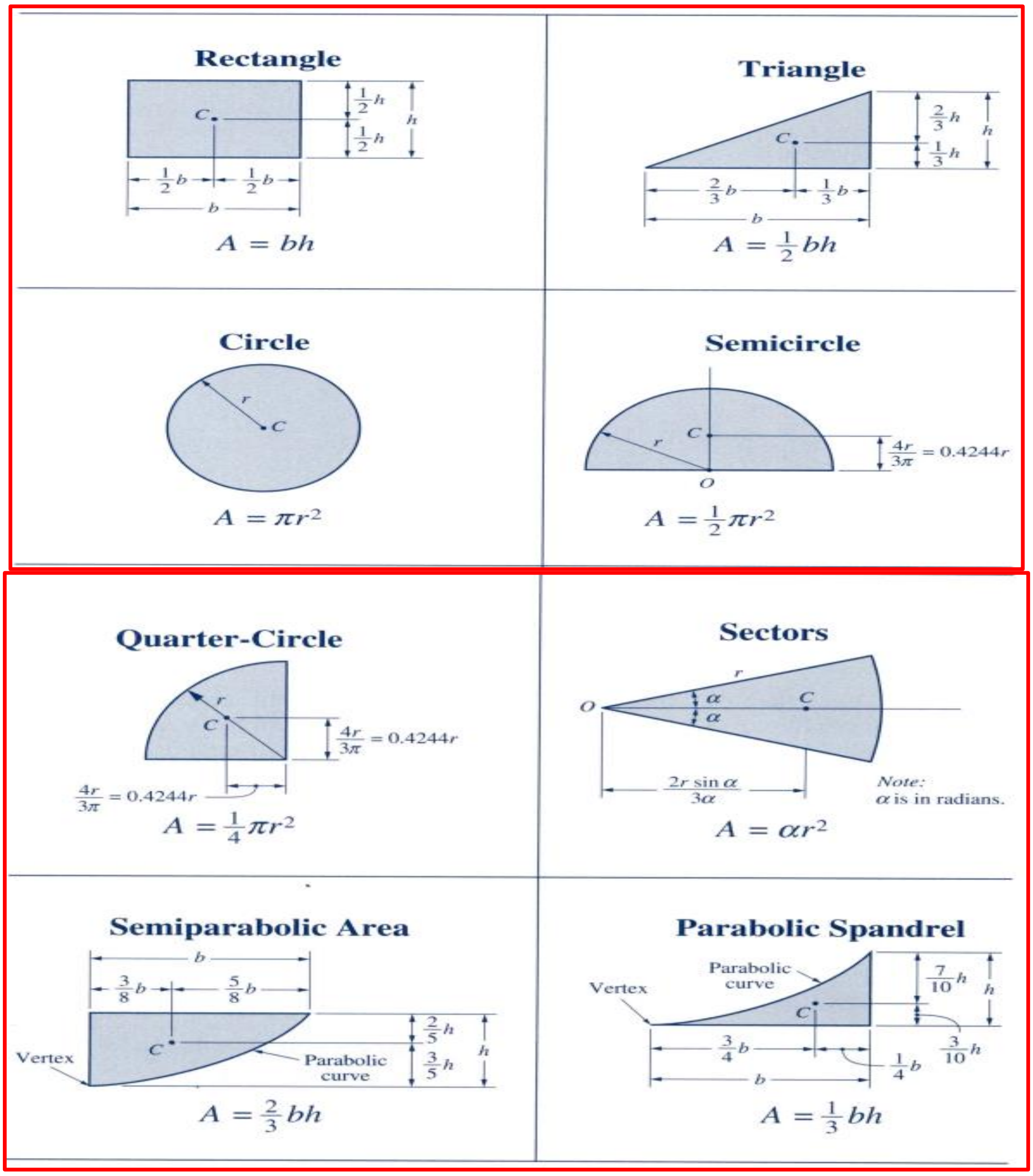
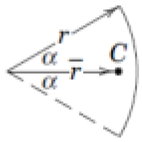
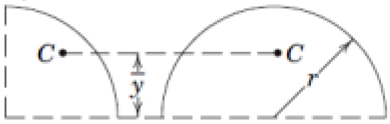
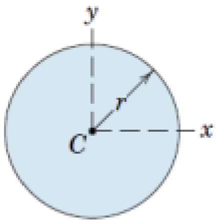
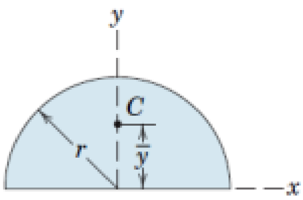
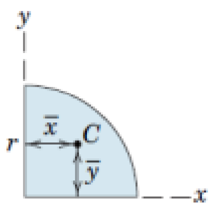
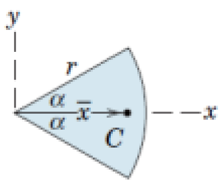
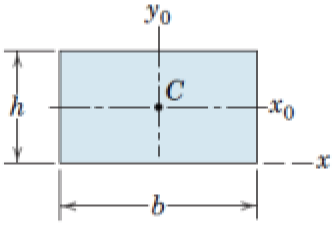
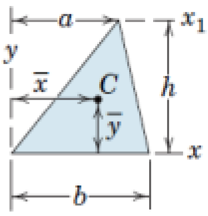
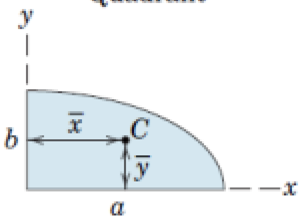
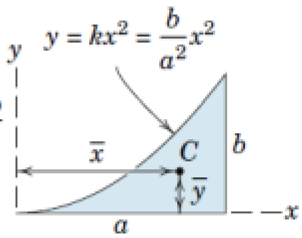
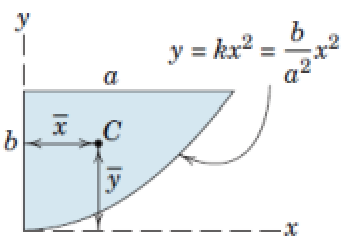


FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \frac{r \sin \alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \frac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{8}$ $\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \frac{2}{3} \frac{r \sin \alpha}{\alpha}$	$I_x = \frac{r^4}{4}(\alpha - \frac{1}{2} \sin 2\alpha)$ $I_y = \frac{r^4}{4}(\alpha + \frac{1}{2} \sin 2\alpha)$ $I_z = \frac{1}{2} r^4 \alpha$

Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2+h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}, \quad \bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$ $I_y = \frac{\pi a^3 b}{16}, \quad \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$ $I_z = \frac{\pi ab}{16}(a^2+b^2)$
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$	$I_x = \frac{ab^3}{21}$ $I_y = \frac{a^3 b}{5}$ $I_z = ab\left(\frac{a^3}{5} + \frac{b^2}{21}\right)$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$	$I_x = \frac{2ab^3}{7}$ $I_y = \frac{2a^3 b}{15}$ $I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$

4.6. Steps for Analysis

1. Divide the body into pieces that are known shapes. Holes are considered as pieces with negative weight or size.
2. Make a table with the first column for segment number, the second column for weight, mass, or size (depending on the problem), the next set of columns for the moment arms, and, finally, several columns for recording results of simple intermediate calculations.
3. Fix the coordinate axes, determine the coordinates of the center of gravity of centroid of each piece, and then fill in the table.
4. Sum the columns to get x, y, and z. Use formulas like.

$$x_C = \frac{\Sigma\, x_i \,.L_i}{\Sigma\, L_i}\,, \qquad y_C = \frac{\Sigma\, y_i \,.L_i}{\Sigma\, L_i}\,, \qquad z_C = \frac{\Sigma\, z_i \,.L_i}{\Sigma\, L_i}$$

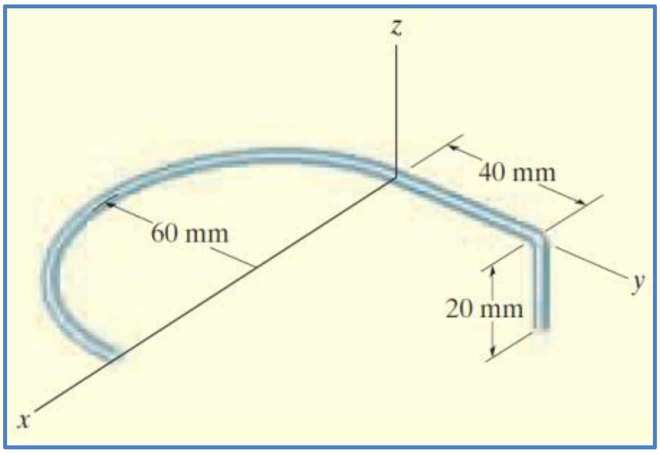
$$x_C = \frac{\Sigma\, x_i \,.A_i}{\Sigma\, A_i}\,, \qquad y_C = \frac{\Sigma\, y_i \,.A_i}{\Sigma\, A_i}\,, \qquad z_C = \frac{\Sigma\, z_i \,.A_i}{\Sigma\, A_i}$$

$$x_C = \frac{\Sigma\, x_i \,.V_i}{\Sigma\, V_i}\,, \qquad y_C = \frac{\Sigma\, y_i \,.V_i}{\Sigma\, V_i}\,, \qquad z_C = \frac{\Sigma\, z_i \,.V_i}{\Sigma\, V_i}$$

$$x_C = \frac{\Sigma\, x_i \,.m_i}{\Sigma\, m_i}\,, \qquad y_C = \frac{\Sigma\, y_i \,.m_i}{\Sigma\, m_i}\,, \qquad z_C = \frac{\Sigma\, z_i \,.m_i}{\Sigma\, m_i}$$

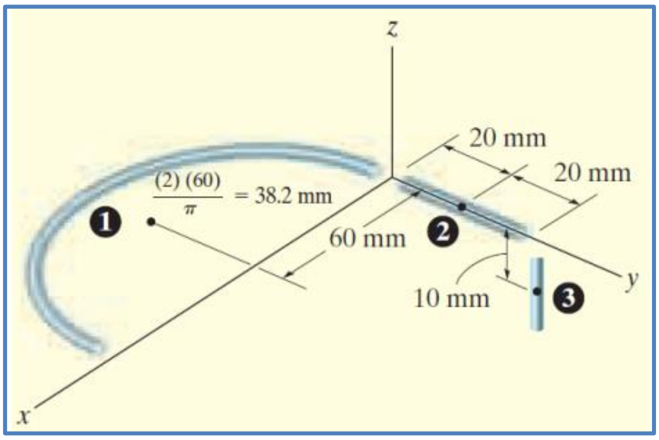
Example - 1

Locate the centroid of the wire shown in the figure below?



Solution:

1. The wire is divided into three segments as shown in the figure below.



2. Moment Arms. The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment (1) is determined either by integration or by using the table .

Seg.	$L_i\ (m^2)$	$x_i\ (m)$	$x_i \, . L_i\ (m^3)$	$y_i\ (m)$	$y_i \, . L_i\ (m^3)$	$z_i\ (m)$	$z_i \, . L_i\ (m^3)$
1	$\pi(60) = 188.5$	60	11310	-38.2	-7200	0	0
2	40	0	0	20	800	0	0
3	20	0	0	40	800	-10	-200
Sum	$\mathbf{248.5}$		$\mathbf{11310}$		$\mathbf{-5600}$		$\mathbf{-200}$

$$x_C = \frac{\Sigma\, x_i\, .L_i}{\Sigma\, L_i} = \frac{11310}{248.5} = 45.5\ mm$$

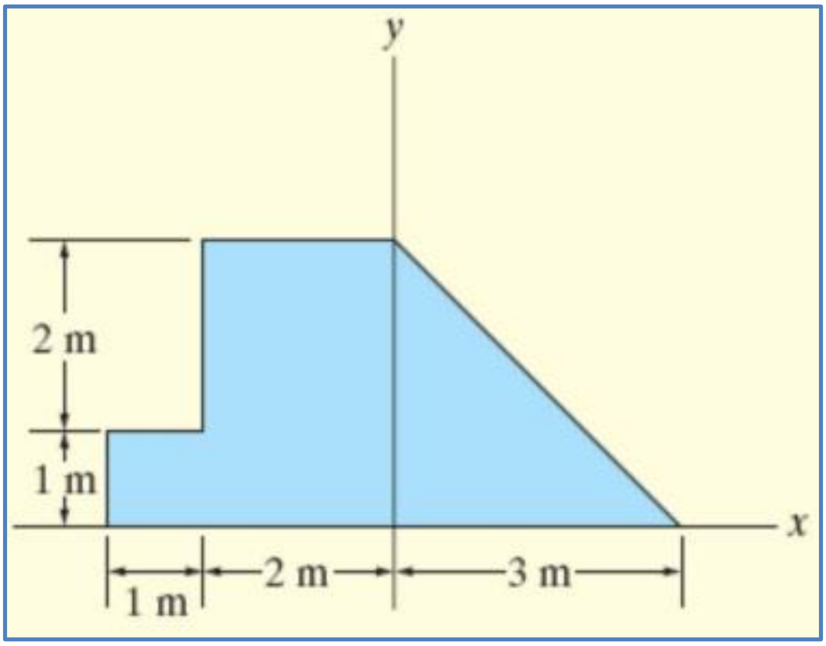
$$y_C = \frac{\Sigma\, y_i\, .L_i}{\Sigma\, L_i} = \frac{-5600}{248.5} = -22.5\ mm$$

$$z_C = \frac{\Sigma\, z_i\, .L_i}{\Sigma\, L_i} = \frac{-200}{248.5} = -0.805\ mm$$

$$C = (45.5, -22.5, -0.805)$$

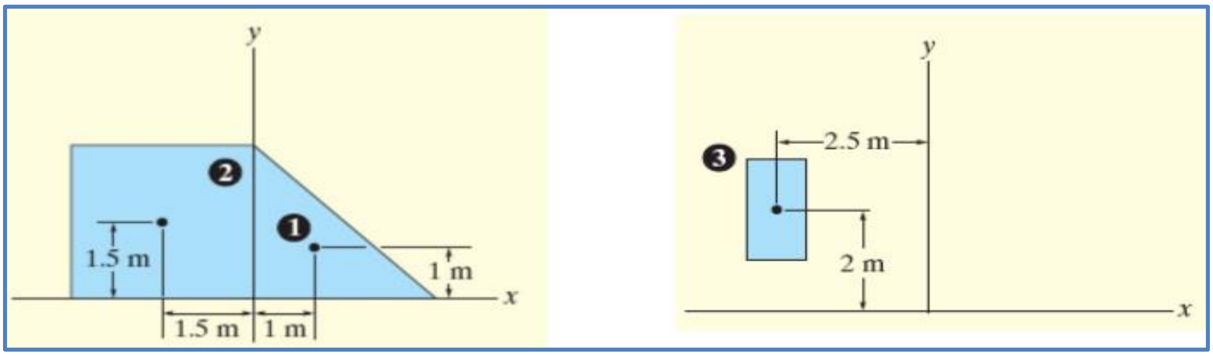
Example - 2

Locate the centroid of the plate area?



Solution:

Plate divided into 3 segments. Area of small rectangle considered "negative".



Solution Moment

Arm Location of the centroid for each piece is determined and indicated in the diagram.

Segment	$A_i\ (m^2)$	$x_i\ (m)$	$x_i\,.A_i\ (m^3)$	$y_i\ (m)$	$y_i\,.A_i\ (m^3)$
1	$\frac{1}{2}(3)(3) = 4.5$	1	4.5	1	4.5
2	$(3)(3) = 9$	− 1.5	− 13.5	1.5	13.5
3	$-(2)(1) = -2$	− 2.5	5	2	− 4
Sum	**11.5**		**− 4**		**14**

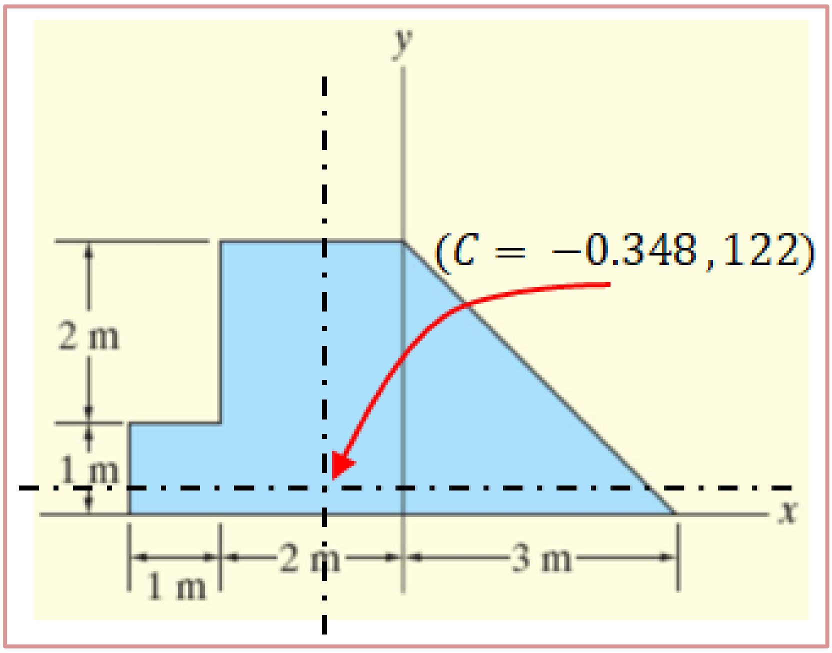
Summations

$$x_C = \frac{\Sigma\, x_i\,.A_i}{\Sigma\, A_i} = \frac{-4}{11.5} = -0.348\ mm$$

$$y_C = \frac{\Sigma\, y_i\,.A_i}{\Sigma\, A_i} = \frac{14}{11.5} = 1.22\ mm$$

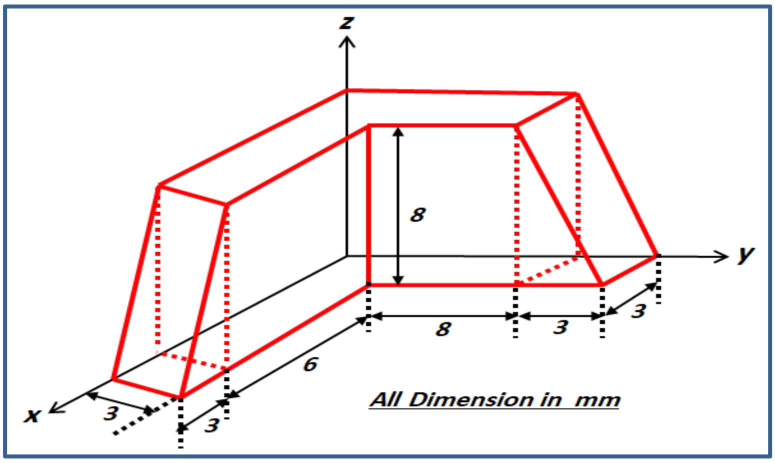
$$z_C = \frac{\Sigma\, z_i\,.A_i}{\Sigma\, A_i} = 0$$

$$C = (-0.348, 1.22)$$



Example - 3

Locate the center of mass of the bracket and shaft combination. The vertical face is made from sheet metal which has a mass of 25 kg/m^2. The material of the horizontal base has a mass of 40 kg/m^2, and the steel shaft has a density of 7830 Kg/m^3. (All dimensions in the figure are in millimeters)?



Solution

Seg.	$V_i\ (mm^2)$	$x_i\ (mm)$	$x_i\,.V_i\ (mm^3)$	$y_i\ (mm)$	$y_i\,.V_i\ (mm^3)$	$z_i\ (mm)$	$z_i\,.V_i\ (mm^3)$
1	36	10	360	1.5	54	$\frac{8}{3}$	96
2	216	4.5	972	1.5	324	4	864
3	192	1.5	288	7	1344	4	768
4	36	1.5	54	12	432	$\frac{8}{3}$	96
Sum	**480**		**1674**		**2154**		**1824**

$$x_C = \frac{\Sigma\, x_i\,.V_i}{\Sigma\, V_i} = \frac{1674}{480} = 3.49\ mm$$

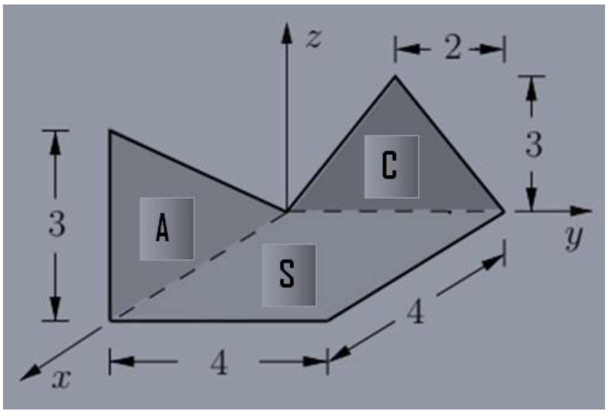
$$y_C = \frac{\Sigma\, y_i\,.V_i}{\Sigma\, V_i} = \frac{2154}{480} = 4.49\ mm$$

$$z_C = \frac{\Sigma\, z_i\,.V_i}{\Sigma\, V_i} = \frac{1824}{480} = 3.80\ mm$$

$$C = (3.49, 4.49, 3.80)\ mm$$

Example - 4

A thin sheet, copper and aluminum with thicknesses ($t_S = 0.03\ m\ , t_C = 0.02\ m, t_A = 0.04\ m$ and densities, ($\rho_S = 7850\ Kg/m^3\ ,\ \rho_C = 8960\ Kg/m^3, \rho_A = 2700\ Kg/m^3$, consisting of a square and two triangles, is bent to the depicted figure (measurements in meter). Locate the center of gravity?



Solution

The body is composed by three parts with already known location of centers of mass. The location of the center of mass of the complete system can be determined from:

$$x_C = \frac{\Sigma\, x_i\, .m_i}{\Sigma\, m_i}\ ,\qquad y_C = \frac{\Sigma\, y_i\, .m_i}{\Sigma\, m_i}\ ,\qquad z_C = \frac{\Sigma\, z_i\, .m_i}{\Sigma\, m_i}$$

$$x_C = \frac{\Sigma\rho_i\, .x_i\, .V_i}{\Sigma\rho_i\, .V_i}\ ,\qquad y_C = \frac{\Sigma\rho_i\, .y_i\, .V_i}{\Sigma\rho_i\, .V_i}\ , z_C = \frac{\Sigma\rho_i\, .z_i\, .V_i}{\Sigma\rho_i\, .V_i}$$

The total area is:

$$V_S = 4 \times 4 \times 0.03 = 0.48\ m^3$$

$$V_C = \frac{1}{2} \times 4 \times 3 \times 0.02 = 0.12\ m^3$$

$$V_A = \frac{1}{2} \times 4 \times 3 \times 0.04 = 0.24\ m^3$$

$$\Sigma V_i = 0.48 + 0.12 + 0.24 = 0.84\ m^3$$

$$\Sigma\rho_i\, .V_i = 7850 \times 0.48 + 8960 \times 0.12 + 2700 \times 0.24$$

$$= 0.48 + 0.12 + 0.24 = 5491\ Kg$$

$$x_C = \frac{\Sigma\rho_i\, .x_i\, .V_i}{\Sigma\rho_i\, .V_i}$$

$$y_C = \frac{\Sigma \rho_i \,. y_i \,. V_i}{\Sigma \rho_i \,. V_i}$$

$$z_C = \frac{\Sigma \rho_i \,. z_i \,. V_i}{\Sigma \rho_i \,. V_i}$$

Calculating the first area moments of the total system about each axis, in each case one first moment of a subsystem drops out because of zero distance: $x_C = 0,\ y_A = 0, and\ z_S = 0$. Thus, we obtain:

$$x_C = \frac{x_s . m_s + x_A . m_A}{\Sigma m_i} = \frac{2 \times 3768 + \left(\frac{2}{3} \times 4\right) \times 648}{5491} = 1.68\, m\,,$$

$$y_C = \frac{x_s . m_s \,. + x_c . m_c \,.}{\Sigma m_i} = \frac{2 \times 3768 + 2 \times 1075}{5491} = 1.76\, m$$

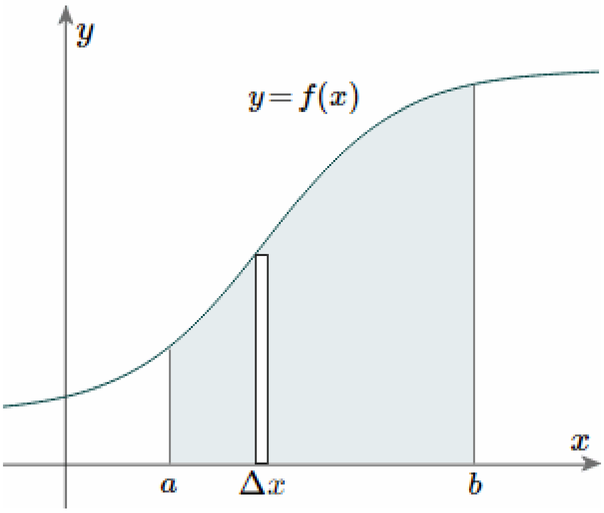
$$z_C = \frac{x_C . m_C + x_A . m_A}{\Sigma m_i} = \frac{1 \times 1075 + 1 \times 648}{5491} = 0.31 m$$

$$C = (1.68, 1.76, 0.31)\, m$$

4.7. Finding the Centroid via the First Moment Integral

Centroid for Curved Areas

Taking the simple case first, we aim to find the centroid for the area defined by a function $f(x)$, and the vertical lines $x = a$ and $x = b$ as indicated in the following figure.



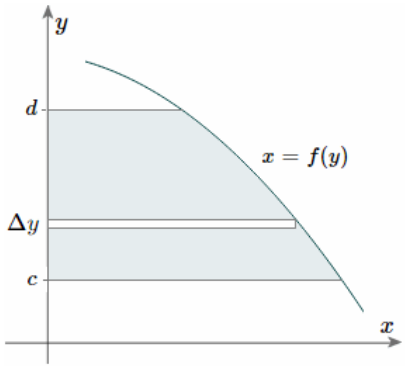
To find the centroid, we use the same basic idea that we were using for the straight-sided case above. The "typical" rectangle indicated is x units from the y-axis, and it has width Δx (which becomes dx when we integrate) and height $y = f(x)$.

Generalizing from the above rectangular areas case, we multiply these 3 values (x, $f(x)$ and Δx, which will give us the area of each thin rectangle times its distance from the x-axis), then add them. If we do this for infinitesimally small strips, we get the x-coordinates of the centroid using the total moments in the x-direction, given by:

$$\overline{x} = \frac{\text{total moments}}{\text{total area}} = \frac{1}{A}\int_a^b x\ f(x)\ dx$$

And, considering the moments in the y-direction about the x-axis and re-expressing the function in terms of y, we have:

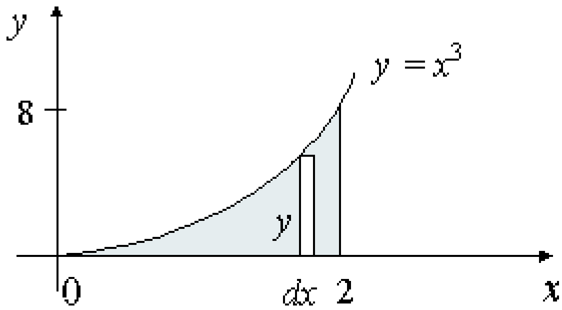
$$\overline{y} = \frac{\text{total moments}}{\text{total area}} = \frac{1}{A}\int_c^d y\ f(y)\ dy$$



Example - 5

Find the centroid of the area bounded by $y=x^3$, $x=2$ and the x-axis.

Here is the area under consideration:



In this case, $y=f(x)=x^3, a=0, b=2.$

We find the shaded area first:

$$A=\int_0^2 x^3\,dx=\left[\frac{x^4}{4}\right]_0^2=\frac{16}{4}=4$$

Next, using the formula for the x-coordinate of the centroid we have:

$$\begin{aligned}\overline{x}&=\frac{1}{A}\int_a^b x f(x)\,dx\\&=\frac{1}{4}\int_0^2 x\left(x^3\right)dx\\&=\frac{1}{4}\int_0^2\left(x^4\right)dx\\&=\frac{1}{4}\left[\frac{x^5}{5}\right]_0^2\\&=\frac{32}{20}\\&=1.6\end{aligned}$$

$$\overline{y} = \frac{1}{A}\int_c^d y(x_2 - x_1)\,dy$$

$$= \frac{1}{4}\int_0^8 y\left(2 - y^{\frac{1}{3}}\right)dy$$

$$= \frac{1}{4}\int_0^8 \left(2y - y^{\frac{4}{3}}\right)dy$$

$$= \frac{1}{4}\left[y^2 - \frac{3y^{\frac{7}{3}}}{7}\right]_0^8$$

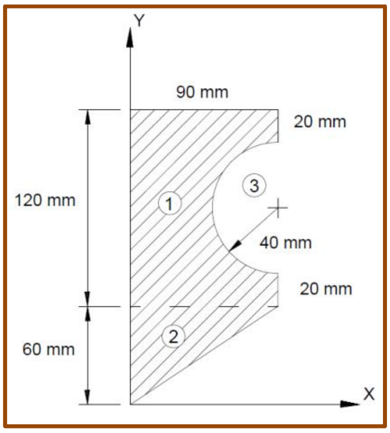
$$= \frac{1}{4}\left[64 - \frac{3 \times 128}{7}\right]$$

$$= 2.29$$

So the centroid for the shaded area is at (1.6, 2.29).

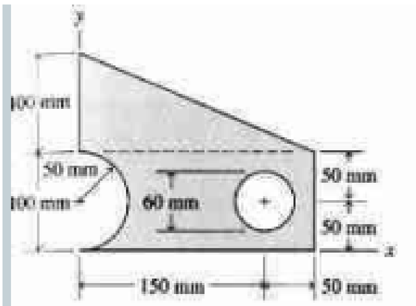
4.8. Chapter Questions

Q₁: Locate the centroid of the area shown in the figure below?



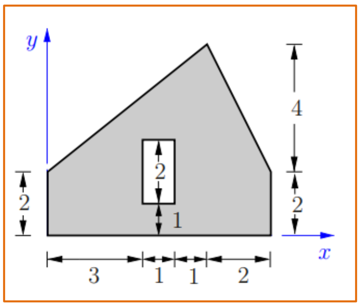
{Answer: $x_C = 34.9\ mm\ , and\ y_C = 100.4\ mm$}

Q₂: Locate the centroid of the area shown in the figure below?



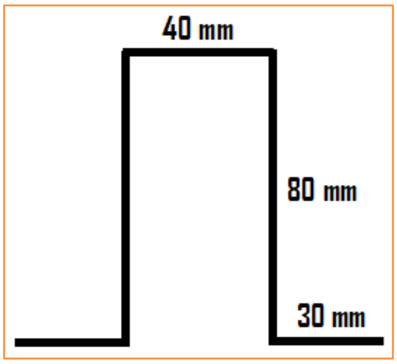
{Answer: $x_C = 92.9\ mm\ , and\ y_C = 85.8\ mm$}

Q₃: Locate the centroid of the depicted area with a rectangular cutout. The measurements are given in meter?



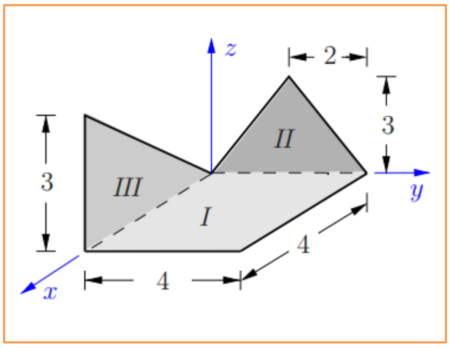
{Answer: $x_C = 3.77\ m\ \ , and\ y_C = 2.18\ m$ }

Q$_4$: A wire with constant thickness is deformed into the depicted figure. The measurements are given in mm. Find the Locate of the centroid?



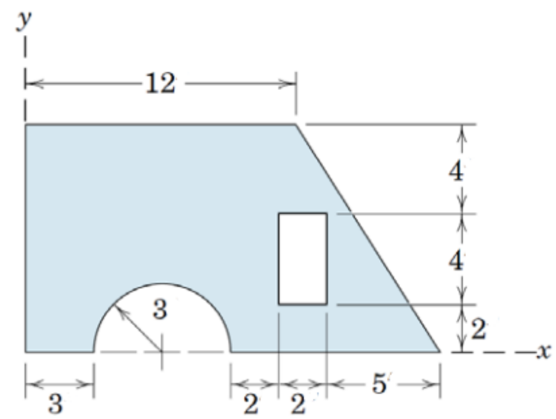
{*Answer:* $y_C = 5.08\ mm$ }

Q$_5$: A thin sheet with constant thickness and density, consisting of a square and two triangles, is bent to the depicted figure (measurements in meter). Locate the center of gravity?



$x_C = 1.71\ m, \qquad y_C = 1.57, and\ z_C = 0.43\ m$ }

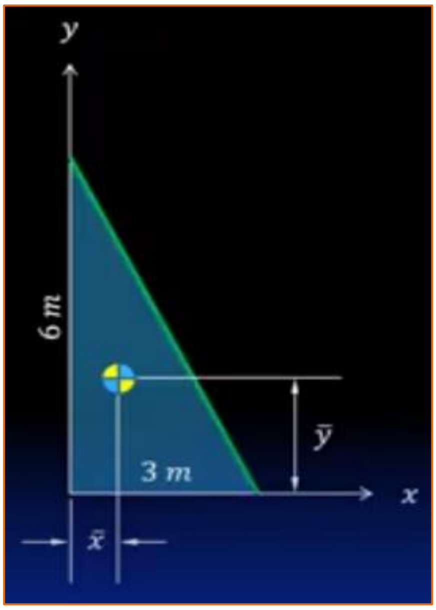
Q$_6$: Locate the centroid of the area shown in the figure below?



{***Answer:*** $x_C = 7.5\ m\ \ , and\ y_C = 5.08\ m$ }

Q_7: Find the Locate of the centroid of the area shown in the figure below, by using integration?

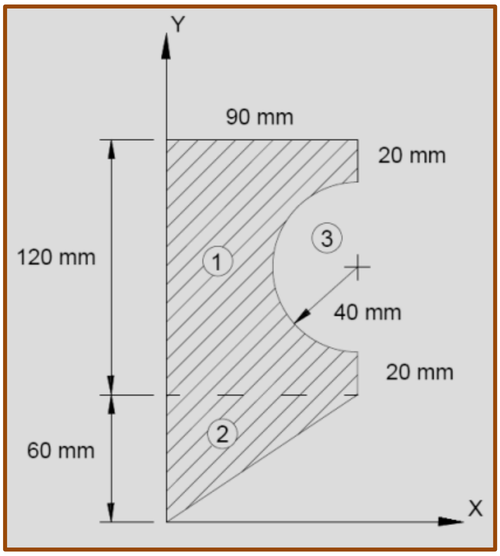
$$y = 2(3 - x)$$



Solve Question Home Work - 4

4.8 Chapter Questions

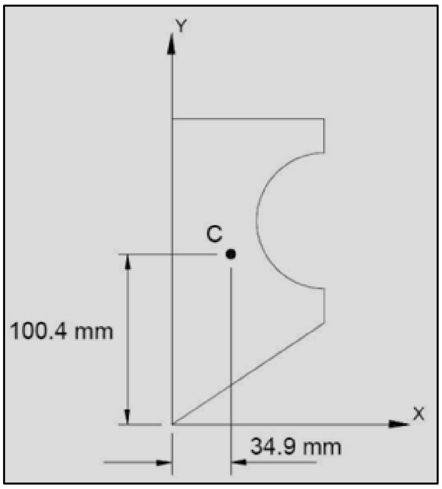
Q₁: Locate the centroid of the area shown in the figure below?



Solution

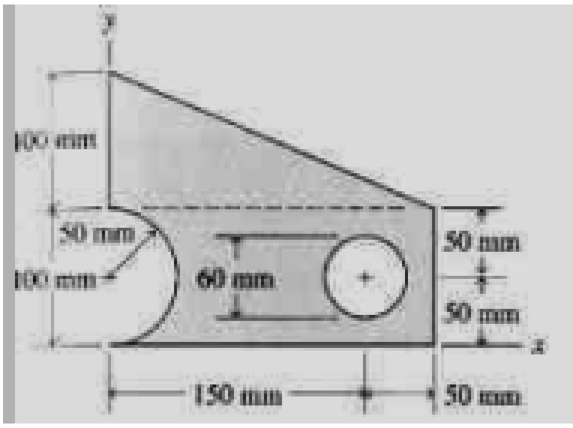
Part	Area, A_i	$\bar{x}_i$	$\bar{y}_i$	$\bar{x}_i A_i$	$\bar{y}_i A_i$
1	10,800	45.0	120.0	486,000	1,296,000
2	2,700	30.0	40.0	81,000	108,000
3	- 2,510	73.0	120.0	- 183,000	- 301,000
Totals	10,990			384,000	1,103,000

$$x_C = \frac{\Sigma x_i . A_i}{A} = \frac{384000}{10990} = 34.94\ mm\ , \qquad y_C = \frac{\Sigma y_i . A_i}{A} = \frac{110300}{10990} = 100.4\ mm$$

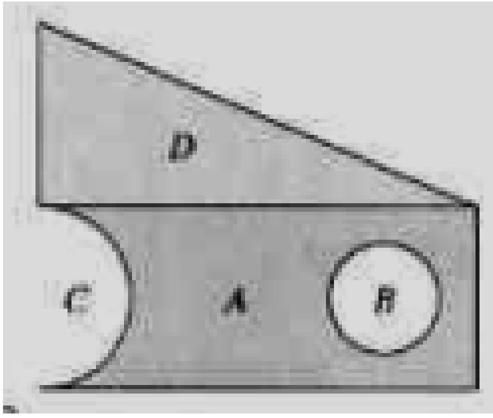


{Answer: $x_C = 34.9\ mm$, and $y_C = 100.4\ mm$}

Q₂: Locate the centroid of the area shown in the figure below?



Solution



shape	A (mm^2)	$\bar{x}$ (mm)	$\bar{x}A$ (mm^3)	$\bar{y}$ (mm)	$\bar{y}A$ (mm^3)
A	20000	100	2000000	50	1000000
B	-2827.43	150	-424115	50	-141372
C	-3926.99	21.22066	-83333.3	50	-196350
D	10000	66.66667	666666.7	133.3333	1333333
	23245.58		2159218		1995612

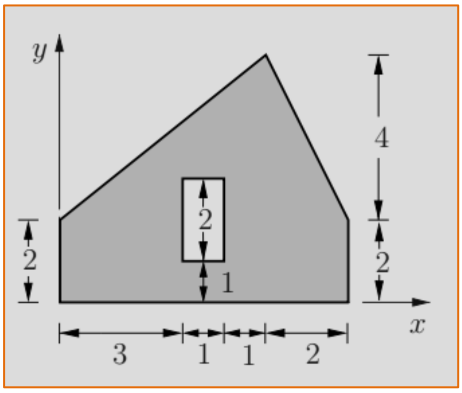
$$\hat{x} = \frac{2159218 mm^3}{23245.58 mm^2} = 92.9\ mm$$

$$\hat{y} = \frac{1995612\ mm^3}{23245.58\ mm^2} = 85.8\ mm$$

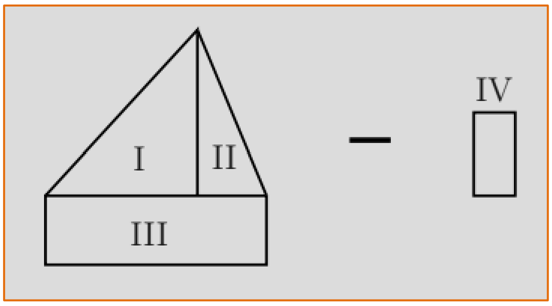
$$x_C = \frac{\Sigma x_i . A_i}{A} = \frac{2159218}{23245.58} = 92.89\ mm\ ,\ y_C = \frac{\Sigma y_i . A_i}{A} = \frac{1995612}{23245.58} = 85.85\ mm$$

{Answer: $x_C = 92.9\ mm$, and $y_C = 85.8\ mm$}

Q_3: Locate the centroid of the depicted area with a rectangular cutout. The measurements are given in meter?



Solution



The calculation is conveniently done by using a table.

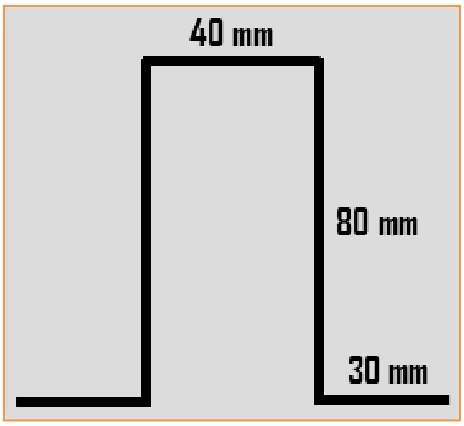
Segment	$A_i\ (m^2)$	$x_i\ (m)$	$x_i \, . A_i\ (m^3)$	$y_i\ (m)$	$y_i \, . A_i\ (m^3)$
I	10	3.33	33.3	3.33	33.3
II	4	5.67	22.68	3.33	13.32
III	14	3.5	49	1	14
IV	-2	3.5	-7	2	-4
Sum	**26**		**97.98**		**56.62**

Thus, we obtain:

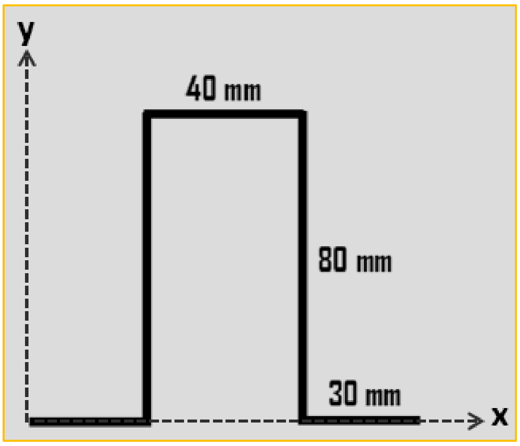
$$x_C = \frac{\Sigma x_i . A_i}{A} = \frac{97.98}{26} = 3.77\ m\,, \quad y_C = \frac{\Sigma y_i . A_i}{A} = \frac{56.62}{26} = 2.18\ m$$

{Answer: $x_C = 3.77\ m$, and $y_C = 2.18\ m$ }

Q$_4$: A wire with constant thickness is deformed into the depicted figure. The measurements are given in mm. Find the Locate of the centroid?



