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Ministry of Higher Education and Scientific Research
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ENGINEERING ANALYSIS

For Third-Year Students

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Linear and Non-Linear Equations

At the heart of linear algebra and much of applied mathematics is the problem of solving system of linear equations. The **standard form** for **single linear equation** in two variables is:

$$ax + by = c$$

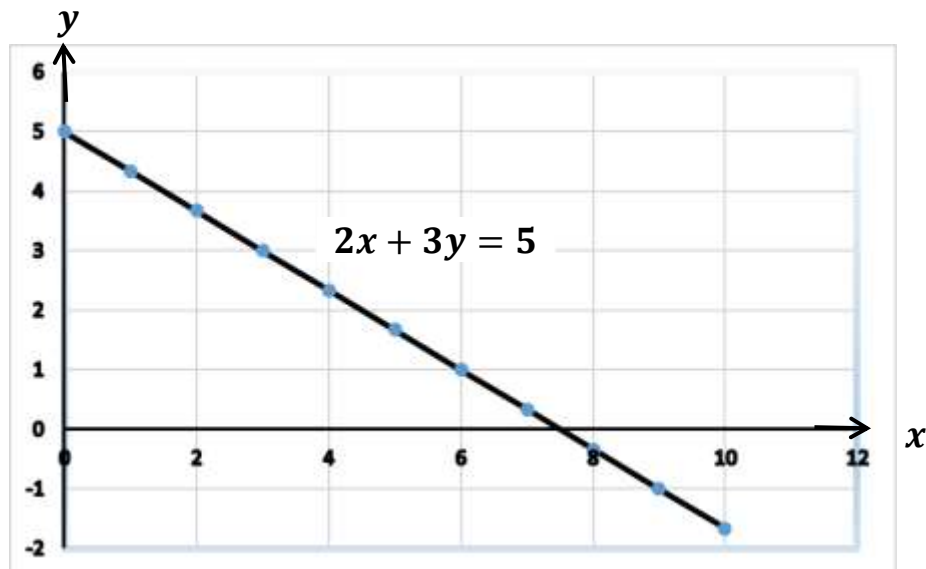
where x and y are **real variables** and a , b , and c are **real numbers**.

If a and b are **not** both **zero**, then the **graph** of this **equation** is a **straight line**; this is **why** is called **linear**.

A simple example of **linear equation** in **standard form** is:

$$2x + 3y = 5$$

and the **graph** of **this equation** is shown in figure below



Also, above equation is called **one linear equation** in the **two unknowns** x and y .

The **linear equation** with **three variables** x , y , and z has form

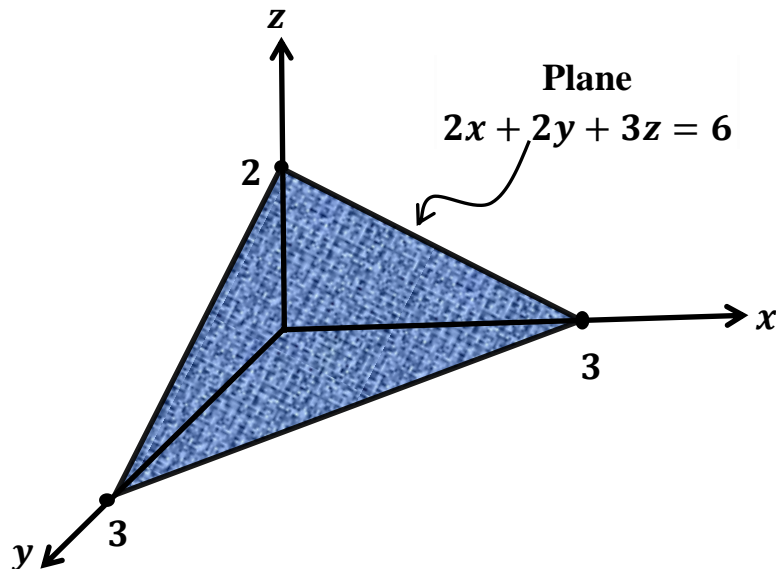
$$ax + by + cz = d$$

where a , b , c , and d are **real numbers**, not all zero, **represents a plane** in the standard xyz coordinate system.

An **example** of a **linear equation** in **three unknowns** is

$$2x + 2y + 3z = 6$$

The **graph** of this **equation** (in space) is a **plane** as shown in figure below.



Of course, an **equation** can contain **more than three variables**.

For a **nonscientific example**, a **person** may have **five employees**, each paid a **different hourly wage** and each **working a variable number of hours** each week, as shown in the following table.

| Employee | Hourly Wage | Hours Worked |
|----------|-------------|--------------|
| 1 | \$4.50 | x_1 |
| 2 | \$3.75 | x_2 |
| 3 | \$5.00 | x_3 |
| 4 | \$6.15 | x_4 |
| 5 | \$5.75 | x_5 |

The weekly payroll P is given by

$$P = 4.50x_1 + 3.75x_2 + 5x_3 + 6.15x_4 + 5.75x_5$$

This is one **linear equation** in **five variables**.

Definition 1: A linear equation in the n variables $x_1, x_2, x_3, \dots, x_n$ is any equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

where $a_1, a_2, a_3, \dots, a_n, b$ are constant real or complex number. The constant a_i is called the coefficient of x_i ; and b is called the constant term of the equation.

Definition 2: A solution of the single linear equation for real numbers $s_1, s_2, s_3, \dots, s_n$, if

$$a_1s_1 + a_2s_2 + a_3s_3 + \dots + a_ns_n = b$$

Which when substituted can be said that

$$s_1 = x_1, \quad s_2 = x_2, \quad s_3 = x_3, \dots, s_n = x_n$$

is a solution of this equation.

An **example** of **Linear equation** involving **complex numbers** is:

$$(h + i) x_n + x_{n-1} = 0 \quad n = 1, 2, 3, \dots$$

Where ***h*** is a **real** and ***i*** is the **imaginary unit** ($i^2 = -1$).

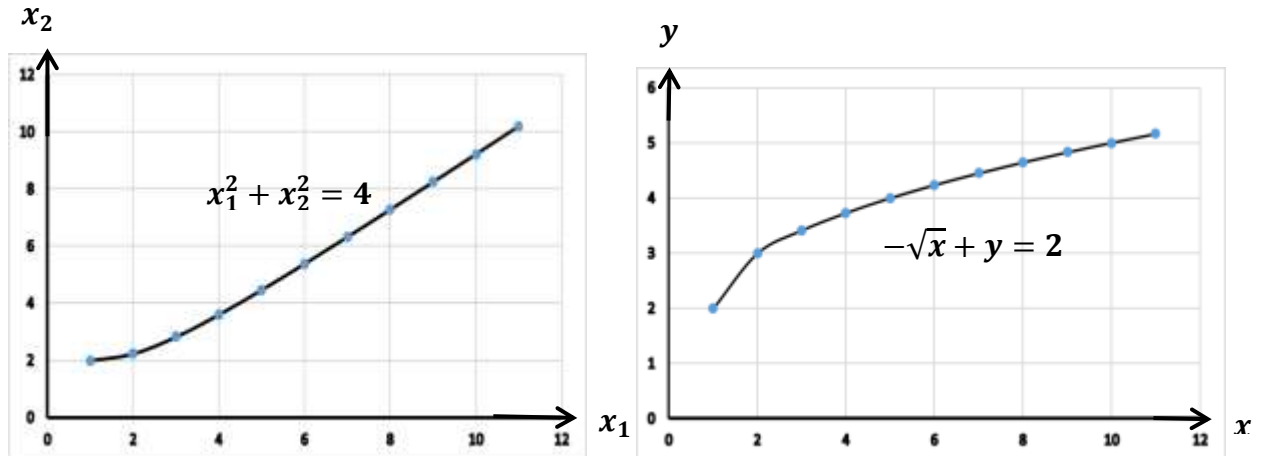
Any **equation** which is **not** in the **form** given in **Definition 1** is called **nonlinear equation**. For example,

$$x_1^2 + x_2^2 = 4$$

$$-\sqrt{x} + y = 2$$

$$x_1 + |x_2| - 3x_3 = 7$$

are **nonlinear equations**. **Graph** of the **first two** are **shown below**



Example 1: Write a **general solution** of the **equation:**

$$2x_1 - 4x_2 = 8$$

Solution:

Dividing both sides of the **equation by 2**, yields

$$x_1 - 2x_2 = 4$$

Solving for x_1 with **ordinary algebra** gives

$$x_1 = 4 + 2x_2$$

The **solution** can be **written** as **pair ordered** as:

$$(x_1, x_2) \text{ or } (4 + 2x_2, x_2)$$

Letting x_2 take on the specific values $0, 1, \dots$, then will have several sets of solutions.

- One is $x_2 = 0$, $x_1 = 4$; another is
- $x_2 = 1$, $x_1 = 6$; and yet another is
- $x_2 = -4$, $x_1 = -4$.

Writing as **ordered pair**, these are $(4, 0)$, $(6, 1)$, and $(4, -4)$, respectively.

However, in case of **complex number**, if we put $x_2 = 2 + i$, then $x_1 = 8 + 2i$, and we'll have a **complex solutions**. If the **unknowns** were **restricted beforehand** to be **real number**, then **complex solutions** would have ruled out.

If **solving** for x_2 with **ordinary algebra**, then would have

$$x_2 = \frac{x_1}{2} - 2$$

The **solution** will be as **pair ordered**:

$$(x_1, x_2) \text{ or } \left(x_1, \frac{x_1}{2} - 2\right)$$

Example 2: Show that $x_1 = 1 + i, x_2 = 1 - i$ is a **solution** of the **equation**:

$$\begin{aligned}(3 + i)x_1 + (1 + i)x_2 &= 4 + 4i \\ x_1 - x_2 &= 2i\end{aligned}$$

Solution:

Substituting the **given numbers** into the **given equations** yields

$$\begin{aligned}(3 + i)(1 + i) + (1 + i)(1 - i) &= 3 + 3i + i^2 + i + 1 + i - i - i^2 \\ &= 3 + 4i + (-1) + 1 - (-1) \\ &= 3 + 4i - 1 + 1 + 14 + 4i\end{aligned}$$

$$(1 + i) - (1 - i) = 2i$$

Example 3: Solve

$$\begin{aligned}ix_1 + (2 - i)x_2 &= 4 - 3i \\(1 + i)x_1 - x_2 &= -3 - i\end{aligned}$$

Solution:

Solving the second equation for x_2 , we have

$$x_2 = (1 + i)x_1 + 3 + i$$

Substituting in the first equation, we find

$$\begin{aligned}ix_1 + (2 - i)[(1 + i)x_1 + 3 + i] &= 4 - 3i \\ix_1 + (2 - i)(1 + i)x_1 + 3(2 - i) + i(2 - i) &= 4 - 3i \\ix_1 + (2 + i - i^2)x_1 + 6 - 3i + 2i - i^2 &= 4 - 3i \\ix_1 + 2x_1 + ix_1 - i^2x_1 + 6 - i - i^2 &= 4 - 3i \\2ix_1 + 2x_1 + x_1 + 6 - i + 1 &= 4 - 3i \\2ix_1 + 3x_1 + 7 - i &= 4 - 3i \\(2i + 3)x_1 = -3 - 2i = -(2i + 3)\end{aligned}$$

$$\Rightarrow x_1 = -\frac{(2i + 3)}{(2i + 3)} = -1$$

Substituting into equation for x_2 , leads to

$$\Rightarrow x_2 = (1 + i)(-1) + 3 + i = -1 - i + 3 + i = 2$$

The solution will be as pair ordered: $(x_1, x_2) = (1, 2)$

A system of Linear Equations

A **system of linear equations** (or **linear system**) is a **finite collection** of **linear equations** in **same variables**. For instance, a **linear system** of m equations in n variables $x_1, x_2, x_3, \dots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

A **solution** of a **linear system** is a set $(s_1, s_2, s_3, \dots, s_n)$ of **numbers** that makes **each equation** true statement when the **values** $s_1, s_2, s_3, \dots, s_n$ are **substituted** for $x_1, x_2, x_3, \dots, x_n$, respectively.

$$s_1 = x_1, s_2 = x_2, s_3 = x_3, \dots, s_n = x_n$$

The **set of all solutions** of a **linear system** is called the **solution set** of the system.

Classification of linear systems

Theorem 1: A **linear system** is said to be **consistent** if it has at least **one solution**; and is said to be **inconsistent** if it has **no solution**.

In **two variables**, here is an **example** of a **system of two equations**:

$$\begin{aligned} 2x + y &= 3 \\ x - 9y &= -8 \end{aligned}$$

Clearly, $x = 1, y = 1$ is the **(only) solution** to this system.

Such a system would be called a **consistent. Geometrically**, solution given by **precisely the point** where the **graphs (two lines)** of these two equations **meet (intersect)**.

- If $b_1 = b_2 = \dots = b_m = 0$, Such a **linear system** is called a **homogeneous linear system**.
- Two systems of **linear equations** are called **equivalent**, if they have precisely the **same set of solutions**.
- **Following operations** on a system **produces** an **equivalent system**:
 - i. **Interchange of any two equations.**
 - ii. **Multiplication of any equation by a nonzero constant.**
 - iii. **Addition to an equation the result of multiplying another equation by a constant.**

These three operations are sometimes **known** as **basic** or **elementary operations**.

- A linear system of the form

$$\begin{array}{r}
 x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\
 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\
 + \phantom{a_{23}x_3} + \dots + a_{3n}x_n = b_3 \\
 \vdots \phantom{a_{23}x_3} \dots \phantom{a_{3n}x_n} = \\
 \phantom{a_{23}x_3} \dots \phantom{a_{3n}x_n} =
 \end{array}$$

is said to be in **row-echelon form**. The point is:

- (a) drop one variable in each successive equation (step),
- (b) The coefficient of the "leading variable" in equation is 1.

In two variables x, y the **row-echelon form** would (sometime) look like

$$\begin{array}{r}
 x + a_{12}y = b_1 \\
 + \phantom{a_{12}y} = b_2
 \end{array}$$

In three variables x , y , z this row-echelon form would (sometime) look like

$$\begin{aligned}x + a_{12}y + a_{13}z &= b_1 \\y + a_{23}z &= b_2 \\z &= b_3\end{aligned}$$

Theorem 2: The following are some facts:

1. Any system of linear equations is equivalent to a linear system in row echelon form.
2. This can be achieved by a sequence of application of the three basic elementary operation described in the next example
3. This process is known as Gaussian elimination

Example 4: Reduce and solve the following system by Gaussian elimination method

$$4x - 5y = 3 \quad (1)$$

$$-8x + 10y = 14 \quad (2)$$

Solution:

Multiply Eq. (1) by 2 and add to Eq. (2), yields

$$4x - 5y = 3 \quad (1)$$

$$0 = 20 \quad (3)$$

The Eq. (3) is absurd. So, the system has no solution.

Thus, the system is inconsistent.

Example 5: Reduce the following system and solve:

$$9x - 4y = 5 \quad (1)$$

$$\frac{1}{2}x + \frac{1}{3}y = 0 \quad (2)$$

Solution:

Multiplying Eq. (2) by 18 and subtract Eq. (1) from Eq. (2), yields

$$9x - 4y = 5 \quad (1)$$

$$10y = -5 \quad (3)$$

Dividing Eq. (3) by 10 and Eq. (1) by 9, yields:

$$x - \frac{4}{9}y = \frac{5}{9} \quad (4)$$

$$y = -\frac{1}{2} \quad (5)$$

This is an **equivalent row-echelon form** of the **given system**

Now substitute $y = -\frac{1}{2}$ in Eq. (4) yields

$$x - \frac{4}{9}\left(-\frac{1}{2}\right) = \frac{5}{9} \Rightarrow x = \frac{5 - 2}{9} = \frac{1}{3}$$

Thus, the **solution** is the pair ordered:

$$(x, y) = \left(\frac{1}{3}, -\frac{1}{2}\right)$$

Thus, the **system** is **consistent**.

Example 6: Deduce an equivalent row-echelon form and solve the following system

$$2x + y - z = -1$$

$$x - 2y + z = 5$$

$$3x - y - 2z = 0$$

Solution:

Interchange the first and second equation, yields

$$x - 2y + z = 5$$

$$2x + y - z = -1$$

$$3x - y - 2z = 0$$

Multiply the first equation by (-2) and **add** to the second equation, yields

$$2x + y - z = -1$$

$$\underline{-2x + 4y - 2z = -10}$$

$$5y - 3z = -11$$

The system becomes

$$x - 2y + z = 5$$

$$5y - 3z = -11$$

$$3x - y - 2z = 0$$

Multiply the first equation by (-3) and **add** to the third equation, yields

$$3x - y - 2z = 0$$

$$\underline{-2x + 6y - 3z = -15}$$

$$5y - 5z = -15$$

The system becomes

$$x - 2y + z = 5$$

$$5y - 3z = -11$$

$$5y - 5z = -15$$

Multiply the second equation by (-1) and **add** to the third equation to eliminate y in the third equation :

$$\begin{array}{r}
 5y - 5z = -15 \\
 -5y + 3z = 11 \\
 \hline
 -2z = -4
 \end{array}$$

Thus, the **row-echelon form** which is an **equivalent system** to the **given system** is

$$\begin{array}{r}
 x - 2y + z = 5 \\
 5y - 3z = -11 \\
 -2z = -4
 \end{array}$$

From the **last equation** of the row echelon form yields

$$z = 2,$$

and **substituting** into **second equation** will give

$$y = -1.$$

Then **substituting** the **obtained values** of **z** and **y** into **first equation** will get

$$x = 1.$$

Therefore, the **solution** is:

$$x = 1, \quad y = -1, \quad \text{and} \quad z = 2$$

Gaussian, Gauss-Jordan Elimination

Definition 4: Given a system of m linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the **augmented matrix** of the system is defined as

$$\left(\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array} \right)$$

and the **coefficient matrix** is defined as

$$\left(\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & \\ \vdots & \vdots & \vdots & & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \end{array} \right)$$

❖ Consider the **linear system** of two equations in two variables

$$\begin{aligned} 4x - 5y &= 3 \\ -8x + 10y &= 14 \end{aligned}$$

The **augmented matrix** of the system is:

$$\left(\begin{array}{ccc} 4 & -5 & 3 \\ -8 & 10 & 14 \end{array} \right)$$

and the **coefficient matrix** is:

$$\left(\begin{array}{cc} 4 & -5 \\ -8 & 10 \end{array} \right)$$

Example: For the following **linear system** of **three equations** in **three variables**, deduce an **equivalent row-echelon form** and solve

$$5x_1 - 3x_2 + 2x_3 = 3 \quad (1)$$

$$2x_1 + 4x_2 - x_3 = 7 \quad (2)$$

$$x_1 - 11x_2 + 4x_3 = 3 \quad (3)$$

The **augmented** and **coefficient matrices** of the system are:

$$\left(\begin{array}{cccc} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \end{array} \right); \left(\begin{array}{ccc} 5 & -3 & 2 \\ 2 & 4 & -1 \\ 1 & -11 & 4 \end{array} \right)$$

