The Republic of Iraq
Ministry of Higher Education and Scientific Research
Northern Technical University
Engineering Technical College / Mosul
Department of Power Mechanics Techniques Engineering

ENGINEERING ANALYSIS

For Third-Year Students

Dr. Qais A. Yousif

<u>Linear and Non-Linear Equations</u>

At the heart of linear algebra and much of applied mathematics is the problem of solving system of linear equations. The **standard form** for **single linear equation** in **two variables** is:

$$\boldsymbol{ax + by = c}$$

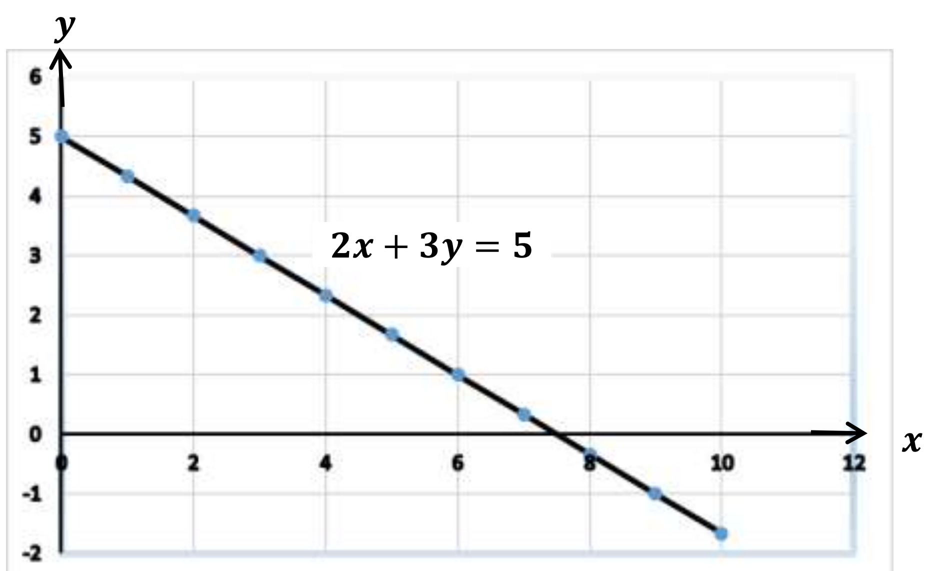
where $\boldsymbol{x}$ and $\boldsymbol{y}$ are **real variables** and $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are **real numbers**.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are **not** both **zero**, then the **graph** of this **equation** is a **straight line**; this is **why** is **called** **<u>linear</u>**.

A simple example of **linear equation** in **standard form** is:

$$\boldsymbol{2x + 3y = 5}$$

and the **graph** of **this equation** is shown in figure below



Also, above equation is called **one linear equation** in the **two unknowns** $\boldsymbol{x}$ and $\boldsymbol{y}$.

The **linear equation** with **three variables** $\boldsymbol{x}, \boldsymbol{y},$ and $\boldsymbol{z}$ has form

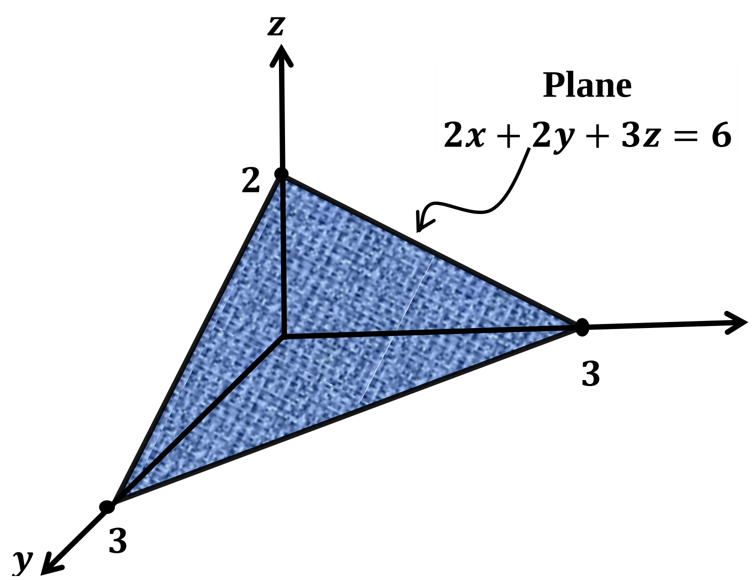
$$\boldsymbol{ax + by + cz = d}$$

where $\boldsymbol{a}$, $\boldsymbol{b}, \boldsymbol{c},$ and $\boldsymbol{d}$ are **real numbers**, not all zero, **represents a plane** in the standard $\boldsymbol{xyz}$ **coordinate system**.

An **example** of a **linear equation** in **three unknowns** is

$$\boldsymbol{2x + 2y + 3z = 6}$$

The **graph** of this **equation** (in **space**) is a **plane** as shown in figure below.



Of course, an **equation** can **contain more than three variables**.

For a **nonscientific example**, a **person** may have **five employees**, **each paid** a **different hourly wage** and each **working** a **variable number** of **hours each week**, as **shown** in the **following table**.

Employee	**Hourly Wage**	**Hours Worked**
1	**$4.50**	x_1
2	**$3.75**	x_2
3	**$5.00**	x_3
4	**$6.15**	x_4
5	**$5.75**	x_5

The **weekly payroll** $\boldsymbol{P}$ is given by

$$P = 4.50x_1 + 3.75x_2 + 5x_3 + 6.15x_4 + 5.75x_5$$

This is one **liner equation** in **five variables**.

<u>Definition</u> 1: A **linear equation** in the $\boldsymbol{n}$ **variables** $x_1, x_2, x_3, \cdots, x_n$ is **any equation** of the **form**

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where $a_1, a_2, a_3, \cdots, a_n, b$ are **constant real** or **complex number**. The **constant** a_i is **called** the **coefficient** of x_i; and $\boldsymbol{b}$ is **called** the **constant term** of the **equation**.

<u>Definition 2</u>: A **solution** of the **single linear equation f**or **real numbers** $s_1, s_2, s_3, \cdots, s_n$, if

$$a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$$

Which when substituted can be said that

$$s_1 = x_1 , \quad s_2 = x_2 = , \quad s_3 = x_3 , \cdots, s_n = x_n$$

is a **solution** of **this equation**.

An **example** of **Linear equation** involving **complex numbers** is:

$$(h + i)\, x_n + x_{n-1} = 0 \qquad n = 1, 2, 3, \cdots$$

Where h is a **real** and i is the **imaginary unit** $(i^2 = -1)$**.**

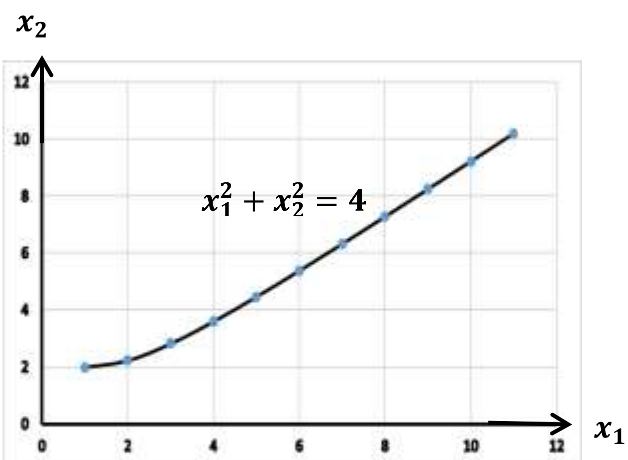
Any **equation** which is **not** in the **form** given in **Definition 1** is **called** **<u>nonlinear equation</u>**. For example,

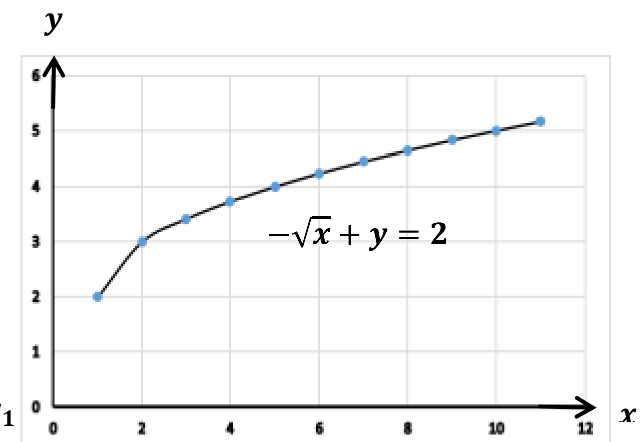
$$x_1^2 + x_2^2 = 4$$

$$-\sqrt{x} + y = 2$$

$$x_1 + |x_2| - 3x_3 = 7$$

are **<u>nonlinear equations</u>**. **Graph** of the **first two** are **shown below**





<u>Example</u> 1: Write a **general solution** of the **equation**:

$$2x_1 - 4x_2 = 8$$

<u>Solution</u>:

Dividing both sides of the **equation by 2**, yields

$$x_1 - 2x_2 = 4$$

Solving for x_1 with **ordinary algebra** gives

$$x_1 = 4 + 2x_2$$

The **solution** can be **written** as **pair ordered** as:

$$(x_1, x_2) \text{ or } (4 + 2x_2, x_2)$$

Letting x_2 **take on** the **specific values** $0, 1, \dots$, then **will have several sets** of **solutions**.