Solution

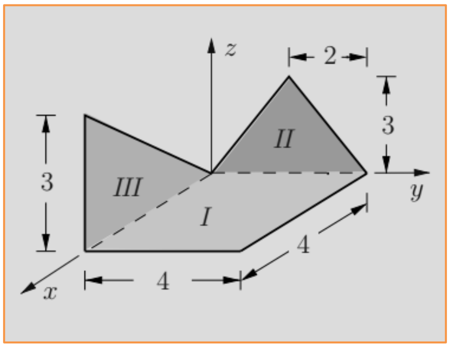


$$x_C = \frac{\Sigma\, x_i \,.L_i}{\Sigma\, L_i} = \frac{30 \times 15 + 80 \times 30 + 40 \times 50 + 80 \times 70 + 30 \times 85}{30 + 80 + 40 + 80 + 30} = 50\ mm$$

$$y_C = \frac{\Sigma\, y_i \,.L_i}{\Sigma\, L_i} = \frac{80 \times 40 + 40 \times 80 + 80 \times 40}{30 + 80 + 40 + 80 + 30} = 36.92\ mm$$

{Answer: $x_C = 50\ mm, \quad y_C = 36.92\ mm$ }

Q_5: A thin sheet with constant thickness and density, consisting of a square and two triangles, is bent to the depicted figure (measurements in meter). Locate the center of gravity?



Solution

The body is composed by three parts with already known location of centers of mass. The location of the center of mass of the complete system can be determined from

$$x_C = \frac{\sum \rho_i x_i V_i}{\sum \rho_i V_i}, \quad y_C = \frac{\sum \rho_i y_i V_i}{\sum \rho_i V_i}, \quad z_C = \frac{\sum \rho_i z_i V_i}{\sum \rho_i V_i}.$$

Since the thickness and the density of the sheet is constant, these terms cancel out and we obtain:

$$x_C = \frac{\sum x_i A_i}{\sum A_i}, \quad y_C = \frac{\sum y_i A_i}{\sum A_i}, \quad z_C = \frac{\sum z_i V_i}{\sum A_i}.$$

The total area is:

$$A = \Sigma A_i = 4 \times 4 + \frac{1}{2} \times 4 \times 3 + \frac{1}{2} \times 4 \times 3 = 28\ m^2$$

Calculating the first area moments of the total system about each axis, in each case one first moment of a subsystem drops out because of zero distance: $x_{II} = 0,\ y_{III} = 0, and\ z_I = 0$. Thus, we obtain:

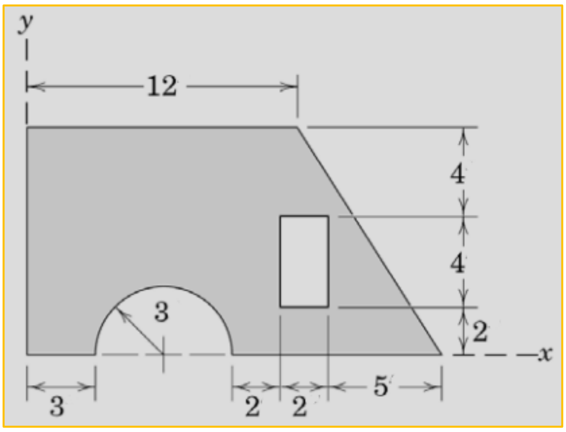
$$x_C = \frac{x_I.A_I + x_{III}.A_{III}}{A} = \frac{2 \times 16 + \left(\frac{2}{3} \times 4\right) \times 6}{28} = 1.71\ m\,,$$

$$y_C = \frac{y_I.A_I + y.A_{III}}{A} = \frac{2 \times 16 + 2 \times 6}{28} = 1.57\ m$$

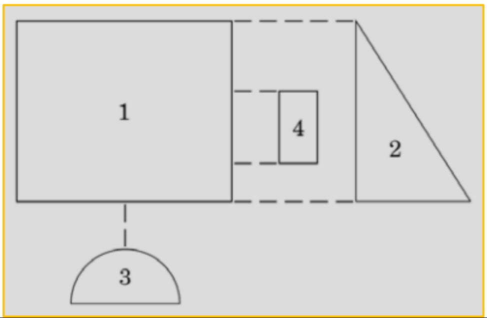
$$z_C = \frac{z_{II}.A_{II} + z_{III}.A_{III}}{A} = \frac{(\frac{1}{3} \times 3) \times 6 + (\frac{1}{3} \times 3) \times 6}{28} = 0.43\ m$$

{*Answer:* $x_C = 1.71\ m,\ y_C = 1.57, and\ z_C = 0.43\ m$ }

Q$_6$: Locate the centroid of the area shown in the figure below, all dimension in m?



Solution



Segment	$A_i\ (m^2)$	$x_i\ (m)$	$x_i\,.A_i\ (m^3)$	$y_i\ (m)$	$y_i\,.A_i\ (m^3)$
1	120	6	720	5	600
2	30	14	420	3.33	100
3	-14.14	6	-84.8	1.273	-18
4	-8	12	-96	4	-32
Sum	**127.9**		**959**		**650**

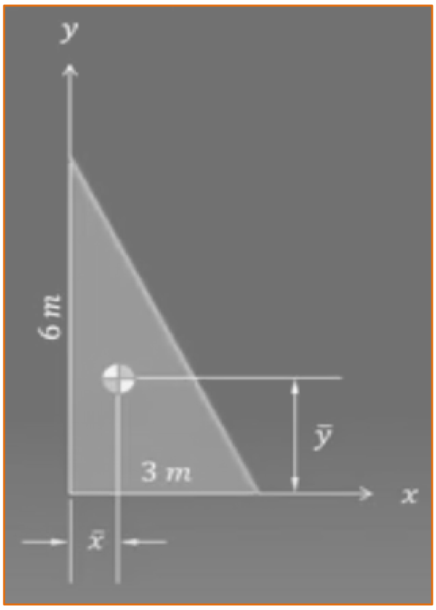
Thus, we obtain:

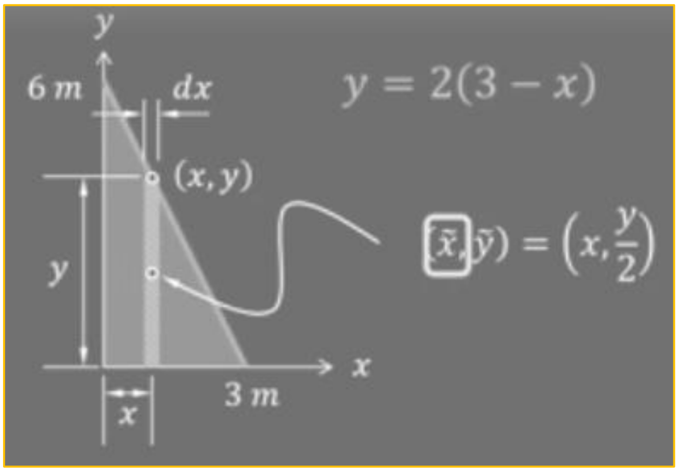
$$x_C = \frac{\Sigma x_i . A_i}{A} = \frac{959}{127.9} = 7.50\ \ m\,, \quad y_C = \frac{\Sigma y_i . A_i}{A} = \frac{650}{127.9} = 5.08\ m$$

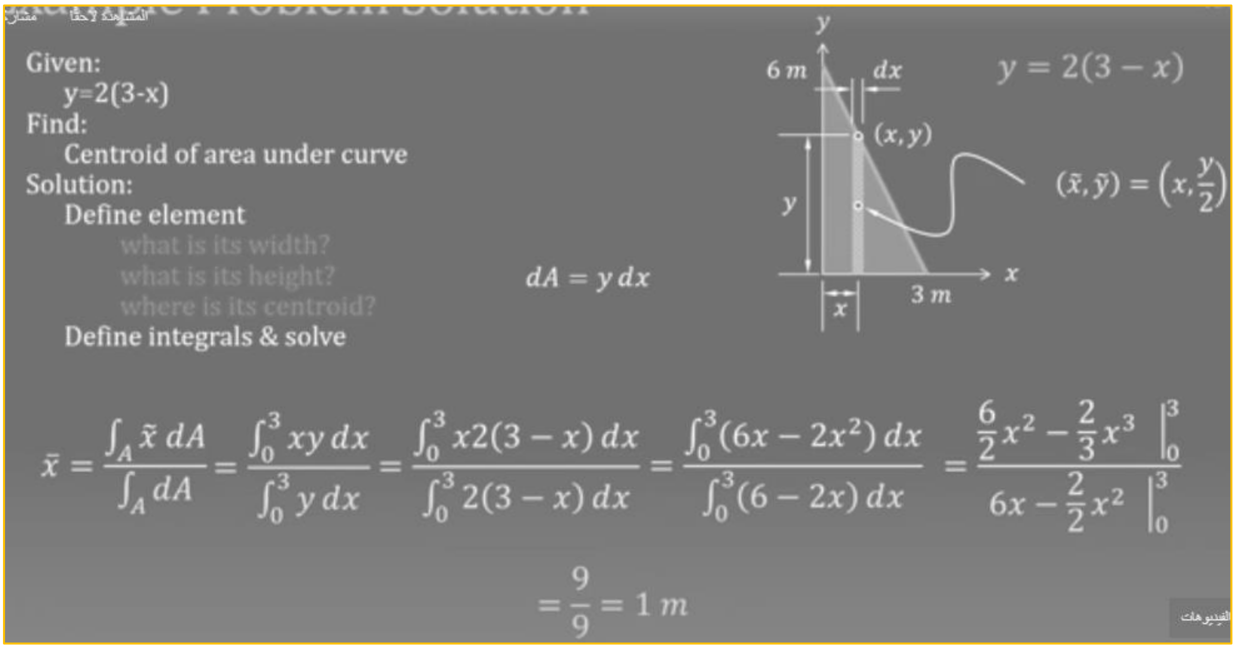
{Answer: $x_C = 7.5\ m\ \ , and\ y_C = 5.08\ m$ }

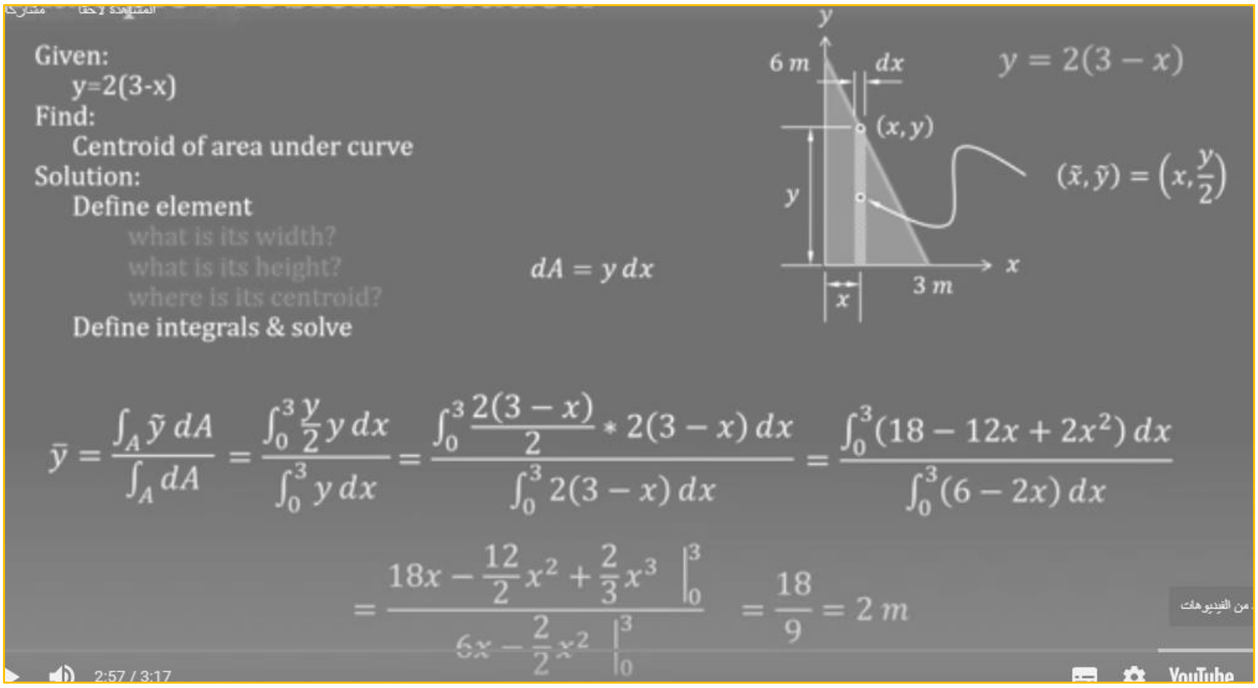
Q$_7$: Find the Locate of the centroid of the area shown in the figure below, by using integration?

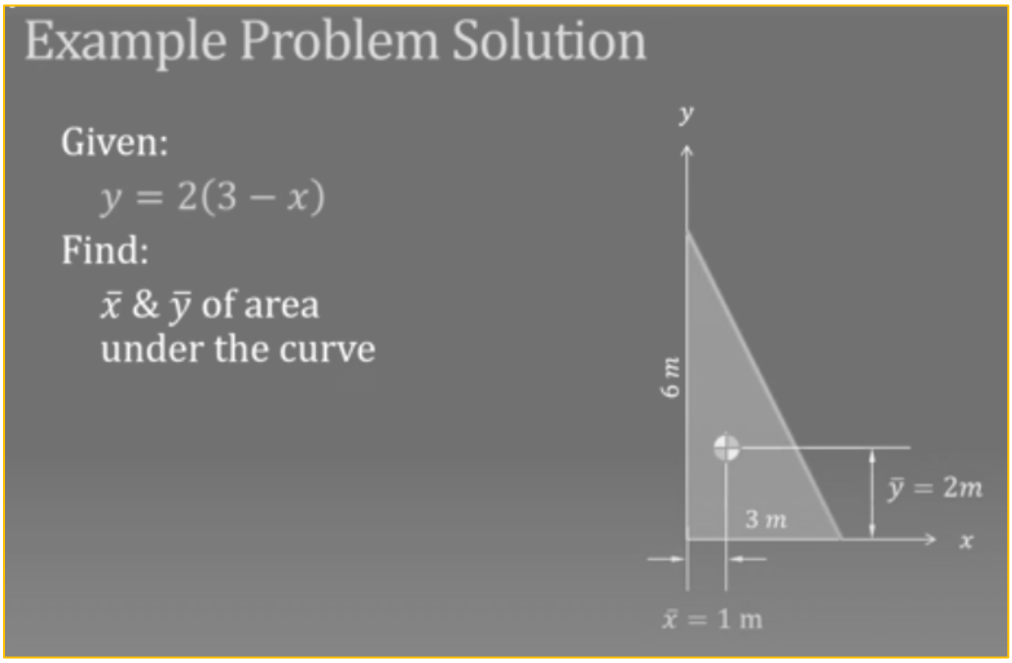
$$y = 2(3 - x)$$






Given:
y=2(3-x)
Find:
Centroid of area under curve
Solution:
Define element
what is its width?
what is its height?
where is its centroid?
Define integrals & solve
$dA = y\,dx$
$y = 2(3-x)$
6 m
dx
(x, y)
y
x
3 m
$(\tilde{x},\tilde{y}) = \left(x, \frac{y}{2}\right)$
$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^3 xy\,dx}{\int_0^3 y\,dx} = \frac{\int_0^3 x2(3-x)\,dx}{\int_0^3 2(3-x)\,dx} = \frac{\int_0^3 (6x-2x^2)\,dx}{\int_0^3 (6-2x)\,dx} = \frac{\frac{6}{2}x^2 - \frac{2}{3}x^3 \Big|_0^3}{6x - \frac{2}{2}x^2 \Big|_0^3}$
$= \frac{9}{9} = 1\,m$


Given:
y=2(3-x)
Find:
Centroid of area under curve
Solution:
Define element
what is its width?
what is its height?
where is its centroid?
Define integrals & solve
$dA = y\,dx$
$y = 2(3-x)$
6 m
dx
(x, y)
y
x
3 m
$(\tilde{x},\tilde{y}) = \left(x, \frac{y}{2}\right)$
$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^3 \frac{y}{2} y\,dx}{\int_0^3 y\,dx} = \frac{\int_0^3 \frac{2(3-x)}{2} * 2(3-x)\,dx}{\int_0^3 2(3-x)\,dx} = \frac{\int_0^3 (18-12x+2x^2)\,dx}{\int_0^3 (6-2x)\,dx}$
$= \frac{18x - \frac{12}{2}x^2 + \frac{2}{3}x^3 \Big|_0^3}{6x - \frac{2}{2}x^2 \Big|_0^3} = \frac{18}{9} = 2\,m$
2:57 / 3:17
YouTube



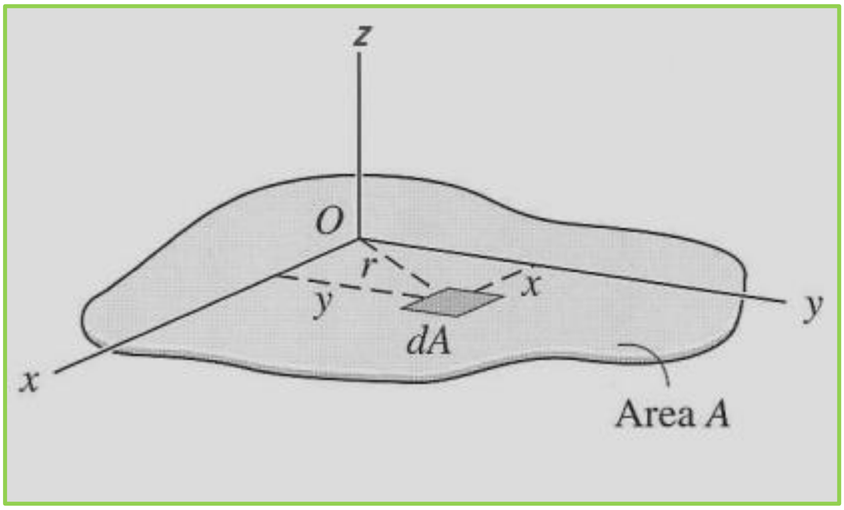
{Answer: $x_C = 1\ m$, $and\ y_C = 2\ m$ }

Chapter 5

Moments of Inertia

5.1. Moment of Inertia and Properties of Plane Areas

The Moment of Inertia (I) is a term used to describe the capacity of a cross-section to resist bending. It is always considered with respect to a reference axis such as (X-X) or (Y-Y). It is a mathematical property of a section concerned with a surface area and how that area is distributed about the reference axis (axis of interest). The reference axis is usually a centroid axis. The moment of inertia is also known as the Second Moment of the Area and is expressed mathematically as:



$$I_x = \int_A y^2 dA$$

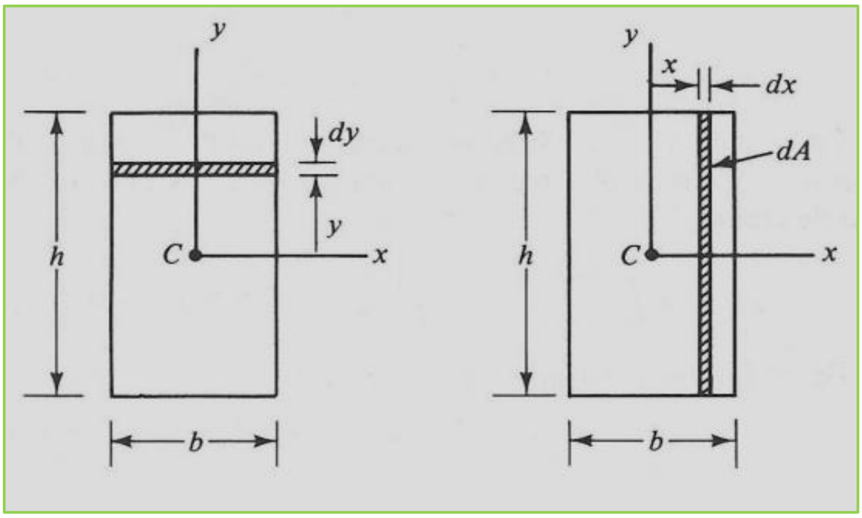
$$I_y = \int_A x^2 dA$$

Where:

y = distance from the x axis to area *dA*

x = distance from the y axis to area d*A*

Example



5.2. Application of moment of inertia

The crank on the oil-pump rig undergoes rotation about a fixed axis that is not at its mass center. The crank develops a kinetic energy directly related to its mass moment of inertia. As the crank rotates, its kinetic energy is converted to potential energy and vice versa.

Is the mass moment of inertia of the crank about its axis of rotation smaller or larger than its moment of inertia about its center of mass.

5.3. Radius of Gyration

The radius of gyration of an area with respect to a particular axis is the square root of the quotient of the moment of inertia divided by the area. It is the distance at which the entire area must be assumed to be concentrated in order that the product of the area and the square of this distance will equal the moment of inertia of the actual area about the given axis. In other words, the radius of gyration describes the way in which the total cross-sectional area is distributed around its centroidal axis. If more area is distributed further from the axis, it will have greater resistance to buckling. The most efficient column section to resist buckling is a circular pipe, because it has its area distributed as far away as possible from the centroid. Rearranging we have:

$$I_x = k_x^2 \,.A \quad , \qquad I_y = k_y^2 \,.A \quad , \qquad I_z = k_z^2 \,.A$$

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_z = \sqrt{\frac{I_y}{A}}$$

The radius of gyration is the distance k away from the axis that all the area can be concentrated to result in the same moment of inertia.

5.4. Polar Moment of Inertia

$$I_p = \int_A \rho^2 dA$$
$$I_p = \int_A (x^2 + y^2) dA$$
$$I_p = \int_A x^2 dA + \int_A y^2 dA$$
$$I_p = I_x + I_y$$

In many texts, the symbol J will be used to denote the polar moment of inertia.

$$\boldsymbol{J = I_x + I_y}$$

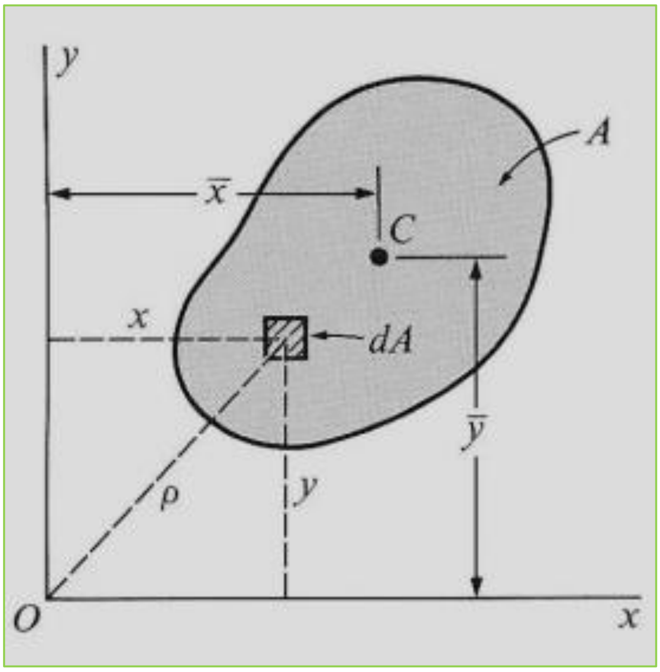
Shear stress formula:

$$\boldsymbol{\tau = \frac{T_r}{J}}$$

5.5. Product of Inertia

$$I_{xy} = \int_A xy\,dA$$

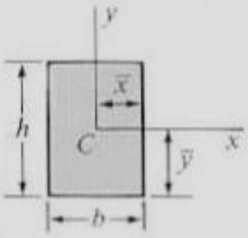
Consider the following:



If an area has at least one axis of symmetry, the product of inertia is zero.

5.6. Properties of Plane Areas

1



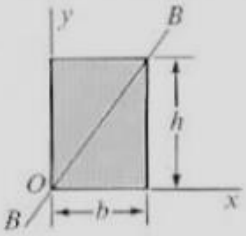
Rectangle (Origin of axes at centroid.)

$$A = bh \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12}$$

$$I_{xy} = 0 \qquad I_p = \frac{bh}{12}(h^2 + b^2)$$

2

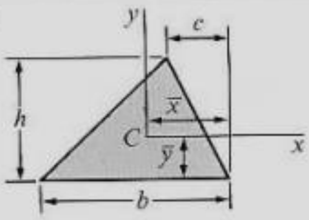


Rectangle (Origin of axes at corner.)

$$I_x = \frac{bh^3}{3} \qquad I_y = \frac{hb^3}{3}$$

$$I_{xy} = \frac{b^2h^2}{4} \qquad I_p = \frac{bh}{3}(h^2 + b^2) \qquad I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

3



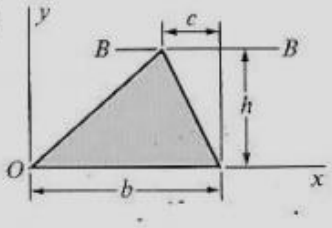
Triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b + c}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{bh}{36}(b^2 - bc + c^2)$$

$$I_{xy} = \frac{bh^2}{72}(b - 2c) \qquad I_p = \frac{bh}{36}(h^2 + b^2 - bc + c^2)$$

4

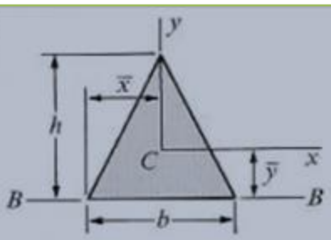


Triangle (Origin of axes at vertex.)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{bh}{12}(3b^2 - 3bc + c^2)$$

$$I_{xy} = \frac{bh^2}{24}(3b - 2c) \qquad I_{BB} = \frac{bh^3}{4}$$

5



Isosceles triangle (Origin of axes at centroid.)

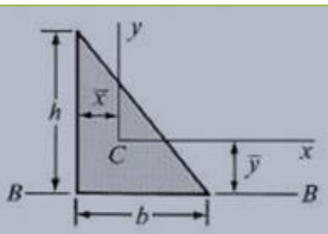
$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{48} \qquad I_{xy} = 0$$

$$I_p = \frac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

(Note: For an equilateral triangle, $h = \sqrt{3}b/2$.)

6



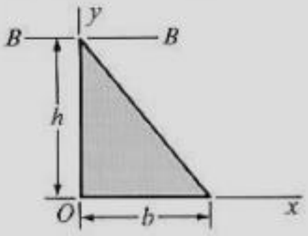
Right triangle (Origin of axes at centroid.)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{36} \qquad I_{xy} = -\frac{b^2h^2}{72}$$

$$I_p = \frac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

7

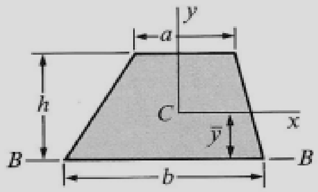


Right triangle (Origin of axes at vertex.)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = \frac{b^2h^2}{24}$$

$$I_p = \frac{bh}{12}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{4}$$

8

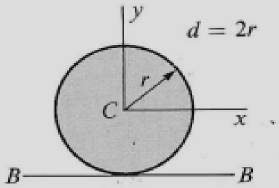


Trapezoid (Origin of axes at centroid.)

$$A = \frac{h(a + b)}{2} \qquad \bar{y} = \frac{h(2a + b)}{3(a + b)}$$

$$I_x = \frac{h^3(a^2 + 4ab + b^2)}{36(a + b)} \qquad I_{BB} = \frac{h^3(3a + b)}{12}$$

9

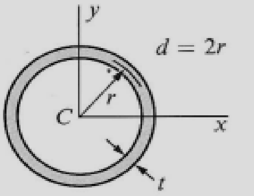


Circle (Origin of axes at center.)

$$A = \pi r^2 = \frac{\pi d^2}{4} \qquad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

$$I_{xy} = 0 \qquad I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$$

10

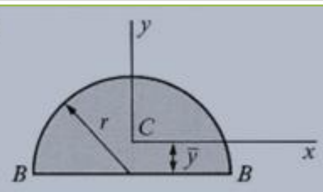


Circular ring (Origin of axes at center.)
Approximate formulas for case when t is small.

$$A = 2\pi rt = \pi dt \qquad I_x = I_y = \pi r^3 t = \frac{\pi d^3 t}{8}$$

$$I_{xy} = 0 \qquad I_p = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

11

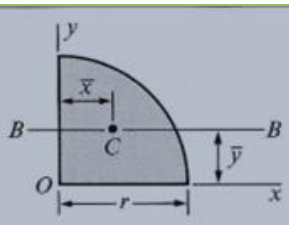
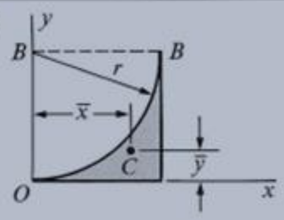


Semicircle (Origin of axes at centroid.)

$$A = \frac{\pi r^2}{2} \qquad \bar{y} = \frac{4r}{3\pi}$$

$$I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4 \qquad I_y = \frac{\pi r^4}{8}$$

$$I_{xy} = 0 \qquad I_{BB} = \frac{\pi r^4}{8}$$

12		**Quarter circle** (Origin of axes at center of circle.) $A = \frac{\pi r^2}{4} \qquad \bar{x} = \bar{y} = \frac{4r}{3\pi}$ $I_x = I_y = \frac{\pi r^4}{16} \qquad I_{xy} = \frac{r^4}{8}$ $I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488 r^4$
13		**Quarter-circular spandrel** (Origin of axes at vertex.) $A = \left(1 - \frac{\pi}{4}\right) r^2$ $\bar{x} = \frac{2r}{3(4-\pi)} \approx 0.7766r \qquad \bar{y} = \frac{(10 - 3\pi)r}{3(4-\pi)} \approx 0.2234r$ $I_x = \left(1 - \frac{5\pi}{16}\right) r^4 \approx 0.01825 r^4 \qquad I_y = I_{BB} = \left(\frac{1}{3} - \frac{\pi}{16}\right) r^4 \approx 0.1370 r^4$

5.7. Moment of Inertia of Composite Areas

In the context of calculating the moment of inertia, a composite area is an area consisting of several non-overlapping (disjoint) sub-areas. The boundaries specifying the sub-areas can be explicitly declared by the geometry or arbitrarily chosen. Considering an area as a composite area is to simplify the calculation of the moment of inertia of the whole area. having *simple* shapes with already known or given formulations of moments of inertia. To find the moment of inertia, the following table and equations are applied:

Part	A (mm)	dx (mm)	dy (mm)	A . dx² (mm⁴)	A . dy² (mm⁴)	Ix0 (mm⁴)	Iy0 (mm⁴)
1							
2							
3							
Total				$\Sigma A \cdot d_x^2$	$\Sigma A \cdot d_y^2$	ΣI_{x0}	ΣI_{y0}

$$I_x = \Sigma I_{x0} + \Sigma A \cdot d_y^2$$
$$I_y = \Sigma I_{y0} + \Sigma A \cdot d_x^2$$
$$I_Z = I_x + I_y$$
$$k_x = \sqrt{\frac{I_x}{A}}$$
$$k_y = \sqrt{\frac{I_y}{A}}$$

5.8. Parallel Axis Theorem

$$I_x = I_{xc} + Ad^2$$

$$I_y = I_{yc} + Ad^2$$

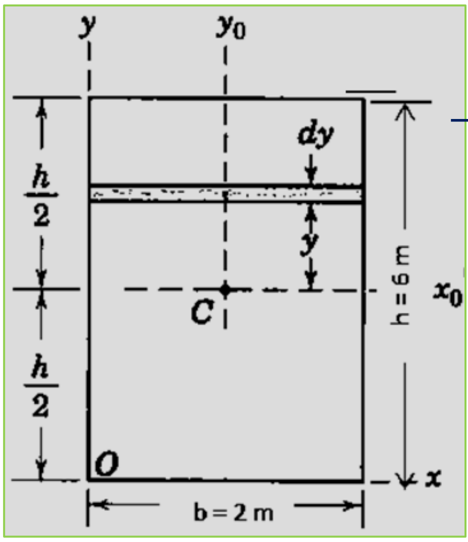
The moment of inertia of an area with respect to any given axis is equal to the moment of inertia with respect to the centroidal axis plus the product of the area and the square of the distance between the 2 axes.

The parallel axis theorem is used to determine the moment of inertia of composite sections.

5.9. Solved Examples

Example 1.

Determine the moments of inertia of the rectangular area about the centoidal (x_0 and y_0) axes, the centroidal polar (z_0) through (C), the x-axes, and the polar axis (z) through (O). If ($h = 6\ m$ and $b = 2\ m$).



<u>Solution:</u>

$$\boldsymbol{A = bh = 2} \times 6 = 12\ m^2$$

By interchange of symbols, the moment of inertia about the centroidal ($x_o - axis$) is:

$$I_x = \int y^2 dA$$

$$\bar{I}_x = \int_{-h/2}^{h/2} y^2 . b\, dy = b. \left[\frac{y^3}{3}\right]_{-h/2}^{h/2} = b . \left[\frac{\left(\frac{h^3}{8}\right)}{3} + \frac{\left(\frac{h^3}{8}\right)}{3}\right] = b . \left[\frac{2h^3}{24}\right] = \frac{1}{12} bh^3$$

$$\bar{I}_x = \frac{1}{12} \times 2 \times 6^3 = \frac{432}{6} = 36\, m^4$$

By interchange of symbols, the moment of inertia about the centroidal ($y_o - axis$) is:

$$I_y = \int x^2 dA$$

$$\bar{I}_y = \int_{-b/2}^{b/2} x^2 . h\, dx = h. \left[\frac{x^3}{3}\right]_{-b/2}^{b/2} = h . \left[\frac{\left(\frac{b^3}{8}\right)}{3} + \frac{\left(\frac{b^3}{8}\right)}{3}\right] = h . \left[\frac{2b^3}{24}\right] = \frac{1}{12} hb^3$$

$$\bar{I}_y = \frac{1}{12} \times 6 \times 2^3 = \frac{48}{12} = 4\, m^4$$

The centroidal polar of inertia is:

$$\bar{I}_z = \bar{I}_x + \bar{I}_y$$

$$\bar{I}_z = \frac{1}{12} bh^3 + \frac{1}{12} hb^3 = \frac{1}{12} hb(h^2 + b^2) = \frac{1}{12} \times 6 \times 2 \times (2^2 + 6^2) = 40\, m^4$$

By the parallel – axis theorem the moment of inertia about the x-axis is:

$$I_x = \bar{I}_x + Ad_x^2$$

$$I_x = \frac{1}{12} bh^3 + A\left(\frac{h}{2}\right)^2 = \frac{1}{12} \times 2 \times 6^3 + 12 \times \frac{6^2}{4} = 144\ m^4$$

Also obtain the polar moment of inertia about (O) by the parallel axis theorem, which gives the following:

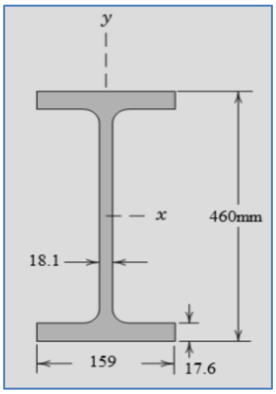
$$I_z = \bar{I}_z + Ad_Z^2$$

$$I_z = \frac{1}{12} A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right] = \frac{1}{3} A(b^2 + h^2)$$

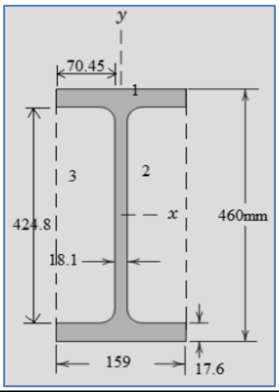
$$= \frac{1}{3} \times 12 \times (2^2 + 6^2) = 272\ m^4$$

Example 2.

The cross-sectional area of a wide-flange I-beam has the dimensions shown. Obtain a close approximation to the handbook value of by treating the section as being composed of three rectangles.



<u>Solution:</u>



Part	A (m)	dx (m)	dy (mm)	A , dx^2 (m^4)	A , dy^2 (m^4)	Ix$_0$ (m^4)	Iy$_0$ (m^4)
1	73140	0	0	0	0	1289702000	154087695
2	-29927	$\frac{70.45}{2}+\frac{18.1}{2}=44.275$	0	-58665168.63	0	-450042237.91	-12377879.61
3	-29927	$\frac{70.45}{2}+\frac{18.1}{2}=44.275$	0	-58665168.63	0	-450042237.91	-12377879.61
Σ	13286			-117330337.26	0	389617524.18	129.331935.78

$$\boldsymbol{I_x} = \boldsymbol{\Sigma I_{x0}} + \boldsymbol{\Sigma A\,.\,d_y^2} = 389617524.18 + 0 = 389617524.18\ mm^4$$

$$\boldsymbol{I_y} = \boldsymbol{\Sigma I_{y0}} + \boldsymbol{\Sigma A\,.\,d_x^2} = 129.331935.78 - 117330337.26 = 12001598.52\ mm^4$$

$$\boldsymbol{I_Z} = \boldsymbol{I_x} + \boldsymbol{I_y} = 389617524.18 + 12001598.52 = 401619122.7\ mm^4$$

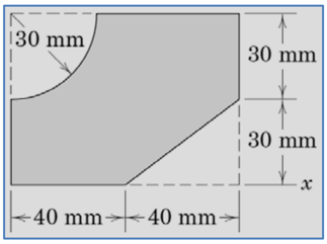
$$\boldsymbol{k_x} = \sqrt{\frac{\boldsymbol{I_x}}{\boldsymbol{A}}} = \sqrt{\frac{389617524.18}{13286}} = 171.25\ mm$$

$$\boldsymbol{k_y} = \sqrt{\frac{\boldsymbol{I_y}}{\boldsymbol{A}}} = \sqrt{\frac{12001598.52}{13286}} = 30.06\ mm$$

$$\boldsymbol{k_z} = \sqrt{\frac{\boldsymbol{I_y}}{\boldsymbol{A}}} = \sqrt{\frac{401619122.7}{13286}} = 173.86\ mm$$

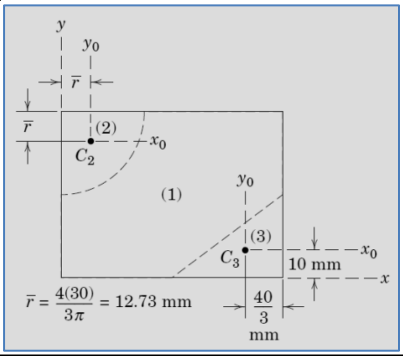
Example 3.

Determine the moments of inertia about the x- and y-axes for the shaded area. Make direct use of the expressions given in Table 1 for the centroidal moments of inertia of the constituent parts.



Solution:

The given area is subdivided into the three subareas shown a rectangular (1), a quarter circular (2), and a triangular (3) area. Two of the subareas are "holes" with negative areas. Centroidal x0 - y0 axes are shown for areas (2) and (3), and the locations of centroids C2 and C3 are from Table 1. The following table will facilitate the calculations.



Part	A (mm)	dx (mm)	dy (mm)	A . dx^2 (mm^3)	A . dy^2 (mm^3)	Ix_0 (mm^4)	Iy_0 (mm^4)
1	4800	40	30	7680000	4320000	1440000	2560000
2	-707.18	12.73	47.27	-114600.57	-1580160.4	-44452.8	-44452.8
3	-600	66.67	10	-2666933.34	-60000	-30000	-53333.33
Total	3492.82			4898466.09	2679839.6	1365547.2	158213.87

$$\boldsymbol{I_x} = \Sigma \boldsymbol{I_{x0}} + \boldsymbol{\Sigma A \,.\, d_y^2} = 1365547.2 + 2679839.6 = 4045386.8\ mm^4$$

$$\boldsymbol{I_y} = \Sigma \boldsymbol{I_{y0}} + \boldsymbol{\Sigma A \,.\, d_x^2} = 158213.87 + 4898466.09 = 7360679.96\ mm^4$$

$$\boldsymbol{I_Z} = \boldsymbol{I_x} + \boldsymbol{I_y} = 4045386.8 + 7360679.96 = 11406066.76\ mm^4$$

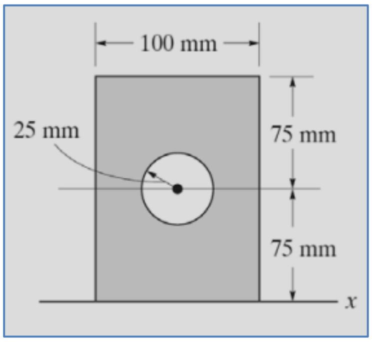
$$\boldsymbol{k_x} = \sqrt{\frac{\boldsymbol{I_x}}{\boldsymbol{A}}} = \sqrt{\frac{4045386.8}{3492.82}} = 34.032\ mm$$

$$\boldsymbol{k_y} = \sqrt{\frac{\boldsymbol{I_y}}{\boldsymbol{A}}} = \sqrt{\frac{7360679.96}{3492.82}} = 45.906\ mm$$

$$k_z = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{11406066.76}{3492.82}} = 57.145\ mm$$

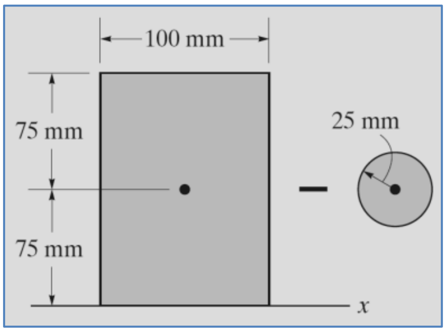
Example 4.

Determine the moment of inertia of the area shown in figure below about the x-axis.



Solution:

Composite Parts. The area can be obtained by subtracting the circle from the rectangle shown in figure below. The centroid of each area is located in the figure. Parallel-Axis Theorem. The moments of inertia about the x-axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas ($I_x = \pi r^4/4$; $I_x = bh^3/12$), found in table 1.