To deduce an **equivalent system** in **row-echelon form**, first **switching Eq. (1) and Eq. (3)**:

$$x_1 - 11x_2 + 4x_3 = 3 \quad (3)$$

$$2x_1 + 4x_2 - x_3 = 7 \quad (2)$$

$$5x_1 - 3x_2 + 2x_3 = 3 \quad (1)$$

Multiplying Eq. (3) by 2 and subtract from Eq. (2),

$$\begin{array}{r} 2x_1 + 4x_2 - x_3 = 7 \\ 2x_1 - 22x_2 + 8x_3 = 6 \\ \hline 26x_2 - 9x_3 = 1 \end{array}$$

The system becomes

$$x_1 - 11x_2 + 4x_3 = 3 \quad (3)$$

$$26x_2 - 9x_3 = 1 \quad (4)$$

$$5x_1 - 3x_2 + 2x_3 = 3 \quad (1)$$

Multiplying Eq. (3) by 5 and subtract from Eq. (1):

$$\begin{array}{r} 5x_1 - 3x_2 + 2x_3 = 3 \\ 5x_1 - 55x_2 + 20x_3 = 15 \\ \hline 52x_2 - 18x_3 = -12 \end{array}$$

The system becomes

$$x_1 - 11x_2 + 4x_3 = 3 \quad (3)$$

$$26x_2 - 9x_3 = 1 \quad (4)$$

$$52x_2 - 18x_3 = -12 \quad (5)$$

Multiplying Eq. (4) by 2 and subtract from Eq. (5)

$$\begin{array}{r} 52x_2 - 18x_3 = -12 \\ 52x_2 - 18x_3 = 2 \\ \hline 0 = -14 \end{array}$$

The system becomes

$$x_1 - 11x_2 + 4x_3 = 3 \quad (3)$$

$$26x_2 - 9x_3 = 1 \quad (4)$$

$$0 = -14 \quad (6)$$

Eq. (6) is absurd, so the system is inconsistent. To obtain the **row-echelon form**, **Eq. (4)** need to be **divided** by **26**:

$$x_1 - 11x_2 + 4x_3 = 3 \quad (3)$$

$$x_2 - \frac{9}{26}x_3 = \frac{1}{26} \quad (7)$$

$$0 = -14 \quad (6)$$

The **augmented** and **matrix** of the system are:

$$\left(\begin{array}{cccc} 1 & -11 & 4 & 3 \\ 0 & 1 & -\frac{9}{26} & \frac{1}{26} \\ 0 & 0 & 0 & -14 \end{array} \right); \left(\begin{array}{ccc} 1 & -11 & 4 \\ 0 & 1 & -\frac{9}{26} \\ 0 & 0 & 0 \end{array} \right)$$

This **examples demonstrates** that the **three basic operations** that were used to **reduce a system of linear equations** to a **row-echelon form**, can be **translated** to a **version** for **matrices**.

Definition 5: By an **elementary row operation** on a **matrix** which are the **following three:**

- iv. **Interchange of any two rows.**
- v. **Multiplication of any row by a nonzero constant**
- vi. **Addition to any row the result of multiplying another row by a constant**

❖ **Two matrices** are said to be **row-equivalent** if **one can be obtained from another** by **application** of a **sequence of elementary row operations**. **Two row-equivalent matrices**, correspond to **two equivalent system of equations**.

Definition 6: A **matrix** is said to be in **row-echelon form**, if it **has the following properties:**

1. **All rows consisting entirely of zeros occur at the bottom of the matrix.**
2. **For each non-zero row, first nonzero entry is 1 (called the leading 1).**
3. **For each successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.**

A **matrix in row-echelon form** is said to be in **reduced row-echelon form**, if **every column that has a leading 1, has zeros in every position above and below the leading 1.**

Theorem 3: Suppose M is a **matrix**. Then, M is **row-equivalent** to a **matrix** B , which is in **row-echelon form**.

❖ The **method of solving** this system by **Gaussian elimination** with **back-substitution equation** is described as follows:

- i. Write the **augmented matrix** of the system.
- ii. Use the **elementary row operations** to **reduce** the **augmented matrix** to a **matrix** in **row-echelon form**.
- iii. Write the **linear system** corresponding to the **row-echelon matrix** and solve by **back-substitution**.

Example: Use the **method of Gaussian elimination** with **beck-substitution** to solve the **following system of linear equations**

$$\begin{aligned}x_1 + 3x_4 &= 4 \\2x_2 - x_3 - x_4 &= 0 \\3x_2 - 2x_4 &= 1 \\2x_1 - x_2 + 4x_3 &= 5\end{aligned}$$

Solution:

The augmented matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 2 & -1 & 4 & 0 & 5 \end{pmatrix}$$

Multiplying row-1 by(2) and subtract from row-4:

$$\begin{array}{ccccc} 2 & -1 & 4 & 0 & 5 & \text{Row-4} \\ 2 & 0 & 0 & 6 & 8 & \text{Row-1 multiplied by 2} \\ \hline 0 & -1 & 4 & -6 & -3 & \text{Row-4} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Multiplying row-2 by $\frac{1}{2}$ and add to row-4:

$$\begin{array}{ccccc} 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \text{Row-2 multiplied by } \frac{1}{2} \\ 0 & -1 & 4 & -6 & -3 & \text{Row-4} \\ \hline 0 & 0 & \frac{7}{2} & \frac{13}{2} & -3 & \text{Row-4} \end{array}$$

The augmented matrix becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & 0 & \frac{7}{2} & \frac{13}{2} & -3 \end{pmatrix}$$

Multiplying row-2 by 3 and subtract from row-3:

$$\begin{array}{ccccc} 0 & 3 & 0 & -2 & 1 & \text{Row-3} \\ 0 & 3 & -\frac{3}{2} & -\frac{3}{2} & 0 & \text{Row-2 multiplied by 3} \\ \hline 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & \text{Row-3} \end{array}$$

The augmented matrix becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Multiplying row-3 by $\frac{2}{3}$, the **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{7}{2} & -\frac{13}{2} & -3 \end{pmatrix}$$

Multiplying row-3 by $\frac{7}{2}$ and **subtract** from row-4:

$$\begin{array}{ccccc} 0 & 0 & \frac{7}{2} & -\frac{13}{2} & -3 & \text{Row-4} \\ 0 & 0 & \frac{7}{2} & -\frac{7}{6} & -\frac{14}{6} & \text{Row-3 multiplied by } \frac{7}{2} \\ \hline 0 & 0 & 0 & -\frac{16}{3} & -\frac{16}{3} & \text{Row-4} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & -\frac{16}{3} & -\frac{16}{3} \end{pmatrix}$$

Multiplying row-4 by $-\frac{3}{16}$, the **augmented matrix** becomes

The **augmented matrix** becomes

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & -\frac{\mathbf{1}}{\mathbf{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{3}} & \frac{\mathbf{2}}{\mathbf{3}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

The above is a **matrix** in **row-echelon form row-equivalent** to the **augmented matrix**. Now the **system of linear equations** corresponding this **row-echelon matrix** is:

$$\begin{aligned} x_1 + 3x_4 &= 4 \\ x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 &= 0 \\ x_3 - \frac{1}{3}x_4 &= \frac{2}{3} \\ x_4 &= 1 \end{aligned}$$

Substituting $x_4 = 1$ into **third** and **first equation** we have

$$x_3 = \frac{2}{3} + \frac{1}{3} = 1 \quad \text{and} \quad x_1 = 1$$

Substituting $x_4 = 1$ and $x_3 = 1$ into **second equation** we have

$$x_2 = 1$$

By **back-substitution**, we have

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 1$$

Gauss-Jordan elimination

Definition 5: A matrix in row-echelon form is said to be in **Gauss – Jordan form**, if **all the entries above leading entries are zero**. The **method of Gaussian elimination with back substitution to solve system of linear equations can be refined by first further reducing the augmented matrix to a Gauss-Jordan form and work with the system corresponding to it**. This method is called **Gauss-Jordan elimination method of solving linear systems**.

Considering previous example, the matrix in the row-echelon form, equivalent to the augmented matrix, is

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

All the **entries above the leading 1 in row 2 is zero**. So, **trying to achieve the same above the leading 1 in row 3:**

Multiply row-3 by $\frac{1}{2}$ and add from row-2 yields

$$\begin{array}{ccccc} 0 & 0 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} & \text{Row-3 multiplied by } \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \text{Row-2} \\ \hline 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \text{Row-2} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Now we want to **get zeros above the leading 1 in row-4.**

Multiply row-4 by **3** and **subtract** from **row-1**, yields

$$\begin{array}{r} \begin{array}{ccccc} 1 & 0 & 0 & 3 & 4 & \text{Row-1} \\ 0 & 0 & 0 & 3 & 3 & \text{Row-4 multiplied by 3} \\ \hline 1 & 0 & 0 & 0 & 1 & \text{Row-1} \end{array} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Multiply row-4 by $\frac{2}{3}$ and **add** to **row-2**, yields

$$\begin{array}{r} \begin{array}{ccccc} 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \text{Row-4 multiplied by } \frac{2}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \text{Row-2} \\ \hline 0 & 1 & 0 & 0 & 1 & \text{Row-2} \end{array} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Multiply row-4 by $\frac{1}{3}$ and **add** to row-3, yields

$$\begin{array}{ccccc}
 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \text{Row-4 multiplied by } \frac{1}{3} \\
 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & \text{Row-3} \\
 \hline
 0 & 0 & 1 & 0 & 1 & \text{Row-3}
 \end{array}$$

The **augmented matrix** becomes

$$\left(\begin{array}{ccccc}
 1 & 0 & 0 & 0 & 1 \\
 0 & 1 & 0 & 0 & 1 \\
 0 & 0 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 1
 \end{array} \right)$$

This **matrix** is in **Gauss-Jordan form**.

The **system of linear equation** corresponding to **this one** is:

$$x_1 + 0 + 0 + 0 = 1$$

$$0 + x_2 + 0 + 0 = 1$$

$$0 + 0 + x_3 + 0 = 1$$

$$0 + 0 + 0 + x_4 = 1$$

So, the **solution** to the **system** is:

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 1$$

Example 2: Solve the following **system of linear equations** using **Gaussian elimination** or **Gauss-Jordan elimination**:

$$2x_1 - 2x_2 + 3x_3 = 24 \quad \text{Eq. (1)}$$

$$2x_2 - x_3 = 14 \quad \text{Eq. (2)}$$

$$7x_1 - 5x_2 = 6 \quad \text{Eq. (3)}$$

Solution:The **augmented matrix** is:

$$\begin{pmatrix} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{pmatrix}$$

Divide row-1 by 2 and row-2 by 2, the **augmented matrix** becomes:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 7 & -5 & 0 & 6 \end{pmatrix} \begin{array}{l} \text{Row-1 divided by 2} \\ \text{Row-2 divided by 2} \end{array}$$

Multiply row-1 by 7 and **subtract** from row 3, yields

$$\begin{array}{r} 7 \quad -5 \quad 0 \quad 6 \quad \text{Row-3} \\ 7 \quad -\frac{7}{2} \quad \frac{21}{2} \quad 84 \quad \text{Row-1 multiplied by 7} \\ \hline 0 \quad -\frac{3}{2} \quad -\frac{21}{2} \quad -78 \quad \text{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & -\frac{3}{2} & -\frac{21}{2} & -78 \end{pmatrix}$$

Multiply row-2 by $\frac{3}{2}$ and **add** to row-3:

$$\begin{array}{r} 0 \quad \frac{3}{2} \quad -\frac{3}{4} \quad \frac{21}{2} \quad \text{Row-3} \\ 0 \quad -\frac{3}{2} \quad -\frac{21}{2} \quad -78 \quad \text{Row-2 multiplied by } \frac{3}{2} \\ \hline 0 \quad 0 \quad -\frac{45}{4} \quad -\frac{135}{2} \quad \text{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & -\frac{45}{4} & -\frac{135}{2} \end{pmatrix}$$

Multiply row-3 by $-\frac{4}{45}$, the **augmented matrix** becomes:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix} \leftarrow \text{Row-3 multiplied by } -\frac{4}{45}$$

The **above matrix** is in **row-echelon form**. So, using **back substitution** and solve the system. The system corresponding to this matrix is:

$$x_1 - \frac{1}{2}x_2 + \frac{3}{2}x_3 = 12$$

$$x_2 - \frac{1}{2}x_3 = 7$$

$$x_3 = 6$$

By **back-substitution**, we have

$$x_3 = 6, x_2 = 7 + \frac{1}{2} \cdot 6 = 10, \text{ and } x_1 = 12 - \frac{3}{2} \cdot 6 + \frac{1}{2} \cdot 10 = 8$$

Alternately, the **row-echelon matrix**

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

could be **reduced** to a **Gauss-Jordan form**. To do this, **multiply row-2** by $\frac{1}{2}$ and **add** to **row-1**, yields

$$\begin{array}{cccccl}
 1 & -\frac{1}{2} & \frac{3}{2} & 12 & \text{Row-1} \\
 0 & \frac{1}{2} & -\frac{1}{4} & \frac{7}{2} & \text{Row-2 multiplied by } \frac{1}{2} \\
 \hline
 1 & 0 & \frac{5}{4} & \frac{31}{2} & \text{Row-1}
 \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & \frac{5}{4} & \frac{31}{2} \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

Multiply row-3 by $\frac{5}{4}$ and **subtract** from **row-1**, yields

$$\begin{array}{cccccl}
 0 & 1 & \frac{5}{4} & \frac{31}{2} & \text{Row-1} \\
 0 & 0 & \frac{5}{4} & \frac{15}{2} & \text{Row-3 multiplied by } \frac{5}{4} \\
 \hline
 0 & 1 & 0 & 8 & \text{Row-1}
 \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

Multiply row-3 by $\frac{1}{2}$ and **add** from **row-2**, yields

$$\begin{array}{cccccl}
 0 & 1 & -\frac{1}{2} & 7 & \text{Row-2} \\
 0 & 0 & \frac{1}{2} & 3 & \text{Row-3 multiplied by } \frac{1}{2} \\
 \hline
 0 & 1 & 0 & 10 & \text{Row-2}
 \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

This **matrix** is in **Gauss-Jordan form**. The system of linear equations corresponding to this matrix is:

$$x_1 + 0 + 0 = 8$$

$$0 + x_2 + 0 = 10$$

$$x_3 = 6$$

This gives the solution of this system:

$$x_1 = 8 \quad , \quad x_2 = 10 \quad , \quad \text{and} \quad x_3 = 6$$

Example 3: Solve the following system of linear equations using Gaussian elimination or Gauss-Jordan elimination:

$$2x_1 + 3x_3 = 3 \quad \text{Eq. (1)}$$

$$4x_1 - 3x_2 + 7x_3 = 5 \quad \text{Eq. (2)}$$

$$8x_1 - 9x_2 + 15x_3 = 10 \quad \text{Eq. (3)}$$

Solution:

The augmented matrix is:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{pmatrix}$$

First, reduce this matrix to row-echelon form.

Multiply row-1 by 2 and **subtract** from row-2, yields

$$\begin{array}{cccc} 4 & -3 & 7 & 5 & \text{Row-2} \\ 4 & 0 & 6 & 6 & \text{Row-1 multiplied by 2} \\ \hline 0 & -3 & 1 & -1 & \text{Row-2} \end{array}$$

The augmented matrix becomes:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 8 & -9 & 15 & 10 \end{pmatrix}$$

Multiply row-1 by 4 and **subtract** from row-3, yields

$$\begin{array}{cccc} 8 & -9 & 15 & 10 & \text{Row-3} \\ 8 & 0 & 12 & 12 & \text{Row-1 multiplied by 4} \\ \hline 0 & -9 & -9 & -14 & \text{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{pmatrix}$$

Multiply row-2 by 3 and **subtract** from row-3, yields

$$\begin{array}{cccc|l} 0 & -9 & 3 & -2 & \text{Row-3} \\ 0 & -9 & 3 & -3 & \text{Row-1 multiplied by 2} \\ \hline 0 & 0 & 0 & 1 & \text{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Divide row-1 by 2 and **row-2** by -3 , the **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The **matrix** is in **row-echelon form**. The **system** of these equations corresponding to this matrix is:

$$\begin{aligned} x_1 + \frac{3}{2}x_3 &= \frac{3}{2} \\ x_2 - \frac{1}{3}x_3 &= \frac{1}{3} \\ 0 &= 1 \end{aligned}$$

The last equation is **absurd**. So, the system is **inconsistent**.

Example 4: Solve the **homogeneous linear system** corresponding to the **coefficient matrix**:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

Since it is a **homogeneous system**, i.e. it has **all the constant terms equal zero** and they system is:

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

The system is **already** in **row-echelon form**. So, by **back substitution** yields

$$x_1 = 0 \text{ and } x_2 = -x_3$$

With $x_3 = t$ a parametric solution is:

$$x_1 = 0, \quad x_2 = -t, \quad \text{and} \quad x_3 = t$$

Example 5: Consider the system of linear equations

$$\begin{aligned} x + y &= 0 \\ y + z &= 0 \\ x + z &= 0 \\ ax + by + cz &= 0 \end{aligned}$$

Find the values of a , b , and c such that the system has:

- (a) a unique solution,
- (b) no solution,
- (c) an infinite number of solution.

Solution:

The **augmented matrix** of the given system is:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ a & b & c & 0 \end{pmatrix}$$

Subtract row-1 from **row-3**, and **multiply row-1** by a and **subtract** from **row-4**. the **augmented matrix** will become:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & b-a & c & 0 \end{pmatrix}$$

Add row-2 to row-3, the **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & b-a & c & 0 \end{pmatrix}$$

Divide row-3 by 2, the **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & b-a & c & 0 \end{pmatrix}$$

Multiply row-2 by $(a - b)$ and add to row-4, the **augmented matrix** will become:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c+a-b & 0 \end{pmatrix}$$

Multiply row-3 by $(c + a - b)$ and subtract from row-4, the **augmented matrix** will become:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The **matrix** is in **row-echelon form**. The corresponding linear system is:

$$\begin{aligned} x + y &= 0 \\ y + z &= 0 \\ z &= 0 \\ 0 &= 0 \end{aligned}$$

The system is **consistent** for all values of a, b, c , and by **back substitution** the system has **unique solution**

$$x = y = z = 0$$

Iterative Methods for Solving System of Linear Equations

An “iterative method” is one which is used repeatedly until the results obtained acquire a pre-assigned degree of accuracy. For example, if results are required to five places of decimals, the number of “iterations” is continued until two consecutive iterations give the same result when rounded off to that number of decimal places. The two methods that are used for solving system of linear equations are:

❖ The Jacobi`s or Gauss- Jacobi Method

The first iterative technique is called the Gauss-Jacobi or Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851).