- **One** is $x_2 = 0$, $x_1 = 4$; **another** is
- $x_2 = 1$, $x_1 = 6$; and **yet another** is
- $x_2 = -4$, $x_1 = -4$.

Writing as **ordered pair**, these are $(4, 0), (6, 1),$ and $(4, -4)$, respectively.

However, in **case** of **complex number**, if we put $x_2 = 2 + i$, then $x_1 = 8 + 2i$, and **we`ll have** a **complex solutions**. If the **unknowns** were **restricted beforehand** to **be real number**, then **complex solutions** would have **<u>ruled out</u>**.

If **solving** for x_2 with **ordinary algebra**, then would have

$$x_2 = \frac{x_1}{2} - 2$$

The **solution** will be as **pair ordered**:

$$(x_1, x_2) \text{ or } \left(x_1, \frac{x_1}{2} - 2\right)$$

<u>Example</u> 2: Show that $x_1 = 1 + i, x_2 = 1 - i$ is **a solution** of the **equation**:

$$(3 + i)x_1 + (1 + i)x_2 = 4 + 4i$$
$$x_1 - x_2 = 2i$$

<u>Solution</u>:

Substituting the **given numbers** into the **given equations** yields

$$\begin{aligned}(3 + i)(1 + i) + (1 + i)(1 - i) &= 3 + 3i + i^2 + i + 1 + i - i - i^2 \\ &= 3 + 4i + (-1) + 1 - (-1) \\ &= 3 + 4i - 1 + 1 + 14 + 4i\end{aligned}$$

$$(1 + i) - (1 - i) = 2i$$

<u>Example</u> 3: Solve

$$\boldsymbol{ix_1 + (2-i)\,x_2 = 4-3i}$$
$$\boldsymbol{(1+i)x_1 - x_2 = -3-i}$$

<u>Solution</u>:

Solving the **second equation** for $\boldsymbol{x_2}$, we have

$$\boldsymbol{x_2 = (1+i)x_1 + 3 + i}$$

Substituting in the **first equation**, we find

$$\boldsymbol{ix_1 + (2-i)\,[(1+i)x_1 + 3 + i] = 4-3i}$$
$$\boldsymbol{ix_1 + \ (2-i)\,(1+i)x_1 + 3(2-i) + i(2-i) = 4-3i}$$
$$\boldsymbol{ix_1 + \ (2+i-i^2)\,x_1 + 6 - 3i + 2i - i^2 = 4-3i}$$
$$\boldsymbol{ix_1 + \ 2x_1 + ix_1 - i^2x_1 + 6 - i - i^2 = 4-3i}$$
$$\boldsymbol{2ix_1 + 2x_1 + x_1 + 6 - i + 1 = 4-3i}$$
$$\boldsymbol{2ix_1 + \ 3x_1 + 7 - i = 4-3i}$$
$$\boldsymbol{(2i+\ 3)x_1 = -3-2i = -(2i+\ 3)}$$

$$\Longrightarrow \quad \boldsymbol{x_1 = -\frac{(2i+\ 3)}{(2i+\ 3)} = -1}$$

Substituting into **equation** for $\boldsymbol{x_2}$, leads to

$$\Longrightarrow \quad \boldsymbol{x_2 = (1+i)(-1) + 3 + i = -1 - i + 3 + i = 2}$$

The **solution** will be as pair ordered:$\boldsymbol{(x_1, x_2) = \ (1, 2)}$

<u>A system of Linear Equations</u>

A **system of linear equations** (or **linear system**) is a **finite collection** of **linear equations** in <u>**same variables**</u>. For instance, a **linear system** of $\boldsymbol{m}$ **equations** in $\boldsymbol{n}$ variables $\boldsymbol{x_1, x_2, x_3, \cdots, x_n}$ can be **written as**

$$\begin{gathered}
\boldsymbol{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1} \\
\boldsymbol{a_{21}x_1 + a_{21}x_1 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2} \\
\vdots \\
\boldsymbol{a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m}
\end{gathered}$$

A **solution** of a **linear system** is a **set** $(\boldsymbol{s_1, s_2, s_3, \cdots, s_n})$ of **numbers** that makes **each equation** true statement when the **values** $\boldsymbol{s_1, s_2, s_3, \cdots, s_n}$ are **substituted** for $\boldsymbol{x_1, x_2, x_3, \cdots, x_n}$, respectively.

$$\boldsymbol{s_1 = x_1, s_2 = x_2, s_3 = x_3, \cdots, s_n = x_n}$$

The **set of all solutions** of a **linear system** is called the **solution set** of the **system**.

<u>Classification of linear systems</u>

<u>**Theorem**</u> **1:** A **linear system** is said to be <u>**consistent**</u> if it **has** at least **one solution**; and is said to be **inconsistent** if it has **no solution**.

In **two variables**, here is an **example** of a **system of two equations**:

$$\begin{aligned}
\boldsymbol{2x + y} &= \boldsymbol{3} \\
\boldsymbol{x - 9y} &= \boldsymbol{-8}
\end{aligned}$$

Clearly, $\boldsymbol{x = 1}$, $\boldsymbol{y = 1}$ is the **(only) solution** to this system.

Such a system would be **called** a <u>**consistent**</u>**.** <u>**Geometrically**</u>, **solution** given by **precisely** the **point where** the **graphs** (**two lines**) of **these two equations** <u>**meet**</u> **(intersect)**.

Also note that the system

$$\begin{aligned} \mathbf{2x + y = 3} \\ \mathbf{2x + y = 7} \end{aligned}$$

does **not have any solution**.

Such a system would be **called** an **<u>inconsistent</u>** **system**. Geometrically, **these two equations** in the system **represent** **<u>two parallel lines</u>** (they **<u>never meet</u>**).

In **three variables**, the following is an **example** of a **system of two equations**:

$$\begin{aligned} \mathbf{2x + \quad y + 2z} &= \mathbf{\quad 3} \\ \mathbf{x - 9y + 2z} &= \mathbf{-8} \end{aligned}$$

Clearly, $\mathbf{x = 1}$, $\mathbf{y = 1}$, and $\mathbf{z = 0}$ is a **solution** to this system. This system **has many more solutions**.

For example,

$$\mathbf{x = 11}\,, \qquad \mathbf{y = 0}\,, \qquad \text{and} \qquad \mathbf{z = -\frac{19}{2}}$$

is **also a solution** of this **system**.

This **system would be called** a **<u>consistent</u>**. **Geometrically**, **solution given** by **precisely** the **<u>points</u>** where the **<u>graphs</u>** of the **<u>two planes</u>** in **space** of **these two equations meet**.

<u>Definition</u> 3: A **system** of $\boldsymbol{m}$ **linear equations** in these $\boldsymbol{n}$ **variables** can be written as

$$\begin{array}{ccccccccc} \boldsymbol{a_{11}x_1} & + & \boldsymbol{a_{12}x_2} & + & \boldsymbol{a_{13}x_3} & + \cdots + & \boldsymbol{a_{1n}x_n} & = & \boldsymbol{b_1} \\ \boldsymbol{a_{21}x_1} & + & \boldsymbol{a_{21}x_2} & + & \boldsymbol{a_{23}x_3} & + \cdots + & \boldsymbol{a_{2n}x_n} & = & \boldsymbol{b_2} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \boldsymbol{a_{m1}x_1} & + & \boldsymbol{a_{m2}x_2} & + & \boldsymbol{a_{m3}x_3} & + \cdots + & \boldsymbol{a_{mn}x_n} & = & \boldsymbol{b_m} \end{array}$$

Where $\boldsymbol{a_{ij}}$ and $\boldsymbol{b_i}$ are all real numbers.