Part	A (mm)	dx (mm)	dy (mm)	A . dx² (mm³)	A . dy² (mm³)	Ix0 (mm⁴)	Iy0 (mm⁴)
1	15000	50	75	37500000	84375000	28125000	12500000
2	-1964.29	50	75	-4910725	-11049131.25	-306919.64	-306919.64
Total	13035.71			32589275	73325868.75	27818080.36	12193080.36

$$I_x = \Sigma I_{x0} + \Sigma A \,.\, d_y^2 = 27818080.36 + 73325868.75 = 101143949.11\ mm^4$$

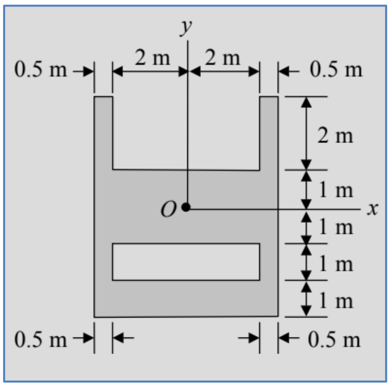
$$I_y = \Sigma I_{y0} + \Sigma A \,.\, d_x^2 = 12193080.36 + 32589275 = 44782355.36\ mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{101143949.11}{13035.71}} = 88.09\ mm$$

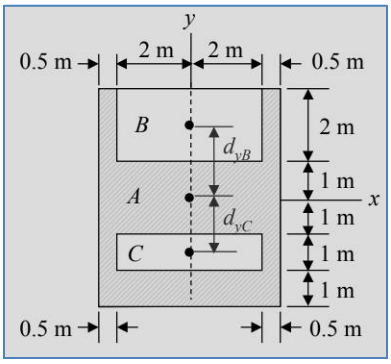
$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{44782355.36}{13035.71}} = 58.61 mm$$

Example 5.

Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes.



Solution:



Part	A (m)	dx (m)	dy (mm)	A . dx^2 (m^4)	A . dy^2 (m^4)	I_{x0} (m^4)	I_{y0} (m^4)
A	30	0	0	0	0	90	62.5
B	-8	0	2	0	-32	-2.67	-10.67
C	-4	0	1.5	0	-9	-0.33	-5.33
Σ	18			0	-41	87	46.5

$$I_x = \Sigma I_{x0} + \Sigma A \,.\, d_y^2 = 87 - 41 = 46mm^4$$

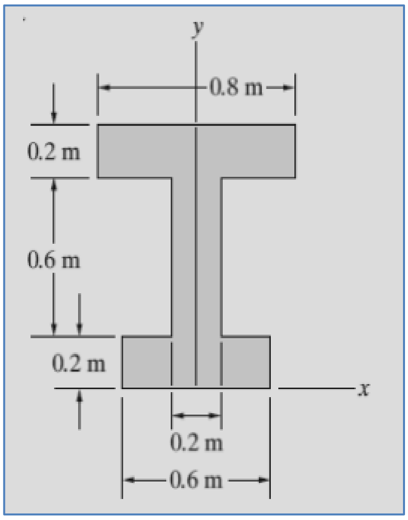
$$I_y = \Sigma I_{y0} + \Sigma A \,.\, d_x^2 = 78.5 - 24 = 54.4\ mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{46}{18}} = 1.599\ mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{46.5}{18}} = 1.607\ mm$$

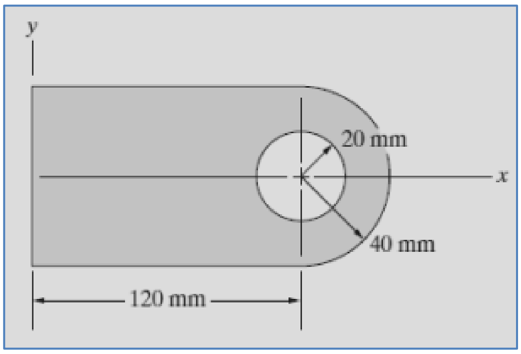
5.10. . Chapter Questions

Q₁: Determine the moment of inertia of the section relative to the x-axis?



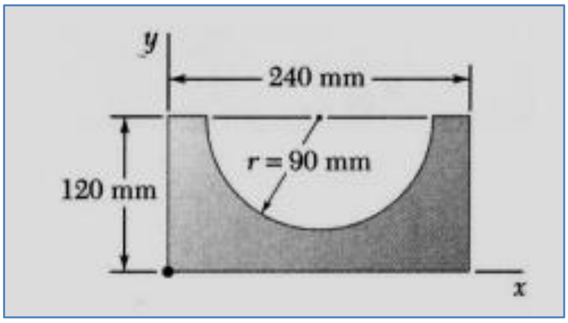
A) I_x = 109.6 (10^9) mm^4
B) I_x = 163.6 (10^9) mm^4
C) I_x = 224.0 (10^9) mm^4
D) I_x = 298.5 (10^9) mm^4

Q₂: Determine the moment of inertia of the section relative to the x-axis?



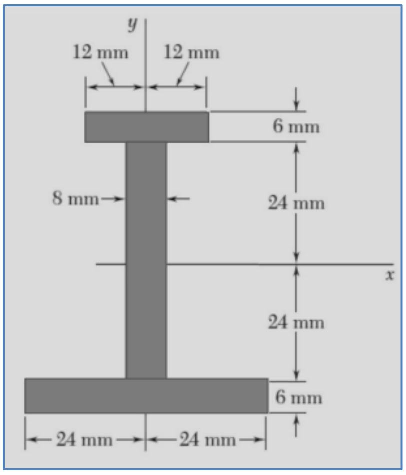
A) I_x = 6.0 (10^6) mm^4
B) I_x = 9.0 (10^6) mm^4
C) I_x = 12.0 (10^6) mm^4
D) I_x = 15.0 (10^6) mm^4

Q₃: Determine the moment of inertia of the shaded area with respect to the x - axis?



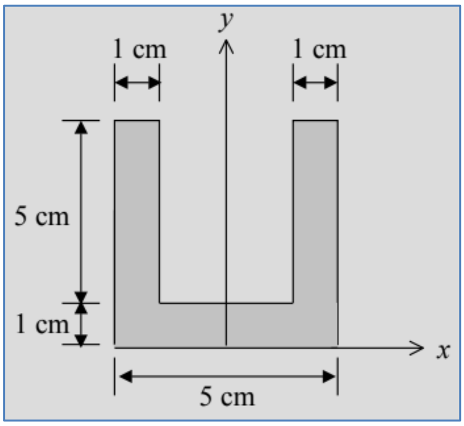
{Answer: $I_x = 92.3 \times 10^6 m^4$ }

Q₄: Determine the moment of inertia of the area shown in the figure?



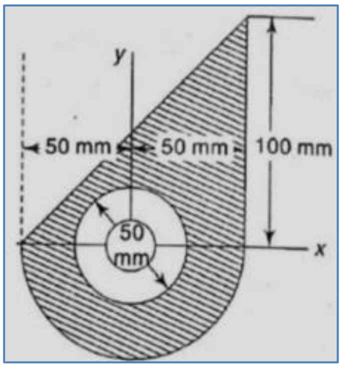
{Answer: $I_x = 39 \times 10^4\ mm^4$ }

Q$_5$: Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroidal axes?



{Answer: $I_x = 145\ cm^4$, $\breve{I_x} = 51.25\ cm^4$, $\breve{I_y} = 51.25\ cm^4$, $\breve{K_x} = \breve{K_y} = 1.848\ cm$}

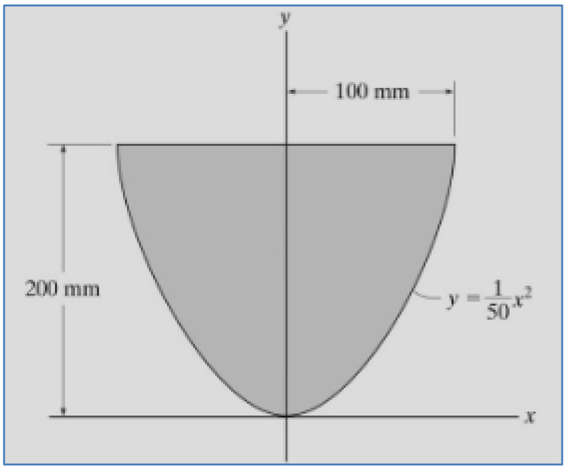
Q$_6$: : Determine the moment of inertia of the area and the radius of gyration shown in the figure?



{Answer: $I_x = 10.47 \times 10^6\ mm^4$, $K_x = 1.848\ cm$}

Q₇: Determine the moment of inertia of the shaded area with respect to the x – axis and y-axis?

$$y = \frac{1}{50}x^2$$



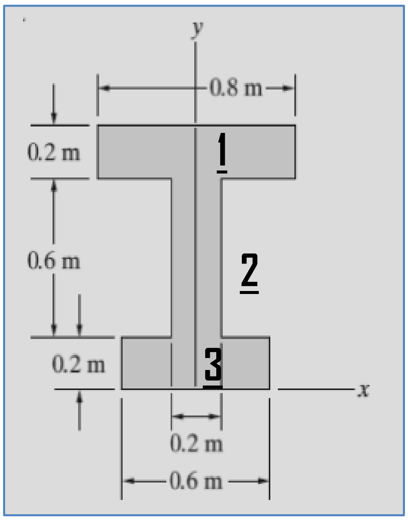
{*Answer:* $I_x = 45.7 \times 10^6\ mm^4$, $I_y = 53 \times 10^6\ mm^4$}

Solve Question

Home Work - 5

5.10 Chapter Questions

Q1: Determine the moment of inertia of the section relative to the x-axis?



Solution:

$$A = w \, . \, L$$

$$I_{x_o} \frac{1}{12} bh^3 \; ; k_x = \sqrt{\frac{I_x}{A}} \quad ; \; k_y = \sqrt{\frac{I_y}{A}}$$

Part	A (m^2)	dx (m)	dy (mm)	A , dx^2 (m^4)	A , dy^2 (m^4)	Ix_0 (m^4)	Iy_0 (m^4)
1	0.16	0	0.9	0	0.1296	0.00053	0.00853
2	0.12	0	0.5	0	0.03	0.0036	0.0004
3	0.12	0	0.1	0	0.0012	0.0004	0.0036
Σ	0.4			0	0.1608	0.00453	0.01253

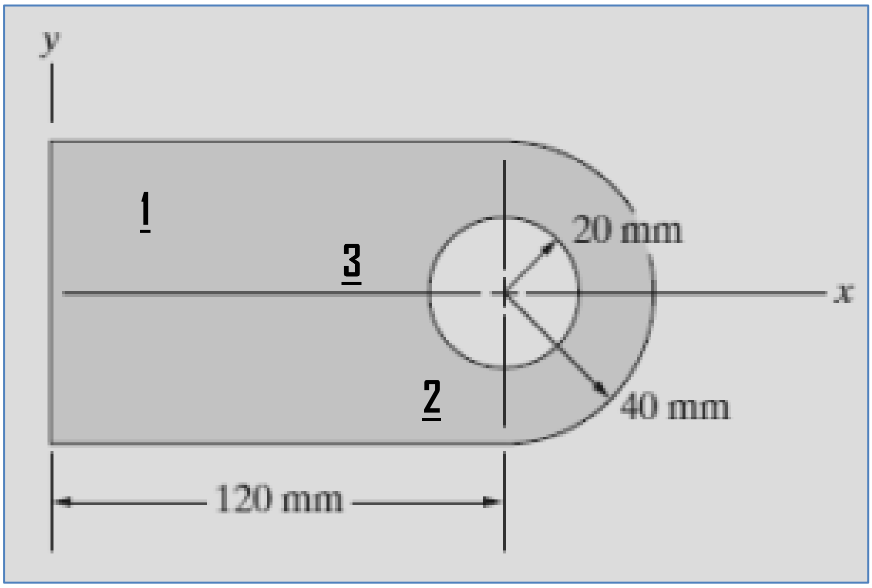
$$I_x = \Sigma I_{x0} + \Sigma A \, . \, d_y^2 = 0.00453 + 0.1608 = 0.1733 \text{ m}^4 = 173.3 \, (10^9) \, mm^4$$

$$I_y = \Sigma I_{y0} + \Sigma A \, . \, d_x^2 = 0.01253 + 0 = 0.01253 \, m^4 = 125.3 \, (10^9) \, mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{0.1733}{0.4}} = 0.43325 \, m = 433.25 \, mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{0.01253}{0.4}} = 0.177 \, m = 177 \, mm$$

Q2: Determine the moment of inertia of the section relative to the x-axis?



A) I_x = 6.0 (10^6) mm^4
B) I_x = 9.0 (10^6) mm^4
C) I_x = 12.0 (10^6) mm^4
D) I_x = 15.0 (10^6) mm^4

Solution:

$$A = w\,.\,L \;\;;\;\; A = \pi\,.\,r^2 \;\;;\;\; x = 0.424\,r$$

$$I_{x_o} = \frac{1}{12}bh^3\,;\;\; I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \;\;;\;\; I_{y_o} = \frac{(9\pi^2-64)d^4}{72\pi} = 0.1098\,r^4\,;\, I_{x_o} = \frac{\pi r^4}{8}$$

Part	A (mm^2)	dx (m)	dy (mm)	A . dx^2 (m^4)	A . dy^2 (m^4)	Ix0 (m^4)	Iy0 (m^4)
1	9600	60	0	34560000	0	5120000	138240000
2	2514.4	136.96	0	47165219.8	0	1005760	281088
3	-1257.2	120	0	-18103680	0	-125720	-125720
Σ	10857.2			63621539.8	0	6000040	138395368

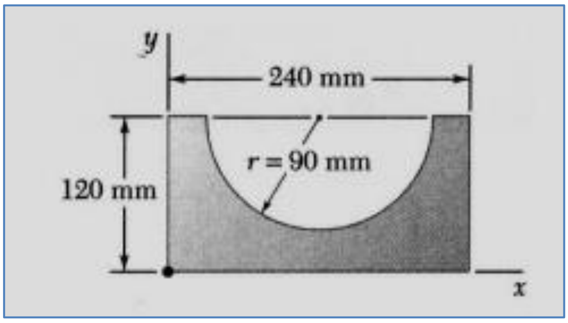
$$I_x = \Sigma I_{x0} + \Sigma A\,.\,d_y^2 = 6000040 + 0 = 6\,(10^6)\,mm^4$$

$$I_y = \Sigma I_{y0} + \Sigma A\,.\,d_x^2 = 138395368 + 63621539.8 = 202.02\,(10^6)\,mm^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{6000040}{10857.2}} = 23.51\,mm$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{202016907}{10857.2}} = 136.41\,mm$$

Q₃: Determine the moment of inertia of the shaded area with respect to the x - axis?



Solution:

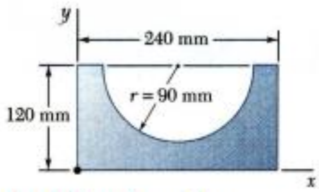
$$A = w\,.L \;\; ; \;\; A = \pi\,.r^2 \;\; ; \;\; x = 0.424\,r$$

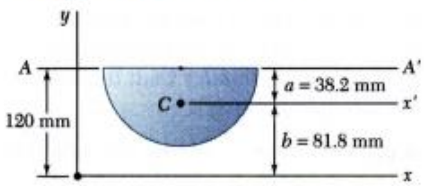
$$I_{x_o} = \frac{1}{12}bh^3 \,; \;\; I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \;\; ; \;\; I_{x_o} = \frac{(9\pi^2-64)d^4}{72\pi} = 0.1098\,r^4$$

$$; I_{y_o} = \frac{\pi r^4}{8}$$

Area Moments of Inertia

Example: Solution





$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\,mm$$

$$b = 120 - a = 81.8\,mm$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$

$$= 12.72\times10^3\,mm^2$$

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the x axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120)^3 = 138.2\times10^6\,mm^4$$

Half-circle:

moment of inertia with respect to AA',

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76\times10^6\,\text{mm}^4$$

Moment of inertia with respect to x',

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = (25.76\times10^6) - (12.72\times10^3)(38.2)^2$$

$$= 7.20\times10^6\,mm^4$$

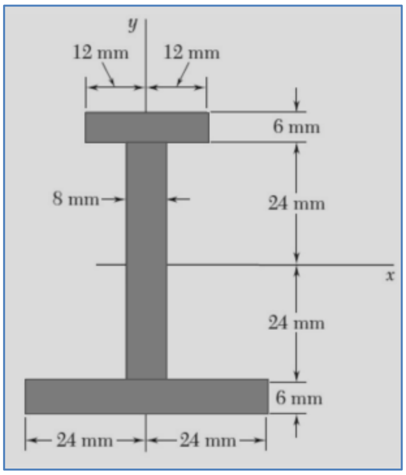
moment of inertia with respect to x,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20\times10^6 + (12.72\times10^3)(81.8)^2$$

$$= 92.3\times10^6\,mm^4$$

{Answer: $I_x = 92.3 \times 10^6 m^4$ }

Q$_4$: Determine the moment of inertia of the area shown in the figure?

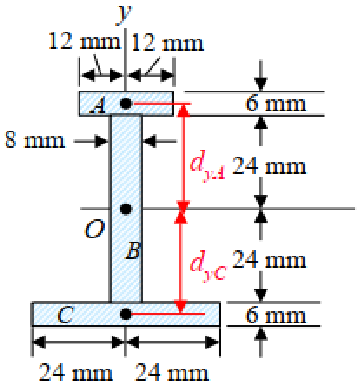


Solution:

$$A = w \, . L$$

$$I_{x_o} \frac{1}{12} bh^3 \; ; k_x = \sqrt{\frac{I_x}{A}} \quad ; \; k_y = \sqrt{\frac{I_y}{A}}$$

SOLUTION



$$I_x = (\bar{I}_x + Ad_y^{\,2})_A + (\bar{I}_x + Ad_y^{\,2})_B + (\bar{I}_x + Ad_y^{\,2})_C$$

(with $d_y = 0$ for B)

$$= [\frac{1}{12}(24)(6)^3 + (24\times6)(27)^2]_A + [\frac{1}{12}(8)(48)^3 + 0]_B + [\frac{1}{12}(48)(6)^3 + (48\times6)(27)^2]_C$$

$$I_x = 390\text{x}10^3\ \text{mm}^4$$

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{390\times10^3}{[(24\times6)+(8\times48)+(48\times6)]}} = 21.9\ mm$$

$$I_y = (\bar{I}_y + Ad_x^{\,2})_A + (\bar{I}_y + Ad_x^{\,2})_B + (\bar{I}_y + Ad_x^{\,2})_C$$

(with $d_x = 0$ for A, B and C)

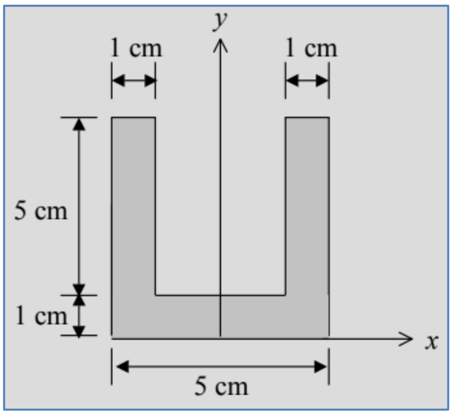
$$= [\frac{1}{12}(6)(24)^3]_A + [\frac{1}{12}(48)(8)^3]_B + [\frac{1}{12}(6)(48)^3]_C$$

$$I_y = 64.3\text{x}10^3\ \text{mm}^4$$

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{64.3\times10^3}{[(24\times6)+(8\times48)+(48\times6)]}} = 8.87\ mm$$

{Answer: $I_x = 39 \times 10^4\ mm^4$ }

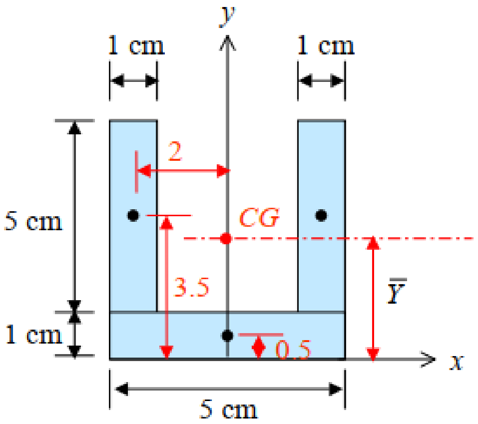
Q5: Determine the moments of inertia and the radius of gyration of the shaded area with respect to the x and y axes and at the centroid axes?



Solution:

$$A = w\ .L$$

$$I_{x_o}\frac{1}{12}bh^3\ ;k_x = \sqrt{\frac{I_x}{A}}\quad ;\ k_y = \sqrt{\frac{I_y}{A}}$$



$\bar{Y}\sum A = \sum \bar{y}A$

$$\bar{Y} = \frac{2[(3.5)(5\times 1)] + (0.5)(1\times 5)}{3(5\times 1)}$$

$= 2.5\ cm$

- **Moments of inertia about *x* axis**

$$I_x = 2[(\frac{1}{12}(1)(5)^3 + (5\times 1)(3.5)^2] + \frac{1}{3}(5)(1)^3$$

$= 145\ cm^4$

- **Moments of inertia about centroid**

$$\bar{I}_x = I_x - Ad_y{}^2$$

$= 145 - (15)(2.5)^2$

$= 51.25\ cm^4$

OR

$$\bar{I}_x = 2[(\frac{1}{12}(1)(5)^3 + (5\times 1)(1)^2]$$

$$+[(\frac{1}{12}(5)(1)^3 + (5\times 1)(2)^2]$$

$= 51.25\ cm^4$

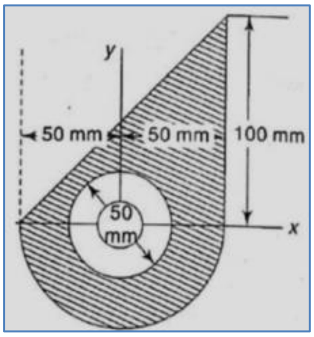
$$\bar{I}_y = I_y = 2[(\frac{1}{12}(5)(1)^3 + (5\times 1)(2)^2] + \frac{1}{12}(1)(5)^3$$

$= 51.25\ cm^4$

$$\bar{k}_x = \bar{k}_y = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{51.25}{15}} = 1.848\ cm$$

{*Answer:* $I_x = 145\ cm^4$, $\breve{I}_x = 51.25\ cm^4$, $\breve{I}_y = 51.25\ cm^4$, $\breve{K}_x = \breve{K}_y = 1.848\ cm$}

Q₆: : Determine the moment of inertia of the area and the radius of gyration shown in the figure?



Solution:

$$A = w\,.\,L \ \ ;\ \ A = \pi\,.\,r^2 \ \ ;\ \ x = 0.424\,r$$

$$A_T = A_1 + A_2 - A_3$$

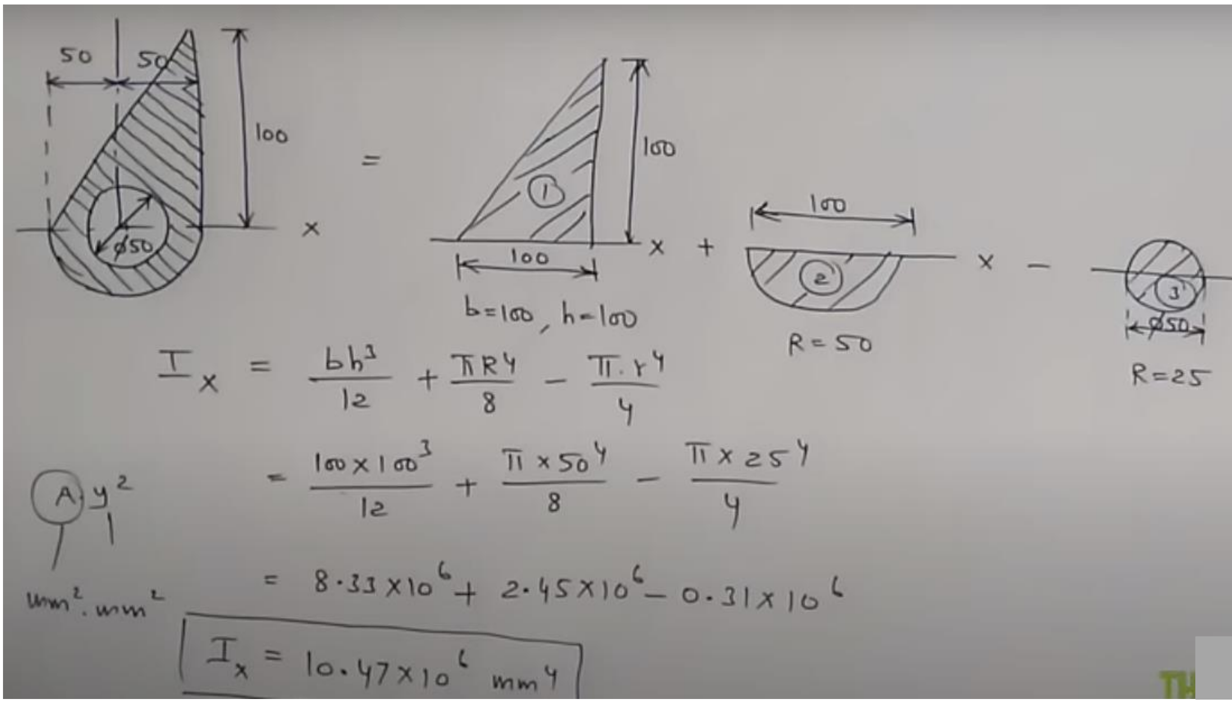
$$A_T = \frac{bh}{2} - \frac{\pi d^2}{8} - \frac{\pi d^2}{4}$$

$$A_T = \frac{100 \times 100}{2} + \frac{3.143 \times (100)^2}{8} - \frac{3.143 \times (50)^2}{4}$$

$$A_T = 5000 - 3928.75 - 1964.375 = 89\ mm^2$$

$$I_{x_o} = \frac{1}{12} bh^3 ;\ I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \quad ;\ I_{x_o} = \frac{(9\pi^2 - 64)d^3}{72\pi} = 0.1098\, r^4 ; I_{y_o} = \frac{\pi r^4}{8}$$

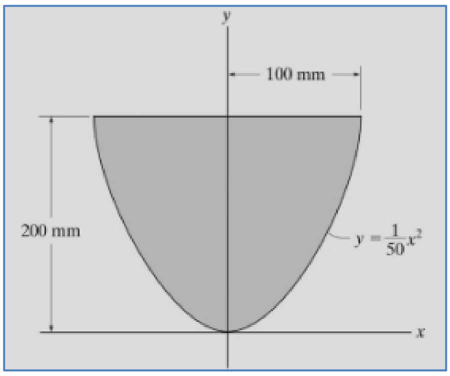
$$k_x = \sqrt{\frac{I_x}{A}} \quad ;\ k_y = \sqrt{\frac{I_y}{A}}$$

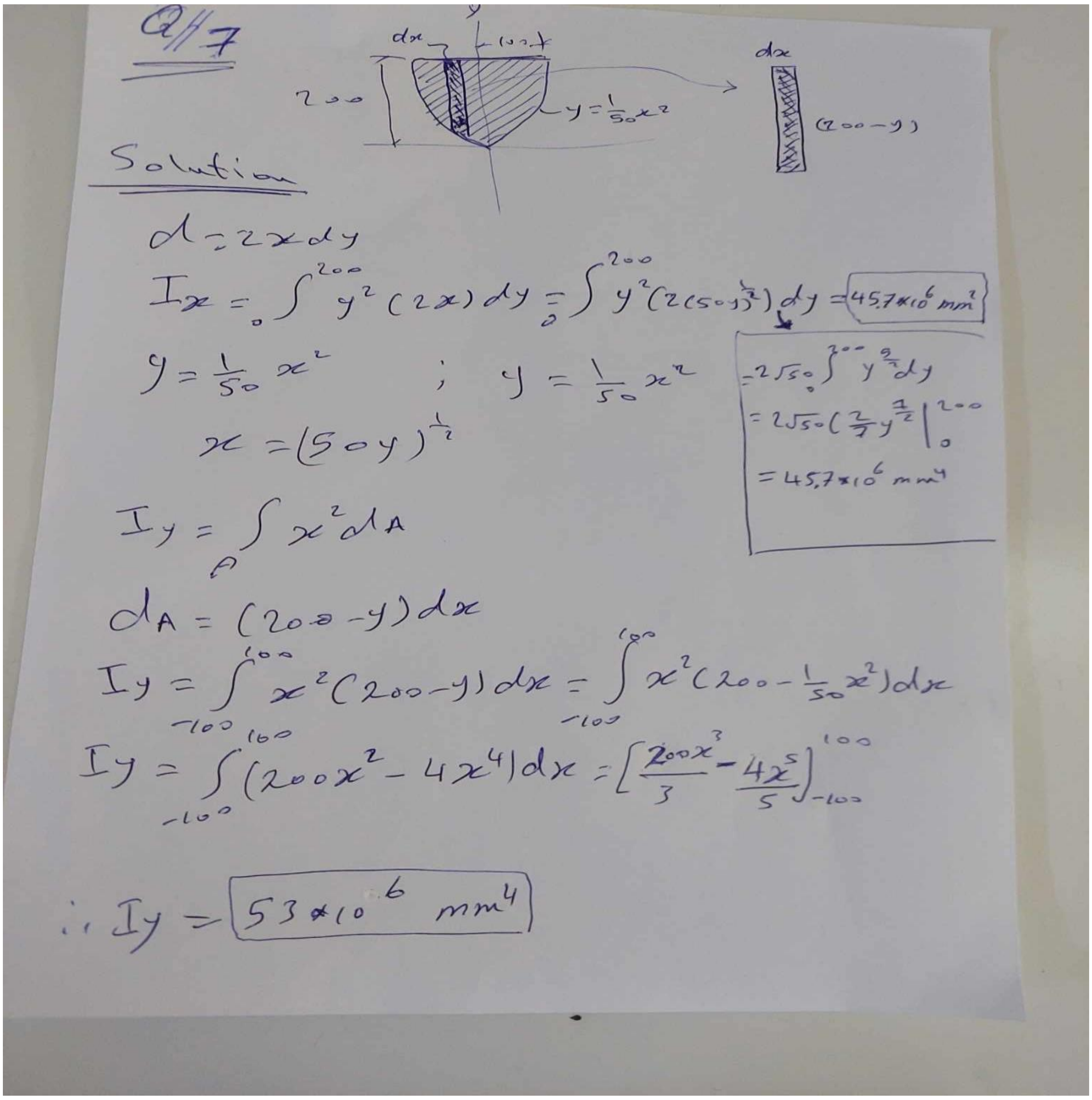


{Answer: $I_x = 10.47 \times 10^6\ mm^4$ }

Q₇: Determine the moment of inertia of the shaded area with respect to the x – axis and y-axis?

$$y = \frac{1}{50} x^2$$





Q/7

Solution

$dA = 2x\,dy$

$I_x = \int_0^{200} y^2 (2x)\,dy = \int_0^{200} y^2 (2(50y)^{\frac{1}{2}})\,dy = 45.7 \times 10^6\ mm^4$

$y = \frac{1}{50}x^2$; $y = \frac{1}{50}x^2$

$x = (50y)^{\frac{1}{2}}$

$= 2\sqrt{50}\int_0^{200} y^{\frac{3}{2}}\,dy$

$= 2\sqrt{50}\left(\frac{2}{5}y^{\frac{5}{2}}\right)\Big|_0^{200}$

$= 45.7 \times 10^6\ mm^4$

$I_y = \int_A x^2\,dA$

$dA = (200 - y)\,dx$

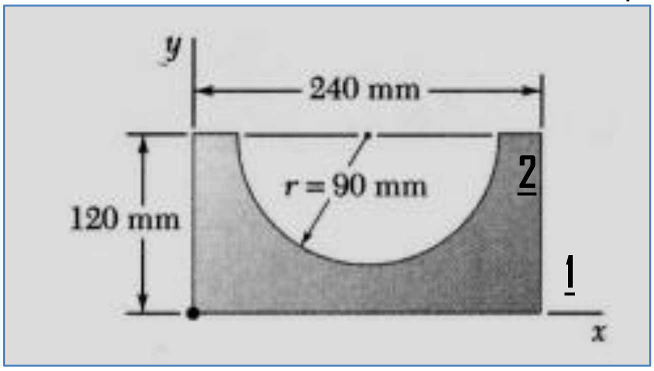
$I_y = \int_{-100}^{100} x^2 (200 - y)\,dx = \int_{-100}^{100} x^2 \left(200 - \frac{1}{50}x^2\right)dx$

$I_y = \int_{-100}^{100} (200x^2 - 4x^4)\,dx = \left[\frac{200x^3}{3} - \frac{4x^5}{5}\right]_{-100}^{100}$

$\therefore I_y = 53 \times 10^6\ mm^4$

{Answer: $I_x = 45.7 \times 10^6\ mm^4$, $I_y = 53 \times 10^6\ mm^4$}

Q$_3$: Determine the moment of inertia of the shaded area with respect to the x - axis?



Solution:

$$\boldsymbol{A = w\,.\,L}\ ;\ \boldsymbol{A = \pi\,.\,r^2}\ ;$$
$$\boldsymbol{x = 0.424\,r = 38.16\ mm}$$

$$I_{x_o} = \frac{1}{12} bh^3 \; ; \; I_{x_o} = I_{y_o} = \frac{\pi d^4}{64} \; ; \; I_{y_o} = \frac{(9\pi^2 - 64) d^4}{72\pi} = 0.1098\, r^4$$
$$; I_{x_o} = \frac{\pi r^4}{8}$$

Part	A (m^2)	dx (m)	dy (mm)	A . dx^2 (m^4)	A . dy^2 (m^4)	Ix_0 (m^4)	Iy_0 (m^4)
1	28800	120	60	864000	216000	34560000	138240000
2	-25458.3	120	81.84	1178496	548146.8	-25776528.75	
Σ	3341.7			2042496	764146.8	8783471.25	

$$I_x = \Sigma I_{x0} + \Sigma A \,.\, d_y^2 = 8783471.25 + 764146.8 = 9547618 \text{ mm}^4$$
$$I_y = \Sigma I_{y0} + \Sigma A \,.\, d_x^2 = 0.01253 + 0 = 0.01253\, m^4 = 125.3\, (10^9)\, mm^4$$
$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{0.1733}{0.4}} = 0.43325\, m = 433.25\, mm$$
$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{0.01253}{0.4}} = 0.177\, m = 177\, mm$$

Chapter 6

Friction

6.1 Introduction

When two surfaces are in contact, these surfaces were either **frictionless** or **rough.**

A. If they were **frictionless**, the force each surface exerted on the other was normal to the surfaces and the two surfaces could move freely with respect to each other.

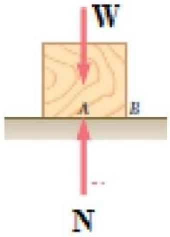
B. If they were **rough**, it was assumed that tangential forces **(Friction force)** could develop to prevent the motion of one surface with respect to the other.

Friction is the force that resists the movement of two contacting surfaces that slide relative to one another. This force always acts tangent to the surface at the point of contact and is directed so as to oppose the possible or existing motion between the surfaces.

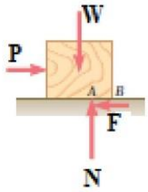
6.2 Law of Friction

The laws of friction are exemplified by the following experiment.

1. A block of weight W is placed on a horizontal plane surface.



2. A horizontal force **P** is applied to the block. If **P** is small, the block will not move; some other horizontal force must therefore exist, which balances **P**.This other force is **the static friction force F**.



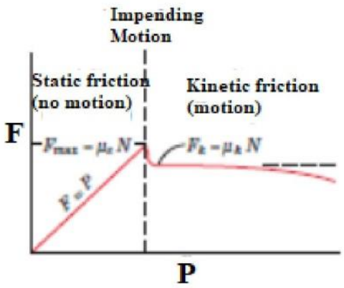
3. If the force **P** is increased, the friction force **F** also increases, continuing to oppose **P**, until its magnitude reaches a certain maximum value $\mathbf{F_m}$.

4.If **P** is further increased, the friction force cannot balance it anymore and the block start sliding.

NOTE:

If N reach the point B before F reaches its maximum value, the block will tip about B before it can start sliding.

5.As soon as the block has been set in motion, the magnitude of **F** drops from $\mathbf{F_m}$ to a lower value $\mathbf{F_k}$(kinetic friction force).



Then from the previous experiment;

The value $\mathbf{F_m}$ of the static friction force is proportional to the normal component **N**

$$F_m = \mu_s N$$

Where

μ_s is the coefficient of static friction

The magnitude $\mathbf{F_k}$ of the kinetic friction force may be put in the form

$$\mathbf{F_k = \mu_k N}$$

Where

μ_k **is the coefficient of kinetic friction**

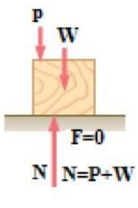
NOTES

1. The maximum frictional force F is proportional to the normal force N.

2. The limiting static friction force is greater than the kinetic frictional force.

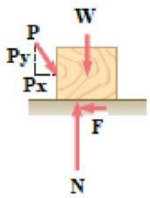
From above, there are four different situations can occur when a rigid body is in contact with a horizontal surface:

1. The forces applied to the body do not tend to move it along the surface of contact; there is no friction force.



a)No friction

2. The applied forces tend to move the body along the surface of contact but are not large enough to set it in motion. The friction force F which has developed can be found by solving equation of equilibrium for the body. Since there is no evidence that **F** has reached its maximum value, the equation $\mathbf{F_m = \mu_s N}$ cannot be used to determine the friction force.



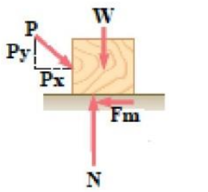
b)No Motion

$$F = P_x$$

$$F < \mu_s N$$

$$N = P_y + W$$

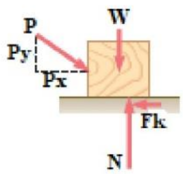
3.The applied forces are such that the body is just about to slide. We say the motion is impending. The friction force **F** has reached its maximum value $\mathbf{F_m}$ and, together with the normal force N, balances the applied force. Both the equations of equilibrium and the equation $\mathbf{F_m = \mu_s N}$ can be used.



c)Motion impending

$$F_m = P_x$$
$$F_m = \mu_s N$$
$$N = P_y + W$$

4. The body is sliding under the action of the applied forces; and the equations of equilibrium do not apply any more. However, **F** is now equal to $\mathbf{F_k}$.



d) Motion

$$F_k < P_x$$
$$F_k = \mu_k N$$
$$N = P_y + W$$

6.3 Coefficient of Friction

The coefficient of static friction μ_s, is defined as the ratio of the magnitude of the maximum static frictional force F, to the magnitude of the normal force N, between the two surfaces. It is depend on the nature of the surfaces in contact.

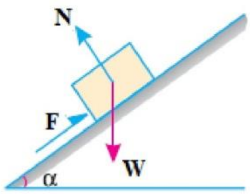
$$\mu_s = \frac{F_m}{N}$$

Approximate values of coefficient of static friction for various dry surfaces are given in the following table. The corresponding values of the coefficient of kinetic friction would be about 25 percent smaller.

Surface in contact	μ_s
Steel on steel	0.4-0.8
Wood on wood	0.2-0.5
Metal on stone	0.3-0.7
Rubber on concrete	0.6-0.8

6.3 Angle of Friction

Consider a body of weight **W** resting on an inclined plane.



Let the angle of inclination (**α**) be gradually increased, till the body just start sliding down the plane. This angle of inclined plane (**φ**), at which a body just begins to slide down the plane, is called the angle of friction. This is also equal to the angle, which the normal reaction makes with the vertical.

$$\tan\varphi = \frac{F}{N} = \mu_s$$

6.4 Types of Friction Problem

Type 1. No apparent impending motion.

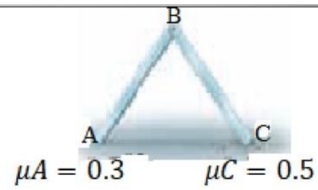
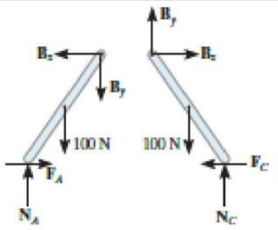
Type 2. Impending motion at all points of contact.

Type 3. Impending motion at some points of contact.

Type 1

Problems in this category are equilibrium problems, which require the number of unknowns to be equal to the number of available equilibrium equations. Once the frictional forces are determined from the solution, their values must be checked to be sure they satisfy $F \leq \mu N$

; Otherwise, slipping will occur and the body will not remain in equilibrium.

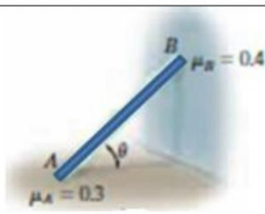
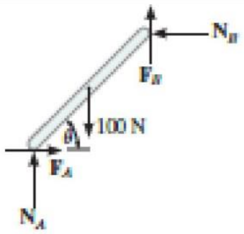
	

Type 2

In this case the total number of unknowns will equal the total number of available equilibrium equations plus the total number of available frictional equations $F = \mu N$.

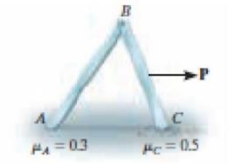
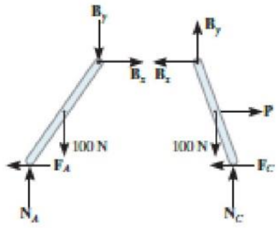
When motion is impending at the points of contact, then $F = \mu N$;

whereas if the body is slipping, then $F_k = \mu_k$

 Finding the small angle at which the 100Nbar can be placed against the wall without slipping.	

Type 3

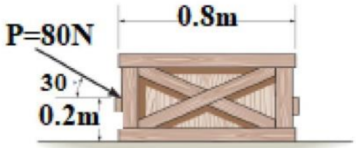
In this type, the number of unknowns will be less than the number of available equilibrium equations plus the number of available frictional equations or conditional equations for tipping. As a result, several possibilities for motion or impending motion will exist and the problem will involve a determination of the kind of motion which actually occurs.

 Determine the horizontal force P needed to cause movement.	

Examples

Example(1):-

The uniform crate shown in figure has a mass of 20kg. If a force P=80N is applied to the crate, determine if it remains in equilibrium. The coefficient of friction is 0.3.



Solution:-

$\rightarrow \sum F_x = 0$

$80\cos 30 - F = 0 \quad F = 69.3N \leftarrow$

$\uparrow \sum F_y = 0$

$-80\sin 30 + Nc - 196.2 = 0 \qquad Nc = 236.2N \uparrow$

$\circlearrowleft \sum M_O = 0$

$80\sin 30(0.4) - 80\cos 30(0.2) + Nc(x) = 0$

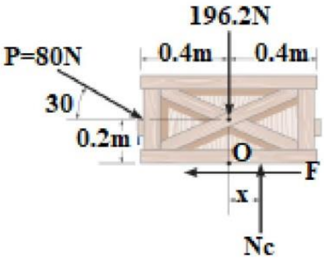
$x = -0.00908m = -9.08mm$

No tipping will occur since:-

1.$\mathbf{x < 0.4\ m}$

2.$\mathbf{F = 69.3 < F_{max}} = \mu Nc = 0.3(236.2) = \mathbf{70.9N}$

∴The crate was still in equilibrium.