Consider that the system given by

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3$$

has a unique solution. To begin the Jacobi method, solve the first equation for x , the second equation for y , and the third equation for z , as follows:

$$x_1 = \frac{k_1 - b_1y_0 - c_1z_0}{a_1}$$

$$y_1 = \frac{k_2 - a_2x_0 - c_2z_0}{b_2}$$

$$z_1 = \frac{k_3 - a_3x_0 - b_3y_0}{c_3}$$

Then using an initial approximation $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$ and substitute into the right-hand side of the rewritten equations to obtain the first approximation. After this procedure has been completed, one iteration has been performed. In the same way, the second approximation is formed by substituting the first approximation of x , y , z values into the right-hand side of the rewritten equations.

By repeating iterations, a sequence of approximations will be formed that often converges to the actual solution. This procedure is illustrated in the following example.

Example 1: Use the Gauss-Jacobi method to approximate the solution of the following system of linear equations

$$5x - 2y + 3z = -1$$

$$-3x + 9y + z = 2$$

$$2x - y - 7z = 3$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Solution:

To begin, writing the system in the form

$$x = \frac{-1 + 2y - 3z}{5}$$

$$y = \frac{2 + 3x - z}{9}$$

$$z = \frac{-3 + 2x - y}{7}$$

Using $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$, as a convenient initial approximation gives the first approximation i:

$$x_1 = \frac{-1 + 2y_0 - 3z_0}{5} = -\frac{1}{5} = -0.2$$

$$y_1 = \frac{2 + 3x_0 - z_0}{9} = \frac{2}{9} = 0.222$$

$$z_1 = \frac{-3 + 2x_0 - y_0}{7} = -\frac{3}{7} = -0.429$$

$$x_2 = \frac{-1 + 2y_1 - 3z_1}{5} = \frac{-1 + 2(0.222) - 3(-0.429)}{5}$$

$$\Rightarrow x_2 = \frac{0.731}{5} = 0.146$$

$$y_2 = \frac{2 + 3x_1 - z_1}{9} = \frac{2 + 3(-0.2) - (-0.429)}{9}$$

$$\begin{aligned} \Rightarrow y_2 &= \frac{1.829}{9} = 0.203 \\ z_2 &= \frac{-3 + 2x_1 - y_1}{7} = \frac{-3 + 2(-0.2) - 0.222}{7} \\ \Rightarrow z_2 &= -\frac{3.622}{7} = -0.517 \end{aligned}$$

Continuing this procedure, the sequence of approximations can be obtained as shown in following table:

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|--------|--------|--------|--------|--------|--------|--------|
| x | -0.2 | 0.146 | 0.192 | 0.181 | 0.185 | 0.186 | 0.186 |
| y | 0.222 | 0.203 | 0.328 | 0.332 | 0.329 | 0.331 | 0.331 |
| z | -0.429 | -0.517 | -0.416 | -0.421 | -0.424 | -0.423 | -0.423 |

Because the last two results are identical, it can be concluded that to three significant digits the solution is:

$$x = 0.186, \quad y = 0.331, \quad z = -0.423$$

❖ The Gauss-Seidel Method

This method differs from the Gauss-Jacobi method in that successive approximations are used within each step as soon as they become available.

The rate of convergence of this method is usually faster than that of the Jacobi method. The scheme of the calculations is according to the following pattern:

$$\begin{aligned} x_1 &= \frac{k_1 - b_1 y_0 - c_1 z_0}{a_1} \\ y_1 &= \frac{k_2 - a_2 x_1 - c_2 z_0}{b_2} \\ z_1 &= \frac{k_3 - a_3 x_1 - b_3 y_1}{c_3} \end{aligned}$$

Example 2: Use the **Gauss-Seidel method** to approximate the solution of the following system of linear equations

$$5x - 2y + 3z = -1$$

$$-3x + 9y + z = 2$$

$$2x - y - 7z = 3$$

Continue the iterations until two successive approximations are **identical** when rounded to **three significant digits**.

Solution:

To begin, writing the system in the form

$$x = \frac{-1 + 2y - 3z}{5}$$

$$y = \frac{2 + 3x - z}{9}$$

$$z = \frac{-3 + 2x - y}{7}$$

Using $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$, as a **convenient initial approximation**, yields the **first approximation** is:

$$x_1 = \frac{-1 + 2y_0 - 3z_0}{5} = -\frac{1}{5} = -0.2$$

$$y_1 = \frac{2 + 3x_1 - z_0}{9} = \frac{2 + 3(-0.2) - 0}{9} = \frac{1.4}{9} = 0.156$$

$$z_1 = \frac{-3 + 2x_1 - y_1}{7} = \frac{-3 + 2(-0.2) - 0.156}{7}$$

$$\Rightarrow z_1 = \frac{-3.556}{7} = -0.508$$

$$x_2 = \frac{-1 + 2y_1 - 3z_1}{5} = \frac{-1 + 2(0.156) - 3(-0.508)}{5}$$

$$\Rightarrow x_2 = \frac{0.838}{5} = 0.167$$

$$y_2 = \frac{2 + 3x_2 - z_1}{9} = \frac{2 + 3(0.167) - (-0.508)}{9}$$

$$\Rightarrow y_2 = \frac{3.009}{9} = 0.334$$

$$z_2 = \frac{-3 + 2x_2 - y_2}{7} = \frac{-3 + 2(0.167) - 0.334}{7}$$

$$\Rightarrow z_2 = -\frac{3}{7} = -0.429$$

Continued iterations procedure, the sequence of approximations can be obtained as shown in following table:

| <i>n</i> | 1 | 2 | 3 | 4 | 5 |
|----------|--------|--------|--------|--------|--------|
| <i>x</i> | -0.2 | 0.167 | 0.191 | 0.186 | 0.186 |
| <i>y</i> | 0.156 | 0.334 | 0.333 | 0.331 | 0.331 |
| <i>z</i> | -0.508 | -0.429 | -0.422 | -0.423 | -0.423 |

Note that after only five iterations of the Gauss-Seidel method, same accuracy achieved the as was obtained with seven iterations of the Gauss-Jacobi method in Example 1.

Example 3: Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$5x + y - z = 4$$

$$x + 4y + 2z = 15$$

$$x - 2y + 5z = 12$$

Obtaining *x*, *y* and *z* correct to the nearest integer.

Solution:

Write the system in the form

$$x_1 = \frac{4 - y + z}{5}$$

$$y_1 = \frac{15 - x - 2z}{4}$$

$$z_1 = \frac{12 - x + 2y}{5}$$

Using an initial approximation $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$, yields

$$x_1 = \frac{4 - y_0 + z_0}{5} = \frac{4 - 0 + 0}{5} = \frac{4}{5} = 0.8$$

$$y_1 = \frac{15 - x_0 - 2z_0}{4} = \frac{15 - 0 - 2(0)}{4} = \frac{15}{4} = 3.75$$

$$z_1 = \frac{12 - x_0 + 2y_0}{5} = \frac{12 - 0 + 2(0)}{5} = \frac{12}{5} = 2.4$$

$$x_2 = \frac{4 - y_1 + z_1}{5} = \frac{4 - 3.75 + 2.4}{5} = 0.53$$

$$y_2 = \frac{15 - x_1 - 2z_1}{4} = \frac{15 - 0.8 + 2(2.4)}{4} = 2.35$$

$$z_2 = \frac{12 - x_1 + 2y_1}{5} = \frac{12 - 0.8 + 2(3.75)}{5} = 3.74$$

$$x_3 = \frac{4 - y_2 + z_2}{5} = \frac{4 - 2.35 + 3.74}{5} = 1.078$$

$$y_3 = \frac{15 - x_2 - 2z_2}{4} = \frac{15 - 0.53 - 2(3.74)}{4} = 1.748$$

$$z_3 = \frac{12 - x_2 + 2y_2}{5} = \frac{12 - 0.53 + 2(2.35)}{5} = 3.234$$

$$x_4 = \frac{4 - y_3 + z_3}{5} = \frac{4 - 1.748 + 3.234}{5} = 1.097$$

$$y_4 = \frac{15 - x_3 - 2z_3}{4} = \frac{15 - 1.078 - 2(3.234)}{4} = 1.864$$

$$z_4 = \frac{12 - x_3 + 2y_3}{5} = \frac{12 - 1.078 + 2(1.748)}{5} = 2.884$$

$$x_5 = \frac{4 - y_4 + z_4}{5} = \frac{4 - 1.864 + 2.884}{5} = 1.004$$

$$y_5 = \frac{15 - x_4 - 2z_4}{4} = \frac{15 - 1.097 - 2(2.884)}{4} = 2.034$$

$$z_5 = \frac{12 - x_4 + 2y_4}{5} = \frac{12 - 1.097 + 2(1.864)}{5} = 2.926$$

$$x_6 = \frac{4 - y_5 + z_5}{5} = \frac{4 - 2.034 + 2.926}{5} = 0.978$$

$$y_6 = \frac{15 - x_5 - 2z_5}{4} = \frac{15 - 1.004 - 2(2.926)}{4} = 2.036$$

$$z_6 = \frac{12 - x_5 + 2y_5}{5} = \frac{12 - 1.004 + 2(2.034)}{5} = 3.013$$

$$x_7 = \frac{4 - y_6 + z_6}{5} = \frac{4 - 2.036 + 3.013}{5} = 0.995$$

$$y_7 = \frac{15 - x_6 - 2z_6}{4} = \frac{15 - 0.978 - 2(3.013)}{4} = 1.999$$

$$z_7 = \frac{12 - x_6 + 2y_6}{5} = \frac{12 - 0.978 + 2(2.036)}{5} = 3.019$$

$$x_8 = \frac{4 - y_7 + z_7}{5} = \frac{4 - 1.999 + 3.019}{5} = 1.004$$

$$y_8 = \frac{15 - x_7 - 2z_7}{4} = \frac{15 - 0.995 - 2(3.019)}{4} = 1.992$$

$$z_8 = \frac{12 - x_7 + 2y_7}{5} = \frac{12 - 0.995 + 2(1.999)}{5} = 3.001$$

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|------|------|-------|-------|-------|-------|-------|-------|
| x | 0.8 | 0.53 | 1.078 | 1.097 | 1.004 | 0.978 | 0.995 | 1.004 |
| y | 3.75 | 2.35 | 1.748 | 1.864 | 2.034 | 2.036 | 1.999 | 1.992 |
| z | 2.4 | 3.74 | 3.234 | 2.884 | 2.926 | 3.013 | 3.019 | 3.001 |

From the table it is evident that the results of the last two iterations when rounded to the nearest integer, both give

$$x = 1 \quad , \quad y = 2 \quad , \quad \text{and} \quad z = 3$$

Example4: Apply the **Jacobi method** to the system

$$x_1 - 5x_2 = -4$$

$$7x_1 - x_2 = 6$$

using the initial approximation $(x_1, x_2) = (0, 0)$ and show that the method diverges.

Solution:

Rewriting the given system in the form

$$x_1 = 5x_2 - 4$$

$$x_2 = 7x_1 - 6$$

Using the initial approximation $x_1 = 0$ and $x_2 = 0$, yields

$$x_1 = 5(0) - 4 = -4$$

$$x_2 = 7(0) - 6 = -6$$

As the first approximation.

Repeating iterations produce the sequence of approximations shown in the following table:

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|----|-----|------|-------|-------|---------|----------|
| x_1 | 0 | -4 | -34 | -174 | -1244 | -6124 | -42.874 | -214.374 |
| x_2 | 0 | -6 | -34 | -244 | -1244 | -8574 | -42.874 | -300.124 |

It can be seen from the table that the approximations given by the **Jacobi method** become progressively worse instead of better, and can be concluded that the method diverges.

The problem of divergence is not resolved by using the **Gauss-Seidel method** rather than the **Jacobi method**. In fact, for this particular system the **Gauss-Seidel method diverges more rapidly**, as shown in the following table.

is **strictly diagonally dominant** because $|3| > |-1|$ and $|5| > |2|$

b) The coefficient matrix is

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 2 \\ 3 & -5 & 1 \end{bmatrix}$$

where,

$$|3| > |2| + |1| = |3|,$$

$$|0| < |1| + |2| = |3|, \text{ and}$$

$$|1| < |3| + |5| = |8|$$

Thus, the **coefficient matrix** is **not strictly diagonally dominant** because the **entries** in the **second** and **third** rows do not conform to the **definition**. For instance, in the **second** row $a_{21} = 1$, $a_{22} = 0$ and $a_{23} = 2$, and it is **not true** that

$$|a_{22}| > |a_{21}| + |a_{23}| \text{ i.e. } |0| < |1| + |2| = |3|$$

Interchanging the **second** and **third** rows in the **original** system of **linear equations** leads to the **coefficient matrix** A to be

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -5 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

where,

$$|4| > |2| + |-1| = |3|,$$

$$|-5| > |3| + |1| = |4|, \text{ and}$$

$$|2| > |1| + |0| = |1|$$

Which is **strictly diagonally dominant**, for which **when apply** the **Gauss-Seidel method** the **solution** can be **approximated**.

Example 6: Check if can apply the Gauss-Jacobi method to the system

$$x_1 - 5x_2 = -4$$

$$7x_1 - x_2 = 6$$

using the initial approximation $(x_1, x_2) = (0, 0)$.

Solution:

The coefficient matrix of the system is:

$$A = \begin{bmatrix} 1 & -5 \\ 7 & -1 \end{bmatrix}$$

Where,

$$|1| < |-5| \quad \text{and} \quad |-1| < |7|$$

Thus, the coefficient matrix is not strictly diagonally dominant matrix because does not conform to the definition

$$|a_{11}| > |a_{12}| \quad \text{and} \quad |a_{22}| > |a_{21}|$$

Interchanging the second and first rows in the original system of linear equations, yields

$$7x_1 - x_2 = 6$$

$$x_1 - 5x_2 = -4$$

Leads the coefficient matrix A to be:

$$A = \begin{bmatrix} 7 & -1 \\ 1 & -5 \end{bmatrix}$$

where,

$$|7| > |-1| \quad \text{and} \quad |-5| > |1|$$

Which is strictly diagonally dominant matrix, and can apply the Gauss-Seidel method to approximate the solution.

Rewriting the given system in the form

$$x_1 = \frac{x_2 + 6}{7}$$

$$x_2 = \frac{x_1 + 4}{5}$$

Using the initial approximation $x_1 = 0$ and $x_2 = 0$, yields

$$x_1^1 = \frac{x_2^0 + 6}{7} = \frac{0 + 6}{7} = \frac{6}{7} = 0.8571$$

$$x_2^1 = \frac{x_1^1 + 4}{5} = \frac{0.8571 + 4}{5} = \frac{4.8571}{5} = 0.9714$$

$$x_1^2 = \frac{x_2^1 + 6}{7} = \frac{0.9714 + 6}{7} = \frac{6}{7} = 0.9959$$

$$x_2^2 = \frac{x_1^2 + 4}{5} = \frac{0.9959 + 4}{5} = \frac{4.8571}{5} = 0.9992$$

Continuing iterations, a sequence of approximations can be obtain as shown in the following table:

| n | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|--------|--------|--------|--------|--------|
| x_1 | 0 | 0.8571 | 0.9959 | 0.9999 | 1.0000 | 1.0000 |
| x_2 | 0 | 0.9714 | 0.9992 | 1.0000 | 1.0000 | 1.0000 |

So, the **solution** is:

$$x_1 = 1 \quad \text{and} \quad x_2 = 1.$$

Application of Linear systems

There is a **wide range** of applications of linear systems, a few of **them** are:

1. **Fitting polynomials,**
2. **Network analysis,**
3. **Kirchhoff 's Laws for electrical networks**

1. Polynomial curve fitting

Recall the facts:

- i. There is **exactly one line** $y = mx + c$ that **passes through two given points**;
- ii. There is **exactly one parabola** $y = ax^2 + bx + c$ that **passes through three given points** (barring some exceptions).
- iii. More generally, **given n number of points** in the **plane** and there is **exactly one polynomials $p(x)$ of degree $n - 1$** , so that the **graph of $y = p(x)$ will pass through these n points describing it as follows:**

Suppose a **collection of data** is **represented by n points**:

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

If the x – **coordinates x_1, x_2, \dots, x_n are distinct**, then there is a **UNIQUE polynomial**

$$p(x) = a_0 + a_1x_1 + a_2x_1^2 + \dots + a_{n-1}x_1^{n-1}$$

of degree $n - 1$ (or less) so that the **graph of $y = p(x)$ passes through these points**. Given n such points, to **determine $p(x)$** it needs to **find the coefficients $a_0, a_1, a_2, \dots, a_{n-1}$** . Since the **points (x_i, y_i) pass through the graph of $y = p(x)$** , we have $y_i = p(x_i)$. **More explicitly,**

$$a_0 + a_1x_1 + a_2x_1^2 + \dots + a_{n-1}x_1^{n-1} = y_1$$

$$a_0 + a_1x_2 + a_2x_2^2 + \dots + a_{n-1}x_2^{n-1} = y_2$$

$$a_0 + a_1x_3 + a_2x_3^2 + \dots + a_{n-1}x_3^{n-1} = y_3$$

$$\vdots \quad \vdots \quad \vdots \quad \quad \quad \vdots \quad \quad \vdots$$

$$a_0 + a_1x_n + a_2x_n^2 + \dots + a_{n-1}x_n^{n-1} = y_n$$

This is a **linear system** of n equations, with n unknowns (variables) $a_0, a_1, a_2, \dots, a_{n-1}$. It is known that, **under our condition** that x_1, x_2, \dots, x_n are **distinct**, the system has a **unique solution**.

The **augmented matrix** of this **linear system** is:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} & y_2 \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} & y_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} & y_n \end{pmatrix}$$

and the **coefficients matrix** is

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

The **coefficients matrix** is called **Vandermonde-matrix** in x_1, x_2, \dots, x_n .

Example 1: Determine the **polynomial function** of **degree 2** that passes through the points $(2, 4), (3, 6), (4, 10)$.