- **If** $b_1 = b_2 = \cdots = b_m = 0$ **,** Such a **linear system** is **called** a **homogeneous linear system**.

- **Two systems** of **linear equations** are called **equivalent**, if they **have precisely** the **same set of solutions**.

- **Following operations** on a **system produces** an **equivalent system**:

  i. **Interchange of any two equations**.
  ii. **Multiplication** of **any equation** by a **nonzero constant**.
  iii. **Addition to an equation the result of multiplying another equation** by a **constant**.

  These three operations are sometimes **known** as **basic** or **elementary operations**.

- A **linear system** of the **form**

$$\begin{aligned} x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ x_3 + \cdots + a_{3n}x_n &= b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad & \vdots \end{aligned}$$

is **said to be** in **row-echelon form**. The point is:

(a) **drop one variable** in **each successive equation** (**step**),

(b) The **coefficient** of the "**leading variable**" in **equation** is **1**.

In **two variables** x, y the **row-echelon form** would (sometime) **look like**

$$\begin{aligned} x + a_{12}y &= b_1 \\ y &= b_2 \end{aligned}$$

In **three variables** x, y, z this **<u>row-echelon form</u>** would (sometime) **look like**

$$\begin{aligned} x + a_{12}y + a_{13}z &= b_1 \\ y + a_{23}z &= b_2 \\ z &= b_3 \end{aligned}$$

<u>Theorem</u> 2: The **following** are **some facts**:

1. **Any system** of **linear equations** is **equivalent** to a **linear system** in **row echelon form**.
2. **This can be achieved** by a **sequence of application** of the **three basic elementary operation** **<u>described</u>** in the **<u>next example</u>**
3. This **process** is **known** as **<u>Gaussian elimination</u>**

<u>Example</u> 4: Reduce and **solve** the **following system by <u>Gaussian elimination method</u>**

$$\begin{aligned} 4x - 5y &= 3 \qquad (1) \\ -8x + 10y &= 14 \qquad (2) \end{aligned}$$

<u>Solution</u>:

<u>Multiply</u> Eq. (1) by **2** and **<u>add</u>** to **Eq. (2), yields**

$$\begin{aligned} 4x - 5y &= 3 \qquad (1) \\ 0 &= 20 \qquad (3) \end{aligned}$$

The **Eq. (3)** is **<u>absurd</u>**. So, the **system** has **<u>no solution</u>**.

Thus, the **system** is **<u>inconsistent</u>**.

<u>Example</u> 5: Reduce the **following system** and **solve**:

$$\mathbf{9x - 4y = 5} \qquad (\mathbf{1})$$

$$\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x} + \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{y} = \mathbf{0} \qquad (\mathbf{2})$$

<u>Solution</u>:

<u>Multiplying</u> Eq. (2) by **18** and **<u>subtract</u> Eq. (1)** from **Eq. (2)**, yields

$$\mathbf{9x - 4y = 5} \qquad (\mathbf{1})$$

$$\mathbf{10y = -5} \qquad (\mathbf{3})$$

<u>Dividing</u> Eq. (3) by **10** and **Eq. (1) by 9**, yields:

$$\boldsymbol{x} - \frac{\mathbf{4}}{\mathbf{9}}\boldsymbol{y} = \frac{\mathbf{5}}{\mathbf{9}} \qquad (\mathbf{4})$$

$$\boldsymbol{y} = -\frac{\mathbf{1}}{\mathbf{2}} \qquad (\mathbf{5})$$

This is an **equivalent row-echelon form** of the **given system**

Now substitute $\boldsymbol{y} = -\frac{\mathbf{1}}{\mathbf{2}}$ in **Eq. (4)** yields

$$\boldsymbol{x} - \frac{\mathbf{4}}{\mathbf{9}}\left(-\frac{\mathbf{1}}{\mathbf{2}}\right) = \frac{\mathbf{5}}{\mathbf{9}} \quad \Rightarrow \quad \boldsymbol{x} = \frac{\mathbf{5-2}}{\mathbf{9}} = \frac{\mathbf{1}}{\mathbf{3}}$$

Thus, the **solution** is the pair ordered:

$$(\boldsymbol{x}, \boldsymbol{y}) = \left(\frac{\mathbf{1}}{\mathbf{3}}, -\frac{\mathbf{1}}{\mathbf{2}}\right)$$

Thus, the **system** is **<u>consistent</u>**.

<u>Example</u> 6: Deduce an **equivalent row-echelon form** and **solve** the **following system**

$$\begin{aligned} 2x + y - z &= -1 \\ x - 2y + z &= 5 \\ 3x - y - 2z &= 0 \end{aligned}$$

<u>Solution</u>:

<u>Interchange</u> the **first** and **second equation**, yields

$$\begin{aligned} x - 2y + z &= 5 \\ 2x + y - z &= -1 \\ 3x - y - 2z &= 0 \end{aligned}$$

<u>Multiply</u> the **first equation** by(-2) and **add** to the **second equation,** yields

$$\begin{array}{r} 2x + y - z = -1 \\ \underline{-2x + 4y - 2z = -10} \\ 5y - 3z = -11 \end{array}$$

The **system becomes**

$$\begin{aligned} x - 2y + z &= 5 \\ 5y - 3z &= -11 \\ 3x - y - 2z &= 0 \end{aligned}$$

<u>Multiply</u> the **first equation** by (-3) and **add** to the **third equation**, yields

$$\begin{array}{r} 3x - y - 2z = 0 \\ \underline{-2x + 6y - 3z = -15} \\ 5y - 5z = -15 \end{array}$$

The **system becomes**

$$\begin{aligned} x - 2y + z &= 5 \\ 5y - 3z &= -11 \\ 5y - 5z &= -15 \end{aligned}$$

<u>Multiply</u> the **second equation** by (-1) and **add** to the **third equation** to **eliminate y** in the **third equation** :

$$\begin{array}{r} \mathbf{5y - 5z = -15} \\ \underline{\mathbf{-5y + 3z = \quad 11}} \\ \mathbf{-2z = -4} \end{array}$$

Thus, the **row-echelon form** which is an **<u>equivalent system</u>** to the **given system** is

$$\begin{aligned} \mathbf{x - 2y + \quad z} &= \mathbf{\quad 5} \\ \mathbf{5y - 3z} &= \mathbf{-11} \\ \mathbf{-2z} &= \mathbf{-4} \end{aligned}$$

From the **last equation** of the **row echelon form** yields

$$\mathbf{z = 2,}$$

and **substituting** into **second equation** will give

$$\mathbf{y = -1}.$$

Then **substituting** the **obtained values** of **z** and **y** into **first equation** will get

$$\mathbf{x = 1.}$$

Therefore, the **solution** is:

$$\mathbf{x = 1}, \qquad \mathbf{y = -1}, \qquad \text{and} \quad \mathbf{z = 2}$$

Gaussian, Gauss-Jordan Elimination

Definition 4: Given a **system** of $\boldsymbol{m}$ **linear equations**

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{21}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{31}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array}$$

the **augmented matrix** of the **system** is defined as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{pmatrix}$$

and the **coefficient matrix** is defined as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

- Consider the **linear system** of **two equations** in **two variables**

$$\begin{aligned} 4x - 5y &= 3 \\ -8x + 10y &= 14 \end{aligned}$$

The **augmented matrix** of the **system** is:

$$\begin{pmatrix} 4 & -5 & 3 \\ -8 & 10 & 14 \end{pmatrix}$$

and the **coefficient matrix** is:

$$\begin{pmatrix} 4 & -5 \\ -8 & 10 \end{pmatrix}$$

<u>Example</u>: For the following **linear system** of **three equations** in **three variables, d**educe an **equivalent row-echelon form** and solve

$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \qquad (1)\\ 2x_1 + 4x_2 - x_3 &= 7 \qquad (2)\\ x_1 - 11x_2 + 4x_3 &= 3 \qquad (3) \end{aligned}$$

The **<u>augmented</u>** and **<u>coefficient matrices</u>** of the **system** are:

$$\begin{pmatrix} 5 & -3 & 2 & 3\\ 2 & 4 & -1 & 7\\ 1 & -11 & 4 & 3 \end{pmatrix} ; \begin{pmatrix} 5 & -3 & 2\\ 2 & 4 & -1\\ 1 & -11 & 4 \end{pmatrix}$$