F.B.D.

Example(2):-

It is observed that when the bed of the dump truck is raised to an angle of θ=25 the vending machines will begin to slide off the bed, determine the static coefficient of friction between a vending machine and the surface of the truck bed.

Solution:-

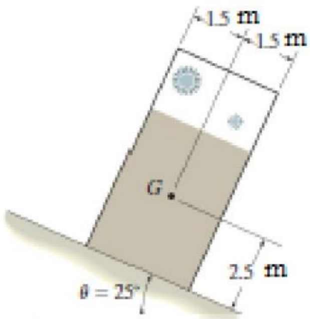
From the F.B.D

$\rightarrow \sum F_x = 0$

$W \sin 25 - F = 0 \qquad (1)$

$\uparrow \sum F_y = 0$

$N - W\cos 25 = 0 \qquad (2)$

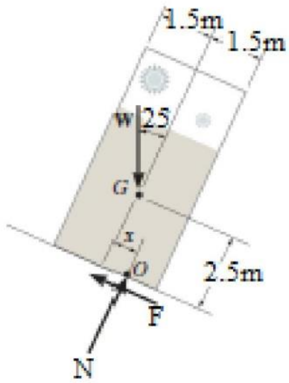


From Eqs (1) and (2)

$F = \mu N \qquad W\sin 25 = \mu(W\cos 25)$

$\mu = \tan 25 = 0.466$

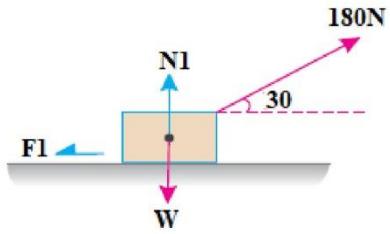
or from angle of friction p153.

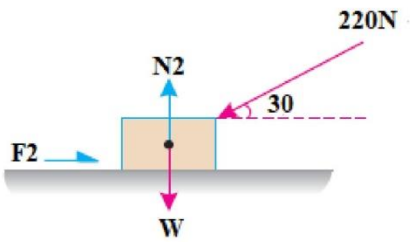


Example(3):-

A body, resting on a rough horizontal plane, required a pull of **180N** inclined at **30º** to the plane just to move it. It was found that a push of **220N** inclined at **30º** to the plane just moved the body. Determine the weight of the body and the coefficient of friction.

Solution:-





From a pull of 180N;

$\uparrow \sum F_y = 0$

N1-W+180 sin 30º=0

N1=W-180 sin 30º=W-90

$\rightarrow \sum F_x = 0$

180 cos 30º-F1=0

F1=180 cos 30º=180x0.866=155.9N

F1=F_{m1}=μ_s N1=μ_s(W-90)

155.9= μ_s(W-90) (1)

From a push of 220N

$\uparrow \sum F_y = 0$

N2-W-220 sin 30º=0

N2=W+220 sin 30º=W+110

$\rightarrow \sum F_x = 0$

F2-220 cos 30º=0

F2=220 cos 30º=220x0.866=190.5N

F2=F_{m2}=μ_s N2=μ_s(W+110)

190.5= μ_s(W+110) (2)

Dividing (1) by (2)

$$\frac{155.9}{190.5} = \frac{\mu_s(W-90)}{\mu_s(W+110)} = \frac{W-90}{W+110}$$

155.9W+17149=190.5W-17145

34.6W=34294

W=991.2N

Now substituting in (1)

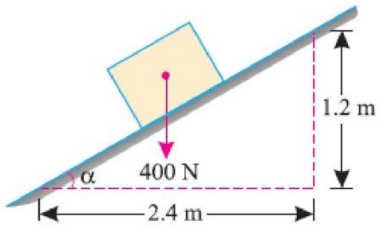
$155.9 = \mu_s(991.2-90) = 901.2\mu_s$

μ_s=0.173

Example(4):-

A n inclined plane is used to unload slowly a body weighing **400N** from a truck **1.2m** high into the ground. The coefficient of friction between the underside of the body and the plank is **0.3**. State whether it is necessary to push the body down the plane or hold it back from sliding down. What minimum force is required parallel to the plane for this purpose.

Solution:-



$\tan\alpha = \frac{1.2}{2.4} = 0.5 \quad \alpha = 26.5°$

1.N=W cos α=400 cos 26.5º=357.9N

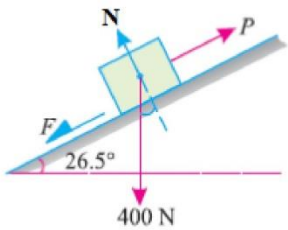
$F_m = \mu_s N = 0.3 \times 357.9 = \mathbf{107.3N}$

Resolving the 400 N along the plane.

=400 sin α=400×sin 26.5º=**178.5N**

The force along the plane (which is responsible for sliding the body) is more than the force of friction; therefore, the body will slide down.

It is not necessary to push the body down the plane; rather it is necessary to hold it back from sliding down.

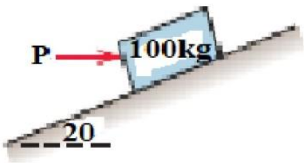


2.**P**=178.5-107.3=**71.2N**

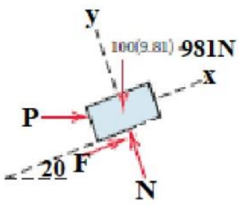
Example(5):-

Determine the magnitude and direction of the friction force acting on the 100kg block shown if, first,P=500N and, second, P=100N.The coefficient of static friction is 0.2,and the coefficient of kinetic friction is 0.17. The force is applied with the block initially at rest.

Solution:-



There is no way of telling from the statement of the problem whether the block will remain in equilibrium or whether it will begin to slip following the application of P, therefore assume the block is in equilibrium;



F.B.D.

$\rightarrow \sum F_x = 0$

$P\cos 20 + F - 981\sin 20 = 0$

$\uparrow \sum F_y = 0$

$N - P\sin 20 - 981\cos 20 = 0$

1.$P = 500$ N subs in above eqs $F = -134.3N$ $N = 1093N$

$F_{max} = \mu N = 0.2(1093) = 219N$

$F < F_{max}$ then the assumption was correct $F = 134.3N$ down the plane

2.$P = 100$ N subs in above eqs $F = 242N$ $N = 956N$

$F_{max} = \mu N = 0.2(956) = 191.2N$

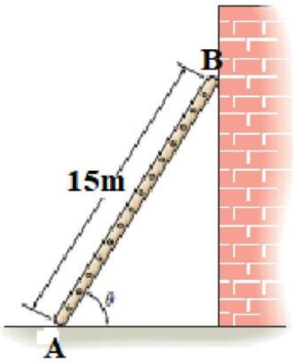
$F > F_{max}$ then the assumption was incorrect

$F_k = \mu_k N = 0.17(956) = 162.5N$ up the plane

Example(6):-

The ladder has a uniform weight of 80N and rests against the wall at B.If the coefficient of friction at A and B is 0.4, determine the smallest angle θ at which the ladder will not slip.

Solution:-



Since the ladder is required to be on the verge to slide down, then:-

$F_A = \mu N_A = 0.4N_A$

$F_B = \mu N_B = 0.4N_B$

From the F.B.D.

$\rightarrow \sum F_x = 0$

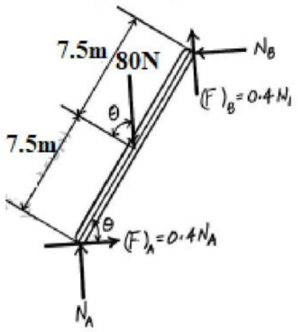
$0.4N_A - N_B = 0 \quad N_B = 0.4N_A \qquad (1)$

$\uparrow \sum F_y = 0$

$N_A + 0.4N_B - 80 = 0 \qquad (2)$

Solving eqs,(1) and (2)

$N_A = 68.97N \qquad N_B = 27.59N$



F.B.D

$\circlearrowleft \sum M_A = 0$

$0.4(27.59)(15\cos\theta) + 27.59(15\sin\theta) - 80\cos\theta(7.5) = 0$

$413.79\sin\theta - 434.48\cos\theta = 0$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{434.48}{413.79} = 1.05 \qquad \theta = 46.4$

Example(7):-

Two blocks A=100N and B=150N are resting on ground. Coefficient of friction between ground and block B is 0.1 and that between block B and A is 0.3. find the minimum value of weight P in that pan so that motion starts. Find whether B is stationary with respect to ground and A moves or B is stationary with respect to A.

Solution:-

1.B is stationary and A moves

F.B.D of block A

$\uparrow \sum F_y = 0$

$N_1 + P\sin 30 - 100 = 0$

$N_1 = 100 - P\sin 30$

$\rightarrow \sum F_x = 0$

$P\cos 30 - 0.3N_1 = 0$

$P\cos 30 - 0.3(100 - P\sin 30) = 0$

$P = 29.53N$

2.A and B are moving together

F.B.D of block A and B

$\uparrow \sum F_y = 0$

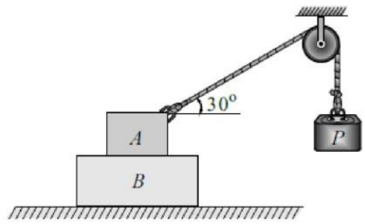
$N_2 - 250 + P\sin 30 = 0$

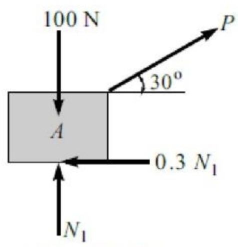
$N_2 = 250 - P\sin 30$

$\rightarrow \sum F_x = 0$

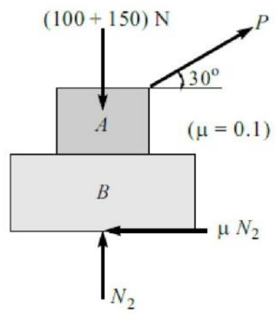
$P\cos 30 - 0.1N_2 = 0$

P=27.29N $\therefore P = 27.29N$





FBD of Block *A*

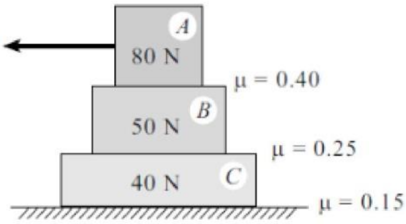


FBD of Block *A* and *B* Together

Example(8):-

Three blocks are placed on the surface one above the other. The static coefficient of friction between the blocks and block C and surface is shown in the figure. Determine the maximum value of P that can be applied before any slipping take place.

Solution:-



1.Block A has impending motion and blocks B and C remain intact

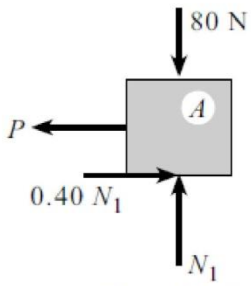
F.B.D of block A

$$\uparrow \sum F_y = 0$$

$$N_1 - 80 = 0 \quad \therefore N_1 = 80N$$

$$\rightarrow \sum F_x = 0$$

$$0.4N_1 - P = 0 \quad \therefore P = 32N \leftarrow$$



FBD of Block *A*

2.Blocks A and B together have impending motion and block C remains intact.

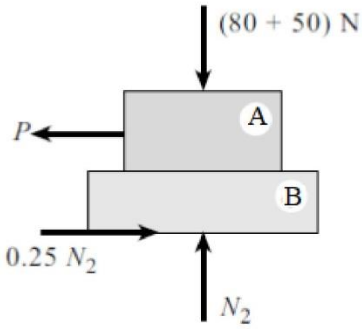
F.B.D of blocks A and B

$$\uparrow \sum F_y = 0$$

$$N_2 - (80 + 50) = 0 \quad \therefore N_2 = 130N$$

$$\rightarrow \sum F_x = 0$$

$$0.25N_2 - P = 0 \quad \therefore P = 32.5N \leftarrow$$



FBD of Block *A* and Block *B* Together

3. All the three blocks together have impending motion.

F.B.D of blocks A, B and C

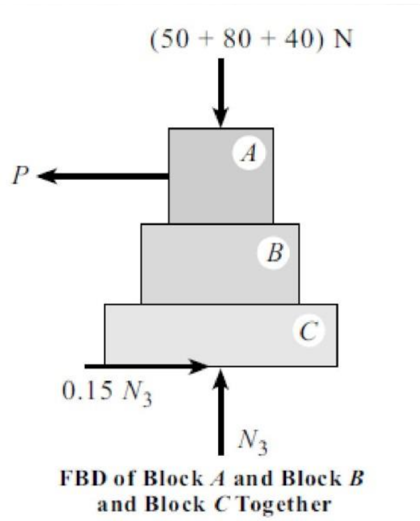
$\uparrow \sum F_y = 0$

$N_3 - (50 + 80 + 40) = 0 \;\; N_3 = 170N$

$\rightarrow \sum F_x = 0$

$0.15N_3 - P = 0 \;\; \therefore N_3 = 25.5N$

Pmax=25.5N before any slipping take place

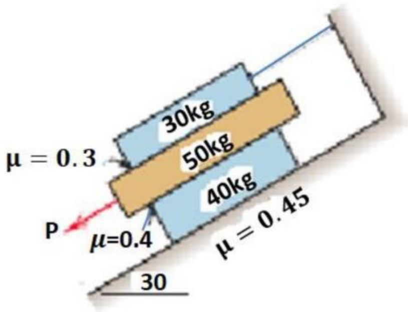


(C)

Example(9):-

The three flat blocks are positioned on the 30 incline as shown, and a force P parallel to the incline is applied to the middle block. Determine the maximum value which P may have before any slipping takes place.

Solution:-



There are two possible conditions for impending motion.

1. The 50kg block slips and the 40kg block remains in place.

F.B.D. (1) and (2)

$\uparrow \sum F_y = 0$

(30 kg)

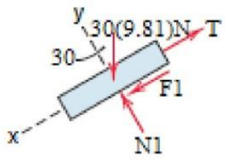
$N_1 - 30(9.81)\cos 30 = 0 \quad N_1 = 255\text{N}$

(50kg)

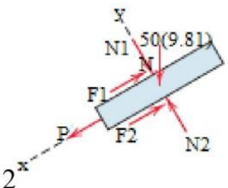
$N_2 - 50(9.81)\cos 30 - 255 = 0 \;\; N_2 = 680\text{N}$

$\text{F1} = 0.3(255) = 76.5\text{N}$

$\text{F2} = 0.4(680) = 272\text{N}$



F.B.D. 1



F.B.D.2

F.B.D.(2)

$\rightarrow \sum F_x = 0$

$\text{P} - 76.5 - 272 + 50(9.81)\sin 30 = 0 \quad \text{P} = 103.1\text{N}$

2 The 50kg and 40kg blocks move together with slipping occurring between the 40kg block and the incline.

$\uparrow \sum F_y = 0$

$N_3 - 255 - 90(9.81)\cos 30 = 0 \;\therefore N_3 = 1019N$

$F_3 = 0.45(1019) = 459N$

$\rightarrow \sum F_x = 0$

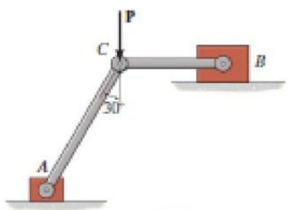
$76.5 + 459 - 90(9.81)\sin 30 - P = 0 \;\therefore P = 94N$

P_{max}=94N, motion impends for the 50kg and 40kg as a unit.

Example(10):-

Blocks A and B have a mass of 3kg and 9kg, respectively, and are connected to the weightless links. Determine the largest vertical force P that can be applied at the pin C without causing any movement. The coefficient of static friction between the blocks and the contacting surfaces is μ=0.3.(the links are two force members such us the truss member).

Solution:-



(a)

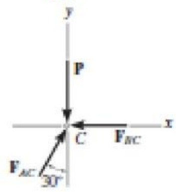
From F.B.D. of pin C

$$\uparrow \sum F_y = 0$$

$F_{AC} \cos 30 - P = 0 \quad F_{AC} = 1.155P$

$$\rightarrow \sum F_x = 0$$

$1.155\ P\sin 30 - F_{BC} = 0 \qquad F_{BC} = 0.5774P$



Pin C

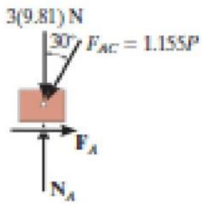
From F.B.D. of block A

$$\rightarrow \sum F_x = 0$$

$(1) F_A - 1.155\ P \sin30 = 0 \qquad F_A = 0.5774P$

$$\uparrow \sum F_y = 0$$

$N_A - 1.155P \cos 30 - 3(9.81) = 0 \quad N_A = P + 29.43N \quad (2)$



F.B.D of block A

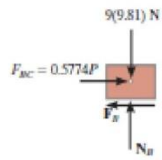
From F.B.D of block B

$$\rightarrow \sum F_x = 0$$

$(3) 0.5774P - F_B = 0 \qquad F_B = 0.5774P$

$$\uparrow \sum F_y = 0$$

$N_B - 9(9.81) = 0 \qquad N_B = 88.29N$



F.B.D of block B

If we assume block A slips first then

$$N_3 - 255 - 90(9.81)\cos 30 = 0 \therefore N_3 = 1019N$$

$$F_3 = 0.45(1019) = 459N$$

$$\rightarrow \sum F_x = 0$$

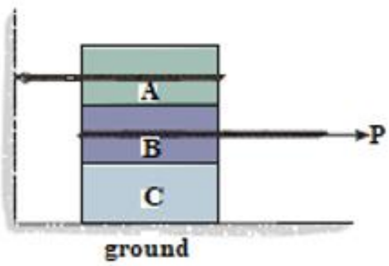
$$76.5 + 459 - 90(9.81)\sin 30 - P = 0 \quad \therefore P = 94N$$

P_{max}=94N, motion impends for the 50kg and 40kg as a unit.

Home Work

H.W(1)

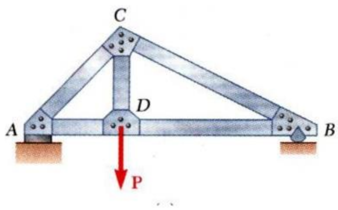
Blocks A,B and C have weights of 50N,25N and 15N respectively. Determine the smallest horizontal force P that will cause impending motion. The coefficient of static friction between A and B is μ=0.3, between B and C is μ=0.4, and between block C and the ground is μ=0.35.(Ans: P=45N)



Chapter 7

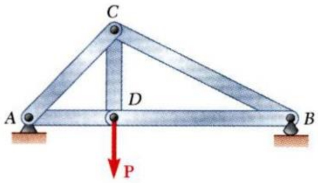
Trusses

7.1 Definition of a Truss

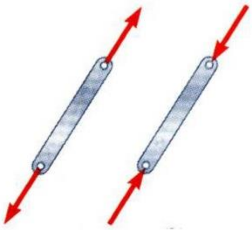


- A truss is an assembly of straight members connected at joints. **No member is continuous through a joint.**

- Most structures are made of several trusses joined together to form a space framework. Each truss carries those loads which act in its plane and may be treated as a two-dimensional structure.

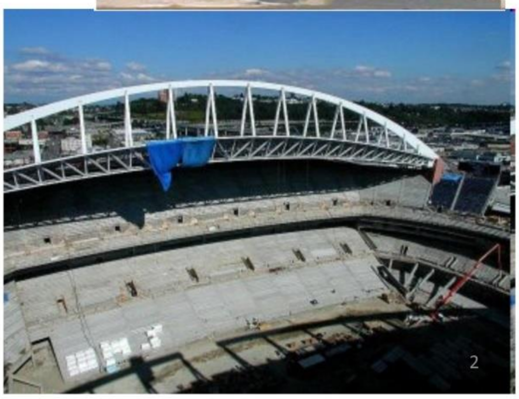


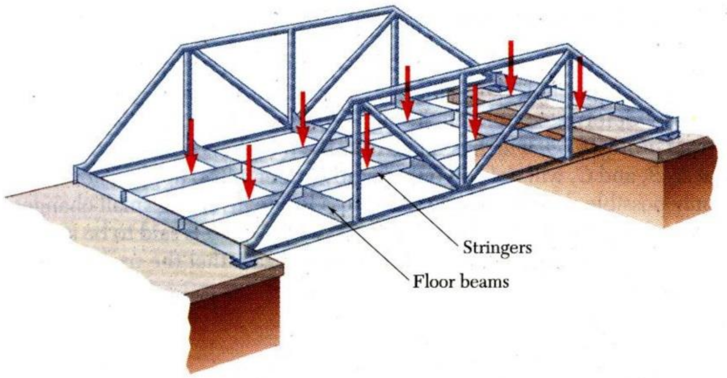
- **Bolted or welded connections are assumed to be pinned together**. Forces acting at the member ends reduce to a single force and no couple. Only two-force members are considered.



- When forces tend to pull the member apart, it is in tension. When the forces tend to compress the member, it is in compression.

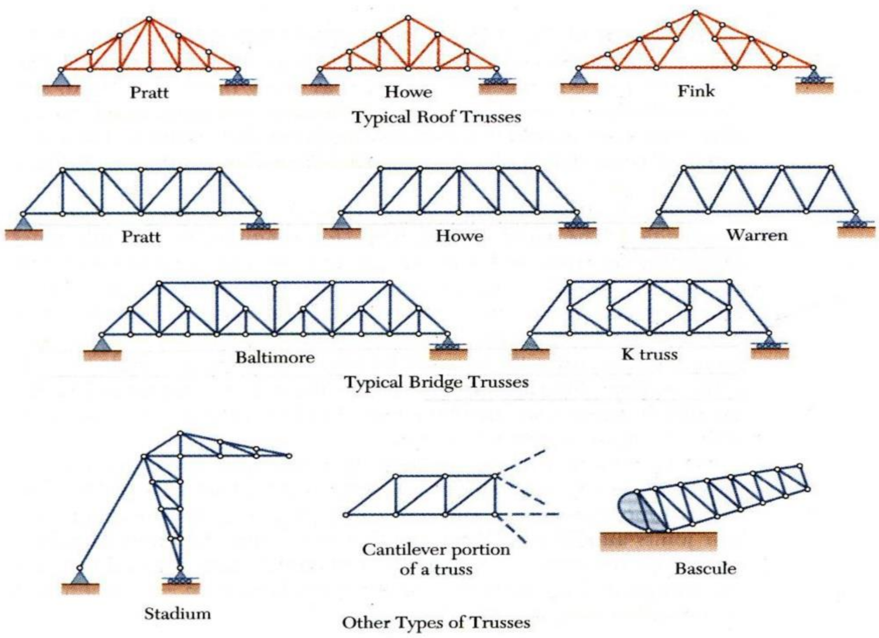
Truss

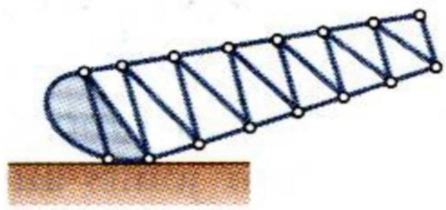




Members of a truss are slender and not capable of supporting large lateral loads. **Loads must be applied at the joints.**

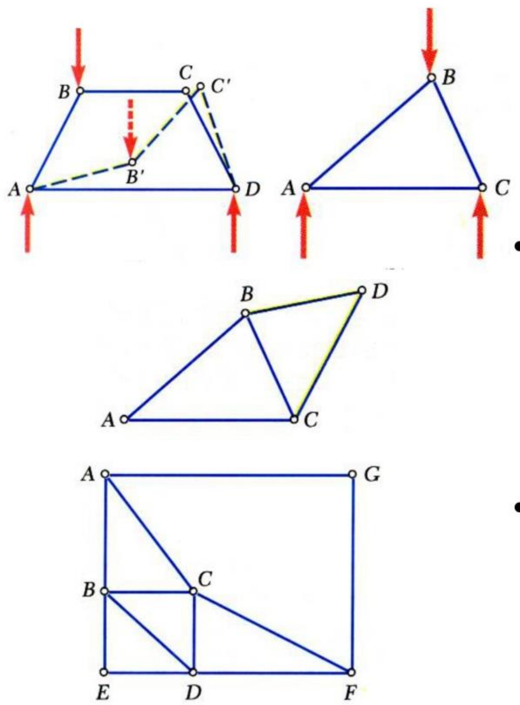
- **Weights are assumed to be distributed to joints.**
- **External distributed loads transferred to joints via stringers and floor beams.**





Bascule

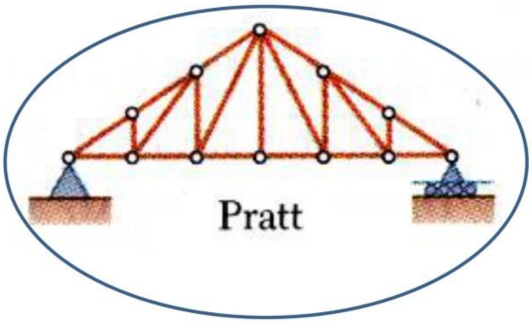
7.2 Simple Truss

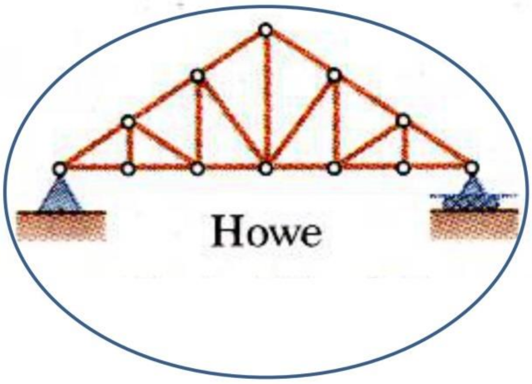


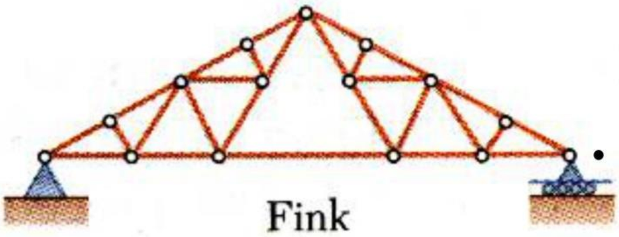
- A rigid truss will not collapse under the application of a load.
- A simple truss is constructed by successively adding two members and one connection to the basic triangular truss.
- In a simple truss, m = 2j - 3 where m is the total number of members and j is the number of joints.

7

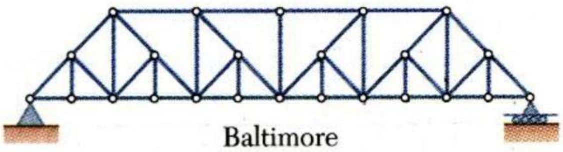
7.2.1 Identify the Simple Truss



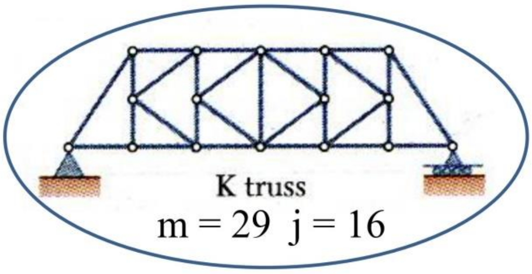




- A simple truss is constructed by successively adding two members and one connection to the basic triangular truss.
- In a simple truss, $m = 2j - 3$ where m is the total number of members and j is the number of joints.



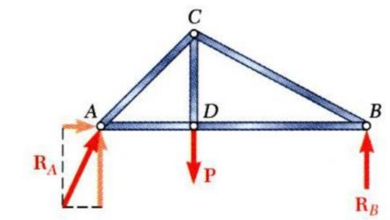
$m = 45 \quad j = 24$

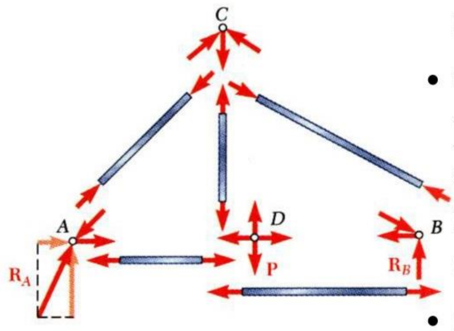


- A simple truss is constructed by successively adding two members and one connection to the basic triangular truss.

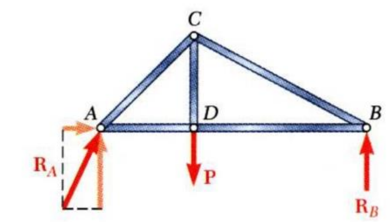
- In a simple truss, $m = 2j - 3$ where m is the total number of members and j is the number of joints.

7.3 Analysis of Trusses by the Method of Joint

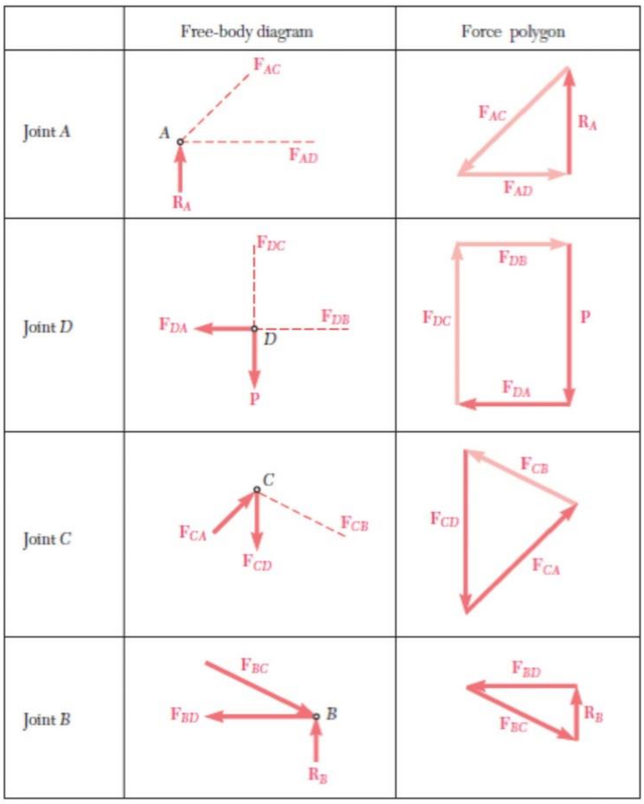


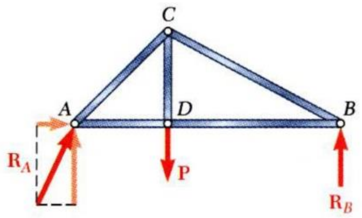


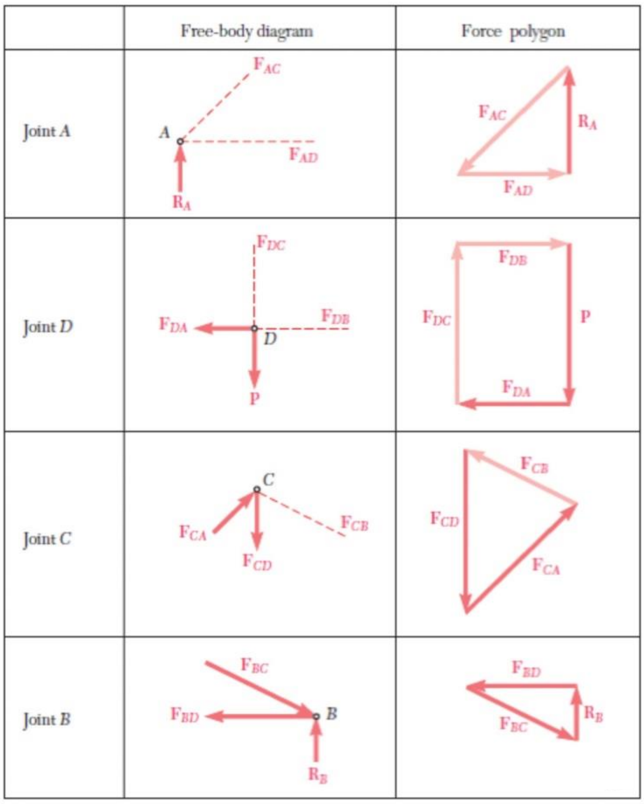
- Dismember the truss and create a free body diagram for each member and pin.
- The two forces exerted on each member are equal, have the same line of action, and opposite sense.
- Forces exerted by a member on the pins or joints at its ends are directed along the member and equal and opposite.
- Conditions of equilibrium on the pins provide 2j equations for 2j unknowns. For a simple truss, $2j = m + 3$. May solve for m member forces and 3 reaction forces at the supports.
- Conditions for equilibrium for the entire truss provide 3 additional equations which are not independent of the pin equations. 10

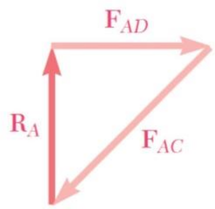


- Use conditions for equilibrium for the entire truss to solve for the reactions R_A and R_B.



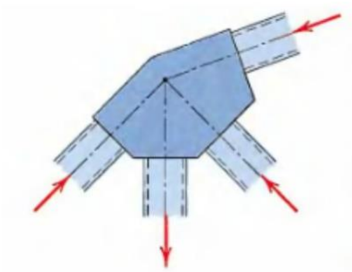




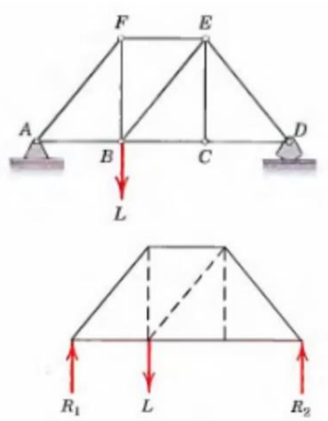


Alternate Force Polygon for Joint A

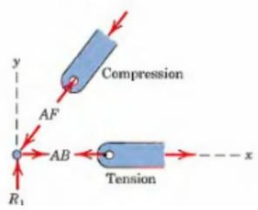
7.4 Truss Connections and Supports



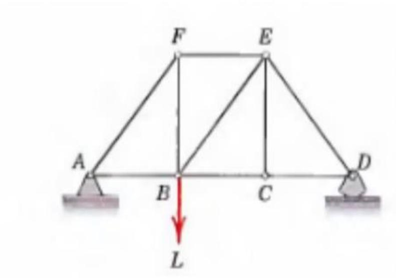
Riveted, Bolted, or Welded

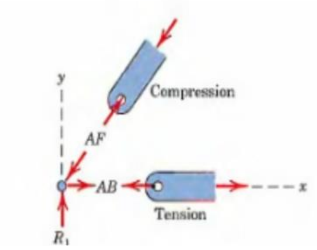


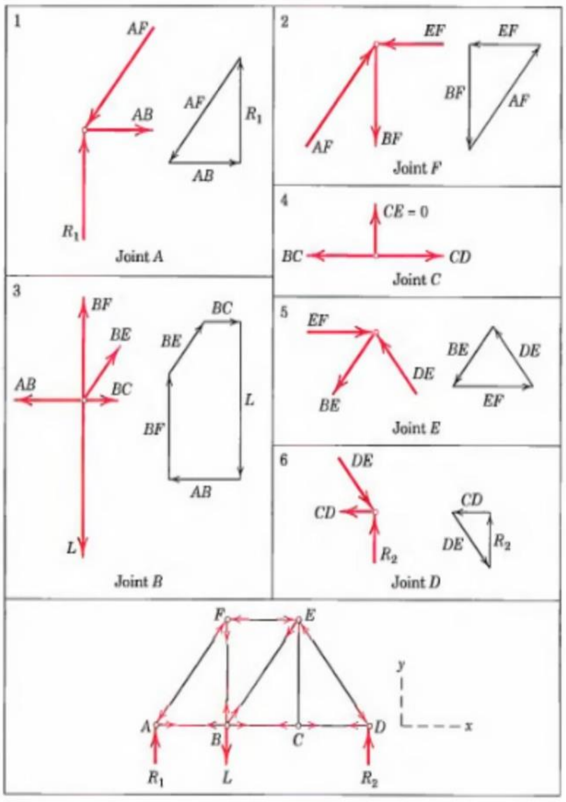
External and internal (dotted) supports



7.5 Method of Joints



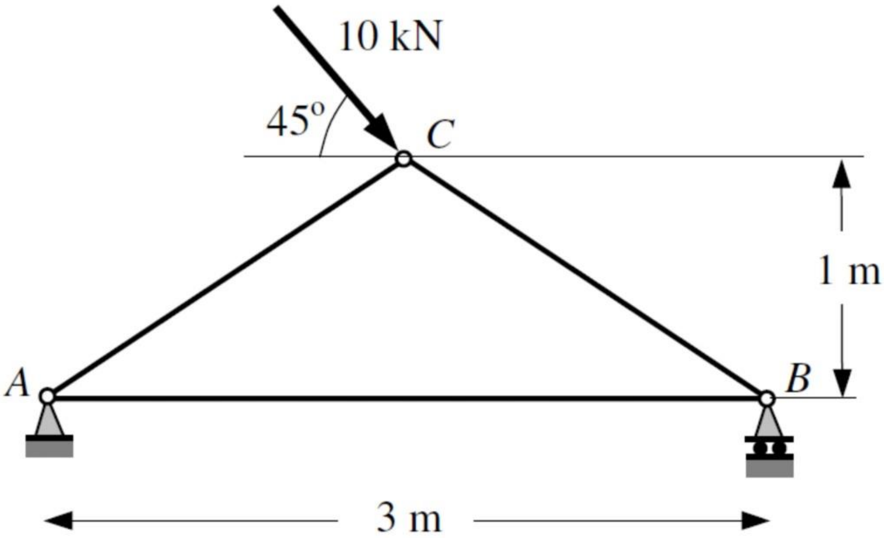




Note the direction of the forces in the members acting on the joints

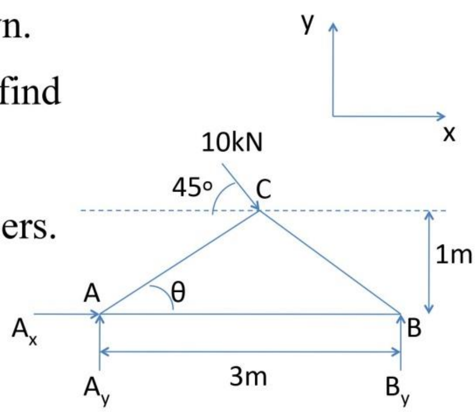
Example - 1

Determine the forces in the members in the following truss



Solution

Consider the FBD of the truss as shown to get the unknown reactions at A and B as shown.
Consider the equilibrium at each hinge to find the force in the members.
Assume tensile forces act on all the members.



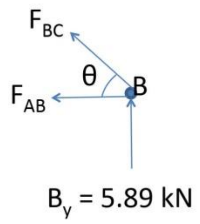
Overall equilibrium $\Rightarrow$

$$\sum F_x = 0 \Rightarrow A_x + 10\cos 45° = 0 \Rightarrow A_x = -7.07kN$$

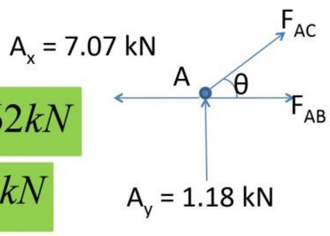
$$\sum M_A = 0 \Rightarrow B_y \times 3 - 10\cos 45° \times 1 - 10\sin 45° \times 1.5 = 0 \Rightarrow B_y = 5.89kN$$

$$\sum F_y = 0 \Rightarrow A_y + B_y = 10\sin 45° \Rightarrow A_y = 1.18kN$$

As we are solving the problem using the method of joints, we take equilibrium at each point. As we have assumed the forces in all the members are tensile, the direction of the reaction force they exert on the hinges are as shown.



$$\tan\theta = \frac{1}{1.5} \Rightarrow \sin\theta = \frac{2}{\sqrt{13}} \Rightarrow \cos\theta = \frac{3}{\sqrt{13}}$$



Equilibrium at B ⇒

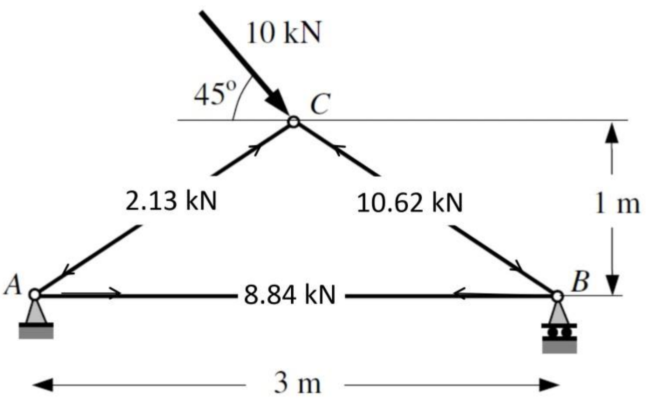
$$\sum F_y = 0 \Rightarrow F_{BC}\sin\theta + B_y = 0 \Rightarrow F_{BC} = -10.62kN$$

$$\sum F_x = 0 \Rightarrow F_{AB} + F_{BC}\cos\theta = 0 \Rightarrow F_{AB} = 8.84kN$$

Equilibrium at A ⇒

$$\sum F_y = 0 \Rightarrow F_{AC}\sin\theta + A_y = 0 \Rightarrow F_{AC} = -2.13kN$$

We assumed that all the forces in the members were tensile. But we got some of them negative. So, the negative sign indicates that the forces in the members are compressive.

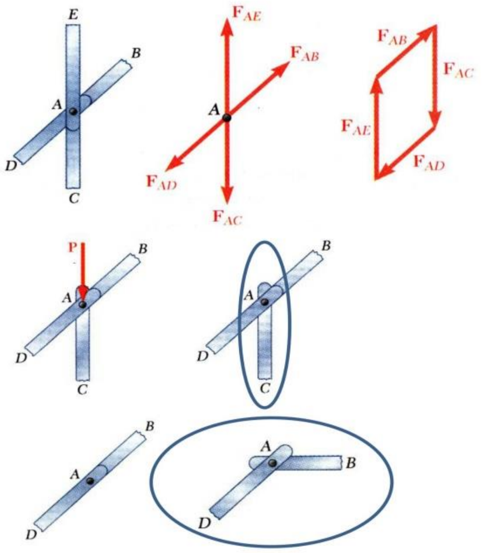


F_{AB}=8.84kN (T)

F_{BC}=10.62kN (C)

F_{AC}=2.13kN (C)

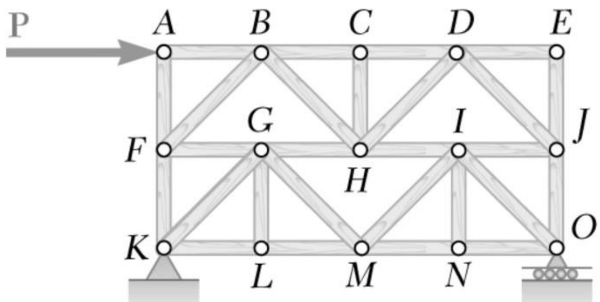
7.6 Joints Under Special Loading Condition

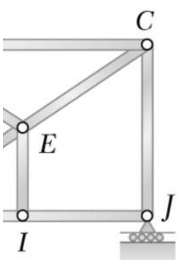


- Forces in opposite members intersecting in two straight lines at a joint are equal.
- The forces in two opposite members are equal when a load is aligned with a third member. The third member force is equal to the load P.
- When a joint connects only two members, the forces in two members are equal when they lie in the same line.
- **Zero force members**
- Recognition of joints under special loading conditions simplifies a truss analysis.