Solution:

Let the **polynomial function** of **degree 2** be:

$$p(x) = a + bx + cx^2$$

Since **these points** pass through the **graph** of

$$y = p(x) = a + bx + cx^2$$

we have

$$a + b2 + c2^2 = 4$$

$$a + 2b + 4c = 4$$

$$a + b3 + c3^2 = 6$$

or

$$a + 3b + 9c = 6$$

$$a + b4 + c4^2 = 10$$

$$a + 4b + 16c = 10$$

The **augmented matrix** of this system is:

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 6 \\ 1 & 4 & 16 & 10 \end{pmatrix}$$

Now **reducing** the **matrix** to the **row-echelon form**. To do this **subtract** **row-1** from **row-2** and **row-3**:

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 5 & 2 \\ 0 & 2 & 12 & 6 \end{pmatrix}$$

Multiply **row-2** by **2** and **Subtract** from **row-3**, yields

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

Divide **row-3** by **2**, yields

$$\begin{pmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The **matrix** is in **row-echelon form**. The **linear system** corresponding to this **matrix** is:

$$a + 2b + 4c = 4$$

$$b + 5c = 2$$

$$c = 1$$

By **back substitution**, yields

$$c = 1 \Rightarrow b = 2 - 5 = -3 \Rightarrow a = 4 - 4 + 6 = 6$$

So,

$$p(x) = a + bx + cx^2 = 6 - 3x + x^2$$

Example 2: Some US census population data is given in the following table

| | | | |
|----------------------------------|-------------|-------------|-------------|
| Year t | 1980 | 1990 | 2000 |
| Population y | 227 | 249 | 281 |

Here **population** is given in **millions**.

1. Fit a second degree polynomial passing through these points.
2. Use it to predict population in year 2010 and 2020.

Solution:

Let t be the variable time and set $t = 0$ for the year 1980. The table reduces to

| | | | |
|-----------------------|------------|------------|------------|
| t | 0 | 10 | 20 |
| y | 227 | 249 | 281 |

Let the polynomial function of degree 2 that fits this data be:

$$p(x) = a + bt + ct^2$$

Since the data points pass through the graph of

$$p(x) = a + bt + ct^2$$

we have

$$a + b0 + c0^2 = 227 \qquad a = 227$$

$$a + b10 + c(10)^2 = 249 \qquad \text{or} \qquad a + 10b + 100c = 249$$

$$a + b20 + c(20)^2 = 281 \qquad a + 20b + 400c = 281$$

The augmented matrix is

$$\left(\begin{array}{cccc} 1 & 0 & 0 & 227 \\ 1 & 10 & 100 & 249 \\ 1 & 20 & 400 & 281 \end{array} \right)$$

Subtract row-1 from **row-2** and **row-3**, yields

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 0 & 10 & 100 & 22 \\ 0 & 20 & 400 & 54 \end{pmatrix}$$

Multiply row-2 by **2** and **subtract** from **row-3**, yields

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 0 & 10 & 100 & 22 \\ 0 & 0 & 200 & 10 \end{pmatrix}$$

Divide row-2 and **row-3** by **10**, yields

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 0 & 1 & 10 & 2.2 \\ 0 & 0 & 20 & 1 \end{pmatrix}$$

Divide row-3 by **2**, yields

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 0 & 1 & 10 & 2.2 \\ 0 & 0 & 10 & 0.5 \end{pmatrix}$$

Subtract row-3 from **row-2**, yields

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 0 & 1 & 0 & 1.7 \\ 0 & 0 & 10 & 0.5 \end{pmatrix}$$

Divide row-3 by **10**, yields

$$\begin{pmatrix} 1 & 0 & 0 & 227 \\ 0 & 1 & 0 & 1.7 \\ 0 & 0 & 1 & 0.05 \end{pmatrix}$$

So, $a = 227$, $b = 1.7$, $c = 0.05$ and

$$y = p(t) = 227 + 1.7t + 0.05t^2$$

For **year 2010**, $t = 30$ and **predicated population** is:

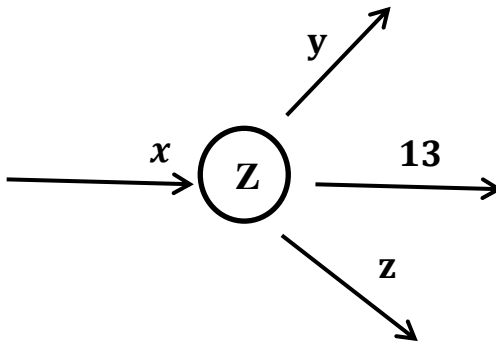
$$p(t) = 227 + 1.7 \times 30 + 0.05 \times (30)^2 = 323 \text{ mi}$$

Similarly, for **year 2020**, $t = 40$ and **predicated population** is:

$$p(t) = 227 + 1.7 \times 40 + 0.05 \times (40)^2 = 375 \text{ mi}$$

2. Network Analysis

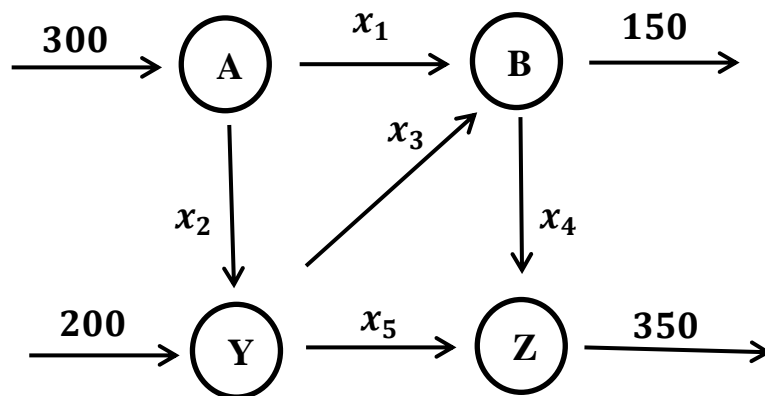
A **network system** consists of **junctions** and **branches**. They are used to **model diverse situations**, including in **economics, traffic, telephone signal and electrical engineering**. Such **models assume**, the **total flow into** a **junction** is **equal** to **total flow out** of the **junction**.



Accordingly, **above network** is **represented** by

$$x = y + 13 + z$$

Example 1: The **flow of traffic** (in **vehicles per hour**) through a **network** of **street** is **shown** in the **following figure**:



1. Solve this system for $x_1, x_2, x_3, x_4,$ and x_5 .
2. Find the traffic flow when $x_2 = 200$ and $x_3 = 50$.
3. Find the traffic flow when $x_2 = 150$ and $x_3 = 0$.

Solution:

From **junction A**, we get $x_1 + x_2 = 300$

From **junction B**, we get $x_1 + x_3 = 150 + x_4 \Rightarrow x_1 + x_3 - x_4 = 150$

From **junction Y**, we get $x_2 + 200 = x_3 + x_5 \Rightarrow x_2 - x_3 - x_5 = -200$

From **junction Z**, we get $x_4 + x_5 = 350$

Thus, the **system of linear equations** is:

$$x_1 + x_2 = 300$$

$$x_1 + x_3 - x_4 = 150$$

$$x_2 - x_3 - x_5 = -200$$

$$x_4 + x_5 = 350$$

To solve this **linear system**, we first of all write the **augmented matrix**:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 1 & 0 & 1 & -1 & 0 & 150 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 1 & 1 & 350 \end{pmatrix}$$

Then, reduce this **matrix** to **row-echelon form**:

Subtract row-1 from **row-2**, yields

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 0 & -1 & 1 & -1 & 0 & -150 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 1 & 1 & 350 \end{pmatrix}$$

Add row-2 to **row-3**, yields

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 0 & -1 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & -1 & -1 & -350 \\ 0 & 0 & 0 & 1 & 1 & 350 \end{pmatrix}$$

Add row-3 to row-4, yields

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 0 & 1 & -1 & 1 & 0 & 150 \\ 0 & 0 & 0 & 1 & 1 & 350 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The **matrix** is in row-echelon form. The **corresponding linear system** is **given by**:

$$x_1 + x_2 = 300$$

$$x_2 - x_3 + x_4 = 150$$

$$x_4 + x_5 = 350$$

$$0 = 0$$

Parametrically, with $x_2 = t$, $x_3 = s$, yields

$$\begin{aligned} x_1 &= 300 - t, & x_2 &= t, & x_3 &= s, \\ x_4 &= 150 - t + s, & x_5 &= 350 - x_4 = 150 + t - s \end{aligned}$$

These **answers** for **part (1)**.

For **part (2)** $t = X_2 = 200$. $s = x_3 = 50$.

So, $x_1 = 100$, $x_2 = 200$, $x_3 = 50$, $x_4 = 0$, $x_5 = 300$

For **part (3)** $t = X_2 = 150$. $s = x_3 = 0$.

So, $x_1 = 150$, $x_2 = 150$, $x_3 = 0$, $x_4 = 0$, $x_5 = 350$

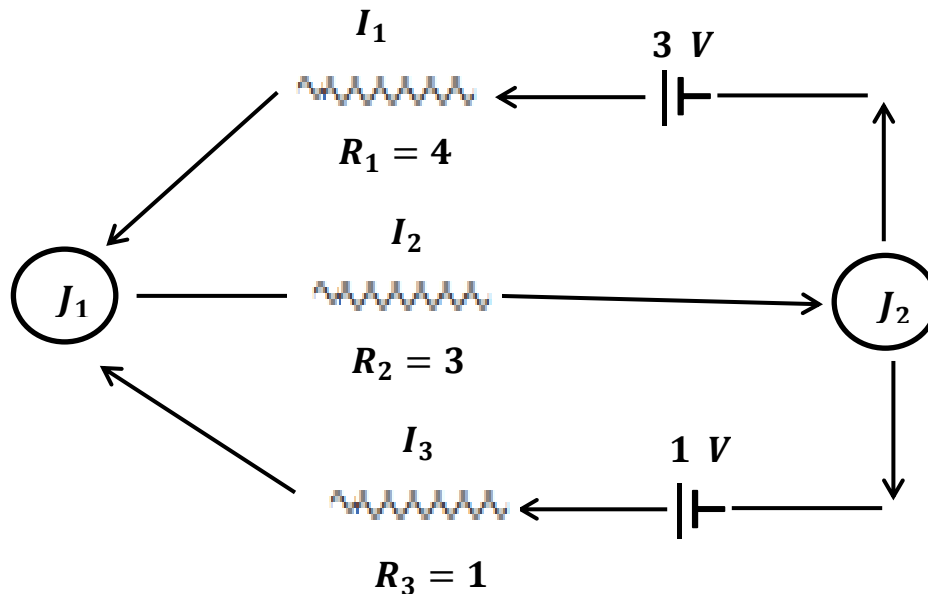
3. Kirchhoff's Laws

System of Linear equations is also applicable in electrical network. Analysis of electrical network is guided by two properties known as Kirchhoff's Laws:

1. All the current flowing into a junction must flow out of it.
2. The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage.

A battery or voltage source is denoted by $| \text{—} |$ or $\text{—} |$ and the resistance is denoted by 

Example 2: Consider the electrical circuit drawn in figure below



Use Kirchhoff-Law to determine I_1 , I_2 , I_3 .

Solution:

- Applying (1) of Kirchhoff-Law to junction J_1 , we have

$$I_1 + I_3 = I_2 \quad \Rightarrow \quad I_1 - I_2 + I_3 = 0 \quad \text{Eq. (1)}$$

- Applying (2) of Kirchhoff-Law:

$$R_1 I_1 + R_2 I_2 = 3 \quad \Rightarrow \quad 4I_1 + 3I_2 = 3 \quad \text{Eq. (2)}$$

$$R_2 I_2 + R_3 I_3 = 1 \quad \Rightarrow \quad 3I_2 + I_3 = 1 \quad \text{Eq. (3)}$$

The network system is given by

$$I_1 - I_2 + I_3 = 0 \quad \text{Eq. (1)}$$

$$4I_1 + 3I_2 = 3 \quad \text{Eq. (2)}$$

$$3I_2 + I_3 = 1 \quad \text{Eq. (3)}$$

First, writing the **augmented matrix** is:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Then, reducing this matrix to **row-echelon form** as follows:

Multiply row-1 by 4 and **subtract** from row-2, yields

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Divide row-2 by 7, yields

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Multiply row-2 by 3 and **subtract** from row-3, yields

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & \frac{19}{7} & -\frac{2}{7} \end{pmatrix}$$

Multiply row-3 by $\frac{7}{19}$, yields

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

Further reducing the matrix to Gauss-Jordan form:

Add row-2 to row-1, yields

$$\begin{pmatrix} 1 & 0 & \frac{3}{7} & \frac{3}{7} \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

Multiply row-3 by $\frac{3}{7}$ and subtract from row-1, yields

$$\begin{pmatrix} 1 & 0 & 0 & \frac{9}{19} \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

Multiply row-3 by $\frac{4}{7}$ and add to row-2, yields

$$\begin{pmatrix} 1 & 0 & 0 & \frac{9}{19} \\ 0 & 1 & 0 & \frac{7}{19} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

The corresponding linear system is:

$$I_1 = \frac{9}{19}, \quad I_2 = \frac{7}{19}, \quad I_3 = -\frac{2}{19}$$

Gaussian Quadrature

The Gaussian quadrature method is an approximate method of calculation of a certain integral

$$I = \int_a^b y(x) dx$$

by replacing the variables

$$x = (b - a) \frac{t}{2} + (a + b) \frac{t}{2} \quad , \quad f(t) = (b - a) \frac{y(x)}{2}$$

the desired integral is reduced to the form

$$I = \int_{-1}^1 f(t) dt$$

The **Gaussian quadrature formula** is:

$$I = \int_{-1}^1 f(t) dt = \sum_{i=1}^n A_i f(t_i)$$

The cusps t_i of the Gaussian quadrature formula are the roots of a Legendre polynomial of degree n , $P_n(t)$. The Legendre polynomial has exactly n real and various roots in the **interval** $(-1, 1)$. The weights A_i of the Gaussian quadrature formula are defined by

$$A_i = \frac{2}{(1 - t_i^2)[P'_n(t_i)]^2}$$

Given in the table are the cusps and weights of the Gaussian quadrature formula for the first five values n .

| n | Cusps | Weight |
|-----|----------------|----------|
| 2 | ± 0.577350 | 1 |
| 3 | 0 | $8/9$ |
| | ± 0.774597 | $5/9$ |
| 4 | ± 0.339981 | 0.652145 |
| | ± 0.861136 | 0.347855 |
| 5 | 0 | 0.568889 |
| | ± 0.538469 | 0.478629 |
| | ± 0.906180 | 0.236927 |

Gaussian quadrature formula is exact for an **arbitrary polynomial** of degree not higher than $(2n - 1)$. The **remainder term** of **Gaussian formula** R_n for the integral $\int_a^b y(x)dx$ is expressed as follows:

$$R_n = \frac{(b - a)^{2n-1} n!}{(2n + 1) [(2n)!]^3} y^{2n}(\xi); a \leq \xi \leq b$$

The Gaussian quadrature method is applied when a subintegral function is smooth enough and a gain in the number of cusps is essential (for instance, in calculating multiple integrals as iterated integrals).

The **Gaussian quadrature formula** is widely used in **solving problems** of radiation heat transfer in **direct integration** of the equation of transfer of radiation over space. The application of **Gaussian formula** in this case works very well especially when the number of intervals of spectrum decomposition is great.

Gaussian quadrature 1-point formula:

$$I = \int_a^b f(x) dx$$

$$x = \left| \frac{b-a}{2} \right| A + \frac{b+a}{2}$$

$$I = \int_{-1}^1 f(x) dx = 2f(0)$$

Gaussian quadrature 2-point formula:

$$I = \int_a^b f(x) dx$$

$$I = \int_{-1}^1 f(x) dx = \frac{b-a}{2} [A_1 f(x_1) + A_2 f(x_2)]$$

where,

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2} \quad , \quad x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2}$$

$$A_1 = A_2 = 1 \quad , \quad z_1 = \frac{-1}{\sqrt{3}} = -0.5774 \quad , \quad z_2 = \frac{1}{\sqrt{3}} = 0.5774$$

$$I = \int_{-1}^1 f(x) dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gaussian quadrature 3-point formula:

$$I = \int_{-1}^1 f(x) dx = \frac{b-a}{2} [A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)]$$

where,

$$A_1 = \frac{5}{9} = 0.5555, \quad A_2 = \frac{8}{9} = 0.8888, \quad A_3 = \frac{5}{9} = 0.5555$$

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2},$$

$$x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2},$$

$$x_3 = \frac{b-a}{2}z_3 + \frac{b+a}{2}$$

$$z_1 = -\sqrt{\frac{3}{5}} = -0.7746, \quad z_2 = 0, \quad z_3 = \sqrt{\frac{3}{5}} = 0.7746$$

$$I = \int_{-1}^1 f(x) dx = \left(\frac{5}{9}\right) f\left(-\sqrt{\frac{3}{5}}\right) + \left(\frac{8}{9}\right) f(0) + \left(\frac{5}{9}\right) f\left(\sqrt{\frac{3}{5}}\right)$$

Example 1: Use Gauss-Legendre two – point and three – point formula to evaluate

$$I = \int_{-2}^2 e^{-\frac{x}{2}} dx$$

Solution:

Given data are:

$$a = -2, \quad b = 2 \quad \text{and} \quad f(x) = e^{-\frac{x}{2}}$$

Gaussian quadrature 2 – point formula:

$$I = \frac{b-a}{2} [A_1 f(x_1) + A_2 f(x_2)]$$

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2}, \quad x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2}$$

$$A_1 = A_2 = 1, \quad z_1 = \frac{-1}{\sqrt{3}}, \quad z_2 = \frac{1}{\sqrt{3}}$$

$$x_1 = \frac{2 - (-2)}{2} \left(\frac{-1}{\sqrt{3}}\right) + \frac{2 - 2}{2} = \frac{-2}{\sqrt{3}},$$

$$x_2 = \frac{2 - (-2)}{2} \left(\frac{1}{\sqrt{3}} \right) + \frac{2 - 2}{2} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow I = \frac{2 - (-2)}{2} \left[(1)f\left(\frac{-2}{\sqrt{3}}\right) + (1)f\left(\frac{2}{\sqrt{3}}\right) \right]$$

$$= 2 \left[e^{-\frac{x_1}{2}} + e^{-\frac{x_2}{2}} \right] = 2 \left[e^{-\frac{(-2)}{\sqrt{3}}} + e^{-\frac{(2)}{\sqrt{3}}} \right] = 4.6854$$

Gaussian quadrature 3-point formula:

$$I = \frac{b - a}{2} [A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)]$$

where, $A_1 = \frac{5}{9}$, $A_2 = \frac{8}{9}$, $A_3 = \frac{5}{9}$,

$$x_1 = \frac{b - a}{2} z_1 + \frac{b + a}{2} = \frac{2 - (-2)}{2} (-0.7746) + \frac{2 - 2}{2} = -1.5492$$

$$x_2 = \frac{b - a}{2} z_2 + \frac{b + a}{2} = \frac{2 - (-2)}{2} (0) + \frac{2 - 2}{2} = 0$$

$$x_3 = \frac{b - a}{2} z_3 + \frac{b + a}{2} = \frac{2 - (-2)}{2} (0.7746) + \frac{2 - 2}{2} = 1.5492$$

$$z_1 = -\sqrt{\frac{3}{5}} = -0.7746, \quad z_2 = 0, \quad z_3 = \sqrt{\frac{3}{5}} = 0.7746$$

$$\Rightarrow I = \frac{2 - (-2)}{2} \left[\left(\frac{5}{9}\right) f\left(-2\sqrt{\frac{3}{5}}\right) + \left(\frac{8}{9}\right) f(0) + \left(\frac{5}{9}\right) f\left(2\sqrt{\frac{3}{5}}\right) \right]$$

$$= 2 \left[\left(\frac{5}{9}\right) e^{-\frac{x_1}{2}} + \left(\frac{8}{9}\right) e^{-\frac{x_2}{2}} + \left(\frac{5}{9}\right) e^{-\frac{x_3}{2}} \right]$$

$$= 2 \left[\left(\frac{5}{9}\right) e^{-\frac{(-2\sqrt{\frac{3}{5}})}{2}} + \left(\frac{8}{9}\right) e^{-\frac{0}{2}} + \left(\frac{5}{9}\right) e^{-\frac{(2\sqrt{\frac{3}{5}})}{2}} \right] = 4.7008$$

Applications of Integrals

- **Arc Length** – determining the length of a curve over a given interval.
- **Surface Area** – determining the surface area of a solid of revolution, i.e. a solid obtained by rotating a region bounded by two curves about a vertical or horizontal axis.
- **Center of Mass** – determining the centre of mass or centroid of a thin plate where the plate can be described as a region bounded by two curves (one of which may be the x or y-axis).
- **Hydrostatic Pressure and Force** – determining the hydrostatic pressure and force on a vertical plate submerged in water. The plates used in the examples can all be described as regions bounded by one or more curves/lines.
- **Probability** – Many quantities can be described with probability density functions. For example, the length of time a person waits in line at a checkout counter or the life span of a light bulb. None of these quantities is fixed values and will depend on a variety of factors. probability density functions and computing the mean (think average wait in line or average life span of a light bulb) of a probability density function.