To **deduced** an **equivalent system** in **row-echelon form**, first **switching Eq. (1)** and **Eq. (3)**:

$$\begin{aligned} x_1 - 11x_2 + 4x_3 &= 3 \qquad (3)\\ 2x_1 + 4x_2 - x_3 &= 7 \qquad (2)\\ 5x_1 - 3x_2 + 2x_3 &= 3 \qquad (1) \end{aligned}$$

Multiplying Eq. (3) by **2** and **subtract** from **Eq. (2)**,

$$\begin{array}{r} 2x_1 + 4x_2 - x_3 = 7\\ \underline{2x_1 - 22x_2 + 8x_3 = 6}\\ 26x_2 - 9x_3 = 1 \end{array}$$

The system becomes

$$\begin{aligned} x_1 - 11x_2 + 4x_3 &= 3 \qquad (3)\\ 26x_2 - 9x_3 &= 1 \qquad (4)\\ 5x_1 - 3x_2 + 2x_3 &= 3 \qquad (1) \end{aligned}$$

Multiplying Eq. (3) by **5** and **subtract** from **Eq. (1)**:

$$\begin{array}{r} 5x_1 - 3x_2 + 2x_3 = 3\\ \underline{5x_1 - 55x_2 + 20x_3 = 15}\\ 52x_2 - 18x_3 = -12 \end{array}$$

The system becomes

$$\begin{aligned} x_1 - 11x_2 + \ 4x_3 &= \ \ 3 \qquad (3) \\ 26x_2 - \ 9x_3 &= \ \ 1 \qquad (4) \\ 52x_2 - 18x_3 &= -12 \qquad (5) \end{aligned}$$

Multiplying Eq. (4) by **2** and **subtract** from **Eq. (5)**

$$\begin{array}{r} 52x_2 - 18x_3 = -12 \\ 52x_2 - 18x_3 = \ \ 2 \\ \hline 0 = -14 \end{array}$$

The system becomes

$$\begin{aligned} x_1 - 11x_2 + \ 4x_3 &= \ \ 3 \qquad (3) \\ 26x_2 - \ 9x_3 &= \ \ 1 \qquad (4) \\ 0 &= -14 \qquad (6) \end{aligned}$$

Eq. (6) is **absurd,** so the system is <u>**inconsistent**</u>. To obtain the **row-echelon form**, **Eq. (4)** need to be **divided** by **26**:

$$\begin{aligned} x_1 - 11x_2 + \ 4x_3 &= \ \ 3 \qquad (3) \\ x_2 - \frac{9}{26}x_3 &= \frac{1}{26} \qquad (7) \\ 0 &= -14 \qquad (6) \end{aligned}$$

The <u>**augmented** and **matrix**</u> of the **system** are:

$$\begin{pmatrix} 1 & -11 & 4 & 3 \\ 0 & 1 & -\frac{9}{26} & \frac{1}{26} \\ 0 & 0 & 0 & -14 \end{pmatrix} ; \begin{pmatrix} 1 & -11 & 4 \\ 0 & 1 & -\frac{9}{26} \\ 0 & 0 & 0 \end{pmatrix}$$

This **examples demonstrates** that the **three basic operations** that **were used** to **reduce** a **system of linear equations** to a **row-echelon form**, can be **translated** to a **version** for **matrices**.

<u>Definition</u> 5: By an **elementary row operation** on a **matrix** which are the **following three**:

iv. **Interchange of any two rows**.
v. **Multiplication** of **any row** by a **nonzero constant**
vi. **Addition** to any **row** the **result** of **multiplying another row** by a **constant**

❖ **Two matrices** are said to be **row-equivalent** if **one can** be **obtained from another** by **application** of a **sequence** of **elementary row operations**. **Two row-equivalent matrices**, correspond to **two equivalent system** of **equations**.

<u>Definition</u> 6: A **matrix** is said to be in **row-echelon form**, if it **has** the **following properties**:

1. **All rows consisting entirely** of **zeros occur** at the **bottom** of the **matrix**.
2. For **each non-zero row**, **first nonzero entry** is **1** (called the **leading 1**).
3. For each **successive nonzero rows**, the **leading 1** in the **higher row** is farther to the **left** than the **leading 1** in the **lower row**.

A **matrix** in **row-echelon form** is said to be in **reduced row-echelon form**, if **every column** that **has** a **leading 1**, has **zeros** in **every position above** and **below** the **leading 1**.

<u>Theorem</u> 3: Suppose ***M*** is a **matrix**. Then, ***M*** is **row-equivalent** to a **matrix** ***B***, which is in **row-echelon form**.

❖ The **method** of **solving** this **system** by **<u>Gaussian elimination</u>** with **back-substitution equation** is described as follows:

i. **Write** the **augmented matrix** of the system.

ii. **Use** the **elementary row operations** to **reduce** the **augmented matrix** to a **matrix** in **row-echelon form**.

iii. **Write** the **linear system** corresponding to the **row-echelon matrix** and **solve** by **back-substitution**.

<u>Example</u>: Use the **method** of **Gaussian elimination** with **beck-substitution** to **solve** the **following system** of **linear equations**

$$\begin{aligned} x_1 + 3x_4 &= 4 \\ 2x_2 - x_3 - x_4 &= 0 \\ 3x_2 - 2x_4 &= 1 \\ 2x_1 - x_2 + 4x_3 &= 5 \end{aligned}$$

<u>Solution</u>:

The augmented matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 2 & -1 & 4 & 0 & 5 \end{pmatrix}$$

Multiplying row-1 by(2) and **subtract** from **row-4:**

$$\begin{array}{rrrrrl} 2 & -1 & 4 & 0 & 5 & \textbf{Row-4} \\ 2 & 0 & 0 & 6 & 8 & \textbf{Row-1}\text{ multiplied by }\mathbf{2} \\ \hline 0 & -1 & 4 & -6 & -3 & \textbf{Row-4} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Multiplying row-2 by $\frac{1}{2}$ **and add to row-4:**

$$\begin{array}{ccccc|l} 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \textbf{Row-2 multiplied by } \frac{1}{2} \\ 0 & -1 & 4 & -6 & -3 & \textbf{Row-4} \\ \hline 0 & 0 & \frac{7}{2} & \frac{13}{2} & -3 & \textbf{Row-4} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 3 & 0 & -2 & 1 \\ 0 & 0 & \frac{7}{2} & -\frac{13}{2} & -3 \end{pmatrix}$$

Multiplying row-2 by **3** and **subtract** from **row-3**:

$$\begin{array}{ccccc|l} 0 & 3 & 0 & -2 & 1 & \textbf{Row-3} \\ 0 & 3 & -\frac{3}{2} & -\frac{3}{2} & 0 & \textbf{Row-2} \text{ multiplied by } \mathbf{3} \\ \hline 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 & \textbf{Row-3} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & -1 & 4 & -6 & -3 \end{pmatrix}$$

Multiplying row-3 by $\frac{2}{3}$, the **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{7}{2} & -\frac{13}{2} & -3 \end{pmatrix}$$

Multiplying row-3 by $\frac{7}{2}$ and **subtract** from **row-4:**

$$\begin{array}{ccccc|l} 0 & 0 & \frac{7}{2} & -\frac{13}{2} & -3 & \textbf{Row-4} \\ 0 & 0 & \frac{7}{2} & -\frac{7}{6} & -\frac{14}{6} & \textbf{Row-3 multiplied by } \frac{7}{2} \\ \hline 0 & 0 & 0 & -\frac{16}{3} & -\frac{16}{3} & \textbf{Row-4} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & -\frac{16}{3} & -\frac{16}{3} \end{pmatrix}$$

Multiplying row-4 by $-\frac{3}{16}$, the **augmented matrix** becomes

The **augmented matrix** becomes

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & -\frac{\mathbf{1}}{\mathbf{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{3}} & \frac{\mathbf{2}}{\mathbf{3}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

The above is a **matrix** in **row-echelon form row-equivalent** to the **augmented matrix**. Now the **system** of **linear equations** corresponding this **row-echelon matrix** is:

$$\boldsymbol{x_1 + 3x_4 = 4}$$

$$\boldsymbol{x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0}$$

$$\boldsymbol{x_3 - \frac{1}{3}x_4 = \frac{2}{3}}$$

$$\boldsymbol{x_4 = 1}$$

Substituting $\boldsymbol{x_4 = 1}$ into **third** and **first equation** we have

$$\boldsymbol{x_3 = \frac{2}{3} + \frac{1}{3} = 1} \quad \text{and} \quad \boldsymbol{x_1 = 1}$$

Substituting $\boldsymbol{x_4 = 1}$ and $\boldsymbol{x_3 = 1}$ into **second equation** we have

$$\boldsymbol{x_2 = 1}$$

By **back-substitution**, we have

$$\boldsymbol{x_1 = 1}, \qquad \boldsymbol{x_2 = 1}, \qquad \boldsymbol{x_3 = 1}, \qquad \boldsymbol{x_4 = 1}$$

Gauss-Jordan elimination

Definition 5: A **matrix** in **row-echelon form** is said to be in **Gauss – Jordan form**, if **all the entries above leading entries** are **zero**. The **method** of **Gaussian elimination** with **back substitution** to **solve system** of **linear equations** can be **refined** by **first further reducing** the **augmented matrix** to a **Gauss-Jordan form** and **work** with the **system corresponding** to it. This **method** is **called Gauss-Jordan elimination method** of **solving linear systems**.