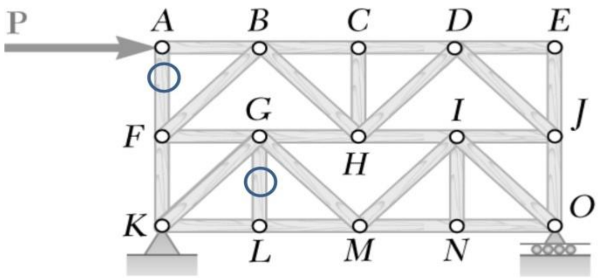
Example - 2

- For the given loading, determine the zero force members in the trusses shown





Solution

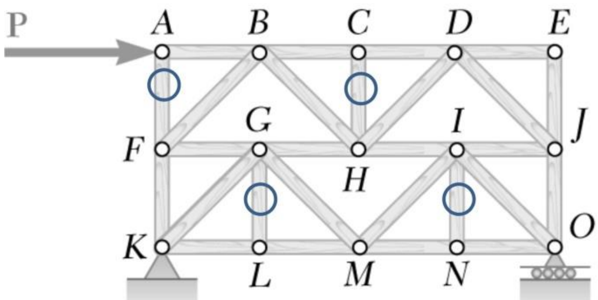


- Joint A is connected to two members and subjected to an external load P oriented along member AB.

$F_{AF} = 0$

- Joint L is connected to three members out of which members LK and LM are in straight lines.

$F_{LG} = 0$

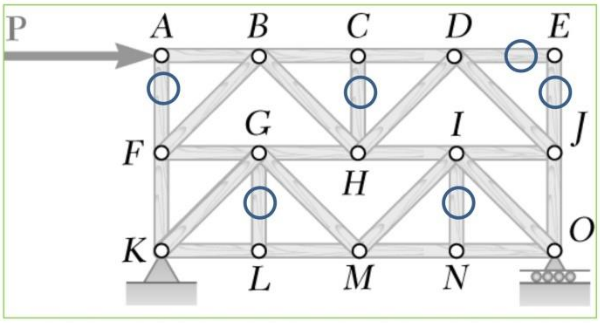


- Joint C is connected to three members out of which members CB and CD are in straight lines.

$F_{CH} = 0$

- Joint N is connected to three members out of which members MN and NO are in straight lines.

$F_{NI} = 0$

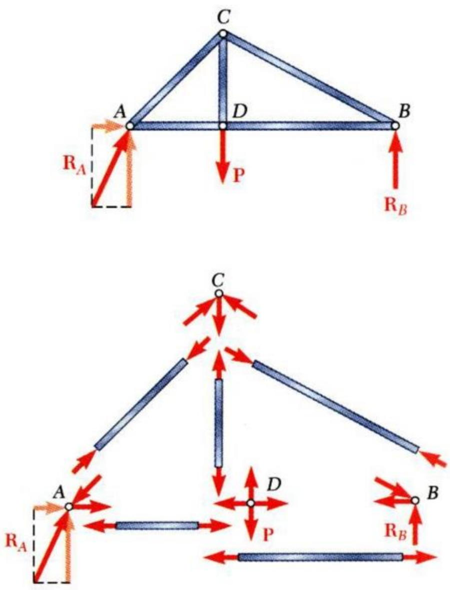


- Joint E is connected to two members which are not in a straight line.

  $F_{EJ} = 0$

  $F_{ED} = 0$

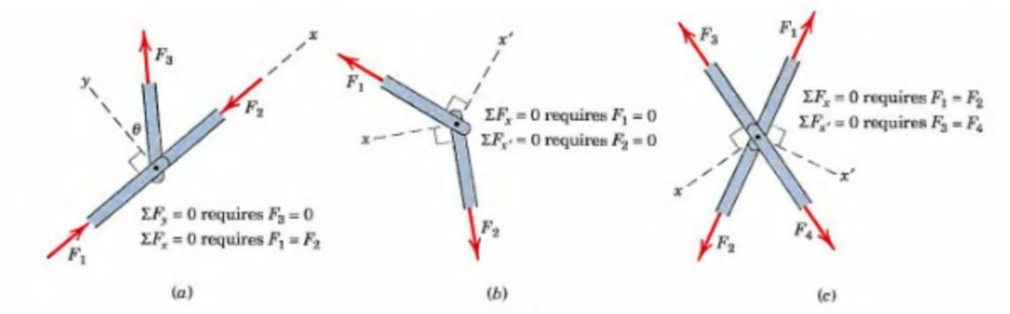
7.7 Analysis of Trusses by the Method of Joints



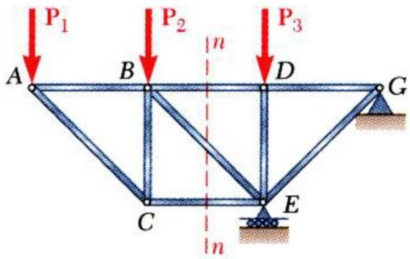
- Dismember the truss and create a freebody diagram for each member and pin.
- The two forces exerted on each member are equal, have the same line of action, and opposite sense.
- Forces exerted by a member on the pins or joints at its ends are directed along the member and equal and opposite.
- Conditions of equilibrium on the pins provide $2n$ equations for $2n$ unknowns. For a simple truss, $2n = m + 3$. May solve for m member forces and 3 reaction forces at the supports.
- Conditions for equilibrium for the entire truss provide 3 additional equations which are not independent of the pin equations.

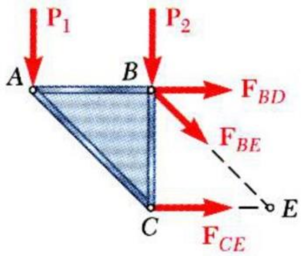
7.8 Special Loading Condition

- Can be recognized by using specialized co-ordinate axes



7.9 Analysis of Trusses by the Method of Sections



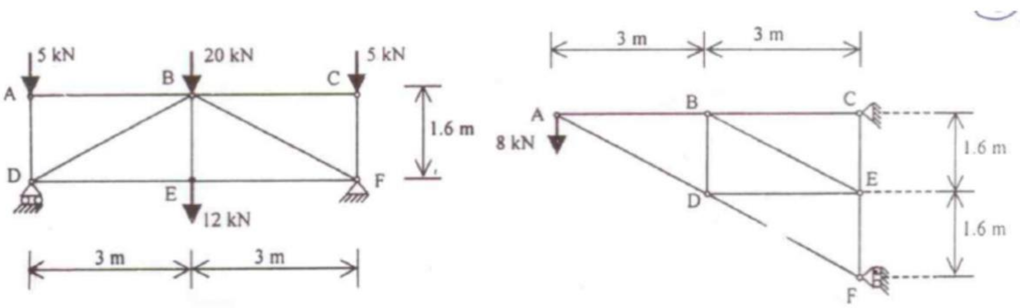


- When the force in only one member or the forces in a very few members are desired, the method of sections works well.
- To determine the force in member BD, pass a section through the truss as shown and create a free body diagram for the left side (or right side).
- With only three members cut by the section, the equations for static equilibrium may be applied to determine the unknown member forces, including F_{BD}.

7.10 PROBLEMS

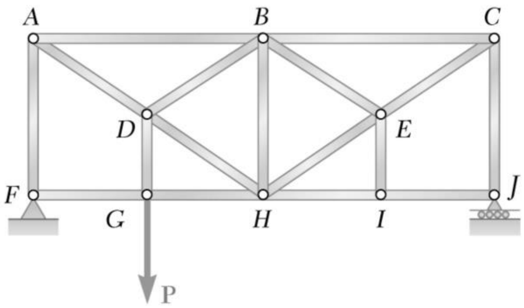
PROBLEM - 1

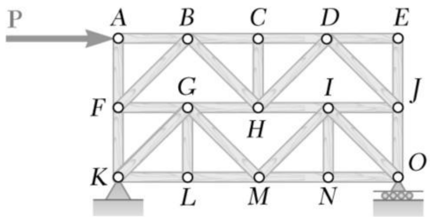
- Using method of joints, determine the forces in the members of the trusses shown



PROBLEM - 2

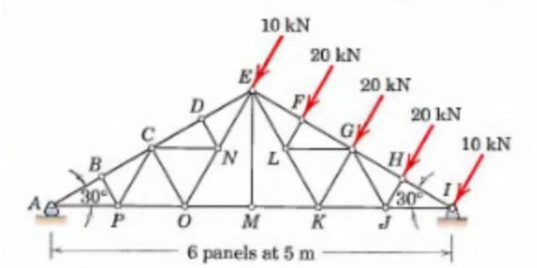
- For the given loading , determine the zero force member in the truss shown





PROBLEM - 3

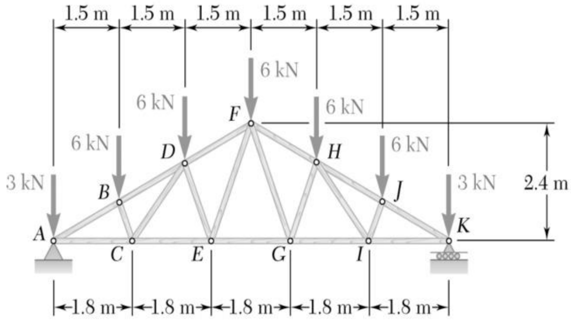
- Find the forces in members EF, KL, and GL for the Fink truss shown. You can use combination of joints + sections.



EF = 75.1 kN (C), KL = 40 kN (T), GL = 20 kN (T)

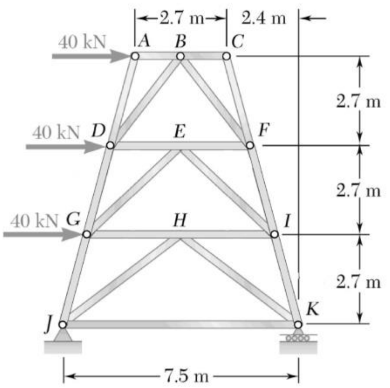
PROBLEM - 4

1. A Fink roof truss is loaded as shown in Fig 5. Use method of section to determine the force in members (a) BD, CD, and CE (b) FH, FG, EG



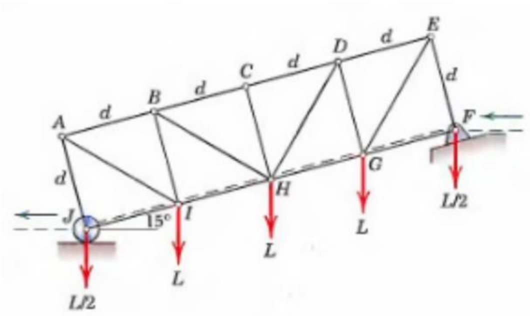
PROBLEM - 5

- Use method of section to determine the force in members IK, HK, FI, EG of the truss shown in the figure.



PROBLEM - 6

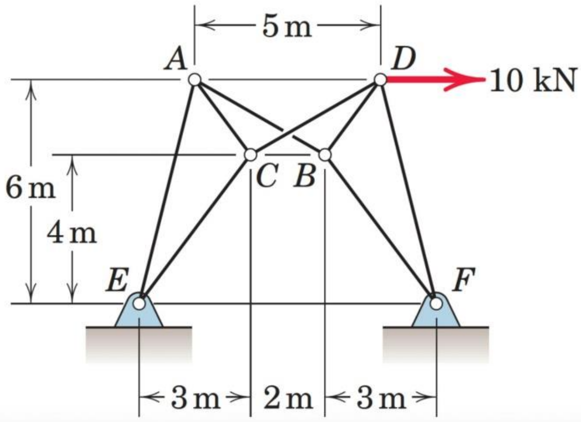
- The truss supports a ramp (shown with a dashed line) which extends from a fixed approach level near joint F to a fixed exit level near J. The loads shown represent the weight of the ramp. Determine the forces in members BH and CD.



Important Notes

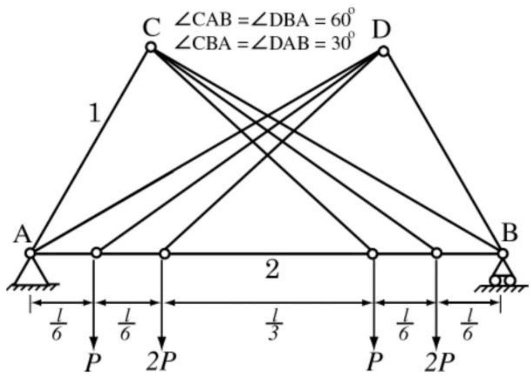
- For a truss to be properly constrained:
  - It should be able to stay in equilibrium for any combination of loading.
  - Equilibrium implies both global equilibrium and internal equilibrium.
- Note that if $2j > m + r$, the truss is most definitely partially constrained (and is unstable to certain loadings). But $2j \geq m + r$, is no guarantee that the truss is stable.
- If $2j < m + r$, the truss can never be statically determinate.

1. The hinged frames *ACE* and *DFB* are connected by two hinged bars, *AB* and *CD*, which cross without being connected. Compute the force in *AB*. Mention clearly if its in tension or compression. **(5 marks)**



PROBLEM - 7

- Determine the forces in bars 1. 2. and 3 of the plane truss supported and loaded as shown in the figure



Chapter 8

Strength of Materials

8. Strength of Materials

8.1. Introduction

Strength of materials, also known as **Mechanics of materials**, is a subject which deals with the behavior of solid subject to stresses and strains.

Stress & Strain

When a force is applied to a structural member, that member will develop both stress and strain as a result of the force.

The applied force will cause the structural member to deform by some length, in proportion to its stiffness.

1. Stress

Stress is the force carried by the member per unit area, and typical units are $[lbf\ /\ in^2\ (psi)]$ for US Customary units and $[N\ /\ m^2\ (Pa)]$ for SI units:

$$\sigma = \frac{F}{A} \qquad (1-1)$$

Where, (F) is the applied force and (A) is the cross-sectional area over which the force acts.

2. Strain

Strain is the ratio of the deformation to the original length of the part:

$$\varepsilon = \frac{L - L_O}{L_O} = \frac{\delta}{L_O} \qquad (1-2)$$

Where (L) is the deformed length, (L_O) is the original unreformed length (ε) is the deformation, and (δ) change in length.

8-2. Types of loading

There are different types of loading which result in different types of stress.

1. Axial Force

Type of stress is called an Axial Stress (general case)

A. Tensile Stress (σ_t): If force is tensile as figure (1-1).

$$\sigma_t = \frac{F}{A} \qquad (1-3)$$

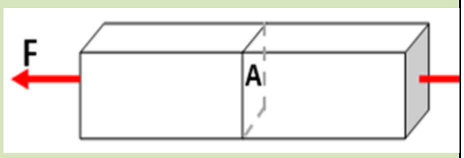


Figure 8-1: Tensile stress

B. Compressive Stress (σ_c): If force is compressive as figure (1-2).

$$\sigma_c = \frac{F}{A} \qquad (1-4)$$

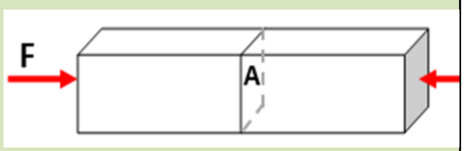


Figure 8-2: Compressive

2. Shear stress (τ)

Type of stress is called a Transverse Shear Stress as figure (1-3).

$$\tau = \frac{F}{A} \qquad (1-5)$$

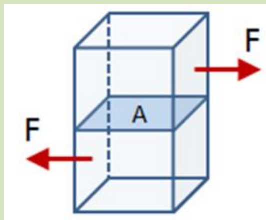


Figure 8-3: Shear stress

3. Bending moment stress (σ_b)

Type of stress is called a Bending Stress as figure (1- 4).

$$\sigma_b = \frac{M\,.\,y}{I_c} \qquad (1-6)$$

Where: (M) is the bending moment, (y) is the distance between the centroid axis and the outer surface, and (I_c) is the centroid moment of inertia of the cross section about the appropriate axis.

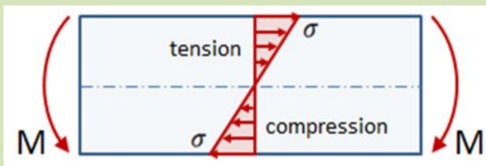


Figure 8-4: Bending stress

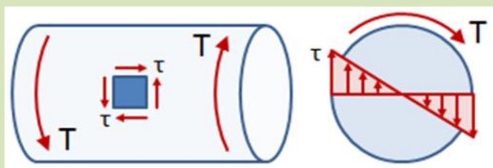


Figure 8-5: Torsional Stress

4. Torsional stress

Type of stress is called a Torsional Stress as figure (1-5). (**Engineer's theory of Torsion (E.T.T.)).**

$$\frac{\tau}{r} = \frac{T}{I} = \frac{G\,.\emptyset}{L} \qquad (1-7)$$

Where: (τ) is the shear stress, (r) is the radius, (T) is the torsion torque, (I) is the polar moment of inertia of the cross section, (G) is modulus of rigidity, $(\emptyset)$ is the torsion angle, and (L) is a length of shaft.

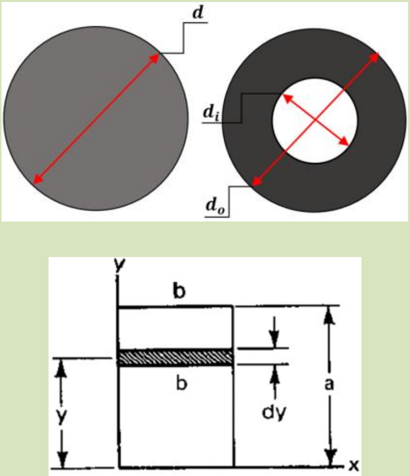


Figure 8-6: Polar moment of inertia

Figure (1-6) shown polar moment of inertia for the following:

$I = \frac{\pi d^4}{32}$ For solid circular section,

$I = \frac{\pi(d_o^4 - d_i^4)}{32}$ For hollow circular section,

$I = \frac{ab^3}{3}$ For solid rectangular section.

8-3. Hooke's Law

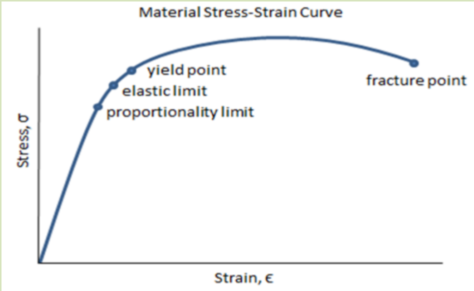


Figure 8-7: Hooke's Law

Stress is proportional to strain in the elastic region of the material's stress-strain curve (below the proportionality limit, where the curve is linear), figure (1-6).

8-3-1. Engineering and True Stress

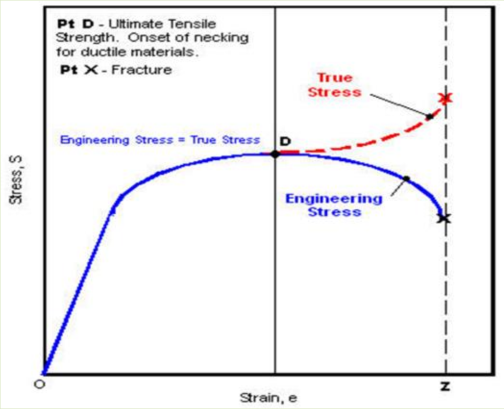


Figure 8-8: Engineering and True Stress

8-3-1-1. Engineering Stress (ES)

ES: is equivalent to the applied uniaxial tensile or compressive force at time, a fraction of the specimen's original cross-sectional area, figure (1-8).

8-3-1-2. True Stress (TS)

TS: is equivalent to the applied uniaxial tensile or compressive force at time, divided by the specimen's cross-sectional area at the moment, figure (1-8).
Normal stress and strain are related by:

$$E = \frac{\sigma}{\varepsilon} \qquad (8-8)$$

Where: (E) is the elastic modulus of the material, (σ) is the normal stress, and (ε) is the normal strain.
Shear stress and strain are related by:

$$G = \frac{\tau}{\gamma} \qquad (8-9)$$

Where: (G) is the shear modulus of the material, (τ) is the shear stress, and (γ) is the shear strain. The elastic modulus and the shear modulus are related by:

$$G = \frac{E}{2(1+\mu)} \qquad (8-10)$$

Where: (μ) is Poisson's ratio.

8-4. Poisson's ratio

Poisson's ratio is the proportion of lateral (transverse) contraction strain to longitudinal extension strain in the direction of stretching force, figure (8-9).

The value of Poisson's ratio varies from 0.25 to 0.33. For rubber its value varies from 0.45 to 0.5. Mathematically:

$$Poisson's\ ratio = \frac{Lateral\ strain}{Longitudinal\ strain}$$

$$\mu = \frac{Lateral\ strain}{Longitudinal\ strain}$$

$$\mu = \frac{\varepsilon_{Lateral}}{\varepsilon_{Long.}} \qquad (8-10)$$

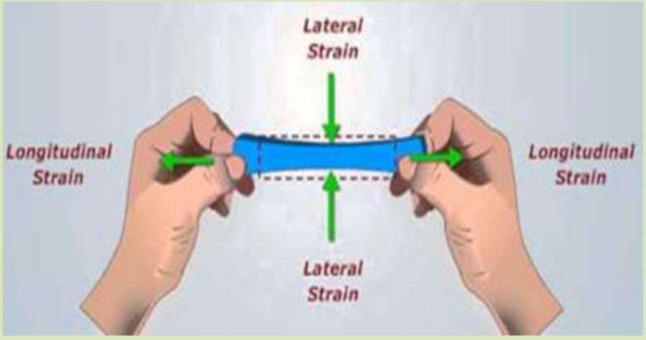


Figure 8-9: Poisson's ratio

8-5. Solve examples

Example 1

A force of (100 KN) is acting on a circular rod, figure (1-10), with diameter (50 mm). The stress in the rod can be calculated as:

Solution:

$\boldsymbol{Given}: F = 100\ KN = 100000\ N, d = 50\ mm, r = \frac{d}{2} = 25\ mm$

$$\boldsymbol{\sigma_t = \frac{F}{A}}$$

$$F = 100 \times 1000 = 100000\ N$$

$$A = \pi\ .\ r^2$$

$$A = \frac{22}{7} \times\ (25)^2 = 1964.286\ mm^2$$

$$\sigma_t = \frac{F}{A} = \frac{100000}{1964.286}$$

$$= 50.909\ \frac{N}{mm^2} (MPa)$$

Figure 8-10: Circular rod

Example 2

A metal shaft diameter (12 mm), and long (1.5 m), figure (1-11). A tensile force of (1000 N) is applied to it and it stretches (0.11 mm). Assume the material is elastic. Determine the stress and strain in the shaft?

Figure 8-11: A metal shaft

Solution:

$\boldsymbol{Given}: d = 12\ mm, r = 6\ mm, L = 1.5\ m = 1500\ mm, F = 1000\ N, \delta = 0.11\ mm$

$$A = \pi\ .\ r^2 = \frac{22}{7} \times (6)^2 = 113.143\ mm^2$$

$$\sigma_t = \frac{F}{A} = \frac{1000}{113.143} = 8.818\ MPa$$

$$\varepsilon = \frac{\delta}{L_O} = \frac{0.11}{1500} = 0.000073 = 73\ \mu\varepsilon$$

Example 3

A steel tensile test specimen, figure (1-12) has an across sectional area of (120 mm^2), and gauge length (50 mm), the gradient of elastic section is (433 KN/mm). Determine the modulus of elasticity?

Solution:

$$\textbf{Given}: A = 120\ mm^2\ , \quad L = 50\ mm,$$

$$Gradient\ ratio\ (\frac{F}{\delta}) = 433\frac{KN}{mm} = 433000\ N/mm$$

$$E = \frac{\sigma}{\varepsilon} = \frac{F}{\delta} . \frac{L}{A} = 433000 \times \frac{50}{120} = 180416.667\ MPa \approx 180.417\ GPa$$

Figure 8-12: A steel tensile test specimen

Example 4

A long of the steel column is (4 m), and diameter (50 cm), figure (1-13). It carries a load of (100 MN). If modulus of elasticity is (210 GPa), calculate the compressive stress and strain and how much the column is compressed?

Solution:

$$\textbf{Given}: L = 4\ m = 4000\ mm,$$

$$d = 50\ cm = 500\ cm, r = 250\ mm, F = 100\ MN = 100000000\ N, E = 210\ GPa = 210000m$$

$$A = \frac{22}{7} \times \ (250)^2 = 196428.571\ mm^2$$

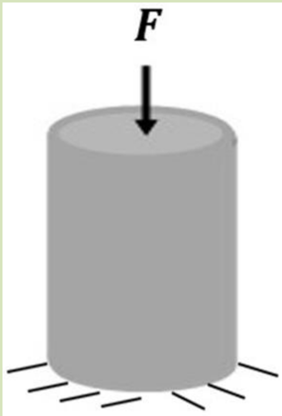


Figure 8-13: A steel column

$$\sigma_c = \frac{F}{A} = \frac{100000000}{196428.571} = 509.091\ MPa$$

$$E = \frac{\sigma}{\varepsilon} \qquad \Rightarrow \qquad \varepsilon = \frac{\sigma}{E} = \frac{509.091}{210000} \approx 0.00242 = 24.2\ \mu\varepsilon$$

$$\varepsilon = \frac{\delta}{L} \qquad \Rightarrow \qquad \delta = \varepsilon\,.L = 0.00242 \times 4000 = 9.68\ mm$$

Example 5

Calculate the force needed to a plate of metal (5 mm) thick and (0.8 m) wide given that the ultimate shear stress (50 MPa), as shown in the figure (1-14)?

Figure 8-14: A plate of metal

Solution:

The area to be cut is a rectangle

$$t = 5\,mm\,;\ w = 0.8\,m = 0.8 \times 1000 = 800\,mm\,;$$

$$\tau = 50\,N\,/\,mm^2$$

$$A = w.t = 5 \times 800 = 4000\ mm^2$$

$$\because\ \tau = \frac{F}{A} \quad \Rightarrow \quad F = \tau.A = 50 \times 4000 = 200000N = 2$$

Example 6

Calculate the force needed to shear a Screw (12 mm) diameter given that the ultimate shear stress is (90 MPa), as shown in the figure (1-15)?

Figure 8-15: A plate of metal

Solution:

The area to be is the circular area:

$$A = \frac{\pi\,d^2}{4} = \frac{3.14 \times (12)^2}{4} = 113.04\ mm^2$$

$$\tau = \frac{F}{A}$$

$$F = \tau.A = 90 \times 113.04 = 10173.6\,N \approx 10.17$$

Example 7

A pin is used to attach a clevis to a rope, figure (1-16). The force in the rope will be a maximum of (60 KN). The maximum permitted shear stress in a pin is (40 MPa). Calculate the diameter of suitable pin?

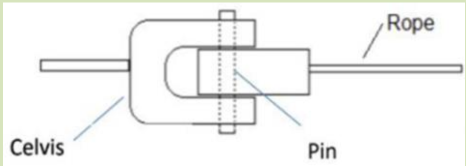


Figure 8-16: A clevis to a rope

Solution

The pin is in double shear so the shear stress is:

$$A = \frac{F}{2\tau} = \frac{60000}{2 \times 40} = 750\ mm^2$$

Also:

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4 \times 750}{3.143}} = 30.9\ mm$$

Example 8

A simply supported beam is subject a point load of (200 N) at the mid - spam of the beam as shown in the figure (1-17). The beam has a circular (50 mm) diameter. Calculate the maximum stress due to bending?

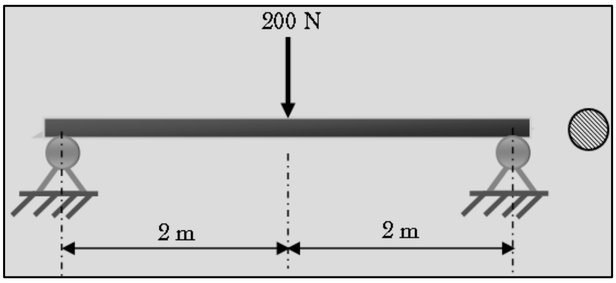
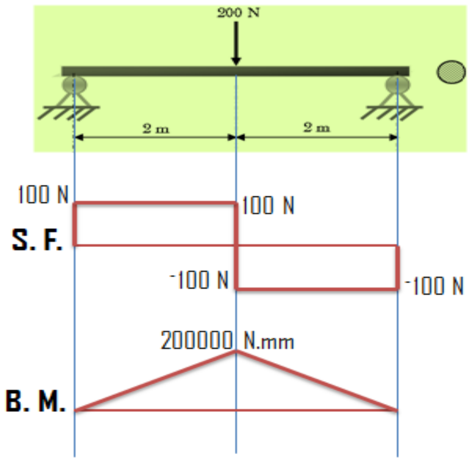


Figure 8-17: A simply supported beam

Solution:

Given:

$$F = 200\ \mathrm{N}, d = 50\ \mathrm{mm}\ , L_1 = 2\ \mathrm{m} = 2000\ \mathrm{mm}\ , L_2 = 2\ \mathrm{m} = 2000\ \mathrm{mm}.$$



$$\sigma_{max.} = \frac{M\ .C}{I}$$

$$M = 100 \times 2000 = 200000\ N.mm$$

$$I = \frac{\pi\ d^4}{64} = \frac{3.143 \times (50)^4}{64} = 306933.59\ mm^4$$

$$C = \frac{d}{2} = \frac{50}{2} = 25\ mm$$

$$\boldsymbol{\sigma_{max.}} = \frac{\boldsymbol{M\ .C}}{\boldsymbol{I}} = \frac{200000 \times 25}{306933.59} = 16.29\ \text{MPa}$$

Example 9

A diameter solid steel shaft (ABCDE), figure (1-18) is (50 mm) see in figure. If have torques (T_1 = 200 N. m, T_2 = 500 N. m and T_3 = 300 N. m), distance between gears (B & C) is (L1 = 200 mm) and distance between gears (C & D) is (L_2 = 300 mm), modulus of rigid is (G = 90 GPa). Determine the maximum shear stress ($\tau_{max.}$) in each part and twisting angle (ϕ_{BD})?

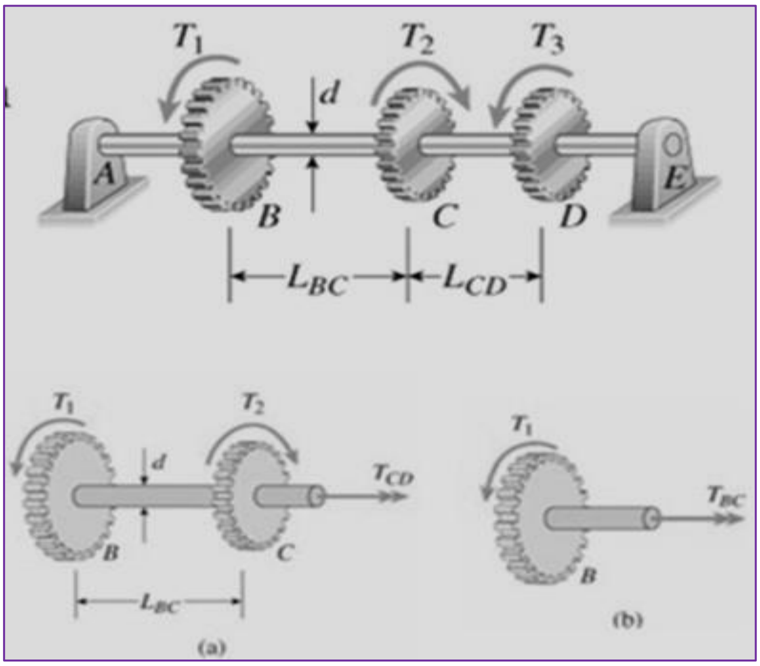


Figure 8-18: A simply supported beam

__*Solution*__

Given: *{d= 50 mm, T_1 = 200 N. m, T_2 = 500 N .m, T_3 = 300 N. m, L_1 = 300 mm, L_2 = 200 mm, G = 90 GPa}.*

$$I_P = \frac{\pi.d^4}{32} = \frac{3.14 \times 50^4}{32} = 613281.25 mm^4 ; \quad r = 25mm$$

$$T_{BC} = -\ T_1 = -200\ N.m$$

$$T_{CD} = T_2 - T_1 = 500 - 200 = 300\, N.m$$

$$\because \frac{\tau}{r} = \frac{T}{I} = \frac{G\varphi}{L} \qquad \Rightarrow \qquad \therefore\ \tau = \frac{T.r}{I}$$

$$\therefore \tau_{BC} = \frac{T_{BC}.r}{I} = \frac{200.10^3 \times 25}{613281.25} = 8.15\ MPa$$

$$\therefore \tau_{CD} = \frac{T_{CD}.r}{I} = \frac{300.10^3 \times 25}{613281.25} = 12.23\ MPa$$

$$\phi_{BD} = \phi_{BC} - \phi_{CD}$$

$$\because \frac{\tau}{r} = \frac{T}{I} = \frac{G\varphi}{L} \qquad \Rightarrow \qquad \therefore\ \varphi = \frac{T.\,L}{I.G}$$

$$\varphi_{BC} = \frac{T_{BC}.\,L_1}{I.G} = \frac{200.10^3 \times 300}{613281.25 \times 90.10^3} \approx 0.00109 \approx 0.0624^\circ$$

$$\varphi_{CD} = \frac{T_{CD}.\,L_2}{I.G} = \frac{300.10^3 \times 200}{613281.25 \times 90.10^3} \approx 0.00109 \approx 0.0624^\circ$$

Example 10

A steel wire having cross sectional area (2 mm^2), figure (1-19). Is stretched by (200 N). Find the lateral strain produced in the wire. If modulus elasticity for steel is (210 GPA) and Poisson's ratio is (0.233)?

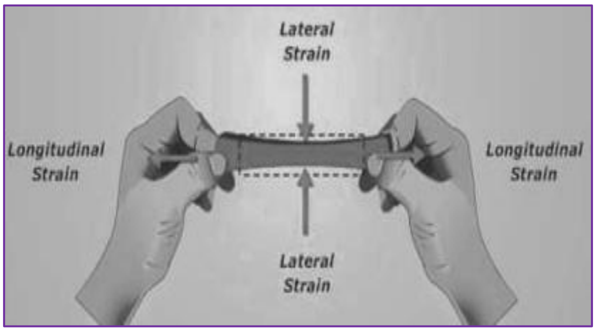


Figure 8-19: A steel wire

Solution

Given: *{A= 2 mm^2, F = 200 N, μ = 0.233, G = 210 GPa}.*

$$E = \frac{\sigma}{\varepsilon_{longitudinal}} = \frac{F}{A.\varepsilon_L} \quad \Rightarrow \quad \varepsilon_L = \frac{F}{A.E} = \frac{200}{2.10^{-6} \times 210.10^9} = 0.00048 \approx 4.8 \times 10^4$$

$$\because \quad \mu = \frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}} \quad \Rightarrow \quad \therefore \varepsilon_{Lateral} = \mu.\varepsilon_L = 0.233 \times 0.00048 = 0.000112 \approx 1.12 \times 10^4$$

8.6. Shear Force and Bending Moment Diagram

The maximum absolute value of the shear force and the bending moment of the beams with regard to the relative load can be determined using the shear force and bending moment diagram in beams.

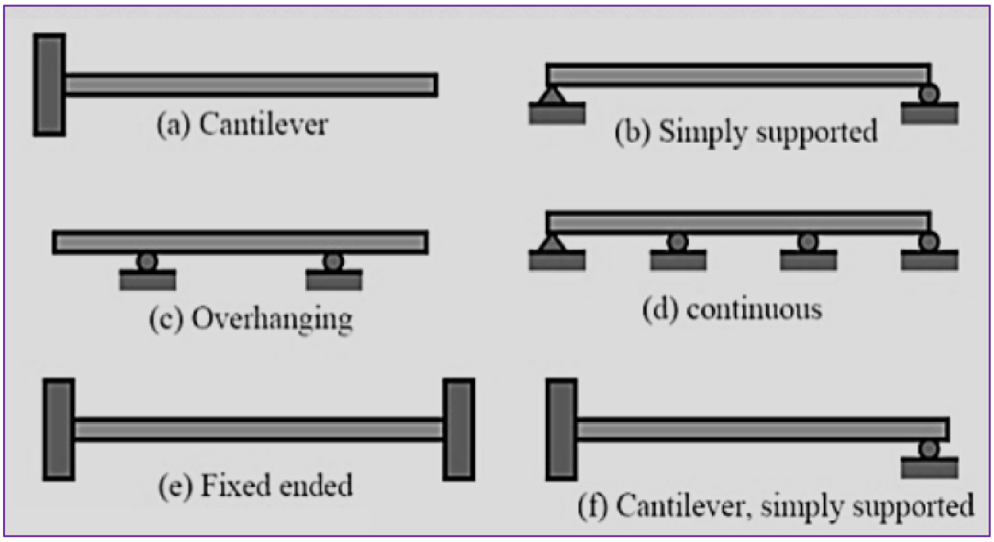
Before we can design the Shear force and bending moment diagram, we must first understand the various types of beams and loads, as well as the reaction forces acting on them.

According to the right or left of the section, the bending moment is the algebraic sum of all the moment of forces. It is the reaction that is induced in a structural element as a result of an external force or moment.

The moment caused by external forces is balanced in the equilibrium position by the couple induced by the internal load; this internal couple is known as a bending moment.

8.7. Type of Beams

a. Cantilever Beam
b. Simply Supported Beam
c. Overhanging Beam
d. Continuous Beam
e. Fixed ended Beam
f. Cantilever, Simply Supported Beam



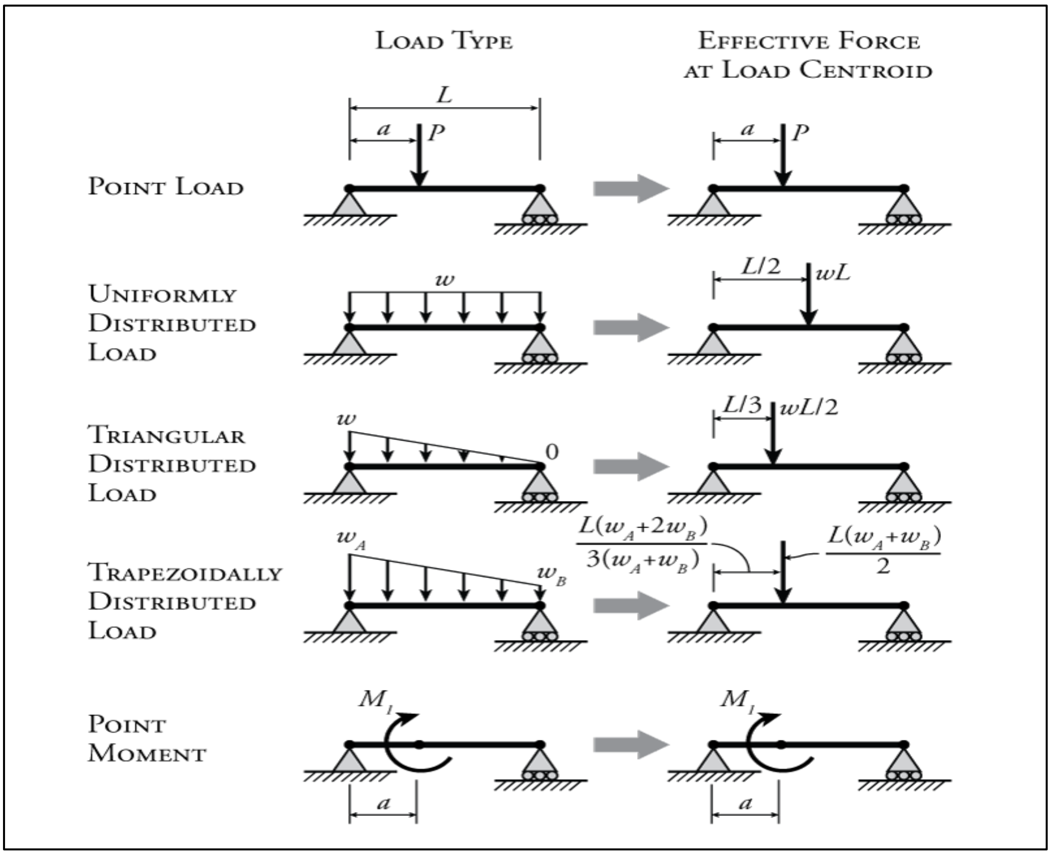
8.7. Type of Loads

The applied weight is normally vertical, whereas the beam is usually horizontal.

Concentrated or Point Load: Act at a point.

Uniformly Distributed Load: The load is evenly distributed along the length of the Beam.

Uniformly Varying Load: Load distribution along the length of the beam, and rate of varying loading from point to point.