Fourier Series

→ Even and Odd Functions

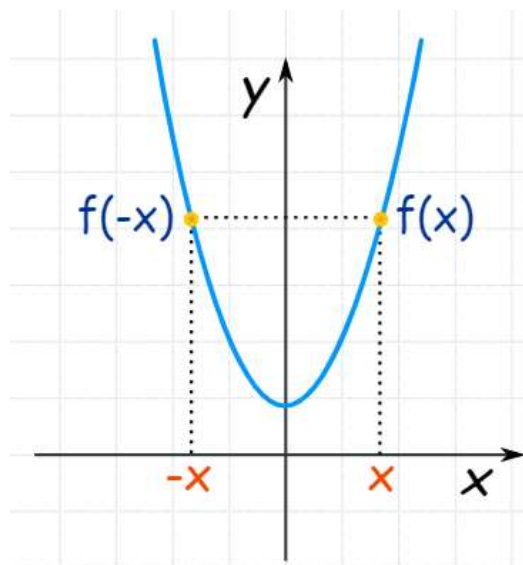
- Even Functions

A function is "even" when:

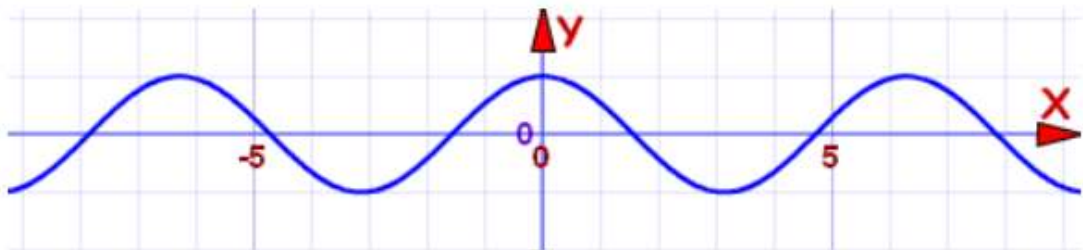
$$f(x) = f(-x) \text{ for all } x$$

In other words, there is **symmetry** about the **y - axis** (like a reflection).

for example, the curve below of function $f(x) = x^2 + 1$



They got called "even" functions because the functions $x^2, x^4, x^6, x^8, \dots$ etc behave like that, but there are other functions that behave like that too, such as $\cos x$:



cosine function: $f(x) = \cos x$ is an even function

But an even exponent does not always make an even function, for example $(x + 1)^2$ is not an even function.

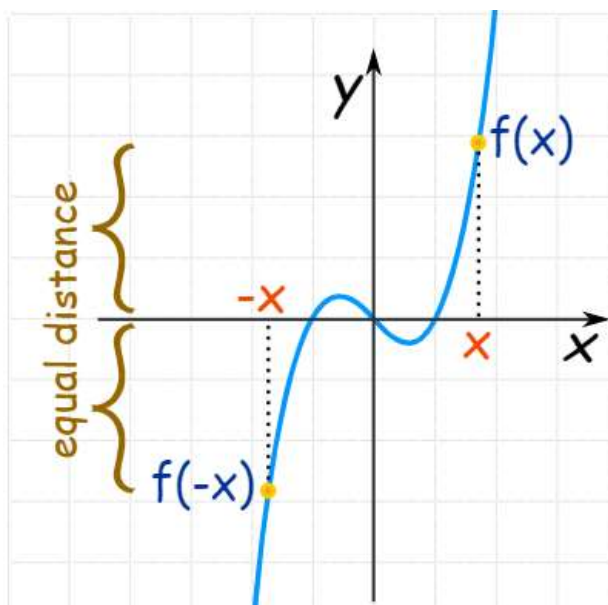
- **Odd Functions**

A function is "odd" when:

$$-f(x) = f(-x) \text{ for all } x$$

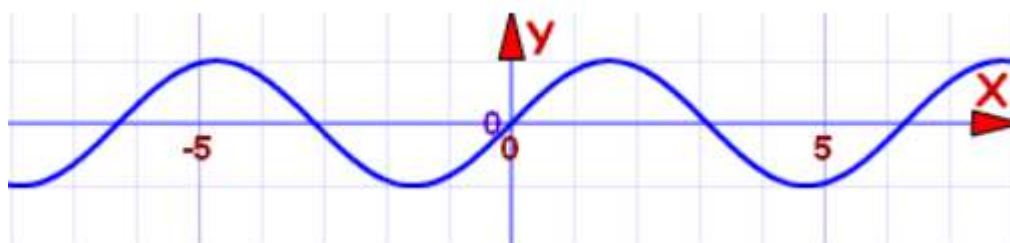
Note the **minus** is in the **front of** $f(x)$: $-f(x)$

The **symmetry** is **about the origin** as shown in the **figure below** of the **curve of the function** $f(x) = x^3 - x$.



Cosine function: $f(x) = x^3 - x$ is an **even function**

They got called "**odd**" **functions** because the functions $x, x^3, x^5, x^7, x^9, \dots$, etc behave like that, but there are other functions that behave like that too, such as **sin x**:

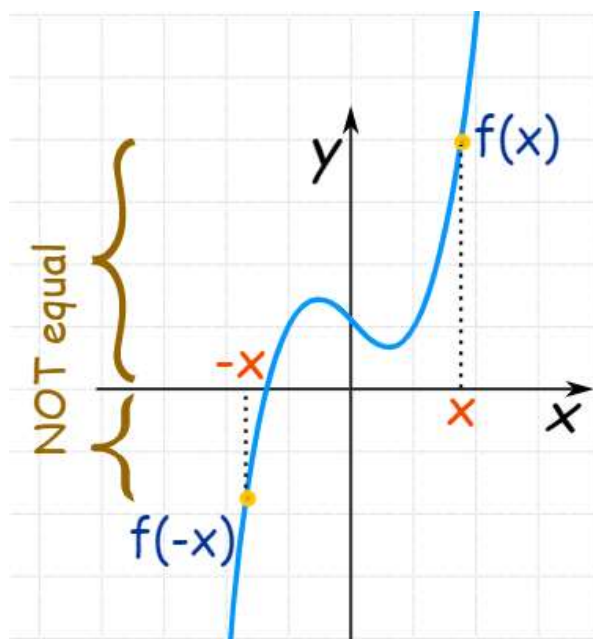


Sine function: $f(x) = \sin x$ is an **odd function**

But an **odd exponent does not always make an odd function**, for example $x^3 + 1$ is **not an odd function**.

- Neither Odd nor Even Functions

Don't be misled by the names "odd" and "even", they are just names and a function **does not have** to be even or odd. In fact, **most functions are neither odd nor even**. For example, just **adding 1** to the function above the curve will be as follows:



Cosine function: $f(x) = x^3 - x + 1$ is **not an odd function**, and it is **not an even function either**.
It is **neither odd nor even function**

Example: is $f(x) = \frac{x}{x^2-1}$ Even or Odd or neither?

Solution:

Substituting $(-x)$ in place of (x) in the given function:

$$f(-x) = \frac{-x}{(-x)^2 - 1} = \frac{-x}{x^2 - 1} = -\frac{x}{x^2 - 1} = -f(x)$$

So, $f(-x) = -f(x)$, which **makes the function an odd function**.

- Even and Odd

The **only function** that is even and odd is , $f(x) = 0$

Special Properties

Adding:

- The **sum of two even functions** is **even**.
- The **sum of two odd functions** is **odd**.
- The **sum of an even and odd function** is **neither even nor odd** (unless one function is zero).

Multiplying:

- The **product of two even functions** is an **even function**.
- The **product of two odd functions** is an **even function**.
- The **product of an even function and an odd function** is an **odd function**.

→ Periodic Functions

Periodic function is a **function** that **repeats itself** at **equal intervals** in the **abscissa**, then the **function** $f(t)$ is said to be a **periodic function** if

$$f(t) = f(t + T) = f(t + 2T) = \dots$$

where T is called the **period** of the **function** $f(t)$. It is a **positive constant** value ($T > 0$). For example, **all trigonometric function** have **period** of $T = 2\pi$. So, the **sine and cosine function** is a **periodic function** with a **period** of 2π :

$$\sin t = \sin(t + 2\pi) \quad \text{or} \quad \sin t = \sin(t + 2n\pi)$$

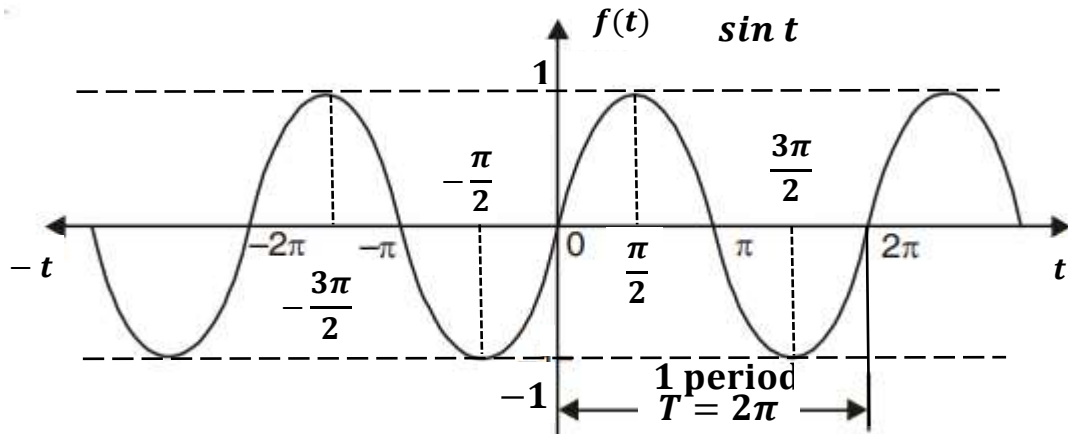
$$\cos t = \cos(t + 2\pi) \quad \text{or} \quad \cos t = \cos(t + 2n\pi)$$

where $n = 1, 2, 3, 4, \dots$

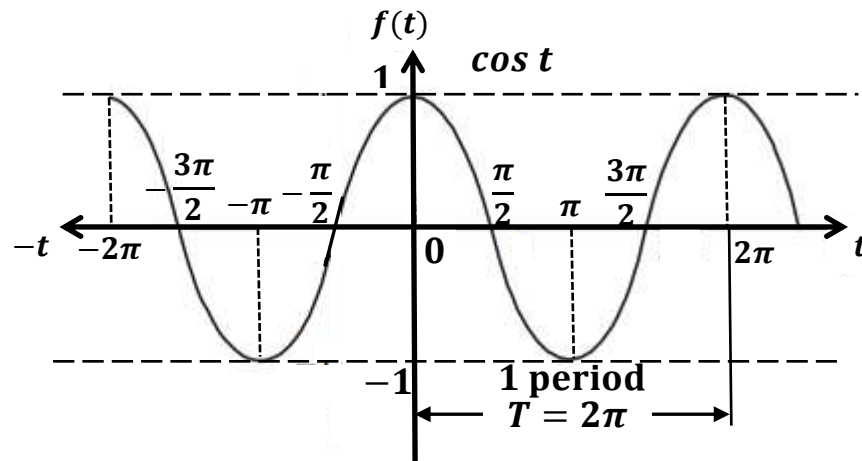
For example:

$$\sin t = \sin(t + 2\pi) = \sin(t + 4\pi) = \sin(t + 6\pi) = \dots$$

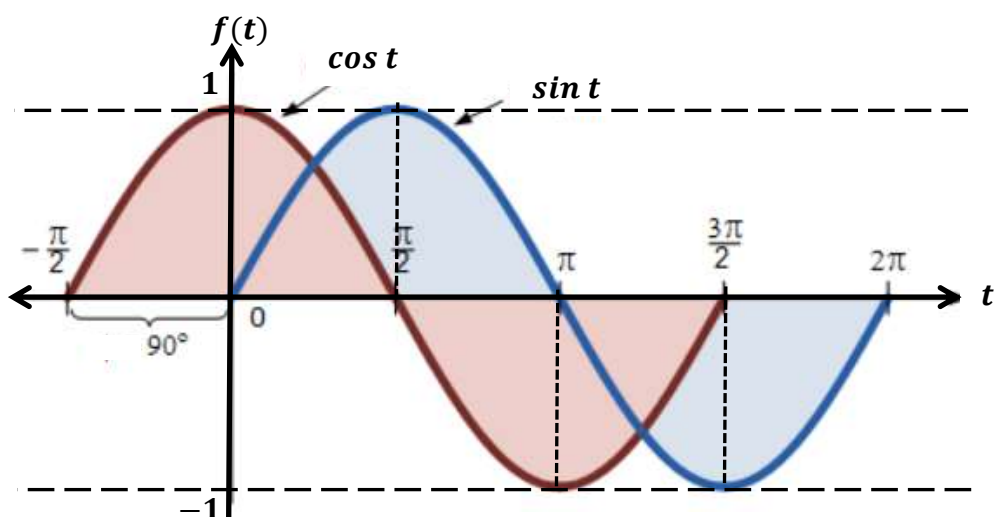
The **Sine Function** has up-down curve which repeats every 2π radians (360°). It starts at 0, heads up to 1 by $\frac{\pi}{2}$ radians (90°) and then heads down to -1 .



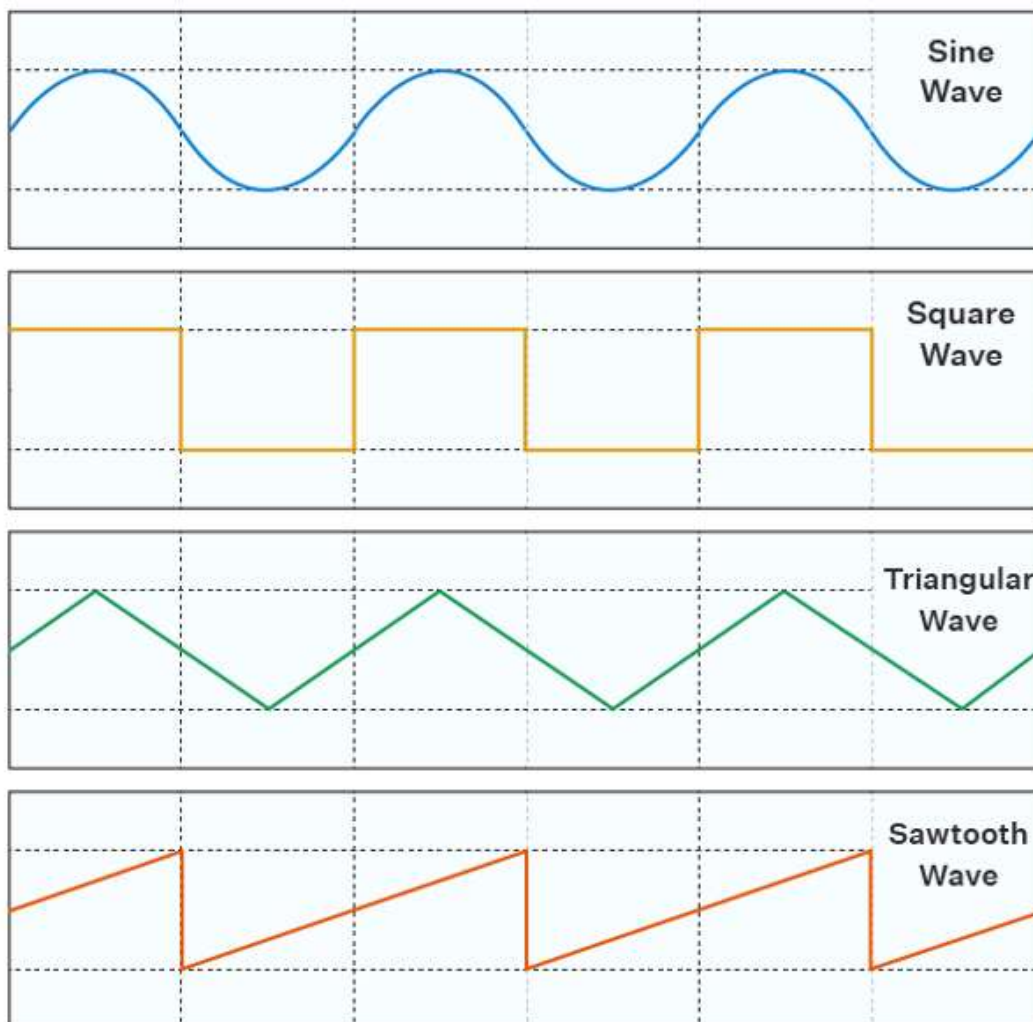
Cosine is just like **Sine**, but it starts at 1 and heads down until π radians (180°) and then heads up again.



In fact, **Sine** and **Cosine** are like good friends follow each other, exactly $\frac{\pi}{2}$ radians (90°) separated by a distance; at a specified distance from each other in **time** or **space**.



The following are the graphs of some of the periodic functions. The graph of each of the below periodic functions has translational symmetry.



The **period** of some of the important periodic functions are as follows.