Considering previous example, the **matrix** in the **row-echelon form**, **equivalent** to the **augmented matrix**, is

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & -\frac{\mathbf{1}}{\mathbf{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{3}} & \frac{\mathbf{2}}{\mathbf{3}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

All the **entries above** the **leading 1** in **row 2** is **zero**. So, **trying** to **achieve** the **same above** the **leading 1** in **row 3**:

Multiply **row-3** by $\frac{\mathbf{1}}{\mathbf{2}}$ and **add** from **row-2** yields

$$\begin{array}{ccccc|l} \mathbf{0} & \mathbf{0} & \frac{\mathbf{1}}{\mathbf{2}} & -\frac{\mathbf{1}}{\mathbf{6}} & \frac{\mathbf{1}}{\mathbf{3}} & \textbf{Row-3 multiplied by } \frac{\mathbf{1}}{\mathbf{2}} \\ \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & -\frac{\mathbf{1}}{\mathbf{2}} & \mathbf{0} & \textbf{Row-2} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & -\frac{\mathbf{2}}{\mathbf{3}} & \frac{\mathbf{1}}{\mathbf{3}} & \textbf{Row-2} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Now we want to **get zeros above** the **leading 1** in **row-4.**
<u>Multiply</u> row-4 by **3** and **<u>subtract</u>** from **row-1,** yields

$$\begin{array}{rrrrrl} 1 & 0 & 0 & 3 & 4 & \textbf{Row-1} \\ 0 & 0 & 0 & 3 & 3 & \textbf{Row-4 multiplied by 3} \\ \hline 1 & 0 & 0 & 0 & 1 & \textbf{Row-1} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

<u>Multiply</u> row-4 by $\frac{2}{3}$ and **add** to **row-2,** yields

$$\begin{array}{rrrrrl} 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} & \textbf{Row-4 multiplied by } \frac{2}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \textbf{Row-2} \\ \hline 0 & 1 & 0 & 0 & 1 & \textbf{Row-2} \end{array}$$

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Multiply **row-4** by $\frac{1}{3}$ and **add** to **row-3**, yields

0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	**Row-4 multiplied by** $\frac{1}{3}$
0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	**Row-3**
0	0	1	0	1	**Row-3**

The **augmented matrix** becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

This **matrix** is in **Gauss-Jordan form**.
The **system of linear equation corresponding** to **this one** is:

$$x_1 + 0 + 0 + 0 = 1$$

$$0 + x_2 + 0 + 0 = 1$$

$$0 + 0 + x_3 + 0 = 1$$

$$0 + 0 + 0 + x_4 = 1$$

So, the **solution** to the **system** is:

$$x_1 = 1\,, \quad x_2 = 1\,, \quad x_3 = 1\,, \quad x_4 = 1$$

Example 2: Solve the following **system** of **linear equations** using **Gaussian elimination** or **Gauss-Jordan elimination**:

$$2x_1 - 2x_2 + 3x_3 = 24 \qquad \text{Eq.}(1)$$

$$2x_2 - x_3 = 14 \qquad \text{Eq.}(2)$$

$$7x_1 - 5x_2 = 6 \qquad \text{Eq.}(3)$$

<u>Solution</u>:

The **augmented matrix** is:

$$\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{3} & \mathbf{24} \\ \mathbf{0} & \mathbf{2} & \mathbf{-1} & \mathbf{14} \\ \mathbf{7} & \mathbf{-5} & \mathbf{0} & \mathbf{6} \end{pmatrix}$$

<u>Divide</u> row-1 by **2** and **row-2** by **2**, the **augmented matrix** becomes:

$$\begin{array}{cl} \begin{pmatrix} \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & \frac{\mathbf{3}}{\mathbf{2}} & \mathbf{12} \\ \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & \mathbf{7} \\ \mathbf{7} & \mathbf{-5} & \mathbf{0} & \mathbf{6} \end{pmatrix} & \begin{array}{l} \textbf{Row-1 divided by 2} \\ \textbf{Row-2 divided by 2} \\ \\ \end{array} \end{array}$$

<u>Multiply</u> row-1 by **7** and **<u>subtract</u>** from **row 3**, yields

$$\begin{array}{ccccl} \mathbf{7} & \mathbf{-5} & \mathbf{0} & \mathbf{6} & \textbf{Row-3} \\ \mathbf{7} & -\frac{\mathbf{7}}{\mathbf{2}} & \frac{\mathbf{21}}{\mathbf{2}} & \mathbf{84} & \textbf{Row-1 multiplied by 7} \\ \hline \mathbf{0} & -\frac{\mathbf{3}}{\mathbf{2}} & -\frac{\mathbf{21}}{\mathbf{2}} & \mathbf{-78} & \textbf{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & \frac{\mathbf{3}}{\mathbf{2}} & \mathbf{12} \\ \mathbf{0} & \mathbf{1} & -\frac{\mathbf{1}}{\mathbf{2}} & \mathbf{7} \\ \mathbf{0} & -\frac{\mathbf{3}}{\mathbf{2}} & -\frac{\mathbf{21}}{\mathbf{2}} & \mathbf{-78} \end{pmatrix}$$

<u>Multiply</u> row-2 by $\frac{\mathbf{3}}{\mathbf{2}}$ and **<u>add</u>** to **row-3**:

$$\begin{array}{ccccl} \mathbf{0} & \frac{\mathbf{3}}{\mathbf{2}} & -\frac{\mathbf{3}}{\mathbf{4}} & \frac{\mathbf{21}}{\mathbf{2}} & \textbf{Row-3} \\ \mathbf{0} & -\frac{\mathbf{3}}{\mathbf{2}} & -\frac{\mathbf{21}}{\mathbf{2}} & \mathbf{-78} & \textbf{Row-2 multiplied by } \frac{\mathbf{3}}{\mathbf{2}} \\ \hline \mathbf{0} & \mathbf{0} & -\frac{\mathbf{45}}{\mathbf{4}} & -\frac{\mathbf{135}}{\mathbf{2}} & \textbf{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & -\frac{45}{4} & -\frac{135}{2} \end{pmatrix}$$

Multiply row-3 by $-\frac{4}{45}$, the **augmented matrix** becomes:

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

Row-3 multiplied by $-\frac{4}{45}$

The **above matrix** is in **row-echelon form**. So, using **back substitution** and **solve** the **system**. The **system corresponding to** this **matrix** is:

$$x_1 - \frac{1}{2}x_2 + \frac{3}{2}x_3 = 12$$

$$x_2 - \frac{1}{2}x_3 = 7$$

$$x_3 = 6$$

By **back-substitution**, we have

$$x_3 = 6 \ , x_2 = 7 + \frac{1}{2}\,6 = 10\,, \ \text{ and } \ x_1 = 12 - \frac{3}{2}\,6 + \frac{1}{2}\,10 = 8$$

Alternately, the **row-echelon matrix**

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

could be **reduced** to a **Gauss-Jordan form**. To do this, **multiply row-2** by $\frac{1}{2}$ and **add** to **row-1**, yields

$$\begin{array}{ccccl} 1 & -\frac{1}{2} & \frac{3}{2} & 12 & \text{Row-1} \\ 0 & \frac{1}{2} & -\frac{1}{4} & \frac{7}{2} & \text{Row-2 multiplied by } \frac{1}{2} \\ \hline 1 & 0 & \frac{5}{4} & \frac{31}{2} & \text{Row-1} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & \frac{5}{4} & \frac{31}{2} \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