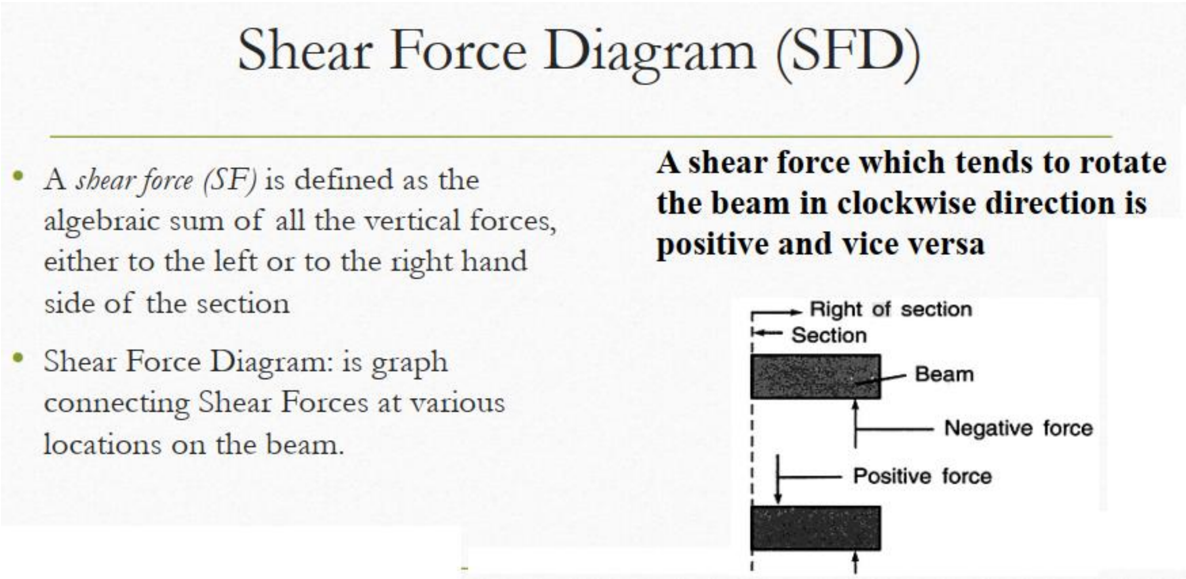


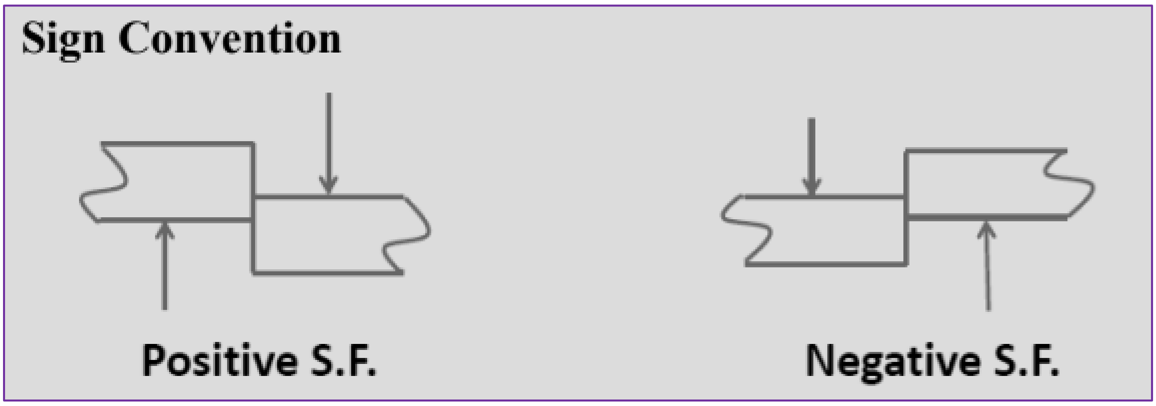
8.8. Sign Convention of Shear Force

Shear force is an imbalanced vertical force that causes one end of the beam to move forward or downward in relation to the other.

When the left-hand portion of a section tends to slide upward and the right-hand half tends to slide downward, the shear force is deemed positive.

When the left-hand portion of a section tends to slide downward, or the right-hand portion tends to slide upward, the shear force at that section is negative.



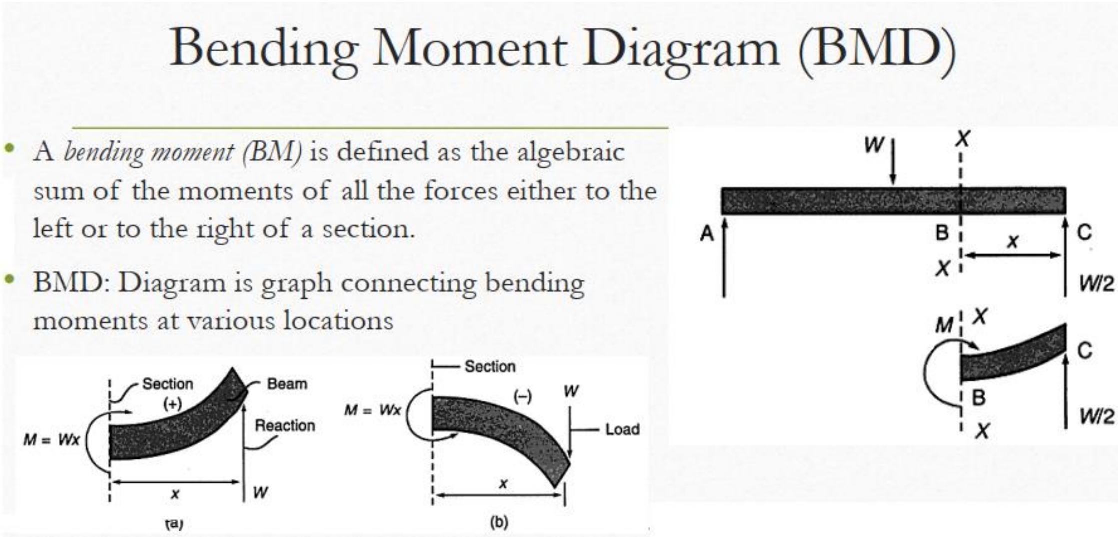


8.10 Sign Convention for Bending Moment

When the bending moment at a section tends to bend the beam at a point to curvature with a concavity at the top, or when the moments are operating clockwise to the left or anti-clockwise to the right, we consider it positive.

On the other hand, the **bending moment** at a section is deemed negative when it tends to bend the beam at a point to curvature with convexity at the top or when moments are taken in an anti-clockwise or clockwise manner.

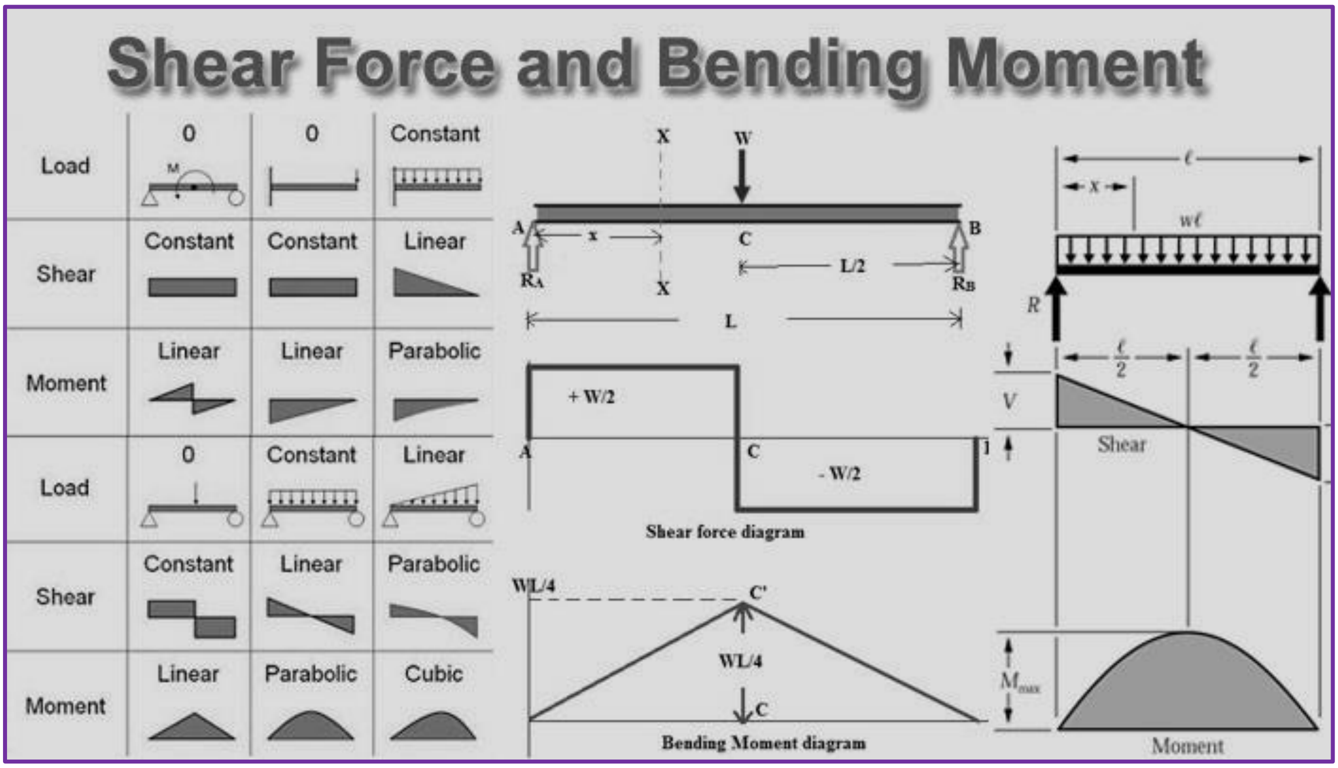
Positive bending moments are sometimes referred to as sagging moments, whereas negative bending moments are referred to as hogging moments.



8.11. Shear Force and Bending Moment Diagram Drawing Instructions

- The ordinates in **SFD and BMD diagrams are shear force or bending moment**, and the abscissa is the length of the beam.
- Take a look at the left or right side of the section.
- On one of the portions, add the forces (including reactions) normal to the beam.
- The force acting downhill is positive, whereas the force acting upwards is negative if the right portion of the section is chosen.
- The force acting downhill is negative, whereas the force acting upwards is positive if the Left component of the section is chosen.
- Shear force and Bending moment positive values are plotted above the baseline, while negative values are plotted below the baseline.

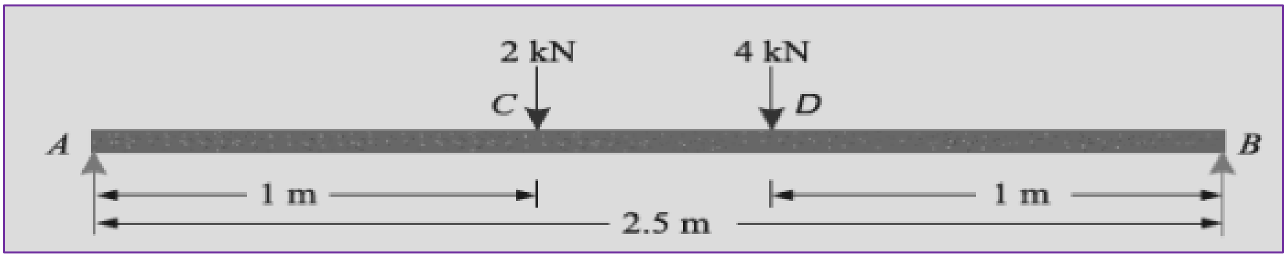
- The sheer force diagram will suddenly increase or decrease. I.e., at a segment where there is a vertical point load, by a vertical straight line.
- Between any two vertical weights, the shear force will be constant. As a result, the horizontal shear force between the two vertical loads will exist.
- At the two ends of a simply supported beam and at the free end of a cantilever, the bending moment will be zero.



8.12. Examples

Example 1.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (2.5 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \mathbf{\Sigma M_A = 0}$$

$$2 \times 1 + 4 \times 1.5 - R_B \times 2.5 = 0$$

$$R_B = \frac{2 \times 1 + 4 \times 1.5}{2.5} = \frac{8}{2.5} = 3.2\ KN$$

$$\curvearrowright_{+}\ \mathbf{\Sigma M_B = 0}$$

$$4 \times 1 + 2 \times 1.5 - R_A \times 2.5 = 0$$

$$R_A = \frac{4 \times 1 + 2 \times 1.5}{2.5} = \frac{7}{2.5} = 2.8\ KN$$

Or

$$\mathbf{\Sigma F_y = 0}$$

$$R_A - 2 - 4 + R_B = 0$$

$$R_A = 2 + 4 - 3.2 = 2.8\ KN$$

2. Shear force diagram:

$$\Sigma F_A = 2.8 \text{ KN}$$

$$\Sigma F_C = 2.8 - 2 = 0.8 \text{ KN}$$

$$\Sigma F_D = 2.8 - 2 - 4 = -3.2 \text{ KN}$$

$$\Sigma F_B = 2.8 - 2 - 4 + 3.2 = 0 \text{ KN}$$

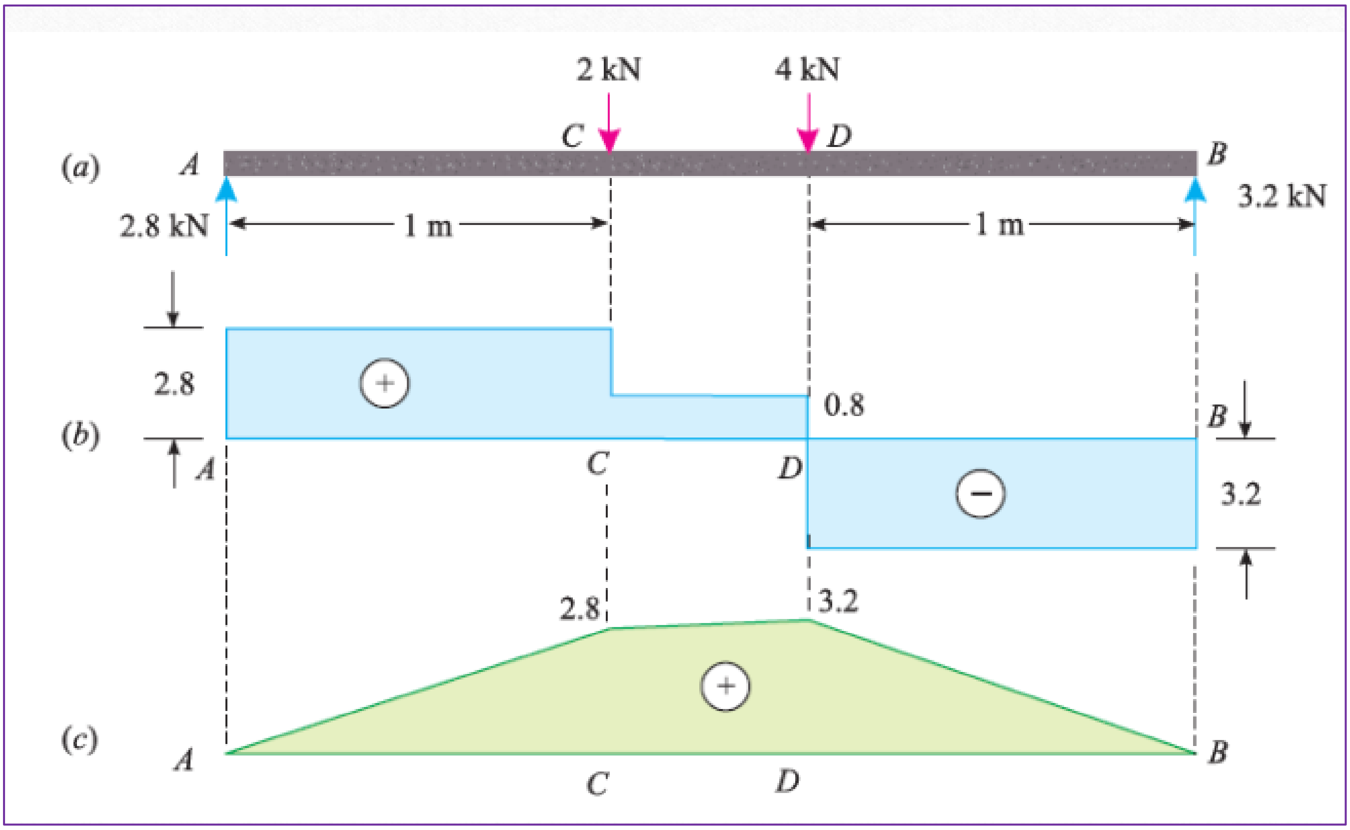
3. Bending moment:

$$\Sigma M_A = 0 \text{ KN.m}$$

$$\Sigma M_C = 2.8 \times 1 = 2.8 \text{ KN.m}$$

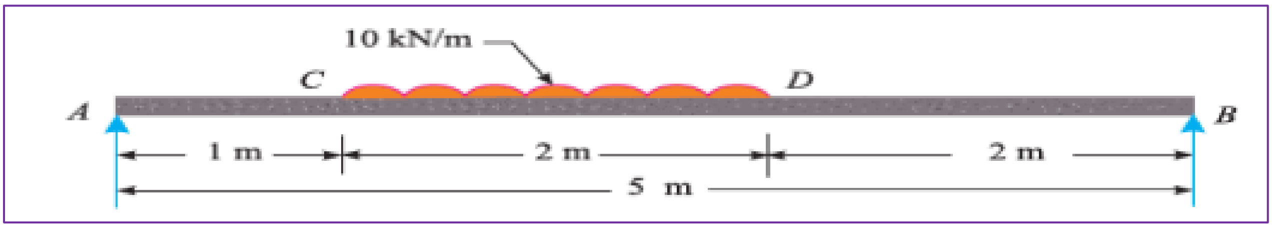
$$\Sigma M_D = 2.8 \times 1.5 - 2 \times 0.5 = 3.2 \text{ KN.m}$$

$$\Sigma M_B = 2.8 \times 2.5 - 2 \times 1.5 - 4 \times 1 = 0 \text{ KN.m}$$



Example 2.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (5 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \mathbf{\Sigma M_A = 0}$$

$$10 \times 2 \times 2 - \mathrm{R_B} \times 5 = 0$$

$$\mathrm{R_B} = \frac{10 \times 2 \times 2}{5} = \frac{40}{5} = 8\ KN$$

$$\curvearrowright_{+}\ \mathbf{\Sigma M_B = 0}$$

$$10 \times 2 \times 3 - \mathrm{R_A} \times 5 = 0$$

$$\mathrm{R_A} = \frac{10 \times 2 \times 3}{5} = 12\ KN$$

Or

$$\mathbf{\Sigma F_y = 0}$$

$$\mathrm{R_A} - 10 \times 2 + R_B = 0$$

$$\mathrm{R_A} = 20 - 8 = 12\ KN$$

2. Shear force diagram:

$$\text{S.F. at point A} = 12 \text{ KN}$$

$$\text{S.F. at point C} = 12 \text{ KN}$$

$$\text{S.F. at point D} = 12 - 10 \times 2 = -8 \text{ KN}$$

$$\text{S.F. at point A} = 12 - 10 \times 2 + 8 = 0\ KN$$

3. Bending moment:

$$B.M.\ at\ point\ A = 0 \text{ KN.m}$$

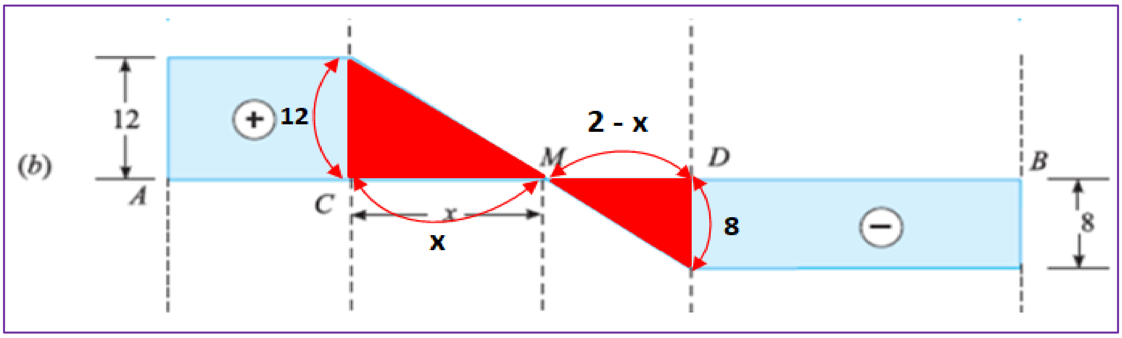
$$B.M.\ at\ point\ C = 12 \times 1 = 12 \text{ KN.m}$$

$$B.M.\ at\ point\ D = 12 \times 3 - 2 \times 10 \times 1 = 16 \text{ KN.m}$$

$$B.M.\ at\ point\ B = 12 \times 5 - 10 \times 2 \times 3 = 0 \text{ KN.m}$$

We follow the following steps to find bending moment at point (M):

From the figure of shear force:



$$\text{S.F. at point M} = 0 \text{ KN}$$

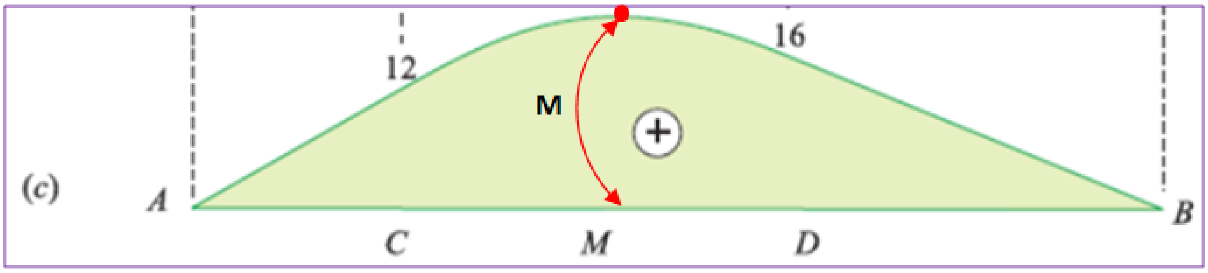
$$\frac{x}{12} = \frac{2 - x}{8}$$

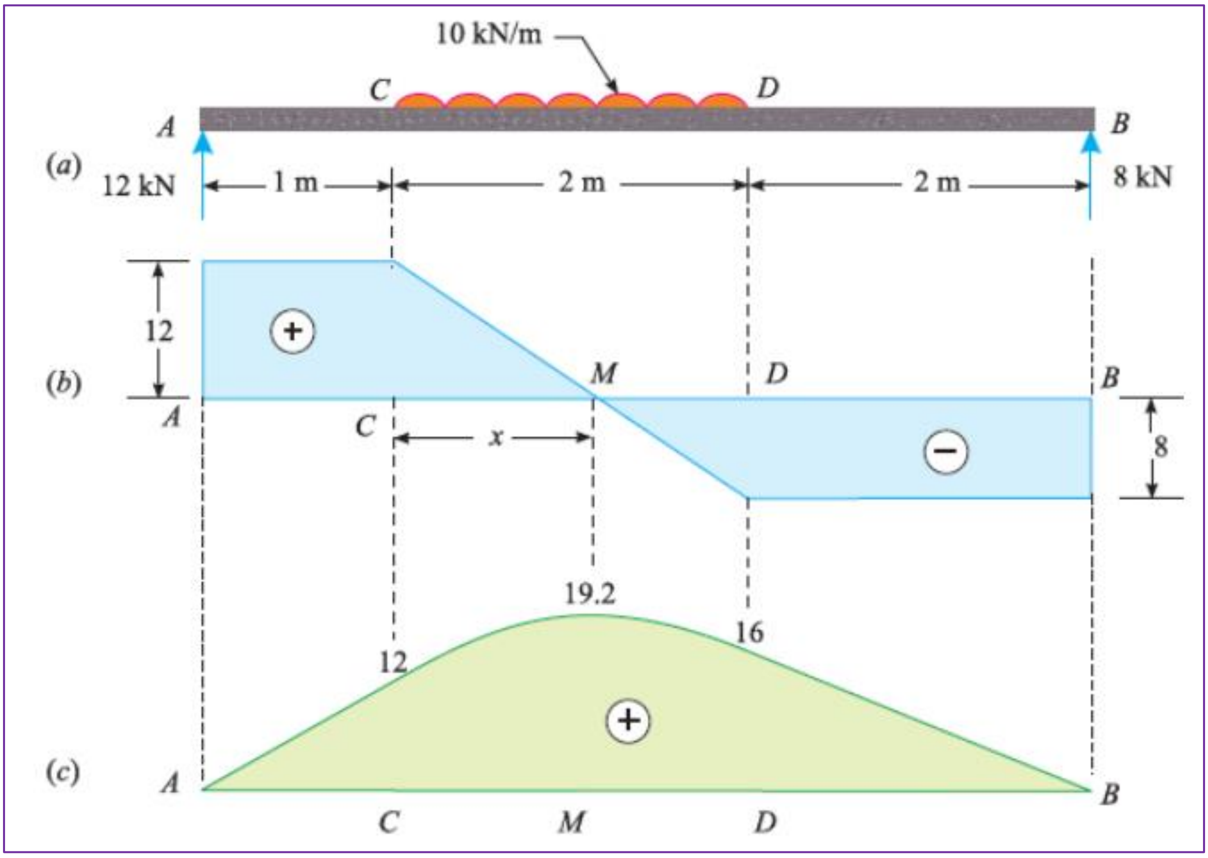
$$8x = 24 - 12x$$

$$x = \frac{24}{20} = 1.2\ m$$

The bending moment: B.M. at point M

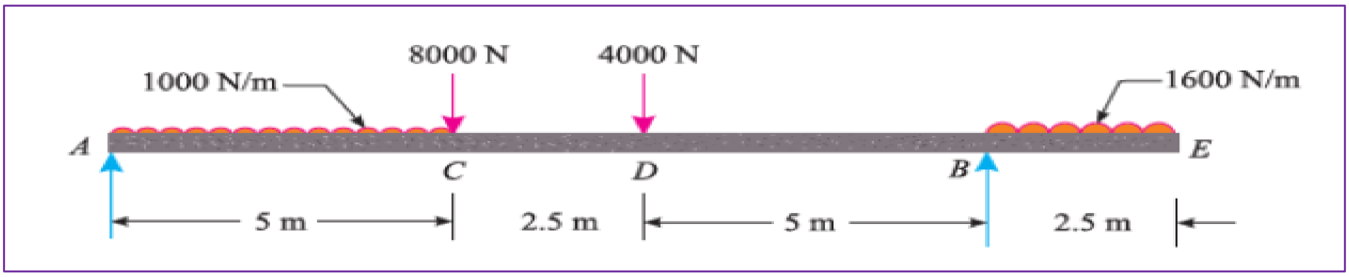
$$B.M.\ at\ point\ B = 12 \times (1 + 1.2) - 10 \times 1.2 \times \frac{1.2}{2} = 19.2\ \text{KN . m}$$





Example 3.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (15 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \boldsymbol{\Sigma M_A = 0}$$

$$1000 \times 5 \times 2.5 + 8000 \times 5 + 400 \times 7.5 - R_B \times 12.5 + 1600 \times 2.5 \times 13.75 = 0$$

$$R_B = \frac{1000 \times 5 \times 2.5 + 8000 \times 5 + 4000 \times 7.5 + 1600 \times 2.5 \times 13.75}{12.5}$$

$$= \frac{137500}{12.5} = \mathbf{11000\ \boldsymbol{N}}$$

$$\boldsymbol{\Sigma F_y = 0}$$

$$R_A - 1000 \times 5 - 8000 - 4000 + R_B - 1600 \times 2.5 = 0$$

$$R_A = 1000 \times 5 + 8000 + 4000 - 11000 + 1600 \times 2.5 = \mathbf{10000\boldsymbol{N}}$$

2. Shear force diagram:

$$\text{S.F. at point A} = 10000\ \text{N}$$

$$\text{S.F. at point C} = 10000 - 1000 \times 5 - 8000 = -3000\ \text{N}$$

$$\text{S.F. at point D} = 10000 - 1000 \times 5 - 8000 - 4000 = -7000\ \text{N}$$

$$\text{S.F. at point B} = 10000 - 1000 \times 5 - 8000 - 4000 + 11000 = 4000\ N$$

$$\text{S.F. at point B} = 10000 - 1000 \times 5 - 8000 - 4000 + 11000 - 1600 \times 2.5$$
$$= 0\ N$$

3. Bending moment:

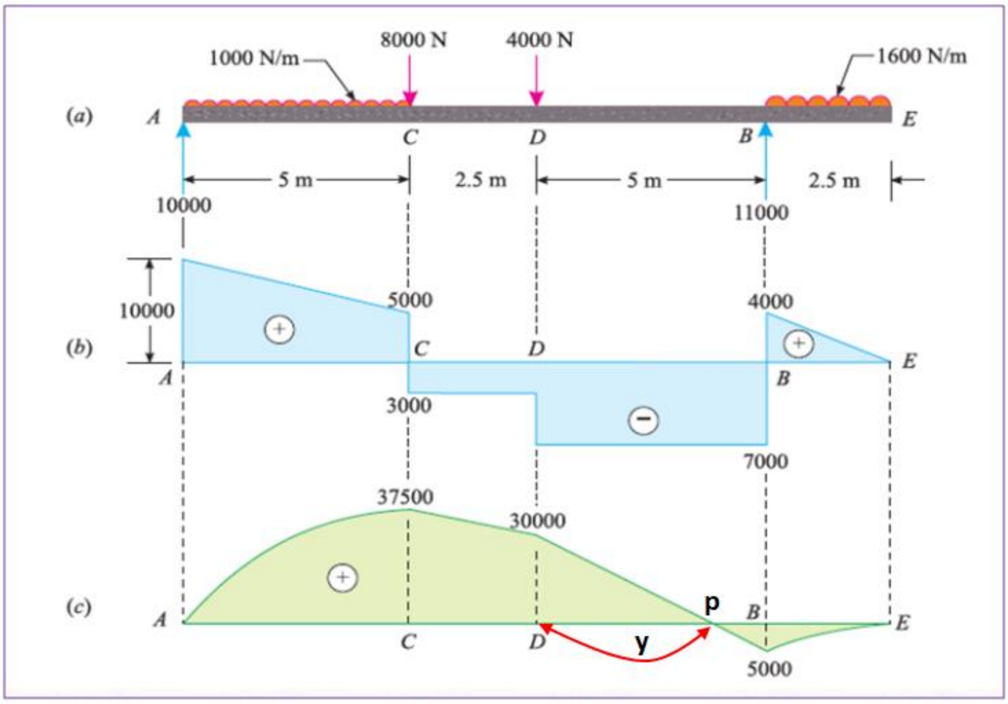
$$B.M.\ at\ point\ A = 0\ \text{KN.m}$$

$$B.M.\ at\ point\ C = 10000 \times 5 - 1000 \times 5 \times 2.5 = 37500\ \text{N.m}$$

$$B.M.\ at\ point\ D = 10000 \times 7.5 - 1000 \times 5 \times 5 - 8000 \times 2.5$$
$$= 30000\ \text{N.m}$$

$$B.M.\ at\ point\ B$$
$$= 10000 \times 12.5 - 1000 \times 5 \times 10 - 8000 \times 7.5 - 4000 \times 5$$
$$= -5000\ \text{N.m}$$

$$B.M.\ at\ point\ E = 0\ \text{N.m}$$

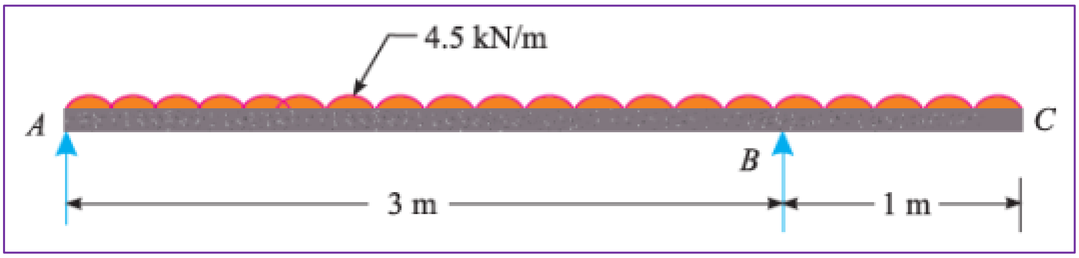


To locate the point (p) of contra flexure:

$$\frac{y}{30000} = \frac{5-y}{5000}$$
$$5000\,y = 150000 - 30000\,y$$
$$y = \frac{150000}{35000} = 4.29\ m$$

Example 4.

Draw shear force and bending moment of a simply supported beam (AB) shown in figure, of span (4 m) is carrying two point loads as.



Solution:

1. Reactions

$$\curvearrowright_{+}\ \mathbf{\Sigma M_A = 0}$$

$$4.5 \times 4 \times 2 - R_B \times 3 = 0$$

$$R_B = \frac{4.5 \times 4 \times 2}{3} = \frac{36}{3} = 12\ KN$$

$$\curvearrowright_{+}\ \mathbf{\Sigma M_B = 0}$$

$$3.5 \times 3 \times 1.5 - R_A \times 3 + 4.5 \times 1 \times 0.5 = 0$$

$$R_A = \frac{3.5 \times 3 \times 1.5 + 4.5 \times 1 \times 0.5}{3} = 6\ KN$$

$$\mathbf{\Sigma F_y = 0}$$

$$R_A - 4.5 \times 4 + R_B = 0$$

$$R_A = 18 - 12 = 6\ KN$$

2. Shear force diagram:

$$\text{S.F. at point A} = 6\ \text{KN}$$

$$\text{S.F. after point B} = 6 - 4.5 \times 3 = -7.5 \text{ KN}$$

$$\text{S.F. at point B} = 6 - 4.5 \times 3 + 12 = 4.5 \text{ KN}$$

$$\text{S.F. at point C} = 6 \times 4 - 4.5 \times 4 + 12 = 0 \text{ KN}$$

3. Bending moment:

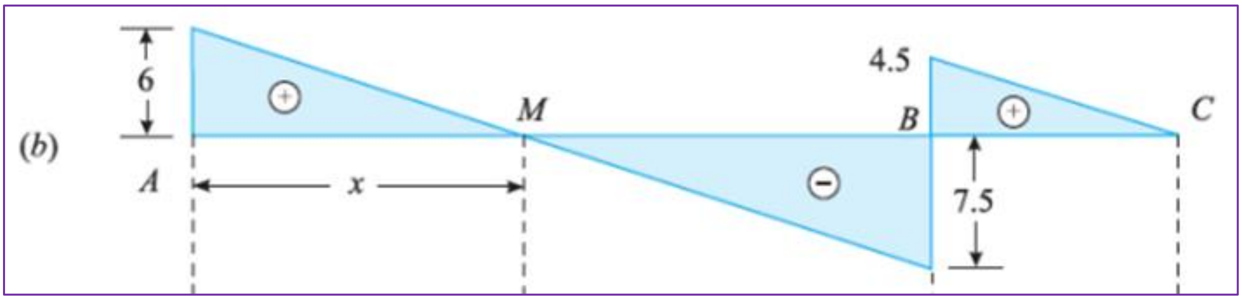
$$B.M. \ at\ point\ A = 0 \text{ KN.m}$$

$$B.M. \ at\ point\ B = 6 \times 3 - 4.5 \times 3 \times 1.5 = -2.25 \text{ KN.m}$$

$$B.M. \ at\ point\ C = 0 \text{ KN.m}$$

We follow the following steps to find bending moment at point (M):

From the figure of shear force:



$$\text{S.F. at point M} = 0 \text{ KN}$$

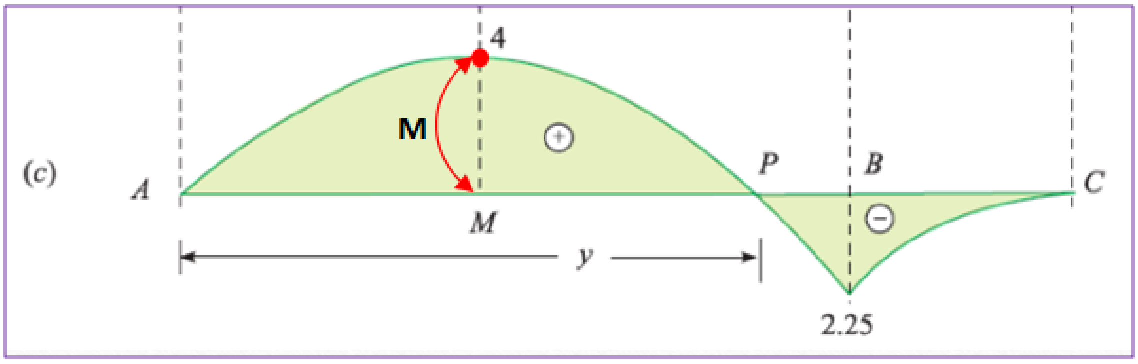
$$\frac{\text{x}}{6} = \frac{3 - x}{7.5}$$

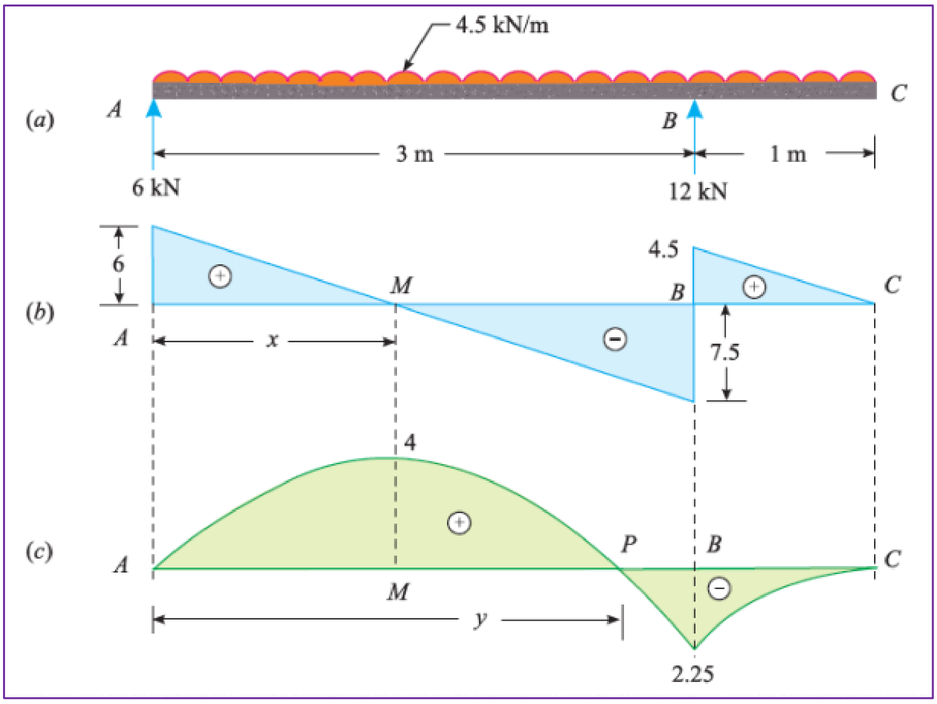
$$7.5\text{ x} = 18 - 6x$$

$$\text{x} = \frac{18}{13.5} = 1.33\ m$$

The bending moment: B.M. at point M

$$B.M. \ at\ point\ B = 6 \times 1.33 - 4.5 \times 1.33 \times \frac{1.33}{2} = 4 \text{ KN.m}$$





To locate the point (p) of contra flexure:

$$\Sigma M_P = 0$$

$$R_A . \mathrm{y} - 4.5 \times \mathrm{y}. \ \frac{y}{2} = 0$$

$$6\mathrm{y} - 2.25\ \mathrm{y}^2 = 0$$

$$\mathrm{y}\ (6 - 2.25\ \mathrm{y}) = 0$$

$$\mathrm{y} = \frac{6}{2.25} = 2.67\ m$$

Chapter 8

Deflection

9. Deflection of Beams

9.1. General Theory

When a beam bends it takes up various shapes such as that illustrated in figure 1. The shape may be superimposed on an x – y graph with the origin at the left end of the beam (before it is loaded). At any distance x metres from the left end, the beam will have a deflection y and a gradient or slope dy/dx and it is these that we are concerned with in this tutorial.

We have already examined the equation relating bending moment and radius of curvature in a beam, namely $\frac{M}{I} = \frac{E}{R}$

M is the bending moment.
I is the second moment of area about the centroid.
E is the modulus of elasticity and
R is the radius of curvature.

Rearranging we have $\frac{1}{R} = \frac{M}{EI}$

Figure 1 illustrates the radius of curvature which is defined as the radius of a circle that has a tangent the same as the point on the x-y graph.

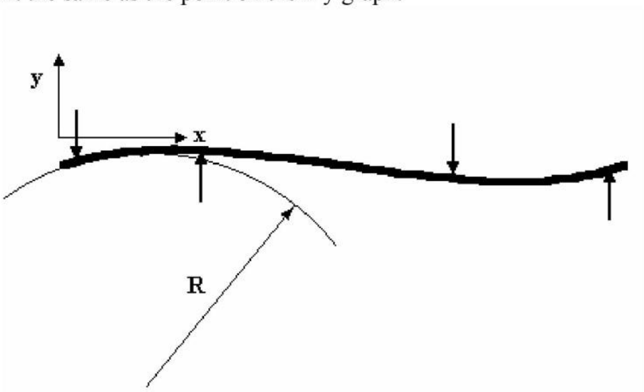


Figure 1

Mathematically it can be shown that any curve plotted on x - y graph has a radius of curvature of defined as

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \frac{dy}{dx}\right]^{\frac{3}{2}}}$$

In beams, R is very large and the equation may be simplified without loss of accuracy to

$$\frac{1}{R} = \frac{d^2x}{dy^2}$$

hence

$$\frac{d^2x}{dy^2} = \frac{M}{EI}$$

or

$$M = EI\frac{d^2x}{dy^2}............(1A)$$

The product EI is called the flexural stiffness of the beam.

In order to solve the slope (dy/dx) or the deflection (y) at any point on the beam, an equation for M in terms of position x must be substituted into equation (1A). We will now examine this for the 4 standard cases.

9.2. Case I: Cantilever with Point Load at Free End

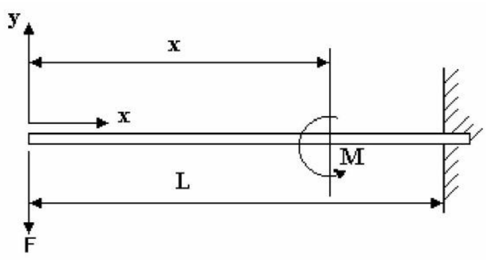


Figure 2

The bending moment at any position x is simply -Fx. Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = -Fx$$

Integrate wrtx and we get $EI\frac{dy}{dx} = -\frac{Fx^2}{2} + A$..................(2A)

Integrate again and we get $EIy = -\frac{Fx^3}{6} + Ax + B$................(2B)

A and B are constants of integration and must be found from the boundary conditions.

These are at $x = L$, $y = 0$ (no deflection)
at $x = L$, $dy/dx = 0$ (gradient horizontal)

Substitute $x = L$ and $dy/dx = 0$ in equation 2A. This gives

$EI(0) = -\frac{FL^2}{2} + A$ hence $A = \frac{FL^2}{2}$

substitute $A = \frac{FL^2}{2}$, $y = 0$ and $x = L$ nto equation 2B and we get

$EI(0) = -\frac{FL^3}{6} + \frac{FL^3}{2} + B$ hence $B = -\frac{FL^3}{3}$

substitute $A = \frac{FL^2}{2}$ and $B = -\frac{FL^3}{3}$ into equations 2A and 2B and the complete equations are

$EI\frac{dy}{dx} = -\frac{Fx^2}{2} + \frac{FL^2}{2}$..................(2C)

$EIy = -\frac{Fx^3}{6} + \frac{FL^2x}{2} - \frac{FL^3}{3}$..................(2D)

The main point of interest is the slope and deflection at the free end where x=0. Substituting x= 0 into (2C) and (2D) gives the standard equations.