- The period of $\sin(x)$ and $\cos(x)$ is 2π .
- The period of $\tan(x)$ and $\cotan(x)$ is π .
- The period of $\sec(x)$ and $\operatorname{cosec}(x)$ is 2π .

The **graph** of a **periodic function** is **symmetric** and **repeats itself along the horizontal axis**.

Example 1: Find the **period** of the **periodic function** $\sin(4x + 5)$.

Solution:

The period of $\sin(x)$ is 2π , and the period of $\sin(4x + 5)$ is :

$$2\pi = 4x \Rightarrow x = \frac{2\pi}{4} = \frac{\pi}{2}$$

Therefore, the **period** of $\sin(4x + 5)$ is $\frac{\pi}{2}$.

Example 2: Find the **period** of the function

$$f(x) = \tan 3x + \sin \frac{5x}{2}$$

Solution:

The **period** of $\tan x$ is π , and the **period** of $\tan x$ is:

$$3x = \pi \Rightarrow x = \frac{\pi}{3}$$

The period of $\sin x$ is 2π , and the period of $\sin \frac{5x}{2}$ is:

$$\frac{5x}{2} = 2\pi \Rightarrow x = \frac{4\pi}{5}$$

Thus, the **period** of the function $f(x) = \tan 3x + \sin \frac{5x}{2}$, is as follows:
Period of $f(x)$ is:

$$f(x) = \frac{\text{LCM of } \pi \text{ and } 4\pi}{\text{HCF of } 3 \text{ and } 5} = \frac{4\pi}{1} = 4\pi$$

Therefore, the **period** of the **function**

$$f(x) = \tan 3x + \sin \frac{5x}{2}$$

is 4π .

The **Fourier series** is a **superposition** of various **periodic wave function series** to form a **complex periodic function**. It is usually **composed** of **sine** and **cosine functions**, and the **summation** of these **wave functions** is taken by **assigning respective weight components** to these series. The **Fourier series** has applications in the **representation** of **heatwaves**, **vibration analysis**, **quantum mechanics**, **electrical engineering**, **signal processing**, **image processing**.

→ Fourier Series

A series of **sines** and **cosines** of an **angle** and its **multiples** of the **form**

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + a_n \cos nx \\ + \dots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \\ + b_n \sin nx + \dots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called the **Fourier series**, where a_1, a_2, \dots, a_n , and $b_1, b_1, b_1, \dots, b_n \dots$ are **constants**.

A **periodic function** $f(x)$ can be **expanded** in a **Fourier Series**.

Useful Integrals

The following integrals are useful in Fourier Series.

| | | | |
|-----|---|-----|---|
| (1) | $\int_0^{2\pi} \sin nx \, dx = 0$ | (5) | $\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0$ |
| (2) | $\int_0^{2\pi} \cos nx \, dx = 0$ | (6) | $\int_0^{2\pi} \cos nx \cdot \cos mx \, dx = 0$ |
| (3) | $\int_0^{2\pi} \sin^2 nx \, dx = \pi$ | (7) | $\int_0^{2\pi} \sin nx \cdot \cos mx \, dx = 0$ |
| (4) | $\int_0^{2\pi} \cos^2 nx \, dx = \pi$ | (8) | $\int_0^{2\pi} \sin nx \cdot \cos nx \, dx = 0$ |
| (9) | $\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$ <p>where $v_1 = \int v \, dx, v_2 = \int v_1 \, dx$ and so on</p> $u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2} \text{ and so on,}$ <p>and $(x)\sin n\pi = 0, \quad \cos n\pi = (-1)^n \text{ where } n \in I$</p> | | |

Determination of Fourier Coefficients (Euler's Formulae)

$$\begin{aligned}
 f(x) = & \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots \\
 & + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx \\
 & + \dots
 \end{aligned} \tag{1}$$

→ To find a_0 , integrating both sides of the Equation (1) from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x) \, dx = \frac{a_0}{2} \int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x \, dx + a_2 \int_0^{2\pi} \cos 2x \, dx$$

$$\begin{aligned}
& + \dots + a_n \int_0^{2\pi} \cos nx \, dx + b_1 \int_0^{2\pi} \sin x \, dx \\
& + b_2 \int_0^{2\pi} \sin 2x \, dx + \dots + b_n \int_0^{2\pi} \sin nx \, dx + \dots
\end{aligned}$$

$$\int_0^{2\pi} f(x) \, dx = \frac{a_0}{2} \int_0^{2\pi} dx \quad [\text{other integrals} = 0 \text{ by formula (1) and (2)}]$$

$$\Rightarrow \int_0^{2\pi} f(x) \, dx = \frac{a_0}{2} 2\pi \quad \Rightarrow \quad a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx$$

→ **To find a_n** , Multiplying each side of the Equation (1) by $\cos nx$ and **integrating** from $x = 0$ to $x = 2\pi$.

→

$$\begin{aligned}
\int_0^{2\pi} f(x) \cos nx \, dx &= \frac{a_0}{2} \int_0^{2\pi} \cos nx \, dx + a_1 \int_0^{2\pi} \cos x \cos nx \, dx \\
&+ a_2 \int_0^{2\pi} \cos 2x \cos nx \, dx + \dots + a_n \int_0^{2\pi} \cos^2 nx \, dx \\
&+ b_1 \int_0^{2\pi} \sin x \cos nx \, dx + b_2 \int_0^{2\pi} \sin 2x \cos nx \, dx \\
&+ \dots + b_n \int_0^{2\pi} \sin nx \cos nx \, dx + \dots \\
&= a_n \int_0^{2\pi} \cos^2 nx \, dx = a_n \pi
\end{aligned}$$

(All other integrals = 0 by formula (1) to (8))

$$\Rightarrow \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

By taking $n = 1, 2, 3, \dots$ the values of a_1, a_2, a_3, \dots can be found.

→ **To find b_n** , Multiplying each side of Equation (1) by $\sin nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\begin{aligned} \int_0^{2\pi} f(x) \sin nx \, dx &= \frac{a_0}{2} \int_0^{2\pi} \sin nx \, dx + a_1 \int_0^{2\pi} \cos x \sin nx \, dx \\ &+ a_2 \int_0^{2\pi} \cos 2x \sin nx \, dx + \dots + a_n \int_0^{2\pi} \cos nx \sin nx \, dx \\ &+ b_1 \int_0^{2\pi} \sin x \sin nx \, dx + b_2 \int_0^{2\pi} \sin 2x \sin nx \, dx \\ &+ \dots + b_n \int_0^{2\pi} \sin^2 nx \, dx + \dots \\ &= b_n \int_0^{2\pi} \sin^2 nx \, dx = b_n \pi \end{aligned}$$

(All other integrals = 0 by formula (1) to (8))

$$\Rightarrow b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

Note: To get similar formula of a_0 , $\frac{1}{2}$ has been written with a_0 in Fourier series.

Example 1: Find the Fourier Series representing

$$f(x) = x \quad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Solution:

Let

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots$$

Hence,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} x dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = \frac{1}{\pi} \left[\frac{4\pi^2}{2} - 0 \right] = 2\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} x \cos nx dx$$

By **integrating** by parts, yields

$$\frac{1}{\pi} \int_0^{2\pi} x \cos nx dx = \frac{1}{\pi} \left[uv - \int_0^{2\pi} v du \right]$$

$$= \left[\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \cos nx dx \Rightarrow v = \frac{\sin nx}{n} \end{array} \right]$$

$$= \frac{1}{\pi} \left[x \frac{\sin nx}{n} - \int_0^{2\pi} \frac{\sin nx}{n} dx \right]$$

$$= \frac{1}{\pi} \left[x \frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[2\pi \frac{\sin 2n\pi}{n} + \frac{\cos 2n\pi}{n^2} - \frac{1}{n^2} \right] = \frac{1}{\pi} \left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2} \right]$$

$$= \frac{1}{n^2\pi} (1 - 1) = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx$$

By **integrating** by parts, yields

$$\frac{1}{\pi} \int_0^{2\pi} x \sin nx dx = \frac{1}{\pi} \left[uv - \int_0^{2\pi} v du \right]$$

$$= \left[\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sin nx = dx \Rightarrow v = -\frac{\cos nx}{n} \end{array} \right]$$

$$\frac{1}{\pi} \int_0^{2\pi} x \sin nx dx = \frac{1}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - \int_0^{2\pi} \left(-\frac{\cos nx}{n} \right) dx \right]$$

$$= \frac{1}{\pi} \left[-\frac{x \cos nx}{n} + \int_0^{2\pi} \frac{\cos nx}{n} dx \right] = \frac{1}{\pi} \left[-x \frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[-2\pi \frac{\cos 2n\pi}{n} + \frac{\sin 2n\pi}{n^2} \right]_0^{2\pi} = \frac{1}{\pi} \left[-2\pi \frac{\cos 2n\pi}{n} \right] = -\frac{2}{n}$$

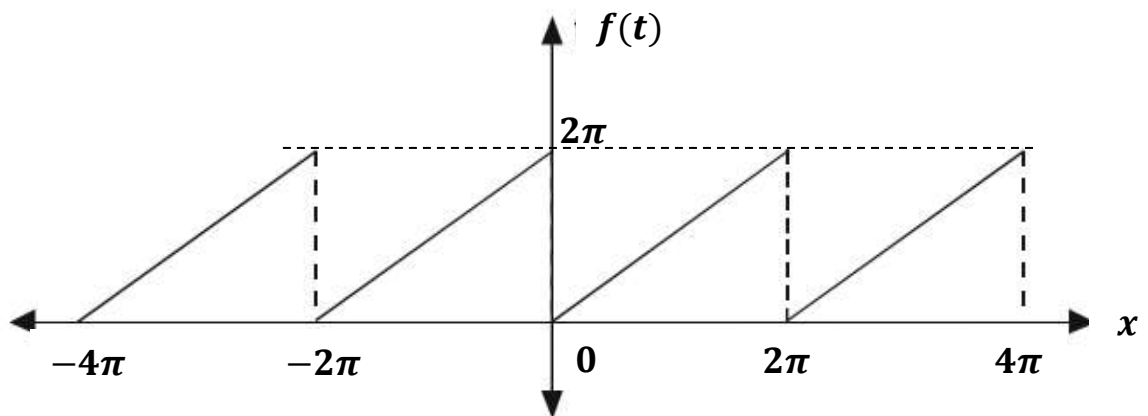
Substituting $a_0 = 2\pi$, $a_n = 0$, $b_n = -\frac{2}{n}$, into Equation of $f(x)$, yields

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots$$

$$\Rightarrow f(x) = x = \frac{2\pi}{2} - \frac{2}{1} \sin x - \frac{2}{2} \sin 2x - \frac{2}{3} \sin 3x - \dots$$

$$\Rightarrow x = \pi - 2 \left(\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right)$$

Now, drawing the graph from $x = -4\pi$ to $x = 4\pi$.



Example 2: Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$. Deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

Solution:

Let

$$x + x^2 = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots$$

$$f(x) = x + x^2 = -x + (-x)^2 = -(x - x^2) = -f(x),$$

so $f(x)$ is an odd function.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$\Rightarrow a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx$$

$$f(x) = x \cos nx = -x \cos(-nx) = -\cos(nx) = -f(x),$$

so $f(x)$ is an odd function.

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[x^2 \frac{\sin nx}{n} - (2x) \frac{(-\cos nx)}{n^2} + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi^2 \frac{\sin n\pi}{n} - (2\pi) \frac{(-\cos n\pi)}{n^2} + 2 \left(-\frac{\sin n\pi}{n^3} \right) \right] = \frac{4(-1)^n}{n^2}$$

$$\Rightarrow a_n = \frac{4}{n^2} (-1)^n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \sin nx dx$$

$$f(x) = x^2 \sin nx = (-x)^2 \sin(-nx) = -x^2 \sin(nx) = -f(x),$$

so $f(x)$ is an odd function.

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) - (1) \left(-\frac{\sin nx}{n^2} \right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[(-\pi) \frac{\cos n\pi}{n} + 2 \frac{\sin n\pi}{n^2} \right] = \frac{2}{\pi} \left[\frac{\pi}{n} \cos n\pi \right] = -\frac{2}{n} (-1)^n \\ &\Rightarrow b_n = -\frac{2}{n} (-1)^{n+1} \end{aligned}$$

Substituting the values of a_0 , a_n , b_n in Fourier series of the function $f(x)$ yields

$$\begin{aligned} x + x^2 &= \frac{\pi^2}{3} + 4 \left[-\cos x + \frac{1}{2^2} \cos 2x - \frac{1}{3^2} \cos 3x + \dots \right] \\ &\quad + 2 \left[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots \right] \end{aligned}$$

Putting $x = \pi$

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right] \quad (1)$$

Putting $x = -\pi$

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right] \quad (2)$$

Adding Equations (1) and (2)

$$\begin{aligned} \frac{4\pi^2}{3} &= 8 \left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right] \\ \Rightarrow \frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

Laplace Transform

The **Laplace transform** is an **integral transform** named **after** its **inventor Pierre-Simon Laplace**. It **transforms a function** of a **real variable t (often time)** to a **function** of a **complex variable s (complex frequency)**. The **transformation** was **named** by this name in **relation to** the **French scientist Laplace** who **lived** in the **nineteenth century**. The **transform** has **many applications** in **science and engineering**.

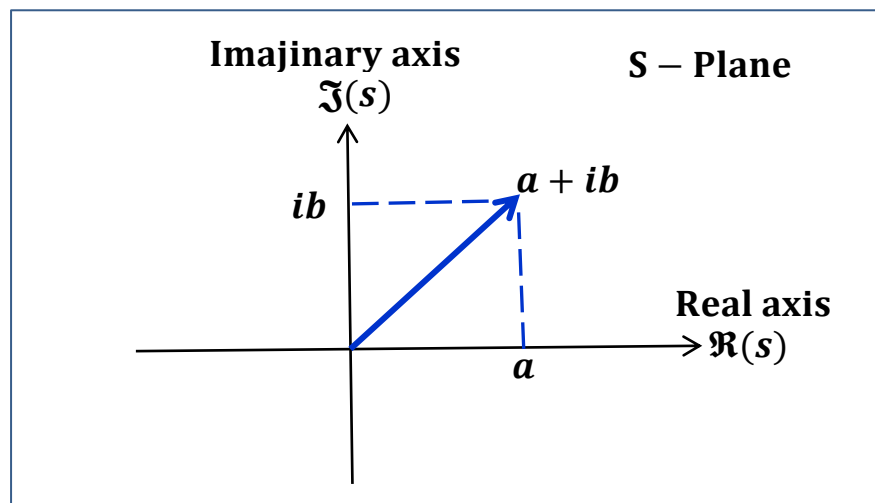
By **Laplace transform**, a **real function f** in **time t** is **translated into** the **s -plane** by taking the **integral** of the **function multiplied** by **e^{-st}** from **0** to **∞** ,

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

where s is a **complex number** with the **form**

$$s = a + bi$$

where **a** and **b** are **real numbers**, and **i** is **called** an **imaginary number** which is a **solution** of the **equation $x^2 = -1$** , because **no real number satisfies** this **equation**. Therefore, **a** is **called** the **real part**, and **ib** is **called** the **imaginary part**.



This **transformation from the t -domain into the s -domain** is known as a **Laplace transform** and the **function $F(s)$** is called the **Laplace transform of f** , and **denoted by $\mathcal{L}[f(t)]$** and **defined as**

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

With this **alternate notation**, note that the **transform is really a function of a new variable, s** , and that all the **t 's will drop out in the integration process**.

Now, the **integral in the definition of the transform is called an improper integral** and it would probably **be best to recall how these kinds of integrals work before jump into computing some transforms**.

Example 1: If $k \neq 0$, evaluate the following integral.

$$\int_0^{\infty} e^{kt} dt$$

Solution:

Convert improper integrals to limits as follows,

$$\begin{aligned} \int_0^{\infty} e^{kt} dt &= \lim_{n \rightarrow \infty} \int_0^n e^{kt} dt = \lim_{n \rightarrow \infty} \left[\frac{1}{k} e^{kt} \right]_0^n \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{k} e^{kn} - \frac{1}{k} e^0 \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{k} e^{kn} - \frac{1}{k} \right] \end{aligned}$$

At this point, careful has to be got. The value of k will affect the answer. k has already **assumed to be non-zero**, now need to be worry about the sign of k .

- If k is **positive** the exponential (e^{kn}) will **go to infinity**.
- if k is **negative** the exponential (e^{kn}) will **go to zero**.