<u>Multiply</u> row-3 by $\frac{5}{4}$ and **<u>subtract</u>** from **row-1**, yields

$$\begin{array}{ccccl} 0 & 1 & \frac{5}{4} & \frac{31}{2} & \text{Row-1} \\ 0 & 0 & \frac{5}{4} & \frac{15}{2} & \text{Row-3 multiplied by } \frac{5}{4} \\ \hline 0 & 1 & 0 & 8 & \text{Row-1} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

<u>Multiply</u> row-3 by $\frac{1}{2}$ and **<u>add</u>** from **row-2**, yields

$$\begin{array}{ccccl} 0 & 1 & -\frac{1}{2} & 7 & \text{Row-2} \\ 0 & 0 & \frac{1}{2} & 3 & \text{Row-3 multiplied by } \frac{1}{2} \\ \hline 0 & 1 & 0 & 10 & \text{Row-2} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$

This **matrix** is in **<u>Gauss-Jordan form</u>**. The **system** of **linear equations corresponding to** this **matrix** is:

$$\boldsymbol{x_1 + 0 + 0 = 8}$$

$$\boldsymbol{0 + x_2 + 0 = 10}$$

$$\boldsymbol{x_3 = 6}$$

This **gives** the **solution** of this **system:**

$$\boldsymbol{x_1 = 8} \quad , \quad \boldsymbol{x_2 = 10} \quad , \quad \text{and} \quad \boldsymbol{x_3 = 6}$$

<u>Example</u> 3: Solve the following **system** of **linear equations** using **Gaussian elimination** or **Gauss-Jordan elimination**:

$$\boldsymbol{2x_1 + 3x_3 = 3} \qquad \boldsymbol{Eq.(1)}$$

$$\boldsymbol{4x_1 - 3x_2 + 7x_3 = 5} \qquad \boldsymbol{Eq.(2)}$$

$$\boldsymbol{8x_1 - 9x_2 + 15x_3 = 10} \qquad \boldsymbol{Eq.(3)}$$

<u>Solution</u>:

The **augmented matrix** is:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{pmatrix}$$

First, **reduce** this **matrix** to **row-echelon form**.

<u>Multiply</u> row-1 by **2** and **<u>subtract</u>** from **row-2,** yields

$$\begin{array}{rrrrl} 4 & -3 & 7 & 5 & \textbf{Row-2} \\ 4 & 0 & 6 & 6 & \textbf{Row-1 multiplied by 2} \\ \hline 0 & -3 & 1 & -1 & \textbf{Row-2} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 8 & -9 & 15 & 10 \end{pmatrix}$$

<u>Multiply</u> row-1 by **4** and **<u>subtract</u>** from **row-3,** yields

$$\begin{array}{rrrrl} 8 & -9 & 15 & 10 & \textbf{Row-3} \\ 8 & 0 & 12 & 12 & \textbf{Row-1 multiplied by 2} \\ \hline 0 & -9 & -9 & -14 & \textbf{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{pmatrix}$$

<u>Multiply</u> row-2 by 3 and **<u>subtract</u>** from **row-3**, yields

$$\begin{array}{rrrrl} 0 & -9 & 3 & -2 & \textbf{Row-3} \\ 0 & -9 & 3 & -3 & \textbf{Row-1 multiplied by 2} \\ \hline 0 & 0 & 0 & 1 & \textbf{Row-3} \end{array}$$

The **augmented matrix** becomes:

$$\begin{pmatrix} 2 & 0 & 3 & 3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Divide</u> row-1 by **2** and **row-2** by **– 3**, the **augmented matrix** becomes:

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The **matrix** is in **<u>row-echelon form</u>**. The **system** of **these equations corresponding to** this **matrix** is:

$$x_1 + \frac{3}{2}x_3 = \frac{3}{2}$$
$$x_2 - \frac{1}{3}x_3 = \frac{1}{3}$$
$$0 = 1$$

The **last equation** is **<u>absurd</u>**. So, the **system** is **<u>inconsistent</u>**.

<u>Example</u> 4: Solve the **homogeneous linear system corresponding to** the **coefficient matrix**:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

<u>Solution</u>:

Since it is a **homogeneous system**, i.e. it has **all** the **constant terms equal zero** and they **system** is:

$$\begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

The **system** is **<u>already</u>** in **<u>row-echelon form</u>**. So, by **<u>back substitution</u>** yields

$$x_1 = 0 \text{ and } x_2 = -x_3$$

With $x_3 = t$ a parametric solution is:

$$x_1 = 0 \quad , \quad x_2 = -t \quad , \quad \text{and} \quad x_3 = t$$

<u>Example</u> 5: Consider the **system** of **linear equations**

$$\begin{aligned} x + y &= 0 \\ y + z &= 0 \\ x + z &= 0 \\ ax + by + cz &= 0 \end{aligned}$$

Find the **values** of ***a***, ***b***, and ***c*** such that **the system has**:

(a) a unique solution,

(b) no solution,

(**c) an infinite number of solution**.

<u>Solution</u>:

The **augmented matrix** of the **given system** is:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ a & b & c & 0 \end{pmatrix}$$

<u>Subtract</u> row-1 from **row-3**, and **<u>multiply</u> row-1** by ***a*** and **<u>subtract</u>** from **row-4**. the **augmented matrix will** become:

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b-a} & \boldsymbol{c} & \mathbf{0} \end{pmatrix}$$

<u>Add</u> row-2 to **row-3**, the **augmented matrix** becomes:

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b-a} & \boldsymbol{c} & \mathbf{0} \end{pmatrix}$$

<u>Divide</u> row-3 by **2**, the **augmented matrix** becomes:

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b-a} & \boldsymbol{c} & \mathbf{0} \end{pmatrix}$$

<u>Multiply</u> row-2 by (***a*** – ***b***) and **<u>add</u>** to **row-4**, the **augmented matrix will** become:

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{c+a-b} & \mathbf{0} \end{pmatrix}$$

<u>Multiply</u> row-3 by (***c*** **+** ***a*** – ***b***) and **<u>subtract</u> from row-4**, the **augmented matrix will** become:

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

The **matrix** is in **<u>row-echelon form</u>**. The **corresponding linear system** is:

$$\begin{aligned} \boldsymbol{x+y} &= \mathbf{0} \\ \boldsymbol{y+z} &= \mathbf{0} \\ \boldsymbol{z} &= \mathbf{0} \\ \mathbf{0} &= \mathbf{0} \end{aligned}$$

The **system** is **<u>consistent</u>** for **all values** of ***a, b, c***, and by **<u>back substitution</u>** the **system** has **unique solution**

$$\boldsymbol{x=y=z=0}$$

Iterative Methods for Solving System of Linear Equations

An "**iterative method**" is **one** which is **used repeatedly until** the **results obtained acquire** a **pre-assigned degree** of **accuracy**. For example, if **results** are **required** to **five places** of **decimals**, the **number** of "**iterations**" is **continued until two consecutive iterations** give the **same result** when **rounded off** to that **number** of **decimal places**. The **two methods** that are **used** for **solving system** of **linear equations are:**

❖ The Jacobi`s or Gauss- Jacobi Method

The **first iterative technique** is **called** the **Gauss-Jacobi** or **Jacobi method**, after **Carl Gustav Jacob Jacobi (1804–1851)**.

Consider that the **system given** by

$$a_1x + b_1y + c_1z = k_1$$

$$a_2x + b_2y + c_2z = k_2$$

$$a_3x + b_3y + c_3z = k_3$$

has a **unique solution**. To **begin** the **Jacobi method**, **solve** the **first equation** for x, the **second equation** for y, and the **third equation** for **z**, as follows:

$$x_1 = \frac{k_1 - b_1y_0 - c_1z_0}{a_1}$$

$$y_1 = \frac{k_2 - a_2x_0 - c_2z_0}{b_2}$$

$$z_1 = \frac{k_3 - a_3x_0 - b_3y_0}{c_3}$$

Then **using** an **initial approximation** $x_0 = 0$, $y_0 = 0$, and $z_0 = 0$ and **substitute** into the **right-hand side** of the **rewritten equations** to **obtain** the **first approximation**. **After this procedure** has been **completed**, **one iteration** has been **performed**. In the **same way**, the **second approximation** is **formed** by **substituting** the **first approximation** of ***x*, y, z values** into the **right-hand side** of the **rewritten equations**.