Slope at free end $\frac{dy}{dx} = \frac{FL^2}{2EI}$..................(2E)

Deflection at free end $y = -\frac{FL^3}{3EI}$..................(2F)

WORKED EXAMPLE No.1

A cantilever beam is 4 m long and has a point load of 5 kN at the free end. The flexural stiffness is 53.3 MNm^2. Calculate the slope and deflection at the free end.

SOLUTION

i. Slope

Using formula 2E we have $\frac{dy}{dx} = \frac{FL^2}{2EI} = \frac{5000 \times 4^2}{2 \times 53.3 \times 10^6} = 750 \times 10^{-6}$ (no units)

ii. Deflection

Using formula 2F we have $y = -\frac{FL^3}{3EI} = -\frac{5000 \times 4^3}{3 \times 53.3 \times 10^6} = -0.002$ m

The deflection is 2 mm downwards.

SELF ASSESSMENT EXERCISE No.1

1. A cantilever beam is 6 m long and has a point load of 20 kN at the free end. The flexural stiffness is 110 MNm^2. Calculate the slope and deflection at the free end. (Answers 0.00327 and -13 mm).

2. A cantilever beam is 5 m long and has a point load of 50 kN at the free end. The deflection at the free end is 3 mm downwards. The modulus of elasticity is 205 GPa. The beam has a solid rectangular section with a depth 3 times the width. (D= 3B). Determine

   i. the flexural stiffness. (694.4 MNm^2)

   ii. the dimensions of the section. (197 mm wide and 591 mm deep).

9.3. Case II: Cantilever with Uniformly Distributed Load

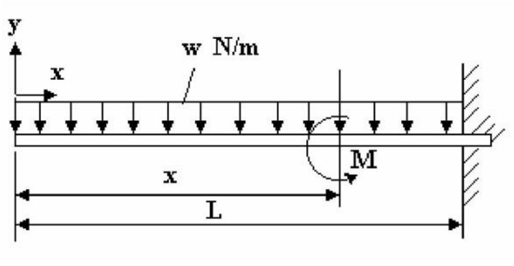


Figure 3

The bending moment at position x is given by M = $-wx^2/2$. Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = -w\frac{x^2}{2}$$

Integrate wrtx and we get $$EI\frac{dy}{dx} = -\frac{wx^3}{6} + A \text{..................(3A)}$$

Integrate again and we get $$EIy = -\frac{wx^4}{24} + Ax + B \text{................(3B)}$$

A and B are constants of integration and must be found from the boundary conditions. These are

at x = L , y = 0 (no deflection)
at x = L, dy/dx = 0 (horizontal)

Substitute x = L and dy/dx = 0 in equation 3A and we get

$$EI(0) = -\frac{wL^3}{6} + A \quad \text{hence} \quad A = \frac{wL^3}{6}$$

Substitute this into equation 3B with the known solution y = 0 and x = L results in

$$EI(0) = -\frac{wL^4}{24} + \frac{wL^4}{6} + B \quad \text{hence } B = -\frac{wL^4}{8}$$

Putting the results for A and B into equations 3A and 3B yields the complete equations

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wL^3}{6} \text{..................(3C)}$$

$$EIy = -\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8} \text{..................(3D)}$$

The main point of interest is the slope and deflection at the free end where x=0. Substituting x= 0 into (3C) and (3D) gives the standard equations.

Slope at free end $$\frac{dy}{dx} = \frac{wL^3}{6EI} \text{..................(3E)}$$

Deflection at free end $$y = -\frac{wL^4}{8EI} \text{..................(3F)}$$

WORKED EXAMPLE No.2

A cantilever beam is 4 m long and has a u.d.l. of 300 N/m. The flexural stiffness is 60 MNm^2. Calculate the slope and deflection at the free end.

SOLUTION

i. Slope

From equation 3E we have $\frac{dy}{dx} = \frac{wL^3}{6EI} = \frac{300 \times 4^3}{6 \times 60 \times 10^6} = 53.3 \times 10^{-6}$ (no units)

ii. Deflection

From equation 3F we have $y = -\frac{wL^4}{8EI} = -\frac{300 \times 4^4}{8 \times 60 \times 10^6} = -0.00016$ m

Deflection is 0.16 mm downwards.

SELF ASSESSMENT EXERCISE No.2

1. A cantilever is 6 m long with a u.d.l. of 1 kN/m. The flexural stiffness is 100 MNm^2. Calculate the slope and deflection at the free end.
   (360×10^{-6} and -1.62 mm)

2. A cantilever beam is 5 m long and carries a u.d.l. of 8 kN/m. The modulus of elasticity is 205 GPa and beam is a solid circular section. Calculate

   i. the flexural stiffness which limits the deflection to 3 mm at the free end.
   (208.3 MNm^2).

   ii. the diameter of the beam. (379 mm).

9.4. Case III: Simply Supported Beam With Point In Middle

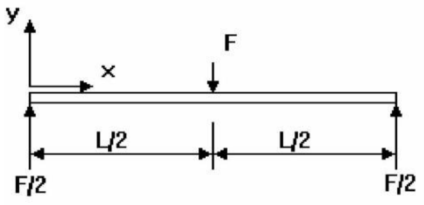


Figure 4

The beam is symmetrical so the reactions are F/2. The bending moment equation will change at the centre position but because the bending will be symmetrical each side of the centre we need only solve for the left hand side.

The bending moment at position x up to the middle is given by M = Fx/2. Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = \frac{Fx}{2}$$

Integrate wrt x once $\quad EI\frac{dy}{dx} = \frac{Fx^2}{4} + A$..................(4A)

Integrate wrt x again $\quad EIy = \frac{Fx^3}{12} + Ax + B$..............(4B)

A and B are constants of integration and must be found from the boundary conditions. These are

at $x = 0$, $y = 0$ (no deflection at the ends)
at $x = L/2$, $dy/dx = 0$ (horizontal at the middle)

putting $x = L/2$ and $dy/dx = 0$ in equation 4A results in

$EI(0) = \frac{FL^2}{16} + A \qquad$ hence $A = -\frac{FL^2}{16}$

substitute $A = -\frac{FL^2}{16}$, $y = 0$ and $x = 0$ nto equation 4B and we get

$EI(0) = B \qquad$ hence $B = 0$

substitute $A = -\frac{FL^2}{16}$ and $B = 0$ into equations 4A and 4B and the complete equations are

$EI\frac{dy}{dx} = \frac{Fx^2}{4} - \frac{FL^2}{16}$..................(4C)

$EIy = \frac{Fx^3}{12} - \frac{FL^3x}{16}$..................(4D)

The main point of interest is the slope at the ends and the deflection at the middle . Substituting x = 0 into (4C) gives the standard equation for the slope at the left end. The slope at the right end will be equal but of opposite sign.

Slope at ends $\qquad \frac{dy}{dx} = \pm\frac{FL^2}{16EI}$........(4E)

The slope is negative on the left end but will be positive on the right end.
Substituting x = L/2 into equation 4D gives the standard equation for the deflection at the middle:

Deflection at middle $\qquad y = -\frac{FL^3}{48EI}$...................................(4F)

WORKED EXAMPLE No.3

A simply supported beam is 8 m long with a load of 500 kN at the middle. The deflection at the middle is 2 mm downwards. Calculate the gradient at the ends.

SOLUTION

From equation 4F we have

$$y = -\frac{FL^3}{48EI} \text{ and y is 2 mm down so } y = -0.002 \text{ m}$$

$$-0.002 = -\frac{500 \times 8^2}{48EI}$$

$$EI = 2.667 \times 10^9 \text{ Nm}^2 \quad \text{or} \quad 2.667 \text{ GNm}^2$$

From equation 4E we have

$$\frac{dy}{dx} = \pm\frac{FL^2}{16EI} = \pm\frac{500\,000 \times 8^2}{16 \times 2.667 \times 10^9} = 750 \times 10^{-6} \text{ (no units)}$$

The gradient will be negative at the left end and positive at the right end.

SELF ASSESSMENT EXERCISE No.3

1. A simply supported beam is 4 m long and has a load of 200 kN at the middle. The flexural stiffness is 300 MNm2. Calculate the slope at the ends and the deflection at the middle.
(0.000667 and -0.89 mm).

2. A simply supported beam is made from a hollow tube 80 mm outer diameter and 40 mm inner diameter. It is simply supported over a span of 6 m. A point load of 900 N is placed at the middle. Find the deflection at the middle if E=200 GPa.
(-10.7 mm).

3. Find the flexural stiffness of a simply supported beam which limits the deflection to 1 mm at the middle. The span is 2 m and the point load is 200 kN at the middle.
(33.3 MNm2).

9.5. Case IV: SIMPLY SUPPORTED BEAM WITH A UNIFORMILY DISTRIBUTED LOAD

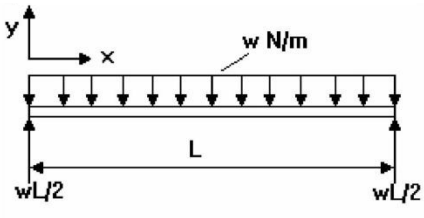


Figure 5

The beam is symmetrical so the reactions are wL/2. The bending moment at position x is

$$M = \frac{wLx}{2} - \frac{wx^2}{2}$$

Substituting this into equation 1A we have

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

Integrate wrt x once $\quad EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A$..................(5A)

Integrate wrt x again $\quad EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B$..............(5B)

A and B are constants of integration and must be found from the boundary conditions. These are at $x = 0$, $y = 0$ (no deflection at the ends)

at $x = L/2$, $dy/dx = 0$ (horizontal at the middle)

Putting $x = L/2$ and $dy/dx = 0$ in equation 5A results in

$EI(0) = \frac{wL^3}{16} - \frac{wL^3}{48} + A \qquad$ hence $A = -\frac{wL^3}{24}$

substitute $A = -\frac{wL^3}{24}$, $y = 0$ and $x = 0$ nto equation 5B and we get

$EI(0) = B \qquad$ hence $B = 0$

substitute $A = -\frac{wL^3}{24}$ and $B = 0$ into equations 5A and 5B and the complete equations are

$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^2}{24}$..................(4C)

$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{wL^3x}{24}$..................(4D)

The main point of interest is the slope at the ends and the deflection at the middle.

Substituting x = 0 into (5C) gives the standard equation for the slope at the left end. The slope at the right end will be equal but of opposite sign.

Slope at free end $\quad \frac{dy}{dx} = \pm\frac{wL^3}{24EI}$..............(5E)

The slope is negative on the left end but will be positive on the right end.

Substituting x= L/2 into equation 5D gives the standard equation for the deflection at the middle:

Deflection at middle $\quad y = -\frac{5wL^4}{384EI}$................(5F)

WORKED EXAMPLE No.4

A simply supported beam is 8 m long with a u.d.l. of 5000 N/m. Calculate the flexuralstiffness which limits the deflection to 2 mm at the middle. Calculate the gradient at the ends.

SOLUTION

Putting y = -0.002 m into equation 5F we have

$$y = -\frac{5wL^4}{384EI}$$

$$-0.002 = -\frac{5 \times 5000 \times 4^4}{384 \times EI} \qquad EI = 133.3 \times 10^6\ Nm^2 \text{ or } 133.3\ MNm^2$$

From equation 5E we have

$$\frac{dy}{dx} = \pm\frac{5000 \times 8^3}{24 \times 133.3 \times 10^6} = \pm 800 \times 10^{-6} \text{ (no units)}$$

The gradient will be negative at the left end and positive at the right end.

SELF ASSESSMENT EXERCISE No.4

1. A simply supported beam is 4 m long with a u.d.l. of 200 N/m. The flexural stiffness is 100 MNm^2. Calculate the slope at the ends and the deflection at the middle.
(5.33×10^{-6}) and -6.67×10^{-6} m).

2. A simply supported beam is made from a hollow tube 80 mm outer diameter and 40 mm inner diameter. It is simply supported over a span of 6 m. The density of the metal is 7300 kg/m^3. E=200 GPa. Calculate the deflection at the middle due to the weight of the beam.
(-12 mm)

3. Find the flexural stiffness of a simply supported beam which limits the deflection to 1 mm at the middle. The span is 2 m and the u.d.l. is 400 N/m.
(83.3 kNm^2)

9.6. The Theory Of Superposition For Combined Loads

This theory states that the slope and deflection of a beam at any point is the sum of the slopes and deflections which would be produced by each load acting on its own. For beams with combinations of loads which are standard cases we only need to use the standard formulae. This is best explained with a worked example.

WORKED EXAMPLE No.5

A cantilever beam is 4m long with a flexural stiffness of 20 MNm2. It has a point load of 1 kN at the free end and a u.d.l. of 300 N/m along its entire length. Calculate the slope and deflection at the free end.

SOLUTION

For the point load only

$$y = -\frac{FL^3}{3EI} = -\frac{1000 \times 4^3}{3 \times 20 \times 10^6} = -0.00106 \text{ m or } -1.06 \text{ mm}$$

For the u.d.l. only

$$y = -\frac{wL^4}{48EI} = -\frac{300 \times 4^4}{8 \times 20 \times 10^6} = -0.00048 \text{ m or } -0.48 \text{ mm}$$

The total deflection is hence **y= - 1.54 mm.**

For the point load only

$$\frac{dy}{dx} = \frac{FL^2}{2EI} = \frac{1000 \times 4^2}{2 \times 20 \times 10^6} = 400 \times 10^{-6}$$

For the u.d.l. only

$$\frac{dy}{dx} = \frac{wL^3}{6EI} = \frac{300 \times 4^3}{6 \times 20 \times 10^6} = 160 \times 10^{-6}$$

The total slope is hence **dy/dx =560 x 10-6.**

9.7. Macaulay's Method

When the loads on a beam do not conform to standard cases, the solution for slope and deflection must be found from first principles. Macaulay developed a method for making the integrations simpler.

The basic equation governing the slope and deflection of beams is

$$EI\frac{d^2y}{dx^2} = M \text{ Where M is a function of x.}$$

When a beam has a variety of loads it is difficult to apply this theory because some loads may be within the limits of x during the derivation but not during the solution at a particular point. Macaulay's method makes it possible to do the integration necessary by placing all the terms containing x within a square bracket and **integrating the bracket, not x.** During evaluation, any bracket with a negative value is ignored because a negative value means that the load it refers to is not within the limit of x. The general method of solution is conducted as follows. Refer to figure 6. In a real example, the loads and reactions would have numerical values but for the sake of demonstrating the general method we will use algebraic symbols. This example has only point loads.

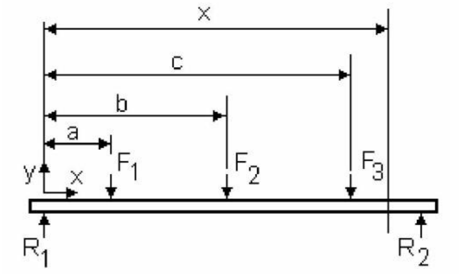


Figure 6

1. Write down the bending moment equation placing x on the extreme right hand end of the beam so that it contains all the loads. write all terms containing x in a square bracket.

$$EI\frac{d^2y}{dx^2} = M = R_1[x] - F_1[x-a] - F_2[x-b] - F_3[x-c]$$

2. Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = R_1\frac{[x]^2}{2} - F_1\frac{[x-a]^2}{2} - F_2\frac{[x-b]^2}{2} - F_3\frac{[x-c]^2}{2} + A$$

3. Integrate again using the same rules.

$$EIy = R_1\frac{[x]^3}{6} - F_1\frac{[x-a]^3}{6} - F_2\frac{[x-b]^3}{6} - F_3\frac{[x-c]^3}{6} + Ax + B$$

4. Use boundary conditions to solve A and B.

5. Solve slope and deflection by putting in appropriate value of x. IGNORE any brackets containing negative values.

WORKED EXAMPLE No.6

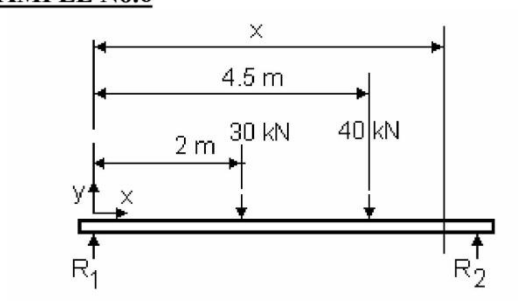


Figure 7

The beam shown is 7 m long with an E I value of 200 MNm^2. Determine the slope and deflection at the middle.

SOLUTION

First solve the reactions by taking moments about the right end.

30 x 5 + 40 x 2.5 = 7 R_1 hence R_1 = 35.71 kN

R_2 = 70 - 35.71 = 34.29 kN

Next write out the bending equation.

$$EI\frac{d^2y}{dx^2} = M = 35710[x] - 30000[x-2] - 40000[x-4.5]$$

Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = 35710\frac{[x]^2}{2} - 30000\frac{[x-2]^2}{2} - 40000\frac{[x-4.5]^2}{2} + A \text{.....(1)}$$

Integrate again

$$EIy = 35710\frac{[x]^3}{6} - 30000\frac{[x-2]^3}{6} - 40000\frac{[x-4.5]^3}{6} + Ax + B \text{......(2)}$$

BOUNDARY CONDITIONS

$x = 0, y = 0$ and $x = 7$ $y = 0$

Using equation 2 and putting x = 0 and y = 0 we get

$$EI(0) = 35710\frac{[0]^3}{6} - 30000\frac{[0-2]^3}{6} - 40000\frac{[0-4.5]^3}{6} + A(0) + B$$

Ignore any bracket containing a negative value.

$0 = 0 - 0 - 0 + 0 + B$ hence $B = 0$

Using equation 2 again but this time x=7 and y = 0

$$EI(0) = 35710\frac{[7]^3}{6} - 30000\frac{[7-2]^3}{6} - 40000\frac{[7-4.5]^3}{6} + A(7) + 0$$

Evaluate A and $A = -187400$

Now use equations 1 and 2 with x = 3.5 to find the slope and deflection at the middle.

$$EI\frac{dy}{dx} = 35710\frac{[3.5]^2}{2} - 30000\frac{[3.5-2]^2}{2} - 40000\frac{[3.5-4.5]^2}{2} - 187400$$

The last bracket is negative so ignore by putting in zero

$$200x10^6\frac{dy}{dx} = 35710\frac{[3.5]^2}{2} - 30000\frac{[3.5-2]^2}{2} - 40000\frac{[0]^2}{2} - 187400$$

$$200x10^6\frac{dy}{dx} = 218724 - 33750 - 187400 = -2426$$

$$\frac{dy}{dx} = \frac{-2426}{200x10^6} - 0.00001213$$ and this is the slope at the middle.

$$EIy = 35710\frac{[3.5]^3}{6} - 30000\frac{[3.5-2]^3}{6} - 40000\frac{[3.5-4.5]^3}{6} - 187400[3.5]$$

$$200x10^6\,y = 255178 - 16875 - 0 - 655900 = -417598$$

$$y = \frac{-417598}{200x10^6} - 0.00209 \text{ m or } 2.09 \text{ mm}$$

WORKED EXAMPLE No.7

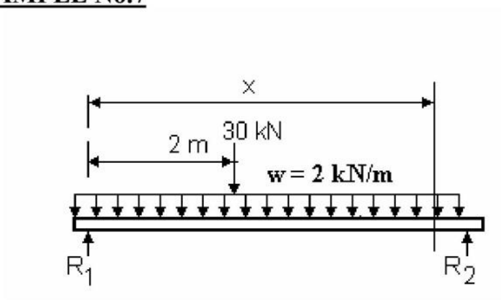


Figure 8

The beam shown is 6 m long with an E I value of 300 MNm^2. Determine the slope at the left end and the deflection at the middle.

SOLUTION

First solve the reactions by taking moments about the right end.

$30 \times 4 + 2 \times 6^2/2 = 6\ R_1 = 156$ hence $R_1 = 26$ kN

Total downwards load is 30 + (6 x 2) = 42 kN

$R_2 = 42 - 26 = 16$ kN

Next write out the bending equation.

$$EI\frac{d^2y}{dx^2} = M = R_1[x] - 30000[x-2] - \frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = 26000[x] - 30000[x-2] - \frac{2000x^2}{2}$$

Integrate once treating the square bracket as the variable.

$$EI\frac{dy}{dx} = 26000\frac{[x]^2}{2} - 30000\frac{[x-2]^2}{2} - \frac{2000[x]^3}{6} + A \ .....(1)$$

Integrate again

$$EIy = 26000\frac{[x]^3}{6} - 30000\frac{[x-2]^3}{6} - \frac{2000[x]^4}{24} + Ax + B \ ......(2)$$

BOUNDARY CONDITIONS

$x = 0, y = 0$ and $x = 6$ $y = 0$

Using equation 2 and putting x = 0 and y = 0 we get

$$EI(0) = 26000\frac{[0]^3}{6} - 30000\frac{[0-2]^3}{6} - \frac{2000[0]^4}{24} + A(0) + B$$

Ignore any bracket containing a negative value.

$0 = 0 - 0 - 0 + 0 + B$ hence $B = 0$

Using equation 2 again but this time x = 6 and y = 0

$$EI(0) = 26000\frac{[6]^3}{6} - 30000\frac{[6-2]^3}{6} - \frac{2000[6]^4}{24} + A(6) + 0$$

$$EI(0) = 936000 - 320000 - 108000 + (6) + 6A$$

$$6A = -508000$$

$$A = -84557$$

Now use equations 1 with x = 0 to find the slope at the left end.

$$EI\frac{dy}{dx} = 260000\frac{[0]^2}{2} - 30000\frac{[0-2]^2}{2} - 2000\frac{[0]^3}{6} - 84557$$

Negative brackets are made zero

$$300 \text{ x}10^6 \frac{dy}{dx} = -84557$$

$$\frac{dy}{dx} = \frac{-84557}{300\text{x}10^6} - 0.000282$$ and this is the slope at the left end.

Now use equations 2 with x = 3 to find the deflection at the middle.

$$EIy = 26000\frac{[3]^3}{6} - 30000\frac{[3.5-2]^3}{6} - \frac{2000[3]^4}{24} - 84557[3]$$

$$300\text{x}10^6 y = 117000 - 16875 - 6750 - 253671 = -160296$$

$$y = \frac{-160296}{300\text{x}10^6} - 0.000534 \text{ m or } 0.534 \text{ mm}$$

SELF ASSESSMENT EXERCISE No.5

1. Find the deflection at the centre of the beam shown. The flexural stiffness is 20 MNm^2. **(0.064 mm)**

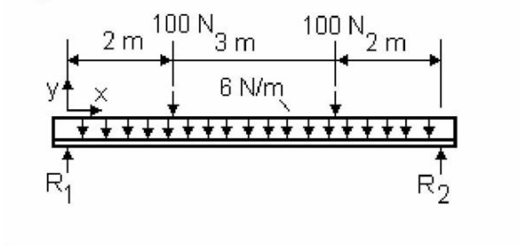


Figure 9

2. Find the deflection of the beam shown at the centre position. The flexural stiffness is 18 MNm^2. **(1.6 mm)**

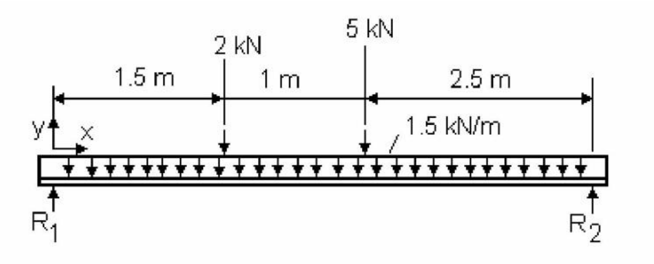


Figure 10

3. Find value of E I which limits the deflection of the beam shown at the end to 2 mm. **(901800 Nm^2)**

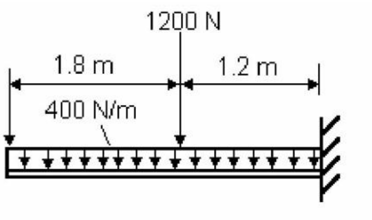


Figure 11

4. A cantilever is 5m long and has a flexural stiffness of 25 MNm^2. It carries a point load of 1.5 kN at the free end and a u.d.l. of 500 N/m along its entire length. Calculate the deflection and slope at the free end.
   (-4.06 mm and 1.167×10^{-3})

5. A cantilever beam is 6 m long and has a point load of 800 N at the free end and a u.d.l. of 400 N/m along its entire length. Calculate the flexural stiffness if the deflection is 1.5 mm downwards at the free end.
   (81.6 MNm^2).

6. A simply supported beam is 6 m long and has a flexural stiffness of 3 MNm^2. It carries a point load of 800 N at the middle and a u.d.l. of 400 N/m along its entire length. Calculate the slope at the ends and the deflection at the middle.
   (1.8 x 10^{-3} and 3.45 mm).

7. Calculate the flexural stiffness of a simply supported beam which will limit the deflection to 2 mm at the middle. The beam is 5 m long and has a point load of 1.2 kN at the middle and a u.d.l. of 600 N/m along its entire length.
   (4 MNm^2).

   The beam has a solid rectangular section twice as deep as it is wide. Given the modulus of elasticity is 120 GPa, calculate the dimensions of the section.
   (168 mm x 84 mm).

9.8. Encastre Beams

An encastré beam is one that is built in at both ends. As with a cantilever, there must be a bending moment and reaction force at the wall. In this analysis it is assumed that

- there is no deflection at the ends.
- the ends are horizontal.
- the beam is free to move horizontally.

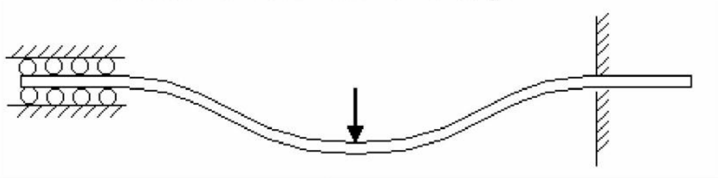


Figure 12

First let us consider two standard cases, one with a point load at the middle and one with a uniformly distributed load. In both cases there will be a reaction force and a fixing moment at both ends. We shall use Macaulay's method to solve the slope and deflection.

9.8.1. Point Load

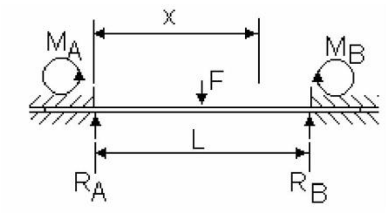


Figure 13

In this case $R_A = R_B = F/2$

The bending moment at distance x from the left end is :

$$M = EI\frac{d^2y}{dx^2} = R_A x - F\left[x - \frac{L}{2}\right] + M_A$$

$$EI\frac{d^2y}{dx^2} = F\frac{x}{2} - F\left[x - \frac{L}{2}\right] + M_A$$

Integrate $$EI\frac{dy}{dx} = F\frac{x^2}{4} - \frac{F\left[x - \frac{L}{2}\right]^2}{2} + M_A x + A$$

Since the slope is zero at both ends it then putting $\frac{dy}{dx} = 0$ and $x = 0$ yields that $A = 0$

$$EI\frac{dy}{dx} = F\frac{x^2}{4} - \frac{F\left[x - \frac{L}{2}\right]^2}{2} + M_A x \text{..............(1)}$$

Integrate again $$EIy = F\frac{x^3}{12} - \frac{F\left[x - \frac{L}{2}\right]^3}{6} + M_A\frac{x^2}{2} + B$$

Since the deflection is zero at both ends then putting $y = 0$ and $x = 0$ yields that $B = 0$

$$EIy = F\frac{x^3}{12} - \frac{F\left[x - \frac{L}{2}\right]^3}{6} + M_A\frac{x^2}{2} \text{..............(2)}$$

The constants of integration A and B are always zero for an encastré beam but the problem is not made easy because we now have to find the fixing moment M.

Equations 1 and 2 give the slope and deflection. Before they can be solved, the fixing moment must be found by using another boundary condition. Remember the slope and deflection are both zero at both ends of the beam so we have two more boundary conditions to use. A suitable condition is that y = 0 at x = L. From equation 2 this yields

$$EI(0) = \frac{FL^3}{12} - \frac{F\left[L - \frac{L}{2}\right]^3}{6} + \frac{M_A L^2}{2}$$

$$0 = \frac{FL^3}{12} - \frac{F\left[\frac{L}{2}\right]^3}{6} + \frac{M_A L^2}{2}$$

$$0 = \frac{FL^3}{12} - \frac{FL^3}{48} + \frac{M_A L^2}{2}$$

$$0 = \frac{3FL^3}{148} + \frac{M_A L^2}{2}$$

$$\frac{M_A L^2}{2} = -\frac{3FL^3}{148}$$

$$M_A = -\frac{FL}{8}$$

If we substitute x = L/2 and M_A = -FL/8 the slope and deflection at the middle from equations 1 and 2 becomes :

$$\frac{dy}{dx} = 0 \qquad y = -\frac{FL^3}{192EI}$$

9.8.2. Uniformly Distributed Load

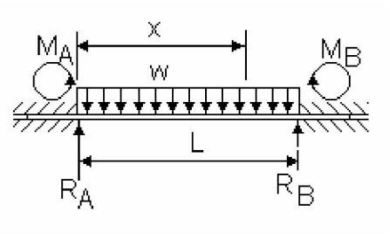


Figure 14

In this case $R_A = R_B = wL/2$

The bending moment at distance x from the left end is :

$$M = EI\frac{d^2y}{dx^2} = R_A x - \frac{wx^2}{2} + M_A$$

$$EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2} + M_A$$

Integrate $\quad EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + M_A x + A$

Since the slope is zero at both ends it then putting $\frac{dy}{dx} = 0$ and $x = 0$ yields that $A = 0$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + M_A x \text{.............(1)}$$

Integrate again $\quad EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{M_A x^2}{2} + B$

Since the deflection is zero at both ends then putting $y = 0$ and $x = 0$ yields that $B = 0$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{M_A x^2}{2} \text{.............(2)}$$

As in the other case, A and B are zero but we must find the fixing moment by using the other boundary condition of $y = 0$ when $x = L$

$$EI(0) = \frac{wL^4}{12} - \frac{wL^4}{24} + \frac{M_A L^2}{2}$$

$$0 = \frac{wL^4}{24} + \frac{M_A L^2}{2}$$

$$M_A = -\frac{wL^2}{12}$$

If we substitute $x = L/2$ and $M_A = -wL^2/12$ into equations 1 and 2 we get the slope and deflection at the middle to be

$$\frac{dy}{dx} = 0 \text{ and } y = -\frac{wL^4}{384EI}$$

The same approach may be used when there is a combination of point and uniform loads.

SELF ASSESSMENT EXERCISE No. 6

1. Solve the value of EI which limits the deflection under the load to 0.05 mm.
   (Ans. 1.53 GNm^2)

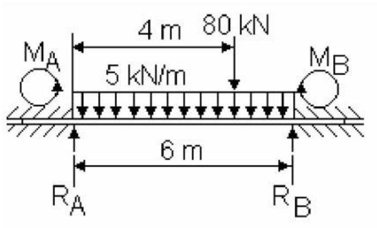


Figure 15

2. Solve the value of EI which limits the deflection under at the middle to 0.2 mm.
   (Ans. 11 MNm^2)

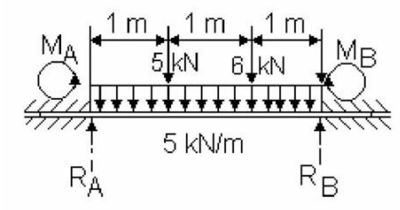


Figure 16

Chapter 10

Theories of Failures

Chapter 10

Theories of Failure

Theories of failure, also called failure criteria, attempt to answer the question: *Can data obtained from a uniaxial tension or compression test be used to predict failure under more complex loadings?* By failure we mean either yielding (resulting in excessive permanent deformation) or actual rupture, whichever occurs first. Failure resulting from local buckling or elastic instability is not considered here.

In general, there are two groups of failure criteria: one for brittle materials that fail by rupture, and the other for ductile materials that exhibit yielding. In this chapter, we consider only failure criteria for plane stress; that is, we assume that $\sigma_3 = 0$.

10.1 The maximum stress theory

The maximum stress theory (sometimes call the maximum principal stress theory) proposed by W.J.M. Rankine (1820-1872) is the oldest as well as the simplest of all the theories. It is based on the assumption that a material fails by fracturing when the largest principal stress exceeds a limiting value, the limit being the yield point in a simple tension test (or ultimate strength, σ_u , if the material is brittle). Although the maximum tensile or compressive stress alone is not sufficient to define yielding, this theory does give results that agree well with test results from brittle materials, such as cast iron or concrete.

$$\frac{\sigma_1}{\sigma_{ult}} = \pm 1 \quad \text{or} \quad \frac{\sigma_2}{\sigma_{ult}} = \pm 1 \qquad (10.1)$$

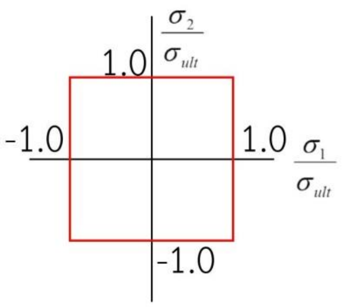
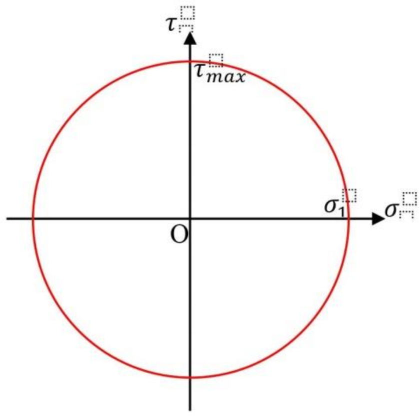
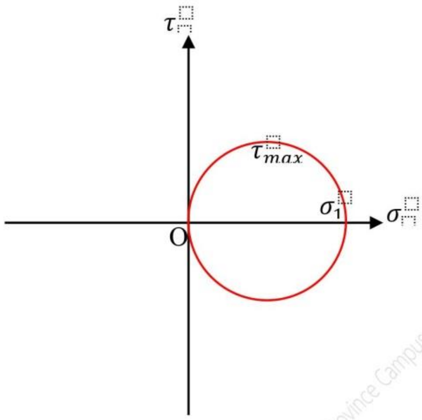


Figure 10.1 Fracture criterion based on maximum stress.



(a) Pure shear



(b) Pure tension

Figure 10.2 Mohr's circle of material element

Although the maximum principal stresses are equal in Figure 10.2(a) and (b), (a) has twice the shearing stress of (b).

10.2 The Maximum Strain Theory

According to the maximum strain theory, which is credited to B. de Saint Venant, a ductile material begins to yield when the maximum principal strain reaches the strain at which yielding occurs in simple tension or when the minimum principal strain (i.e. the compressive strain) equals the yield point strain in simple compression.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \tag{10.2}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right] \tag{10.3}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \tag{10.4}$$

We see that when $\sigma_x = -\sigma_y = -\sigma_z$, the maximum strain is $\frac{\sigma}{E}[1+2\nu]$. Vice versa, when $-\sigma_x = -\sigma_y = -\sigma_z$, as in hydrostatic compression, the maximum strain is $\frac{\sigma}{E}[1-2\nu]$.

∴ Different strains may appear with the same maximum stress.

10.3 The Maximum Shear Theory

The maximum shear theory, also known as the Guest's theory or the Tresca yield criterion in recognition of the contribution of H.E. Tresca (1814-1885) to its application, was proposed by C.A. Coulomb (1736-1806). This theory assumes that yielding will start when the maximum shearing stress in the material equals the maximum shearing stress developed at yielding in a simple tension test. Since the maximum shearing stress is equal to one-half the difference between the principal stresses, the condition for yielding is

$$\tau_w = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{\sigma_{yp}}{2} \qquad (10.5)$$

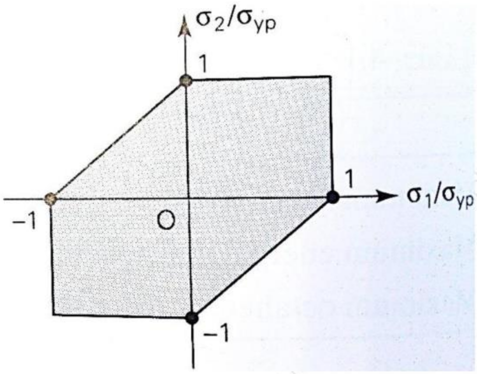


Figure 10.3 Yield criterion based on maximum shearing stress

The maximum shear stress theory for plane stress can be expressed for any two in-plane principal stresses as σ_1 and σ_2 by the following criteria:

$$\left.\begin{aligned} |\sigma_1| &= \sigma_{yp} \\ |\sigma_2| &= \sigma_{yp} \end{aligned}\right\} \quad \sigma_1,\ \sigma_2 \text{ have same signs}$$

$$\left.|\sigma_1 - \sigma_2| = \sigma_{yp}\right\} \quad \sigma_1,\ \sigma_2 \text{ have opposite signs}$$

Experimental work shows best agreement with this theory when applied to ductile materials.

10.4 The Mises Yield Theory

The Mises yield theory, also known as the maximum shear distortion theory, was proposed by M.T. Huber in 1904 and further developed by R. von Mises (1913) and H. Hencky (1925). In this theory, failure by yielding occurs when, at any point in the body, the distortion energy per unit volume in a state of combined stress becomes equal to that associated with yielding in a simple tension test.

If $\sigma_1 > \sigma_2 > \sigma_3$ are the principal stresses and σ_{yp} is the yield strength in simple tension, this concept gives

$$\frac{1}{3}\left\{[\sigma_1-\sigma_2]^2+[\sigma_2-\sigma_3]^2+[\sigma_3-\sigma_1]^2\right\} = \frac{1}{3}\left\{[\sigma_{yp}-0]^2+[0-0]^2+[0-\sigma_{yp}]^2\right\} = \frac{2}{3}\sigma_{yp}^2 \quad (10.6)$$

from which we obtain

$$[\sigma_1-\sigma_2]^2+[\sigma_2-\sigma_3]^2+[\sigma_3-\sigma_1]^2 = 2\sigma_{yp}^2 \quad (10.7)$$

Since $\sigma_3 = 0$ for plane stress problem then the criterion for yielding becomes

$$\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2 = \sigma_{yp}^2 \quad (10.8)$$

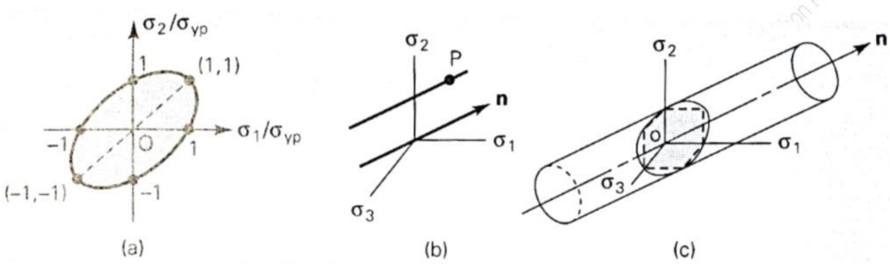


Figure 10.4 Yield criterion based on distortion energy: (a) plane stress yield ellipse; (b) a state of stress defined by position; (c) yield surface for triaxial state of stress.

The Mises yield theory of failure finds considerable experiment support in situations involving ductile materials and plane stress. For this reason, it is in common use in design.

10.5 The Mohr's Theory

The Mohr theory of failure is used to predict the fracture of a material having different properties in tension and compression when results of various types of tests are available for that material. This criterion makes use of the well-known Mohr's circle stress, which relies on stress plots in σ, τ coordinates.

If the data describing states of limiting stress are derived from only simple tension, simple compression, and pure shear tests, the three resulting circles are adequate to construct the envelope. The Mohr envelope thus represents the locus of all possible failure states.

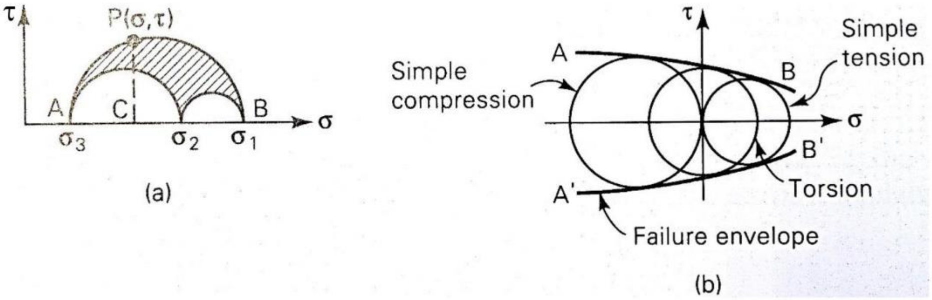


Figure 10.5 (a) Mohr's circles of stress; (b) Mohr's envelopes.

10.6 The Coulomb-Mohr Theory

The Coulomb-Mohr or internal friction theory assumes that the critical shearing stress is related to internal friction. If the frictional force is regarded as a function of the normal stress acting on a shear plane, the critical shearing stress and normal stress can be connected by an equation of the following form

$$\tau = a\sigma + b \qquad (10.9)$$

The constants a and b represent properties of the particular material. This expression may also be viewed as a straight-line version of the Mohr envelope.

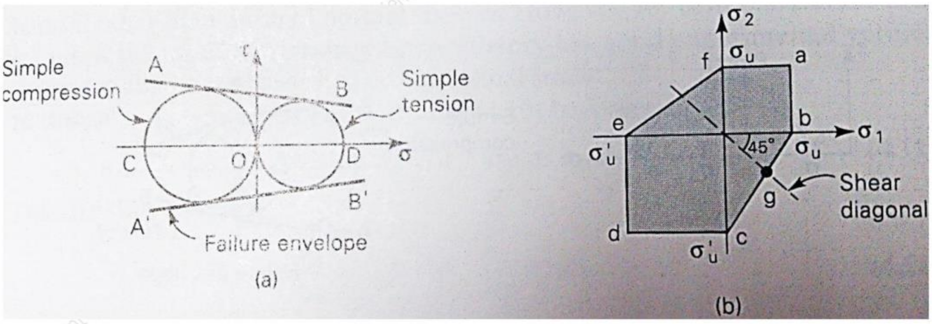


Figure 10.6 (a) Straight-line Mohr's envelopes; (b) Coulomb-Mohr fracture criterion.