So, the **integral** is only **convergent** (i.e. the **limit exists** and is **finite**) provided $k < 0$. In this case yields,

$$\int_0^{\infty} e^{kt} dt = \lim_{n \rightarrow \infty} \left[\frac{1}{k} e^{kn} - \frac{1}{k} \right] = -\frac{1}{k} \quad \text{provided } k < 0$$

Example 2: Find $\mathcal{L}(1)$

Solution:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$f(t) = 1 \Rightarrow F(s) = \mathcal{L}(1) = \int_0^{\infty} (1) e^{-st} dt = -\frac{1}{s} [e^{-st}]_0^{\infty}$$

$$\Rightarrow \mathcal{L}(1) = -\frac{1}{s} [e^{-\infty} - e^0] = -\frac{1}{s} \left(\frac{1}{e^{\infty}} - 1 \right) = -\frac{1}{s} (0 - 1) = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}(1) = \frac{1}{s}$$

Thus, the **Laplace transform** of the **constant** is:

$$\mathcal{L}(k) = \frac{k}{s}, \quad k = 1, 2, 3, \dots, \infty$$

Examples are:

$$\mathcal{L}(2) = \frac{2}{s}, \quad \mathcal{L}(10) = \frac{10}{s}, \quad \mathcal{L}(23) = \frac{23}{s}, \quad \dots$$

Example 3: Find $\mathcal{L}(e^{at})$

Solution:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$\begin{aligned} f(t) = e^{at} \Rightarrow \mathcal{L}(e^{at}) &= \int_0^{\infty} (e^{at}) e^{-st} dt = \int_0^{\infty} e^{at-st} dt \\ &= \int_0^{\infty} e^{(a-s)t} dt \end{aligned}$$

$$\begin{aligned}\Rightarrow \mathcal{L}(e^{at}) &= \frac{1}{a-s} [e^{(a-s)t}]_0^{\infty} = \frac{1}{a-s} [e^{-\infty} - e^0] \\ &= -\frac{1}{a-s} \left(\frac{1}{e^{\infty}} - 1 \right) = \frac{1}{s-a} (0 - 1) = \frac{1}{s-a}\end{aligned}$$

Thus, the Laplace transform of $f(t) = e^{\pm at}$ is:

$$\mathcal{L}(e^{\pm at}) = \frac{1}{s \mp a} \quad a = 1, 2, 3, \dots, \infty$$

$$\mathcal{L}(e^{-2t}) = \frac{1}{s+2}, \quad \mathcal{L}(10) = \frac{10}{s}, \quad \mathcal{L}(23) = \frac{23}{s}, \quad \dots$$

Examples are:

$$\mathcal{L}(e^{2t}) = \frac{1}{s-2}, \quad \mathcal{L}(e^{-3t}) = \frac{1}{s+3}, \quad \dots$$

Example 4: Find $\mathcal{L}[\sin(at)]$

Solution:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$f(t) = \sin(at) \Rightarrow \mathcal{L}[\sin(at)] = \int_0^{\infty} \sin(at) e^{-st} dt$$

$$= \lim_{n \rightarrow \infty} \int_0^n e^{-st} \sin(at) dt$$

Now, if integrate by parts yields,

$$\int u dv = uv - \int v du$$

$$\int_0^n e^{-st} \sin(at) dt$$

$$= \left| \begin{array}{l} u = e^{-st} \quad \Rightarrow \quad du = -s e^{-st} dt \\ dv = \sin(at) dt \quad \Rightarrow \quad v = \int_0^n \sin(at) dt \\ \Rightarrow v = -\frac{\cos(at)}{a} \Big|_0^n = -\left[\frac{\cos(an)}{a} - \frac{\cos 0}{a} \right] \\ \Rightarrow v = -\frac{\cos(an)}{a} + \frac{1}{a} = \frac{1}{a} [1 - \cos(an)] \end{array} \right|$$

$$\begin{aligned}
&= e^{-sn} \frac{1}{a} [1 - \cos(an)] - \int_0^n \frac{1}{a} [1 - \cos(an)] (-s e^{-st} dt) \\
&= \frac{1}{a} e^{-sn} [1 - \cos(an)] + \frac{s}{a} \int_0^n e^{-st} [1 - \cos(an)] dt \\
&= -\frac{1}{a} e^{-sn} [1 - \cos(an)] + \frac{s}{a} \int_0^n e^{-st} dt - \frac{s}{a} \int_0^n e^{-st} \cos(an) dt \\
\frac{s}{a} \int_0^n e^{-st} dt &= \frac{s}{a} \left[-\frac{1}{s} e^{-st} \right]_0^n = -\frac{1}{a} (e^{-sn} - e^0) = \frac{1}{a} (e^{-sn} - 1)
\end{aligned}$$

$$\int_0^n e^{-st} \cos(an) dt = \left[\begin{array}{l} u = e^{-st} \Rightarrow du = -s e^{-st} dt \\ dv = \cos(an) dt \Rightarrow v = \int_0^n \cos(at) dt \\ \Rightarrow v = \frac{\sin(at)}{a} \Big|_0^n = \frac{\sin(an)}{a} + \frac{\sin 0}{a} \\ \Rightarrow v = \frac{\sin(an)}{a} \end{array} \right]$$

$$\begin{aligned}
&= e^{-st} \frac{\sin(an)}{a} - \int_0^n \frac{\sin(an)}{a} (-s e^{-st} dt) \\
&= \frac{1}{a} e^{-st} \sin(an) + \frac{s}{a} \int_0^n e^{-st} \sin(an) dt
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_0^n e^{-st} \sin(at) dt &= \\
&= -\frac{1}{a} e^{-sn} [1 - \cos(an)] + \frac{1}{a} (e^{-sn} - 1) - \frac{s}{a} e^{-st} \sin(an) \\
&\quad - \frac{s^2}{a^2} \int_0^n e^{-st} \sin(an) dt \\
&= -\frac{1}{a} e^{-sn} + \frac{1}{a} e^{-sn} \cos(an) + \frac{1}{a} e^{-sn} - \frac{1}{a} - \frac{s}{a} e^{-st} \sin(an) \\
&\quad - \frac{s^2}{a^2} \int_0^n e^{-st} \sin(an) dt \\
&= \frac{1}{a} e^{-sn} \cos(an) - \frac{1}{a} - \frac{s}{a} e^{-st} \sin(an) - \int_0^n e^{-st} \cos(an) dt
\end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L} [\sin(at)] &= \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{a} e^{-sn} \cos(an) - \frac{1}{a} - \frac{s}{a} e^{-st} \sin(an) \right. \\ &\quad \left. - \int_0^n e^{-st} \cos(an) dt \right] \end{aligned}$$

Now, notice that in the **limits** it **had to assume** that $s > 0$ in order to do the following **two limits**:

$$\lim_{n \rightarrow \infty} e^{-sn} \cos(an) = 0$$

$$\lim_{n \rightarrow \infty} e^{-sn} \sin(an) = 0$$

Without **this assumption**, we get a **divergent integral again**. Also, note that when **got back** to the **integral** it just **converted** the **upper limit back to infinity**. The **reason** for this is that, if you think about it, **this integral** is **nothing more than** the **integral** that **started with**. Therefore, now get,

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2} F(s)$$

Now, **simply solve** for $F(s)$ to get,

$$\mathcal{L} [\sin(at)] = F(s) = \frac{a}{s^2 + a^2}$$

Example 5: Find $\mathcal{L} (\sinh at)$

Solution:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

substituting, yields

$$\begin{aligned} \mathcal{L} (\sinh at) &= \mathcal{L} \left(\frac{e^{at} - e^{-at}}{2} \right) = \frac{1}{2} \mathcal{L}(e^{at} - e^{-at}) \\ &= \frac{1}{2} \left(\frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{1}{2} \left(\frac{s+a - s+a}{(s-a)(s+a)} \right) \\ &= \frac{1}{2} \left(\frac{2a}{s^2 - a^2} \right) = \frac{a}{s^2 - a^2} \end{aligned}$$

Thus,

$$\mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}, \quad \mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}, \quad \mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}$$

Examples are:

$$\mathcal{L}(\sinh 2t) = \frac{2}{s^2 - 2^2} = \frac{2}{s^2 - 4},$$

$$\mathcal{L}(\cosh 3t) = \frac{s}{s^2 - 3^2} = \frac{s}{s^2 - 9}$$

$$\mathcal{L}(\sin 4t) = \frac{4}{s^2 + 4^2} = \frac{4}{s^2 + 16},$$

$$\mathcal{L}(\cos 6t) = \frac{s}{s^2 + 6^2} = \frac{s}{s^2 + 36}$$

Example 6: Find $\mathcal{L}(\sin^2 t)$

Solution:

$$\sin^2 t = \frac{1 - \cos 2t}{2}, \quad \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\begin{aligned} \mathcal{L}(\sin^2 t) &= \mathcal{L}\left(\frac{1 - \cos 2t}{2}\right) = \frac{1}{2} \mathcal{L}(1 - \cos 2t) \\ &= \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2 + 2^2} \right) \\ \Rightarrow \mathcal{L}(\sin^2 t) &= \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2 + 4} \right) \end{aligned}$$

Example 7: Find $\mathcal{L}(t^n)$

Solution:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \dots, \infty$$

Examples are:

$$\mathcal{L}(t) = \mathcal{L}(t^1) = \frac{1!}{s^2} = \frac{1}{s^2},$$

$$\mathcal{L}(t^2) = \frac{2!}{s^3} = \frac{2 \times 1}{s^3} = \frac{2}{s^3},$$

$$\mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{3 \times 2 \times 1}{s^4} = \frac{6}{s^4}$$

Example 8: Find $\mathcal{L}(t^2 + 2)^2$

Solution:

$$\mathcal{L}(t^4 + 4t^2 + 4) = \frac{4!}{s^5} + 4\frac{2!}{s^3} + \frac{4}{s}$$

| | |
|---------------------------|---|
| $\mathcal{L}(k)$ | $= \frac{k}{s}, \quad k = 1, 2, 3, \dots, \infty$ |
| $\mathcal{L}(e^{\pm at})$ | $= \frac{1}{s \mp a}, \quad a = 1, 2, 3, \dots, \infty$ |
| $\mathcal{L}(\sin at)$ | $= \frac{a}{s^2 + a^2}, \quad a = 1, 2, 3, \dots, \infty$ |
| $\mathcal{L}(\cos at)$ | $= \frac{s}{s^2 + a^2}, \quad a = 1, 2, 3, \dots, \infty$ |
| $\mathcal{L}(\sinh at)$ | $= \frac{a}{s^2 - a^2}, \quad a = 1, 2, 3, \dots, \infty$ |
| $\mathcal{L}(\cosh at)$ | $= \frac{s}{s^2 - a^2}, \quad a = 1, 2, 3, \dots, \infty$ |
| $\mathcal{L}(t^n)$ | $= \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \dots, \infty$ |

Example 9:

$$\mathcal{L}[e^{\pm at} f(t)] = \mathcal{L}f(t) \Big|_{s \rightarrow s \mp a}$$

$$\mathcal{L}[e^{3t} \sin(2t)] = \mathcal{L} \sin(2t) \Big|_{s \rightarrow s-3} = \frac{2}{s^2 + 2^2} \Big|_{s \rightarrow s-3} = \frac{2}{(s-3)^2 + 4}$$

$$\Rightarrow \mathcal{L}[e^{3t} \sin(2t)] = \frac{2}{s^2 - 6s + 9 + 4} = \frac{2}{s^2 - 6s + 13}$$

Example 10:

$$\mathcal{L}[e^{-at} f(t)] = \mathcal{L}f(t) \Big|_{s \rightarrow s+a}$$

1. Find $\mathcal{L}[e^{-t} \cos^2(2t)]$

Solution:

$$\begin{aligned}
\mathcal{L}[e^{-t} \cos^2(2t)] &= \mathcal{L} \cos^2(2t) \Big|_{s \rightarrow s+1} = \mathcal{L} \frac{1 + \cos 4t}{2} \Big|_{s \rightarrow s+1} \\
&= \frac{1}{2} \mathcal{L}[1 + \cos 4t] \Big|_{s \rightarrow s+1} = \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 4^2} \right] \Big|_{s \rightarrow s+1} \\
&= \frac{1}{2} \left[\frac{1}{s+1} + \frac{s}{(s+1)^2 + 16} \right] \\
&= \frac{1}{2} \left[\frac{1}{s+1} + \frac{s}{s^2 + 2s + 1 + 16} \right] \\
\Rightarrow \mathcal{L}[e^{-t} \cos^2(2t)] &= \frac{1}{2} \left[\frac{1}{s+1} + \frac{s}{s^2 + 2s + 17} \right]
\end{aligned}$$

2. Find $\mathcal{L}[\sinh t \cos(2t)]$

Solution:

$$\sinh t = \frac{e^t - e^{-t}}{2}$$

$$\begin{aligned}
\Rightarrow \mathcal{L}[\sinh t \cos(2t)] &= \mathcal{L} \left[\frac{e^t - e^{-t}}{2} \cos(2t) \right] \\
&= \frac{1}{2} \mathcal{L}[(e^t - e^{-t}) \cos(2t)] = \frac{1}{2} \mathcal{L}[e^t \cos(2t) - e^{-t} \cos(2t)] \\
&= \frac{1}{2} \left[\left(\frac{s}{s^2 + 2^2} \right) \Big|_{s \rightarrow s-1} - \left(\frac{s}{s^2 + 2^2} \right) \Big|_{s \rightarrow s+1} \right] \\
&= \frac{1}{2} \left[\left(\frac{s-1}{(s-1)^2 + 2^2} \right) - \left(\frac{s+1}{(s+1)^2 + 2^2} \right) \right]
\end{aligned}$$

Homework:

1. $\mathcal{L}(\cosh t \cos^2 t)$
2. $\mathcal{L}(\sinh 3t \cosh 2t)$
3. $\mathcal{L}(\sin^2 t \cos^2 t)$

Example 11:

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \mathcal{L} f(t)$$

$$\begin{aligned}
\mathcal{L}[t \sin(2t)] &= (-1)^1 \frac{d^1}{ds^1} \mathcal{L}(\sin 2t) \\
&= -\frac{d}{ds} \frac{2}{s^2 + 2^2} = -\frac{d}{ds} \left(\frac{2}{s^2 + 4} \right) \\
&= -\frac{0 - 2s \times s}{(s^2 + 4)^2} = \frac{4s^2}{(s^2 + 4)^2}
\end{aligned}$$

Example 12:

$$\begin{aligned}
\mathcal{L}[t^3 e^{-2t}] &= (-1)^3 \frac{d^3}{ds^3} \mathcal{L}(e^{-2t}) = -\frac{d^3}{ds^3} \frac{1}{s+2} = -\frac{d^3}{ds^3} (s+2)^{-1} \\
&= -\frac{d^2}{ds^2} (-1) (s+2)^{-2} \\
&= -\frac{d}{ds} (2) (s+2)^{-3} = -(-6) (s+2)^{-4} \\
&= \frac{6}{(s+2)^4}
\end{aligned}$$

or

$$\mathcal{L}[t^3 e^{-2t}] = \mathcal{L}[t^3]_{s \rightarrow s+2} = \frac{3!}{s^4} \Big|_{s \rightarrow s+2} = \frac{3 \times 2 \times 1}{(s+2)^4} = \frac{6}{(s+2)^4}$$

Example 13:

$$\begin{aligned}
\mathcal{L}[t^2 \cos 2t] &= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}(\cos 2t) = -\frac{d^2}{ds^2} \left(\frac{s}{s^2 + 2^2} \right) \\
&= -\frac{d^2}{ds^2} \left(\frac{s}{s^2 + 4} \right) = -\frac{d}{ds} \left[\frac{s^2 + 4 - 2s^2}{(s^2 + 4)^2} \right] \\
&= -\frac{d}{ds} \left[\frac{4 - s^2}{(s^2 + 4)^2} \right] \\
&= \frac{(-2s)(s^2 + 4)^2 - 2(s^2 + 4)(2s)(4 - s^2)}{(s^2 + 4)^4} \\
&= \frac{(-2s)(s^2 + 4)[(s^2 + 4) + 2(4 - s^2)]}{(s^2 + 4)^4} \\
&= \frac{(-2s)[s^2 + 4 + 8 - 2s^2]}{(s^2 + 4)^3} = \frac{(-2s)(12 - s^2)}{(s^2 + 4)^3}
\end{aligned}$$

Example 14:

$$\begin{aligned}
\mathcal{L}[4t e^{3t} \sin 2t] &= 4\mathcal{L}[t e^{3t} \sin 2t] = 4(-1)^1 \frac{d^1}{ds^1} \mathcal{L}(e^{3t} \sin 2t) \\
&= -4 \frac{d}{ds} \mathcal{L}[\sin 2t]_{s \rightarrow s-3} = -4 \frac{d}{ds} \left[\frac{2}{s^2 - 2^2} \right]_{s \rightarrow s-3}
\end{aligned}$$

$$\begin{aligned}
&= -4 \frac{d}{ds} \left[\frac{2}{(s-3)^2 - 4} \right] = -4 \frac{d}{ds} \left[\frac{2}{s^2 - 6s + 9 - 4} \right] \\
&= -4 \frac{d}{ds} \left[\frac{2}{s^2 - 6s + 5} \right] = -4 \left[\frac{0 - 2(2s - 6)}{(s^2 - 6s + 5)^2} \right] \\
&= \frac{16(s-3)}{(s^2 - 6s + 5)^2}
\end{aligned}$$

Example 15:

$$\begin{aligned}
\mathcal{L} \frac{f(t)}{t} &= \int_s^\infty \mathcal{L}f(t) ds \\
\mathcal{L} \frac{\sin t}{t} &= \int_s^\infty \mathcal{L} \sin t ds = \int_s^\infty \frac{1}{s^2 + 1^2} ds = [\tan^{-1}]_s^\infty \\
&= \tan^{-1} \infty - \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s
\end{aligned}$$

Example 16:

$$\begin{aligned}
\mathcal{L} \left(\frac{e^{3t} - e^{2t}}{t} \right) &= \int_s^\infty \mathcal{L}(e^{3t} - e^{2t}) ds \\
&= \left(\int_s^\infty \frac{1}{s-3} - \frac{1}{s-2} \right) ds = [\ln(s-3) - \ln(s-2)]_s^\infty \\
&= \left[\ln \frac{(s-3)}{(s-2)} \right]_s^\infty \\
&= \ln \frac{(\infty-3)}{\infty-2} - \ln \frac{(s-3)}{s-2} = \ln 1 - \ln \frac{(s-3)}{s-2} \\
&= -\ln \frac{(s-3)}{s-2} \quad \text{or} \quad \ln \frac{(s-2)}{(s-3)}
\end{aligned}$$

Properties of Laplace Transform

i. Linearity

$$ax + by \Leftrightarrow aX + bY$$

proof:

$$\begin{aligned}\mathcal{L} [ax(t) + by(t)] &= \int_0^{\infty} [ax(t) + by(t)] e^{-st} dt \\ &= a \int_0^{\infty} x(t) e^{-st} dt + b \int_0^{\infty} y(t) e^{-st} dt = aX(s) + bY(s)\end{aligned}$$

ii. Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad , \quad x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

proof:

$$\begin{aligned}\mathcal{L} [x(at)] &= \int_0^{\infty} [x(at)] e^{-st} dt \\ &= \int_0^{\infty} x(\tau) e^{-s\frac{\tau}{a}} \frac{d\tau}{|a|} = \int_0^{\infty} x(\tau) e^{-\frac{s}{a}\tau} \frac{d\tau}{|a|} = \frac{1}{|a|} X\left(\frac{s}{a}\right)\end{aligned}$$

iii. Modulation in Time / Time – Shift

$$\mathcal{L} [x(t - t_0) u(t - t_0)] = X(s) e^{-st_0}$$

proof:

$$\begin{aligned}\mathcal{L} [x(t - t_0) u(t - t_0)] &= \int_0^{\infty} [\mathcal{L} [x(t - t_0)]] e^{-st} dt \\ &= \int_0^{\infty} x(\tau) e^{-s(\tau+t_0)} d\tau \\ &= e^{-st_0} \int_0^{\infty} x(\tau) e^{-s\tau} d\tau = e^{-st_0} X(s)\end{aligned}$$

iv. Modulation in Frequency / Frequency – Shift

$$e^{at} x(t) \Leftrightarrow X(s - a)$$

proof:

$$\begin{aligned}\mathcal{L}[x(t) e^{at}] &= \int_0^{\infty} [x(t) e^{at}] e^{-st} dt = \int_0^{\infty} x(t) e^{at} e^{-st} dt \\ &= X(s - a)\end{aligned}$$

v. Time – Reverse

$$x(-t) \Leftrightarrow X(-s)$$

proof:

$$\begin{aligned}\mathcal{L}[x(-t)] &= \int_0^{\infty} [x(-t)] e^{-st} dt = \int_0^{\infty} x(\tau) e^{st} d\tau \\ &= \int_0^{\infty} x(\tau) e^{-(-st)} d\tau = X(-s)\end{aligned}$$

vi. Time Differentiation

$$\frac{d}{dt}x(t) \Leftrightarrow sX(s) - X(0)$$

proof:

$$\begin{aligned}\mathcal{L}\left[\frac{d}{dt}x(t)\right] &= \int_0^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \int_0^{\infty} e^{st} dx(t) \\ &= \underbrace{[e^{st} x(t)]_0^{\infty}}_{x(0)} - \int_0^{\infty} \underbrace{x(t) de^{-st}}_{sX(s)} = sX(s) - x(0)\end{aligned}$$

Remark. Repeat,

$$\begin{aligned}\mathcal{L}\left[\frac{d}{dt}\left(\frac{dx(t)}{dt}\right)\right] &= s \mathcal{L}\left(\frac{dx(t)}{dt}\right) - \left.\frac{dx(t)}{dt}\right|_{t=0} \\ &= s [sX(s) - x(0)] - x(0) \\ &= s^2X(s) - sx(0) - x(0)\end{aligned}$$

The **general form**, which can be **prove by Mathematical Induction**, is:

$$\mathcal{L} \left[\frac{d^n x(t)}{dt^n} \right] = s^n X(s) - s^{n-1}x(0) - \dots - x^{n-1}(0)$$

vii. Time Integration

$$\int_0^t x(\tau) d\tau \Leftrightarrow \frac{X(s)}{s}$$

proof:

$$\begin{aligned} \mathcal{L} \left[\int_0^t x(\tau) d\tau \right] &= \int_0^\infty \left[\int_0^t x(\tau) d\tau \right] e^{-st} dt \\ &= \int_0^\infty \left[\int_0^t x(\tau) d\tau \right] \frac{de^{-st}}{-s} dx(t) \end{aligned}$$

Integrating by parts:

$$\underbrace{\left[\frac{e^{-st} \int_0^t x(\tau) d\tau}{-s} \right]_0^\infty}_{= 0} + \frac{1}{s} \int_0^\infty e^{-st} d \left[\int_0^t x(\tau) d\tau \right]$$

By fundamental Theorem of Calculus

$$\frac{d}{dt} \left[\int_0^t x(\tau) d\tau \right] = x(t) \Leftrightarrow d \left[\int_0^t x(\tau) d\tau \right] = x(t) dt$$

The Laplace Theorem then becomes

$$\Rightarrow \mathcal{L} \left[\int_0^t x(\tau) d\tau \right] = \frac{1}{s} \int_0^\infty e^{-st} x(t) dt = \frac{X(s)}{s}$$

The Inverse Laplace Transform

Given the **Laplace Transform** $F(s)$, it can be obtained the **corresponding time domain function** $f(t)$ via the **Inverse Laplace Transform**:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} ds$$

This is **quite tedious** to apply **in practice**. Instead, it can be simplified by using the Laplace Transform tables to obtain the corresponding functions.