By repeating iterations, a **sequence** of **approximations** will be **formed** that often **converges** to the **actual solution**. This **procedure** is **illustrated** in the **following example**.

<u>Example</u> **1: Use** the **Gauss-Jacobi method** to **approximate** the **solution** of the **following system** of **linear equations**

$$\begin{aligned} 5x - 2y + 3z &= -1 \\ -3x + 9y + z &= 2 \\ 2x - y - 7z &= 3 \end{aligned}$$

Continue the **iterations until two successive approximations** are **identical** when **rounded** to **three significant digits**.

<u>**Solution**</u>**:**

To begin, writing the **system** in the **form**

$$x = \frac{-1 + 2y - 3z}{5}$$

$$y = \frac{2 + 3x - z}{9}$$

$$z = \frac{-3 + 2x - y}{7}$$

Using $x_0 = 0, y_0 = 0,$ and $z_0 = 0$, as a **convenient initial approximation** gives the **first approximation** i:

$$x_1 = \frac{-1 + 2y_0 - 3z_0}{5} = -\frac{1}{5} = -0.2$$

$$y_1 = \frac{2 + 3x_0 - z_0}{9} = \frac{2}{9} = 0.222$$

$$z_1 = \frac{-3 + 2x_0 - y_0}{7} = -\frac{3}{7} = -0.429$$

$$x_2 = \frac{-1 + 2y_1 - 3z_1}{5} = \frac{-1 + 2(0.222) - 3(-0.429)}{5}$$

$$\Rightarrow \quad x_2 = \frac{0.731}{5} = 0.146$$

$$y_2 = \frac{2 + 3x_1 - z_1}{9} = \frac{2 + 3(-0.2) - (-0.429)}{9}$$

$$\Rightarrow \quad y_2 = \frac{1.829}{9} = 0.203$$

$$z_2 = \frac{-3 + 2x_1 - y_1}{7} = \frac{-3 + 2(-0.2) - 0.222}{7}$$

$$\Rightarrow \quad z_2 = -\frac{3.622}{7} = -0.517$$

Continuing this **procedure**, the **sequence** of **approximations** can be **obtained** as **shown** in **following table**:

n	1	2	3	4	5	6	7
x	-0.2	0.146	0.192	0.181	0.185	0.186	0.186
y	0.222	0.203	0.328	0.332	0.329	0.331	0.331
z	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

Because the **last two results** are **identical**, it can be **concluded** that to **three significant digits** the **solution** is:

$$x = 0.186 \;, \quad y = 0.331 \;, \quad z = -0.423$$

❖ The Gauss-Seidel Method

This **method differs** from the **Gauss-Jacobi method** in that **successive approximations** are **used within each step** as soon as they **become available**.

The **rate of convergence** of this method is usually **faster than** that of the **Jacobi method**. The **scheme** of the **calculations** is **according to** the **following pattern**:

$$x_1 = \frac{k_1 - b_1 y_0 - c_1 z_0}{a_1}$$

$$y_1 = \frac{k_2 - a_2 x_1 - c_2 z_0}{b_2}$$

$$z_1 = \frac{k_3 - a_3 x_1 - b_3 y_1}{c_3}$$

<u>Example</u> **2: Use** the **Gauss-Seidel method** to **approximate** the **solution** of the **following system** of **linear equations**

$$5x - 2y + 3z = -1$$

$$-3x + 9y + z = 2$$

$$2x - y - 7z = 3$$

Continue the **iterations until two successive approximations** are **identical** when **rounded** to **three significant digits**.

<u>**Solution**</u>**:**

To begin, writing the **system** in the **form**

$$x = \frac{-1 + 2y - 3z}{5}$$

$$y = \frac{2 + 3x - z}{9}$$

$$z = \frac{-3 + 2x - y}{7}$$

Using $x_0 = 0, y_0 = 0,$ and $z_0 = 0$, as a **convenient initial approximation**, yields the **first approximation** is:

$$x_1 = \frac{-1 + 2y_0 - 3z_0}{5} = -\frac{1}{5} = -0.2$$

$$y_1 = \frac{2 + 3x_1 - z_0}{9} = \frac{2 + 3(-0.2) - 0}{9} = \frac{1.4}{9} = 0.156$$

$$z_1 = \frac{-3 + 2x_1 - y_1}{7} = \frac{-3 + 2(-0.2) - 0.156}{7}$$

$$\Rightarrow \quad z_1 = \frac{-3.556}{7} = -0.508$$

$$x_2 = \frac{-1 + 2y_1 - 3z_1}{5} = \frac{-1 + 2(0.156) - 3(-0.508)}{5}$$

$$\Rightarrow \quad x_2 = \frac{0.838}{5} = 0.167$$

$$y_2 = \frac{2 + 3x_2 - z_1}{9} = \frac{2 + 3(0.167) - (-0.508)}{9}$$

$$\Rightarrow \quad y_2 = \frac{3.009}{9} = 0.334$$

$$z_2 = \frac{-3 + 2x_2 - y_2}{7} = \frac{-3 + 2(0.167) - 0.334}{7}$$

$$\Rightarrow \quad z_2 = -\frac{3}{7} = -0.429$$

Continued iterations procedure, the **sequence** of **approximations** can be **obtained** as **shown** in **following table**:

n	1	2	3	4	5
x	-0.2	0.167	0.191	0.186	0.186
y	0.156	0.334	0.333	0.331	0.331
z	-0.508	-0.429	-0.422	-0.423	-0.423

Note that after only five iterations of the **Gauss-Seidel method**, **same accuracy achieved** the as was **obtained** with **seven iterations** of the **Gauss-Jacobi method** in **Example 1**.

<u>**Example**</u> **3: Use** the **Gauss-Jacobi method** to solve the **simultaneous linear equations**

$$5x + y - z = 4$$

$$x + 4y + 2z = 15$$

$$x - 2y + 5z = 12$$

Obtaining x, y and z correct to the nearest integer.

<u>**Solution**</u>**:**

Write the **system** in the **form**

$$x_1 = \frac{4 - y + z}{5}$$

$$y_1 = \frac{15 - x - 2z}{4}$$

$$z_1 = \frac{12 - x + 2y}{5}$$

Using an **initial approximation** $\boldsymbol{x_0 = 0}$, $\boldsymbol{y_0 = 0}$, and $\boldsymbol{z_0 = 0}$, yields

$$\boldsymbol{x_1 = \frac{4 - y_0 + z_0}{5} = \frac{4 - 0 + 0}{5} = \frac{4}{5} = 0.8}$$
$$\boldsymbol{y_1 = \frac{15 - x_0 - 2z_0}{4} = \frac{15 - 0 - 2(0)}{4} = \frac{15}{4} = 3.75}$$
$$\boldsymbol{z_1 = \frac{12 - x_0 + 2y_0}{5} = \frac{12 - 0 + 2(0)}{5} = \frac{12}{5} = 2.4}$$

$$\boldsymbol{x_2 = \frac{4 - y_1 + z_1}{5} = \frac{4 - 3.75 + 2.4}{5} = 0.53}$$
$$\boldsymbol{y_2 = \frac{15 - x_1 - 2z_1}{4} = \frac{15 - 0.8 + 2(2.4)}{4} = 2.35}$$
$$\boldsymbol{z_2 = \frac{12 - x_1 + 2y_1}{5} = \frac{12 - 0.8 + 2(3.75)}{5} = 3.74}$$

$$\boldsymbol{x_3 = \frac{4 - y_2 + z_2}{5} = \frac{4 - 2.35 + 3.74}{5} = 1.078}$$
$$\boldsymbol{y_3 = \frac{15 - x_2 - 2z_2}{4} = \frac{15 - 0.53 - 2(3.74)}{4} = 1.748}$$
$$\boldsymbol{z_3 = \frac{12 - x_2 + 2y_2}{5} = \frac{12 - 0.53 + 2(2.35)}{5} = 3.234}$$

$$\boldsymbol{x_4 = \frac{4 - y_3 + z_3}{5} = \frac{4 - 1.748 + 3.234}{5} = 1.097}$$
$$\boldsymbol{y_4 = \frac{15 - x_3 - 2z_3}{4} = \frac{15 - 1.078 - 2(3.234)}{4} = 1.864}$$
$$\boldsymbol{z_4 = \frac{12 - x_3 + 2y_3}{5} = \frac{12 - 1.078 + 2(1.748)}{5} = 2.884}$$