For the case of plane stress, $\sigma_3 = 0$ when σ_1 is tensile and σ_2 is compressive. The maximum shearing stress τ and the normal stress σ acting on the shear plane are given by

$$\tau = \frac{\sigma_1 - \sigma_2}{2}, \ \sigma = \frac{\sigma_1 + \sigma_2}{2} \qquad (10.10)$$

Introducing these expressions into Eq. (10.9), we obtain

$$(\sigma_1)(1-a)-(\sigma_2)(1+a) = 2b \qquad (10.11)$$

To evaluate the material constants, the following conditions are applied:

$$\sigma_1 = \sigma_u \text{ when } \sigma_2 = 0$$

$$\sigma_2 = -\sigma_u' \text{ when } \sigma_1 = 0 \qquad (10.12)$$

Here σ_u and σ_u' represent the ultimate strength of the material in tension and compression, respectively. If now Eqs. (10.12) are inserted into Eq. (10.11), the results are

$$(\sigma_u)(1-a) = 2b \text{ and } (\sigma_u')(1+a) = 2b$$

from which

$$a = \frac{\sigma_u - \sigma_u'}{\sigma_u + \sigma_u'},\ b = \frac{(\sigma_u)(\sigma_u')}{\sigma_u + \sigma_u'} \qquad (10.13)$$

These constants are now introduced into Eq. (10.11) to complete the equation of the envelope of failure by fracturing. When this is done, the following expression is obtained, applicable for $\sigma_1 > 0$, $\sigma_2 < 0$:

$$\frac{\sigma_1}{\sigma_u} - \frac{\sigma_2}{\sigma_u'} = 1 \qquad (10.14)$$

In the case of biaxial tension (now $\sigma_{min} = \sigma_3 = 0$, σ_1 and σ_2 are tensile), fracture occurs if either of the two tensile stresses achieves the value σ_u. That is,

$$\sigma_1 = \sigma_u,\ \sigma_2 = \sigma_u \qquad (10.15)$$

For biaxial compression (now $\sigma_{max} = \sigma_3 = 0$, σ_1 and σ_2 are compressive), fracture occurs if either of the compressive stresses attains the value σ_u':

$$\sigma_2 = -\sigma_u',\ \sigma_1 = -\sigma_u' \qquad (10.16)$$

The figure above is a graphical representation of the Coulomb-Mohr theory plotted in the σ_1, σ_2 plane. Points lying within the shaded area should not represent failure, according to the theory. In the case of pure shear, the corresponding limiting point is g. The magnitude of the limiting shear stress may be graphically determined from the figure or calculated from Eq. (f) by letting $\sigma_1 = -\sigma_2$:

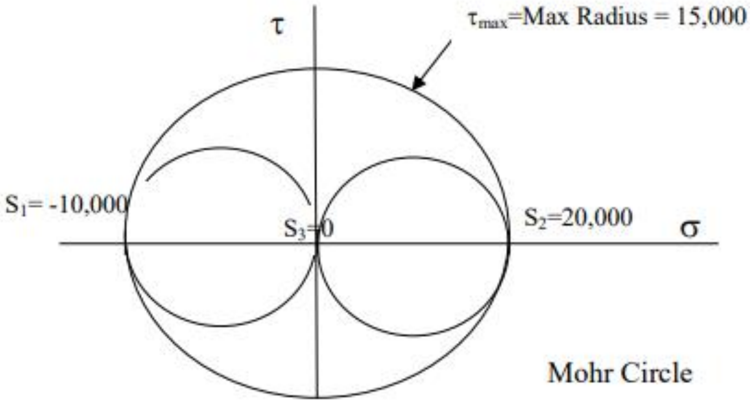
$$\sigma_1 = \tau_u = \frac{(\sigma_u)(\sigma_u')}{\sigma_u + \sigma_u'} \qquad (10.17)$$

10.7 Examples

Example 1: Use Maximum Shear Stress theory to determine the Factor of Safety N_{fs} , when the stress at a point is given by S_1 = -10,000 psi, S_2=20,000 psi, S_3=0, and the yield strength of the part material S_{yp}=51,000 psi.

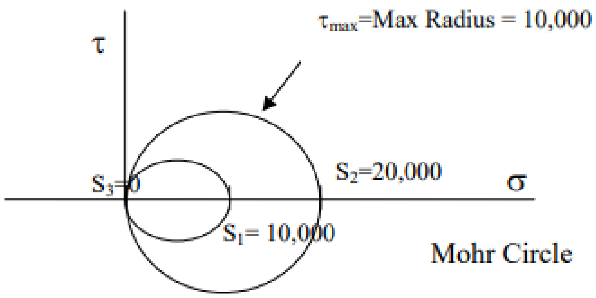
$$\max\{|S_1 - S_2|; |S_2 - S_3|; |S_3 - S_1|\} = \frac{S_{yp}}{N_{fs}}$$

$$N_{fs} = \frac{S_{yp}}{\max\{|S_1 - S_2|; |S_2 - S_3|; |S_3 - S_1|\}} = \frac{51{,}000}{|-10{,}000 - 20{,}000)|} = \frac{51{,}000}{30{,}000} = 1.7$$



Example 2: Use Maximum Shear Stress theory to determine the Factor of Safety N_{fs} , when the stress at a point is given by S_1 = 10,000 psi, S_2=20,000 psi, S_3=0, and the yield strength of the part material S_{yp}=51,000 psi.

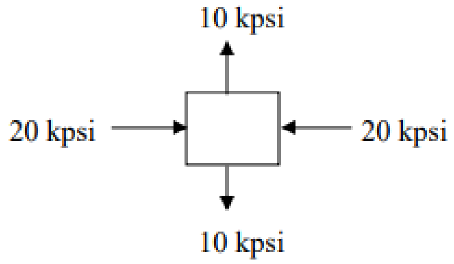
$$N_{fs} = \frac{S_{yp}}{\max\{|S_1 - S_2|; |S_2 - S_3|; |S_3 - S_1|\}} = \frac{51{,}000}{|0 - 20000)|} = \frac{51{,}000}{20{,}000} = 2.76$$



Example 3: The stresses on a Class 25 Gray Cast Iron part is shown below. Find the factor of safety.

From table 14-16 in textbook for Class 25 Grey CI
S_{UT} = 25,000 psi & S_{UC} =100,000 psi

The Max Tensile stress = 10,000,
N_{fsT}= 25,000/10,000= 2.5
The Max Compressive stress = 20,000,
N_{fsC} = 100,000/20,000 = 5



The N_{fs} of the part is 2.5 (smaller of the two).

Example 4:
Use Von Mises-Hencky theory to determine the Factor of Safety N_{fs} , when the stress at a point is given by S_1 = -10,000 psi, S_2=20,000 psi, S_3=0, and the yield strength of the part material S_{yp}=51,000 psi.

$$S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1 = \left(\frac{S_{yp}}{N_{fs}}\right)^2$$

$$N_{fs} = \frac{S_{yp}}{\sqrt{S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1}}$$

$$= \frac{51{,}000}{\sqrt{-10{,}000^2 + 20{,}000^2 + 0^2 - (-10{,}000 \times 20{,}000) - (20{,}000 \times 0) - (0 \times 10{,}000)}}$$

$$= \frac{51{,}000}{\sqrt{-10{,}000^2 + 20{,}000^2 - (-10{,}000 \times 20{,}000)}}$$

$$= \frac{51{,}000}{\sqrt{10^8 + 4 \times 10^8 + 2 \times 10^8)}} = 1.93$$

Example - 5 The ultimate strength of a brittle material is 40 MPa in tension and 50 MPa in compression. Use Mohr's failure criterion to determine whether the plane state of stress in Fig. Ex10.1(a) would result in rupture of this material.

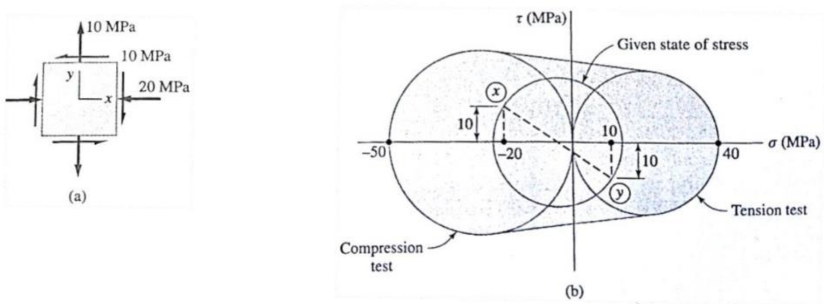


Figure Ex10.1 (a) state of stress; (b) Mohr's failure envelope

Solution: We first draw the Mohr's circles representing the states of stress at rupture for uniaxial tension and for uniaxial compression, as shown in Fig. Ex10.1(b). We complete the failure envelope by drawing tangent lines to the two circles. Any state of stress with a Mohr's circle that lies entirely within the failure envelope (the shaded area in the figure) is deemed to be safe against rupture. Otherwise, failure is predicted. The Mohr's circle representing the given state of stress is also shown in Fig. Ex10.1(b). Because the circle lies within the failure envelope, this state of stress would not cause rupture.

Example - 6 A thin-walled tube is fabricated of a brittle material having ultimate tensile and compressive strengths σ_t = 300 MPa and σ_c = 700 MPa. The radius and thickness of the tube are r = 100 mm and t = 5 mm. Calculate the limiting torque that can be applied without causing failure by fracture. Apply (a) the maximum stress theory and (b) the Coulomb-Mohr theory.

Solution: The torque and maximum shearing stress are related by the torsion formula:

$$T = (\tau)\left(\frac{J}{r}\right) = (2)(\pi)(r)^2(t)(\tau) = 2\pi(100mm)^2(5mm)(\tau)$$

$$= (314.159\times10^3)(\tau)\ N\cdot mm$$

(a) Maximum stress theory:

Because we have $\sigma_t = \sigma_{ult} = 300\ \frac{N}{mm^2} = \tau$

$$T = \left(314.159\times10^{3}\ mm^{3}\right)\left(300\,\frac{N}{mm^{2}}\right) = 94\ 247.70 \qquad kN\cdot mm$$

(b) Coulomb-Mohr theory:

$$\frac{\tau}{300\times10^{6}} - \frac{-\tau}{700\times10^{6}} = 1$$

From which t = 210 MPa, Then

$$T = (2)(\pi)(r)^{2}(t)(\tau)$$

$$= (2\pi)(100\ mm)^{2}\ (5\ mm)\left(210\ \frac{N}{mm^{2}}\right) = 65.9\ kN\cdot m$$

Based on the maximum stress theory, the torque that can be applied to the tube is thus 30% larger than that based on the Coulomb-Mohr theory. To prevent fracture, the torque should not exceed 65.9 kN•m.

Example - 7 A torsion bar-spring made of ASTM grade A-48 cast iron is loaded as shown in figure below. The stress concentration factors are 1.7 for bending and 1.4 for torsion. Determine the diameter d to resist loads P = 25 N and T = 10 N•m, using a factor of safety FS = 2.5. Apply (a) the maximum stress theory and (b) the Coulomb-Mohr theory.

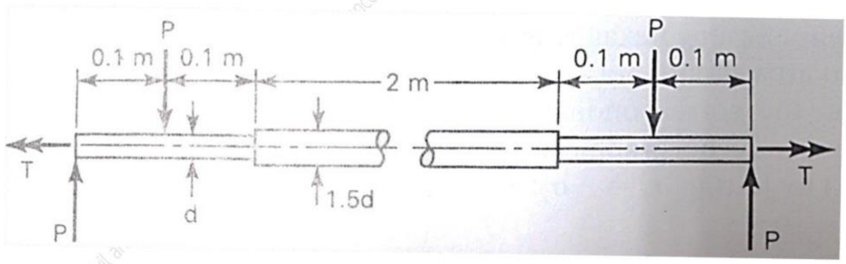


Figure Ex10.3 A torsion-bar spring.

Solution: The stresses produced by bending moment M = 0.1P and torque T at the shoulder are

$$\sigma_x = (k_b)\left(\frac{32M}{\pi d^3}\right)$$

$$\tau = (k_t)\left(\frac{16T}{\pi d^3}\right) \qquad \text{(Ex10.3.a)}$$

The principal stresses are then

$$\sigma_{1,2} = \left(\frac{16}{\pi d^3}\right)\left(k_b\,M \mp \sqrt{k_b^2 M^2 + k_t^2 T^2}\right) \qquad \text{(Ex10.3.b)}$$

Substituting the given data, we have

$$\sigma_{1,2} = \left(\frac{16}{\pi d^3}\right)\left[1.7(0.1 \times 25) \mp \sqrt{(1.7 \times 0.1 \times 25)^2 + (1.4 \times 10)^2}\right]$$

The foregoing results in

$$\sigma_1 = \frac{96.16}{d^3}, \sigma_2 = -\frac{52.87}{d^3} \qquad \text{(Ex10.3.c)}$$

The allowable ultimate strengths of the material in tension and compression are $\frac{170}{2.5}$ = 68 MPa and $\frac{650}{2.5}$ = 260 MPa, respectively.

(a) Maximum stress theory

$$\frac{96.16}{d^3} = 68 \times 10^6$$

$\therefore$ d = 11.2 mm **Ans.**

Similarly, $-\frac{52.87}{d^3} = 68 \times 10^6$ gives d = 9.2 mm.

(b) Coulomb-Mohr theory:

$$\frac{96.16}{(68\times10^6)(d^3)} - \frac{-52.87}{(260\times10^6)(d^3)} = 1$$

$\therefore$ d = 11.7 mm. **Ans.**

The diameter of the spring based on the Coulomb-Mohr theory is therefore about 4.5% larger than that based on the maximum stress theory. A 12-mm-diameter bar, a commercial size, should be used to prevent fracture.

Example - 8 The steel pipe as shown in Figure Ex10.4.a has an inner diameter of 60 mm and an outer diameter of 80 mm. If it is subjected to a torsional moment of 8 kN·m and a bending moment of 3.5 kN·m, determine if these loadings cause failure as defined by the Mises yield theory. The yield stress for the steel found from a tension test is σ_{yp} = 250 MPa.

$$\tau_A = \frac{(T)(c)}{J} = \frac{(8000\, N\cdot m)(0.04\, m)}{\left(\frac{\pi}{2}\right)\left[(0.04\, m)^4 - (0.03)^4\right]} = 116.4 \text{ MPa}$$

$$\sigma_A = \frac{(M)(c)}{I_{NA}} = \frac{(3500\, N\cdot m)(0.04\, m)}{\left(\frac{\pi}{4}\right)\left[(0.04\, m)^4 - (0.03)^4\right]} = 101.9 \text{ MPa}$$

Mohr's circle for this state of plane stress has a center located at

$$\sigma_{avg} = \frac{(0-101.9)}{2} = -50.9 \text{ MPa}$$

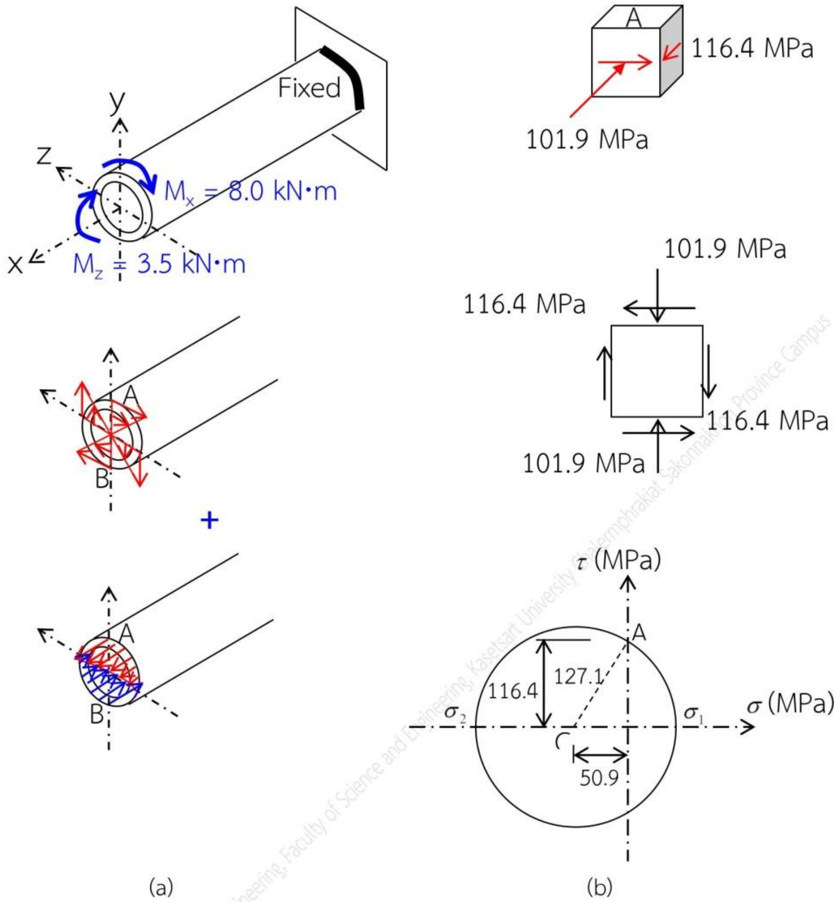


Figure Ex10.4 (a)

σ_1 = -50.9 + 127.1 = 76.2 MPa

σ_2 = -50.9 - 127.1 = -178.0 MPa

$(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2) \leq \sigma_{yp}^2$

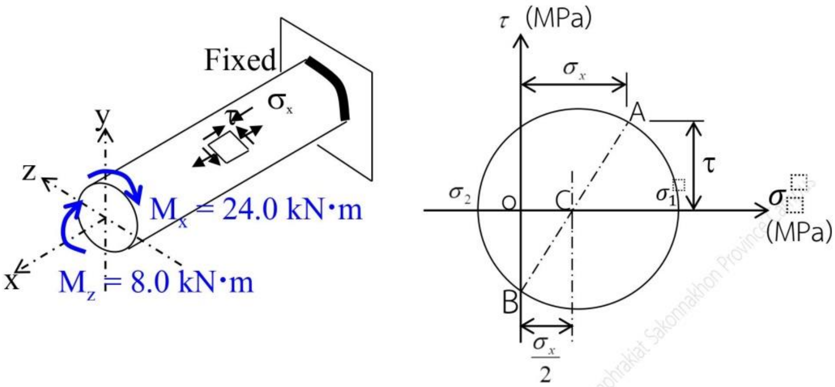
$[(76.2)^2 - (76.2)(-178.0) + (-178.0)^2] \leq (250.0)^2$

51 100 < 62 500 (O.K.)

Since the criterion has been met, the material within the pipe will not yield ("fail") according to the Mises yield theory.

Example - 9 A circular shaft of tensile strength σ_{yp} = 350 MPa is subjected to a combined state of loading defined by bending moment M = 8 kN•m and torque T =

24 kN·m. Calculate the required shaft diameter d in order to achieve a factor of safety FS = 2. Apply (a) the Guest's theory and (b) the maximum shear distortion theory.



Solution: For the situation described, the principal stresses are

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \frac{\sqrt{(\sigma_x^2 + 4\tau^2)}}{2} \qquad \text{(Ex10.5.1)}$$

where

$$\sigma_x = \frac{My}{I_{NA}} = \frac{32M}{\pi d^3} \qquad \text{(Ex10.5.2)}$$

$$\tau = \frac{Tr}{J} = \frac{16T}{\pi d^3} \qquad \text{(Ex10.5.3)}$$

Therefore

$$\sigma_{1,2} = \left[\frac{16}{\pi d^3}\right]\left[M \pm \sqrt{M^2 + T^2}\right] \qquad \text{(Ex10.5.4)}$$

(a) For the state of stress under consideration, it may be observed from Mohr's circle that σ_1 is tensile and σ_2 is compressive. Thus,

$$\frac{\sigma_{yp}}{FS} = \sqrt{(\sigma_x^2 + 4\tau^2)} \qquad \text{(Ex10.5.5)}$$

or

$$\frac{\sigma_{yp}}{FS} = \left[\frac{32}{\pi d^3}\right]\sqrt{(M^2 + T^2)} \qquad \text{(Ex10.5.6)}$$

After substitution of the numerical values, Eq. (Ex10.5.6) gives d = 113.8 mm.

(b)

$$\frac{\sigma_{yp}}{FS} = \sqrt{(\sigma_x^2 + 3\tau^2)} \qquad \text{(Ex.10.5.7)}$$

$$\frac{\sigma_{yp}}{FS} = \left[\frac{16}{\pi d^3}\right]\sqrt{(4M^2+3T^2)} \quad \text{(Ex10.5.8)}$$

Substituting the data into Eq. (Ex10.5.8) and solving for d, we have d = 109 mm. The diameter based on the Guest's theory is thus 4.4% larger than that based on the maximum shear distortion theory. A 114-mm shaft should be used to prevent initiation of yielding.

Example - 10 A steel conical tank, supported at its edges, is filled with a liquid of density γ. The yield point stress (σ_{yp}) of the material is known. The cone angle is 2α. Determine the required wall thickness t of the tank based on a factor of safety FS. Apply (a) the maximum shear theory and (b) the maximum shear distortion theory.

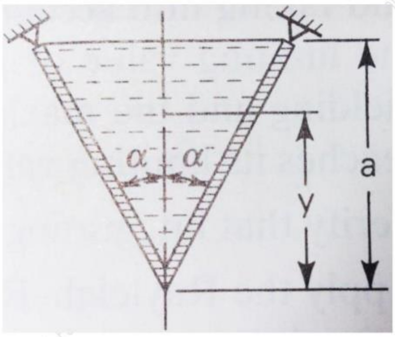


Figure Ex10.6.1 Cross section of conical tank

Solution: The variations of the circumferential and longitudinal stresses in the tank are, respectively

$$\sigma_1 = (\gamma)(a-y)(y)\left[\frac{\tan\alpha}{t\cos\alpha}\right] \quad \text{(Ex10.6.1)}$$

$$\sigma_2 = (\gamma)(a-\frac{2}{3}y)(y)\left[\frac{\tan\alpha}{2t\cos\alpha}\right] \quad \text{(Ex10.6.2)}$$

The principal stresses have the largest magnitude:

$$\sigma_{1,\max} = \left[\frac{\gamma a^2}{4t}\right]\left[\frac{\tan\alpha}{\cos\alpha}\right], \text{ at } y = \frac{a}{2} \quad \text{(Ex10.6.3)}$$

$$\sigma_{2,\max} = \left[\frac{3\gamma a^2}{16t}\right]\left[\frac{\tan\alpha}{\cos\alpha}\right], \text{ at } y = \frac{3a}{4} \quad \text{(Ex10.6.4)}$$

(a) Because σ_1 and σ_2 are of the same sign and $|\sigma_1| > |\sigma_2|$, we have

$$\frac{\sigma_{yp}}{FS} = \left[\frac{\gamma a^2}{4t}\right]\left[\frac{\tan\alpha}{\cos\alpha}\right] \quad \text{(Ex10.6.5)}$$

The thickness of the tank is found from this equation to be

$$t = 0.250\left[\frac{(\gamma)(a^2)(FS)}{\sigma_{yp}}\right]\left[\frac{\tan\alpha}{\cos\alpha}\right] \quad \text{(Ex10.6.6)}$$

(b) It is observed in Eqs. (Ex10.6.3) and (Ex10.6.4) that the largest values of principal stress are found at different locations. We shall therefore first locate the section at which the combined principal stresses are at a critical value. For this purpose, we insert Eq. (Ex10.6.6) into

$$\frac{\sigma_{yp}^2}{FS^2} = \left\{(\gamma)(a-y)(y)\left[\frac{\tan\alpha}{t\cos\alpha}\right]\right\}^2 + \left\{(\gamma)(a-\frac{2}{3}y)(y)\left[\frac{\tan\alpha}{2t\cos\alpha}\right]\right\}^2$$
$$- \left\{(\gamma)(a-y)(y)\left[\frac{\tan\alpha}{t\cos\alpha}\right]\right\}\left\{(\gamma)(a-\frac{2}{3}y)(y)\left[\frac{\tan\alpha}{2t\cos\alpha}\right]\right\} \quad \text{(Ex10.6.7)}$$

Upon differentiating Eq. (Ex10.6.7) with respect to the variable y and equating the result to zero, we obtain

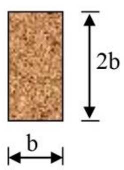
$$y = 0.52a \quad \text{(Ex10.6.8)}$$

Upon substitution of this value of y into Eq. (7), the thickness of the tank is determined:

$$t = 0.225\left[\frac{(\gamma)(a^2)(FS)}{\sigma_{yp}}\right]\left[\frac{\tan\alpha}{\cos\alpha}\right]$$ ***Ans.***

Example - 11 Determine the width b of the cantilever of height 2b and length 0.25 m subjected to a 450-N concentrated force at its free end. Apply the Mises yield theory. The tensile and compressive strengths of the material are both 280 MPa.

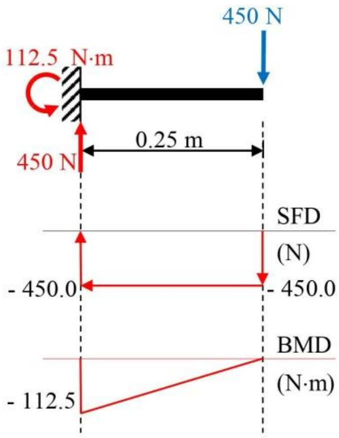
Solution:

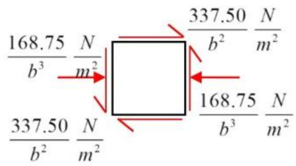


$$I_{NA} = \left(\frac{1}{12}\right)(bm)(2bm)^3 = \left(\frac{2}{3}\right)(b)^4\ m^4$$

$$\sigma = \frac{(112.5\,N\cdot m)(b)}{\left(\frac{2}{3}\right)(b)^4} = \frac{168.75}{b^3}\ \frac{N}{m^2}$$

$$\tau = \frac{(V)(Q)}{(I_{NA})(b)} = \frac{(450\,N)\left[(bm)(bm)\left(\frac{b}{2}m\right)\right]}{\left(\frac{2}{3}\right)(bm)^4(b)} = \frac{337.50}{b^2}\ \frac{N}{m^2}$$





จาก Mohr's circle

$$\sigma_{\substack{\min\\ \max}} = \left[\frac{\sigma_x+\sigma_y}{2}\right] \mp \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\left(\tau_{xy}\right)^2}$$

$$\left[\frac{\sigma_x+\sigma_y}{2}\right] = \left[\frac{\left(\frac{168.75}{b^3}\right)+0}{2}\right] = \frac{84.375}{b^3}\frac{N}{m^2} = \left[\frac{\sigma_x-\sigma_y}{2}\right]$$

$$\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\left(\tau_{xy}\right)^2} = \sqrt{\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2}$$

$$\sigma_1^2 = \left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2$$

$$+\,(2)\left[\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)\right]\left[\sqrt{\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2}\right]$$

$$= (2)\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2$$

$$+\,(2)\left[\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)\right]\left[\sqrt{\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2}\right]$$

$$\sigma_2^2 = \left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2$$

$$-\,(2)\left[\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)\right]\left[\sqrt{\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2+\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2}\right]$$

$$= (2)\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2 + \left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2$$

$$- (2)\left[\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)\right]\left[\sqrt{\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2 + \left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2}\right]$$

$$(\sigma_1)(\sigma_2) = - \left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2$$

$$(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2) \leq \sigma_{yp}^2$$

$$(2)\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2 + \left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2 + (2)\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2 + \left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2$$

$$+ \left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2 \leq \left(280\times10^6\frac{N}{m^2}\right)^2$$

$$(4)\left(\frac{84.375}{b^3}\frac{N}{m^2}\right)^2 + (3)\left(\frac{337.5}{b^2}\frac{N}{m^2}\right)^2 \leq \left(280\times10^6\frac{N}{m^2}\right)^2$$

$$\left(\frac{28.4765625\times10^3}{b^6}\frac{N^2}{m^4}\right) + \left(\frac{341.71875\times10^3}{b^4}\frac{N^2}{m^4}\right) \leq \left(78400\times10^{12}\frac{N^2}{m^4}\right)$$

$$(28.4765625\times10^3) + \left[(341.71875\times10^3)(b^2)\right] \leq \left[(78400\times10^{12})(b^6)\right]$$

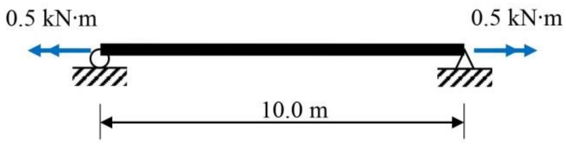
$$\left[(78400\times10^{12})(b^6)\right] - (28.4765625\times10^3) - \left[(341.71875\times10^3)(b^2)\right] \geq 0$$

$$(b^2)\left[(78400\times10^9)(b^4) - (341.71875)\right] \geq (28.4765625)$$

$\therefore$ b $\geq$ 8.448 mm. *Ans.*

Example - 12 Determine the required diameter of a steel transmission shaft 10 m in length and of yield strength 350 MPa in order to resist a torque of up to 500 N·m. The shaft is supported by frictionless bearings at its ends. Design the shaft according to the maximum shear stress theory, selecting a factor of safety of 1.5, (a) neglecting the shaft weight, and (b) including the effect of shaft weight. Use γ = 77 kN/m^3 as the weight per unit volume of steel and G = 70 GPa.

Solution:



(a) $\tau_{\max} = \dfrac{(T)(r)}{J}$

$J = \dfrac{(\pi)(d)^4}{32}$

$\tau_{\max} = \dfrac{(1.5)(0.5\,kN\cdot m)\left(\dfrac{d}{2}\,m\right)}{\left[\dfrac{(\pi)(d\,m)^4}{32}\right]} = \dfrac{12000}{(\pi)(d)^3}\,\dfrac{N}{m^2}$

จากทฤษฎี: $\tau_{\substack{abs\\ \max}} = \dfrac{\sigma_{yp}}{2}$

$\dfrac{12000}{(\pi)(d)^3}\,\dfrac{N}{m^2} = \dfrac{\sigma_{yp}}{2}$

$\dfrac{12000}{(\pi)(d)^3}\,\dfrac{N}{m^2} = \dfrac{350\times10^6}{2}\,\dfrac{N}{m^2}$

$\therefore d = 27.947\ mm.$ *Ans.*

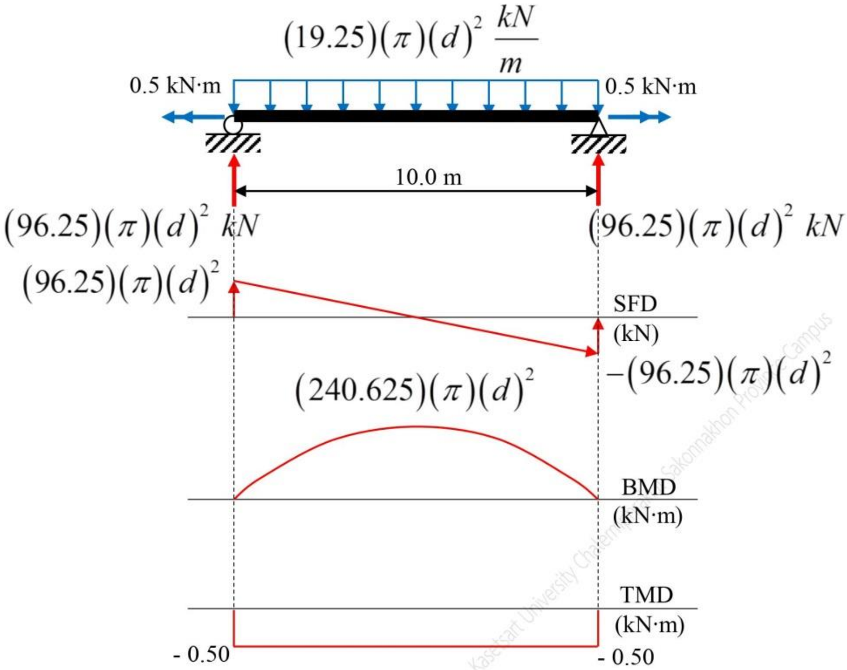
(b) $\pi\left[\dfrac{(d\,m)^2}{4}\right]\left(77\,\dfrac{kN}{m^3}\right) = (19.25)(\pi)(d)^2\,\dfrac{kN}{m}$

$\tau_{\max} = \dfrac{(T)(r)}{J} + \dfrac{(V)(Q)}{(I)(b)}$

$I_{NA} = \dfrac{(\pi)(d\,m)^4}{64} = \dfrac{(\pi)(d)^4}{64}\,m^4$

$Q = \left[\dfrac{(\pi)(d\,m)^2}{8}\right]\left[\dfrac{(4)(d\,m)}{(6)(\pi)}\right] = \dfrac{(d)^3}{12}\,m^3$

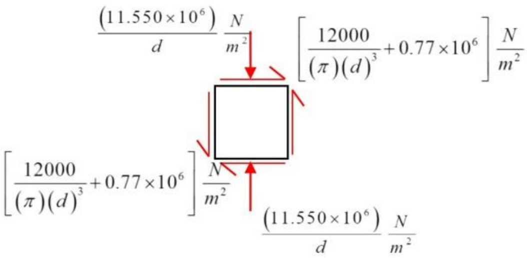
$\tau_{\max} = \dfrac{12000}{(\pi)(d)^3}\,\dfrac{N}{m^2} + \dfrac{(1.5)\left[(96250)(\pi)(d)^2\right]\left(\dfrac{d^3}{12}\right)}{\left[\dfrac{(\pi)(d)^4}{64}\right](d)}\,\dfrac{N}{m^2}$



$$\tau_{\max} = \frac{12000}{(\pi)(d)^3} + \left(0.77\times10^6\right)\frac{N}{m^2}$$

$$\sigma_{\max} = \frac{\left(M_{\max}\right)\left(\dfrac{d}{2}\right)}{I_{NA}} = \frac{(1.5)\left[(240625)(\pi)(d)^2\right]\left(\dfrac{d}{2}\right)}{\left[\dfrac{(\pi)(d)^4}{64}\right]}$$

$$= \frac{\left(11.550\times10^6\right)}{d}\ \frac{N}{m^2}$$



$$\left[\frac{\sigma_x+\sigma_y}{2}\right] = \left[\frac{\left(\frac{11.55\times10^6}{d}\right)+0}{2}\right] = \left(\frac{5.775\times10^6}{d}\right)\frac{N}{m^2} = \left[\frac{\sigma_x-\sigma_y}{2}\right]$$

$$\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\left(\tau_{xy}\right)^2} = \sqrt{\left(\frac{5.775\times10^6}{d}\frac{N}{m^2}\right)^2+\left(\frac{12000}{(\pi)(d)^3}+0.77\times10^6\frac{N}{m^2}\right)^2}$$

$$\frac{\sigma_{max}-\sigma_{min}}{2} = \frac{\sigma_{yp}}{2}$$

$$\frac{(2)\left(\sqrt{\left(\frac{5.775\times10^6}{d}\frac{N}{m^2}\right)^2+\left(\frac{12000}{(\pi)(d)^3}+0.77\times10^6\frac{N}{m^2}\right)^2}\right)}{2} = \frac{350\times10^6}{2}\frac{N}{m^2}$$

$$\left(\frac{5.775\times10^6}{d}\right)^2+\left(\frac{12000}{(\pi)(d)^3}+0.77\times10^6\right)^2 = 30625\times10^{12}$$

$$\frac{33.350625\times10^{12}}{d^2}+\frac{144\times10^6}{(\pi^2)(d^6)}+\frac{18480\times10^6}{(\pi)(d^3)}+0.5929\times10^{12} = 30625\times10^{12}$$

$$\frac{33.350625\times10^{12}}{d^2}+\frac{144\times10^6}{(\pi^2)(d^6)}+\frac{18480\times10^6}{(\pi)(d^3)} = 30624.4071\times10^{12}$$

$$(33.350625)(d^4)+\frac{144\times10^{-6}}{(\pi^2)}+\frac{(18480\times10^{-6})(d^3)}{(\pi)} = (30624.4071)(d^6)$$

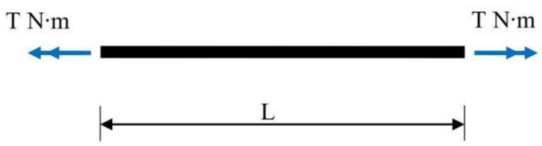
$$(30624.4071)(d^6)-(33.350625)(d^4)-\frac{(18480\times10^{-6})(d^3)}{(\pi)}-\left[\frac{144\times10^{-6}}{(\pi^2)}\right] = 0$$

$$\therefore d = 36.695\ mm.$$ Ans.

Example - 13 A solid cylinder of radius 50 mm is subjected to a twisting moment T and an axial load P. Assume that the energy of distortion theory governs and that the yield strength of the material is σ_{yp} = 280 MPa. Determine the maximum twisting moment consistent with elastic behavior of the bar for (a) P = 0 and (b) P = 400π kN.

Solution:

(a)



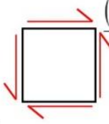
$$J = \frac{(\pi)(d)^4}{32} = \frac{(\pi)(100\,mm)^4}{32} = \left(3.125\times10^6\right)(\pi)\ mm^4$$

$$\tau_{\max} = \frac{(T)(50\,mm)}{J} = \frac{(T)(50\,mm)}{\left[\left(3.125\pi\times10^6\right) mm^4\right]}$$

$$= \frac{\left(16\times10^{-6}\right)(T)}{(\pi)}\ \frac{N}{mm^2}$$

$$\sigma_{\substack{\min\\ \max}} = \left[\frac{\sigma_x+\sigma_y}{2}\right] \mp \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\left(\tau_{xy}\right)^2}$$

$$\frac{\left(16\times10^{-6}\right)(T)}{\pi}\ \frac{N}{mm^2}$$



$$\frac{\left(16\times10^{-6}\right)(T)}{\pi}\ \frac{N}{mm^2}$$

$$\sigma_{\min} = -\sqrt{\left(\tau_{xy}\right)^2} = -\tau_{\max} = \sigma_1$$

$$\sigma_{\max} = \sqrt{\left(\tau_{xy}\right)^2} = \tau_{\max} = \sigma_2$$

$$\left(\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2\right) \le \sigma_{yp}^2$$

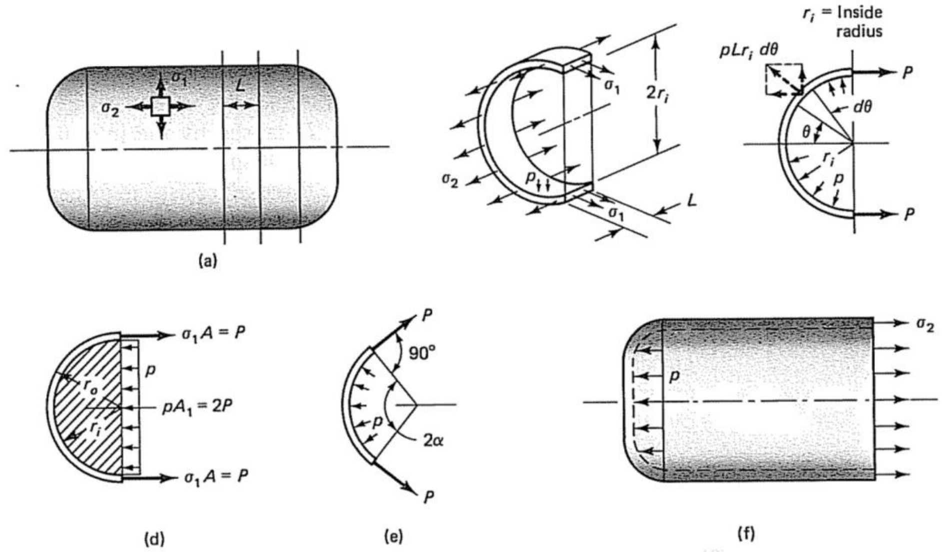
$$\left(-\tau_{\max}\right)^2-\left(-\tau_{\max}\right)\left(\tau_{\max}\right)+\left(\tau_{\max}\right)^2 \le \left(280\ \frac{N}{mm^2}\right)^2$$

$$\left(\sqrt{3}\right)\left[\frac{\left(16\times10^{-6}\right)(T)}{(\pi)}\ \frac{N}{mm^2}\right] \le \left(280\ \frac{N}{mm^2}\right)$$

$\therefore T$ = 31.74 kN·m. *Ans.*

(b) $\therefore T$ = 26.05 kN·m. *Ans.*

Example - 14 A thin-walled cylindrical pressure vessel of diameter d = 0.5 m and wall thickness t = 5 mm is fabricated of a material with 280-MPa tensile yield strength. Determine the internal pressure p required according to the following theories of failure: (a) maximum distortion energy theory and (b) maximum shear stress.

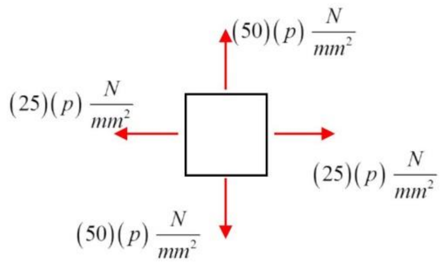


$$\sigma_1 = \frac{(p)(r_i)}{t}, \quad \sigma_2 = \frac{(p)(r_i)}{(2)(t)}$$

(a) p = 6.466 MPa.

(b) $\sigma_1 = \dfrac{(p)(250\,mm)}{(5\,mm)} = (50)(p)\dfrac{N}{mm^2}$

$$\sigma_2 = \frac{(p)(250\,mm)}{(2)(5)} = (25)(p)\frac{N}{mm^2}$$



จาก Mohr’s circle

$$\sigma_{\substack{\min \\ \max}} = \left[\frac{\sigma_x + \sigma_y}{2}\right] \mp \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

$$\sigma_{\max} = \left[\frac{\sigma_x + \sigma_y}{2}\right] + \left(\frac{\sigma_x - \sigma_y}{2}\right) = \sigma_x = \sigma_1$$

$$\sigma_{\min} = \left[\frac{\sigma_x + \sigma_y}{2}\right] - \left(\frac{\sigma_x - \sigma_y}{2}\right) = \sigma_y = \sigma_2$$

Because σ_1 and σ_2 have same signs

$$\tau_{\substack{abs \\ \max}} = \frac{\sigma_{\max}}{2} = \frac{\sigma_{yp}}{2}$$

$$(50)(p) = 280$$

$$\therefore p = 5.60\ MPa$$

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