Example: what is

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = ?$$

Solution:

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \quad \times s(s+1)$$

$$1 = A(s+1) + Bs \Rightarrow 1 = A + s(A+B)$$

$$\Rightarrow A = 1, A+B = 0 \Rightarrow B = -A = -1$$

$$\Rightarrow \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = 1(t) - e^{-t}$$

In this example, we “**broke up**” the function $\frac{1}{s(s+1)}$ into a **sum of simpler functions**, and then applied the **inverse Laplace Transform** (by consulting the **Laplace Transform table**) to each of them. This is a **general technique for inverting Laplace Transforms**.

Note: If the **denominator** is a **second degree** or **more**, **provided that it is more than a term**, then the **fraction partition method** has to be used.

This is **illustrated** in the **next example**

Example: Evaluate

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right]$$

Solution:

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{A(s+3) + Bs}{s+3} = \frac{3A + s(A+B)}{s(s+3)}$$

$$A = \frac{1}{3} \quad \text{and} \quad B = -\frac{1}{3}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3} \mathcal{L}^{-1}\frac{1}{s} - \frac{1}{3} \mathcal{L}^{-1}\frac{1}{s+3} = \frac{1}{3}(1) - \frac{1}{3}e^{-3t}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3}(1 - e^{-3t})$$

Some **examples** are:

$$1. \mathcal{L}^{-1}\left(\frac{3}{s^2 - 9}\right) = \sinh 3t$$

$$2. \mathcal{L}^{-1}\left(\frac{5!}{s^6}\right) = t^5$$

$$3. \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{3!} \mathcal{L}^{-1}\frac{3!}{s^4} = \frac{1}{3!}t^3$$

$$4. \mathcal{L}^{-1}\left(\frac{5}{s^2 + 25}\right) = \sin 5t$$

$$5. \mathcal{L}^{-1}\left(\frac{20}{s^2 + 25}\right) = \frac{20}{5} \mathcal{L}^{-1}\left(\frac{5}{s^2 + 25}\right) = 4\sin 5t$$

$$\begin{aligned}
 6. \mathcal{L}^{-1}\left(\frac{1}{s+5} - \frac{3}{s^3}\right) &= e^{-5t} - 3\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = e^{-5t} - \frac{3}{2!}\mathcal{L}^{-1}\left(\frac{2!}{s^3}\right) \\
 &= e^{-5t} - \frac{3}{2!}\mathcal{L}^{-1}(t^2)
 \end{aligned}$$

$$\begin{aligned}
 7. \mathcal{L}^{-1}\left(\frac{s-6}{s^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - 6\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) \\
 &= \cos 2t - \frac{6}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \cos 2t - 3\sin 2t
 \end{aligned}$$

$$8. \mathcal{L}^{-1}\left[\frac{1}{s^3 + 6s^2 + 5s}\right]$$

$$s^3 + 6s^2 + 5s = s(s^2 + 6s + 5) = s(s+1)(s+5)$$

$$\frac{1}{s^3 + 6s^2 + 5s} = \frac{1}{s(s+1)(s+5)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+5}$$

$$A = \frac{1}{(0+1)(0+5)} = \frac{1}{5},$$

$$B = \frac{1}{(-1)(-1+5)} = -\frac{1}{4},$$

$$C = \frac{1}{(-5)(-5+1)} = \frac{1}{20}$$

$$\begin{aligned}
 \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^3 + 6s^2 + 5s}\right] &= \frac{1}{5}\mathcal{L}^{-1}\frac{1}{s} - \frac{1}{4}\mathcal{L}^{-1}\frac{1}{s+1} + \frac{1}{20}\mathcal{L}^{-1}\frac{1}{s+5} \\
 &= \frac{1}{5}(1) - \frac{1}{4}e^{-t} + \frac{1}{20}e^{-5t}
 \end{aligned}$$

$$9. \mathcal{L}^{-1}\left[\frac{s^2 + 2s + 4}{s(s+3)^3}\right]$$

$$\frac{s^2 + 2s + 4}{s(s+3)^3} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{(s+3)^2} + \frac{D}{(s+3)^3}$$

$$A = \frac{0 \pm 0 + 4}{(0+3)^3} = \frac{4}{27},$$

$$C_k = \frac{1}{(m+k)!} \frac{d^{m-k}}{ds^{m-k}} [(s-a)^m F(s)] ,$$

where, m – current derivative and k – Repetition number

$$\begin{aligned} B &= \frac{1}{(3-1)!} \frac{d^{3-1}}{ds^{3-1}} \left[(s+3)^3 \times \frac{s^2+2s+4}{s(s+3)^3} \right] \\ &= \frac{1}{2!} \frac{d^2}{ds^2} \left[\frac{s^2+2s+4}{s} \right] = \frac{1}{2} \frac{d}{ds} \left[\frac{s(2s+2) - s^2 - 2s - 4}{s^2} \right] \\ &= \frac{1}{2} \frac{d}{ds} \left[\frac{2s^2 + 2s - s^2 - 2s - 4}{s^2} \right] = \frac{1}{2} \frac{d}{ds} \left[\frac{s^2 - 4}{s^2} \right] \\ &= \frac{1}{2} \left[\frac{2s \times s^2 - 2s(s^2 - 4)}{s^4} \right] = \frac{1}{2} \left[\frac{2s^3 - 2s^3 + 8s}{s^4} \right] = \frac{1}{2} \frac{8s}{s^4} = \frac{4}{s^3} \\ \Rightarrow B &= \frac{4}{s^3} = \frac{4}{(-3)^3} = -\frac{4}{27} \end{aligned}$$

$$\begin{aligned} C &= \frac{1}{(3-2)!} \frac{d^{3-2}}{ds^{3-2}} \left[\frac{s^2+2s+4}{s} \right] = \frac{d}{ds} \left[\frac{s^2+2s+4}{s} \right] \\ &= \left[\frac{(2s+2)s - s^2 - 2s - 4}{s^2} \right] = \frac{2s^2 + 2s - s^2 - 2s - 4}{s^2} = \frac{s^2 - 4}{s^2} \\ \Rightarrow C &= \frac{(-3)^2 - 4}{(-3)^2} = \frac{5}{9} \end{aligned}$$

$$\begin{aligned} D &= \frac{1}{(3-3)!} \frac{d^{3-3}}{ds^{3-3}} \left[\frac{s^2+2s+4}{s} \right] = \left[\frac{s^2+2s+4}{s} \right] \\ 0! = 1 \Rightarrow D &= \frac{(-3)^2 + 2(-3) + 4}{(-3)} = \frac{9 - 6 + 4}{-3} = -\frac{7}{3} \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1} \left[\frac{s^2+2s+4}{s(s+3)^3} \right] &= \mathcal{L}^{-1} \left[\frac{4/27}{s} + \frac{-4/27}{s+3} + \frac{5/9}{(s+3)^2} + \frac{-7/3}{(s+3)^3} \right] \\ &= \frac{4}{27} \mathcal{L}^{-1} \frac{1}{s} - \frac{4}{27} \mathcal{L}^{-1} \frac{1}{s+3} + \frac{5}{9} \mathcal{L}^{-1} \frac{1}{(s+3)^2} - \frac{7}{3} \mathcal{L}^{-1} \frac{1}{(s+3)^3} \end{aligned}$$

$$= \frac{4}{27} (1) - \frac{4}{27} e^{-3t} + \frac{5}{9} \mathcal{L}^{-1} \frac{1}{(s+3)^2} - \frac{7}{3} \mathcal{L}^{-1} \frac{1}{(s+3)^3}$$

$$\mathcal{L}^{-1} \frac{1}{(s+3)^2} \quad \text{and} \quad \mathcal{L}^{-1} \frac{1}{(s+3)^3}$$

$$\mathcal{L}^{-1}(s \pm a)^n = e^{\mp at} \mathcal{L}^{-1} s^n$$

$$\mathcal{L}^{-1}(s - 5)^2 = e^{5t} \mathcal{L}^{-1} s^2$$

$$\mathcal{L}^{-1} \frac{1}{(s+3)^2} = e^{-3t} \mathcal{L}^{-1} \frac{1}{s^2} = e^{-3t} \cdot t$$

$$\mathcal{L}^{-1} \frac{1}{(s+3)^3} = e^{-3t} \mathcal{L}^{-1} \frac{1}{s^3} = e^{-3t} \cdot \frac{1}{2!} t^2$$

$$\Rightarrow \mathcal{L}^{-1} \left[\frac{s^2 + 2s + 4}{s(s+3)^3} \right] = \frac{4}{27} - \frac{4}{27} e^{-3t} + \frac{5}{9} e^{-3t} \cdot t - \frac{7}{3} e^{-3t} \cdot t^2$$

❖ Applications of Laplace Transform

The Laplace Transform has many applications. Two of the most important are the solution of:

- Differential equations and
- Convolution.

1. Solving Differential Equations using Laplace transforms

The Laplace Transform can greatly simplify the solution of problems involving differential equations.

The Laplace transform can be employed to solve constant coefficient ordinary differential equations. In particular initial value problems considered, and it shall be found that the initial conditions are automatically included as part of the solution process.

The **idea** is **simple**; the **Laplace transform** of **each term** in the **differential equation** is taken.

If the **unknown function** is $y(t)$ then, **on taking** the **Laplace transform**, an **algebraic equation** involving $Y(s) = \mathcal{L}[y(t)]$ is **obtained**. This **equation** is **solved** for $Y(s)$ which is **then inverted** to **produce** the **required solution** $y(t) = \mathcal{L}^{-1}[Y(s)]$.

Begin with a **straightforward initial value problem** involving a **first order constant coefficient differential equation**. Let's using the **Laplace transform approach** to **find** the **solution** of:

$$\frac{dy}{dt} + 2y = 12e^{3t} \quad y(0) = 3$$

Let's write

$$\mathcal{L}[y(t)] = Y(s)$$

Then, **taking** the **Laplace transform** of **every term** in the **differential equation** gives:

$$\mathcal{L}\left[\frac{dy}{dt}\right] + \mathcal{L}[2y] = \mathcal{L}[12e^{3t}]$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -y(0) + sY(s) = -3 + sY(s)$$

$$\mathcal{L}[2y] = 2Y(s) \quad \text{and}$$

$$\mathcal{L}[12e^{3t}] = \frac{12}{s-3}$$

Substituting these **expressions** into the **transformed version** of the **differential equation** gives

$$-3 + sY(s) + 2Y(s) = \frac{12}{s-3}$$

Solving for $Y(s)$ yields

$$Y(s)(s+2) = \frac{12}{s-3} + 3 = \frac{3+3s}{s-3}$$

Therefore,
$$Y(s) = \frac{3(s+1)}{(s+2)(s-3)}$$

Now, using partial fractions, this last expression can be written in a more convenient form:

$$Y(s) = \frac{3/5}{(s+2)} + \frac{12/5}{(s-3)}$$

and then, taking inverse of the Laplace transform, yields

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{5} \mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right] + \frac{12}{5} \mathcal{L}^{-1}\left[\frac{1}{(s-3)}\right]$$

thus

$$y(t) = \frac{3}{5} e^{-2t} + \frac{12}{5} e^{3t}$$

This is the **solution** to the **given initial value problem**.

Example 1: Solve the second-order initial-value problem

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 2y = e^{-t} \quad y(0) = 0, \quad y'(0) = 0$$

using the Laplace transform method.

Solution:

Taking the Laplace transform of each term:

$$\mathcal{L}\left[\frac{d^2 y}{dt^2}\right] + 2\mathcal{L}\left[\frac{dy}{dt}\right] + 2\mathcal{L}[y] = \mathcal{L}[e^{-t}]$$

$$\mathcal{L}\left[\frac{d^2 y}{dt^2}\right] = -y'(0) - s y(0) + s^2 Y(s) = s^2 Y(s)$$

$$2\mathcal{L}\left[\frac{dy}{dt}\right] = 2[-y(0) + s Y(s)] = 2sY(s)$$

$$\mathcal{L}[2y] = 2Y(s) \quad \text{and}$$

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}$$

$$\Rightarrow s^2 Y(s) + 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

Solving for $Y(s)$ yields

$$Y(s)(s^2 + 2s + 2) = \frac{1}{s + 1}$$

$$\Rightarrow Y(s) = \frac{1}{(s + 1)(s^2 + 2s + 2)}$$

Using partial fraction:

$$\frac{1}{(s + 1)(s^2 + 2s + 2)} \equiv \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 2}$$

$$\equiv \frac{A(s^2 + 2s + 2) + (Bs + C)(s + 1)}{(s + 1)(s^2 + 2s + 2)}$$

$$\Rightarrow 1 \equiv As^2 + 2As + 2A + Bs^2 + Bs + Cs + C$$

$$\Rightarrow 1 \equiv (A+B)s^2 + (2A + B + C)s + 2A + C$$

$$A + B = 0 \Rightarrow A = -B$$

$$2A + B + C = 0 \Rightarrow C = B$$

$$2A + C = 1 \Rightarrow 2A - A = 1$$

$$\Rightarrow A = 1 \Rightarrow B = -1 \Rightarrow C = -1$$

$$\Rightarrow \frac{1}{(s + 1)(s^2 + 2s + 2)} = \frac{1}{s + 1} + \frac{(-1)s - 1}{s^2 + 2s + 2}$$

$$= \frac{1}{s + 1} - \frac{s + 1}{s^2 + 2s + 2}$$

Also, $(s^2 + 2s + 2)$ can be written as:

$$s^2 + 2s + 2 \equiv (s + 1)^2 + 1$$

$$\Rightarrow \frac{1}{(s + 1)(s^2 + 2s + 2)} \equiv \frac{1}{s + 1} - \frac{(s + 1)}{(s + 1)^2 + 1}$$

Taking the Inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s + 1}\right] - \mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 1}\right]$$

thus

$$y(t) = e^{-t} - e^{-t}\cos t$$

Solving systems of differential equations

The **Laplace transform method** is also well suited to solving systems of differential equations. A simple example will illustrate the technique. Let $x(t)$, $y(t)$ be two independent functions which satisfy the coupled differential equation:

$$\frac{dx}{dt} + y = e^{-t}$$

$$\frac{dy}{dt} - x = 3e^{-t}$$

$$x(0) = 0 \quad , \quad y(0) = 1$$

Taking the Laplace transform of every term in the differential equations and use the notation:

$$\mathcal{L}[x(t)] = X(s) \quad \text{and} \quad \mathcal{L}[y(t)] = Y(s)$$

and the initial conditions and rearranging give

$$\mathcal{L}\left[\frac{dx}{dt}\right] = -x(0) + sX(s) = sX(s)$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -y(0) + sY(s) = -1 + sY(s)$$

$$\mathcal{L}[x(t)] = X(s)$$

$$\mathcal{L}[y] = Y(s)$$

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}$$

and

$$3\mathcal{L}[e^{-t}] = \frac{3}{s+1}$$

$$sX(s) + Y(s) = \frac{1}{s+1}$$

$$-X(s) - 1 + sY(s) = \frac{3}{s+1} \implies -X(s) + sY(s) = \frac{s+4}{s+1}$$

these algebraic equations can be solved in $X(s)$ and $Y(s)$ using a variety of techniques (inverse matrix; Cramer's determinant method etc.)

Here Cramer's method will be used:

$$\begin{aligned} X(s) &= \frac{\begin{vmatrix} 1 & 1 \\ s+1 & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{1 \cdot s - (s+1)}{s^2 + 1} = \frac{-4}{(s^2 + 1)(s + 1)} \\ &= \frac{2(s-1)}{s^2 + 1} - \frac{2}{s+1} \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{\begin{vmatrix} s & 1 \\ -1 & s+1 \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{s(s+1) + 1}{s^2 + 1} = \frac{s^2 + 4s + 1}{(s^2 + 1)(s + 1)} \\ &= -\frac{1}{s+1} + \frac{2(s+1)}{s^2 + 1} \end{aligned}$$

The last lines in each case having been obtained using partial fractions.

Taking Inverse Laplace Transform yields

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X(s)] = 2\mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2 + 1}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s+1}\right] \\ &\Rightarrow x(t) = 2 \cos t - 2 \sin t - 2e^{-t} \end{aligned}$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)] = -\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 2\mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right] + 2\mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right] \\ &\Rightarrow y(t) = 2 \cos t + 2 \sin t - e^{-t} \end{aligned}$$