$$\boldsymbol{x_5 = \frac{4 - y_4 + z_4}{5} = \frac{4 - 1.864 + 2.884}{5} = 1.004}$$
$$\boldsymbol{y_5 = \frac{15 - x_4 - 2z_4}{4} = \frac{15 - 1.097 - 2(2.884)}{4} = 2.034}$$
$$\boldsymbol{z_5 = \frac{12 - x_4 + 2y_4}{5} = \frac{12 - 1.097 + 2(1.864)}{5} = 2.926}$$

$$x_6 = \frac{4 - y_5 + z_5}{5} = \frac{4 - 2.034 + 2.926}{5} = 0.978$$
$$y_6 = \frac{15 - x_5 - 2z_5}{4} = \frac{15 - 1.004 - 2(2.926)}{4} = 2.036$$
$$z_6 = \frac{12 - x_5 + 2y_5}{5} = \frac{12 - 1.004 + 2(2.034)}{5} = 3.013$$

$$x_7 = \frac{4 - y_6 + z_6}{5} = \frac{4 - 2.036 + 3.013}{5} = 0.995$$
$$y_7 = \frac{15 - x_6 - 2z_6}{4} = \frac{15 - 0.978 - 2(3.013)}{4} = 1.999$$
$$z_7 = \frac{12 - x_6 + 2y_6}{5} = \frac{12 - 0.978 + 2(2.036)}{5} = 3.019$$

$$x_8 = \frac{4 - y_7 + z_7}{5} = \frac{4 - 1.999 + 3.019}{5} = 1.004$$
$$y_8 = \frac{15 - x_7 - 2z_7}{4} = \frac{15 - 0.995 - 2(3.019)}{4} = 1.992$$
$$z_8 = \frac{12 - x_7 + 2y_7}{5} = \frac{12 - 0.995 + 2(1.999)}{5} = 3.001$$

n	0	1	2	3	4	5	6	7
x	0.8	0.53	1.078	1.097	1.004	0.978	0.995	1.004
y	3.75	2.35	1.748	1.864	2.034	2.036	1.999	1.992
z	2.4	3.74	3.234	2.884	2.926	3.013	3.019	3.001

From the table it is **evident** that the **results** of the **last two iterations** when **rounded** to the **nearest integer,** both **give**

$$x = 1 \quad , \quad y = 2 \quad , \quad \text{and} \quad z = 3$$

<u>**Example**</u>**4**: Apply the **Jacobi method** to the **system**

$$\boldsymbol{x_1 - 5x_2 = -4}$$
$$\boldsymbol{7x_1 - x_2 = \quad 6}$$

using the **initial approximati**on$(x_1, x_2) = (\boldsymbol{0}, \boldsymbol{0})$ and **show** that the **method diverges**.

<u>**Solution**</u>**:**

Rewriting the **given system** in the **form**

$$\boldsymbol{x_1 = 5x_2 - 4}$$
$$\boldsymbol{x_2 = 7x_1 - 6}$$

Using the **initial approximation** $\boldsymbol{x_1 = 0}$ and $\boldsymbol{x_2 = 0}$**, yields**

$$\boldsymbol{x_1 = 5(0) - 4 = -4}$$
$$\boldsymbol{x_2 = 7(0) - 6 = -6}$$

As the **first approximation**.

Repeating iterations produce the **sequence** of **approximations shown** in the **following table**:

n	0	1	2	3	4	5	6	7
x_1	0	-4	-34	-174	-1244	-6124	-42.874	-214.374
x_2	0	-6	-34	-244	-1244	-8574	-42.874	-300.124

It **can be seen** from the **table** that the **approximations given** by the <u>**Jacobi method**</u> **become progressively worse instead** of **better**, and can **be concluded** that the **method diverges**.

The **problem** of **divergence** is **not resolved** by **using** the <u>**Gauss-Seidel method**</u> **rather than** the <u>**Jacobi method**</u>. In fact, for this **particular system** the **Gauss-Seidel method** <u>**diverges**</u> <u>**more rapidly**</u>, as **shown** in the **following table**.

n	0	1	2	3	4	5
x_1	0	-4	-174	-6124	$-214,374$	$-7,503,124$
x_2	0	-34	-1244	$-42,874$	$-1,500,624$	$-52,521,874$

Using a **special type** of **<u>coefficient matrix</u>**, **called** a **<u>strictly diagonally dominant matrix</u>**, can **guarantee** that **both methods** will **converge**.

<u>Strictly Diagonally Dominant Matrix</u>

An $n \times n$ **matrix** A is **strictly diagonally dominant** if the **absolute** value of **each entry** on the main **diagonal** is **greater than** the **sum** of the **absolute values** of the **other entries** in the **same row**. That is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix}$$

$$\begin{array}{l} |a_{11}| > |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\ |a_{22}| > |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad \cdots \qquad \vdots \\ |a_{nn}| > |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}| \end{array}$$

<u>Example</u> 5: Which of the **following systems** of **linear equations** has a **strictly diagonally dominant coefficient matrix?**

a) $\begin{aligned} 3x_1 - x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{aligned}$ **b)** $\begin{aligned} 4x_1 + 2x_2 - x_3 &= -1 \\ x_1 + 2x_3 &= 2 \\ 3x_1 - 5x_2 + x_3 &= 3 \end{aligned}$

<u>Solution</u>:

a) The **coefficient matrix** is:

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$$

is **strictly diagonally dominant** because $|3| > |-1|$ and $|5| > |2|$

b) The **coefficient matrix** is

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 2 \\ 3 & -5 & 1 \end{bmatrix}$$

where,

$$|3| > |2| + |1| = |3|,$$

$$|0| < |1| + |2| = |3|, \text{and}$$

$$|1| < |3| + |5| = |8|$$

Thus, the **coefficient matrix** is **not strictly diagonally dominant** because the **entries** in the **second** and **third rows do not conform** to the **definition**. For instance, in the **second row** $a_{21} = 1$, $a_{22} = 0$ and $a_{23} = 2$, and it is **not true** that

$$|a_{22}| > |a_{21}| + |a_{23}| \;\; i.e. \;\; |0| < |1| + |2| = |3|$$

Interchanging the **second** and **third rows** in the **original system** of **linear equations leads to** the **coefficient matrix** A to be

$$A = \begin{bmatrix} 4 & 2 & -1 \\ 3 & -5 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

where,

$$|4| > |2| + |-1| = |3| \,,$$

$$|-5| > |3| + |1| = |4| \,, \text{and}$$

$$|2| > |1| + |0| = |1|$$

Which is **strictly diagonally dominant**, for which **when apply** the **Gauss-Seidel method** the **solution** can be **approximated**.

<u>Example</u> 6: **Check** if **can apply** the **Gauss-Jacobi method** to the **system**

$$\boldsymbol{x_1 - 5x_2 = -4}$$
$$\boldsymbol{7x_1 - x_2 = \ \ 6}$$

using the **initial approximation** $(\boldsymbol{x_1}, \boldsymbol{x_2}) = (\boldsymbol{0}, \boldsymbol{0})$.

<u>Solution</u>:

The **coefficient matrix** of the **system** is:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{-5} \\ \boldsymbol{7} & \boldsymbol{-1} \end{bmatrix}$$

Where,

$$|\boldsymbol{1}| < |\boldsymbol{-5}| \quad \text{and} \quad |\boldsymbol{-1}| < |\boldsymbol{7}|$$

Thus, the **<u>coefficient matrix</u>** is **<u>not strictly diagonally dominant matrix</u>** because **<u>does not conform</u>** to the **definition**

$$|\boldsymbol{a_{11}}| > |\boldsymbol{a_{12}}| \qquad \text{and} \qquad |\boldsymbol{a_{22}}| > |\boldsymbol{a_{21}}|$$

<u>Interchanging</u> the **second** and **first rows** in the **original system** of **linear equations**, yields

$$\boldsymbol{7x_1 - x_2 = \ \ 6}$$
$$\boldsymbol{x_1 - 5x_2 = -4}$$

Leads the **coefficient matrix *A*** to be**:**

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{7} & \boldsymbol{-1} \\ \boldsymbol{1} & \boldsymbol{-5} \end{bmatrix}$$

where,

$$|\boldsymbol{7}| > |\boldsymbol{-1}| \qquad \text{and} \qquad |\boldsymbol{-5}| > |\boldsymbol{1}|$$

Which is **<u>strictly diagonally dominant matrix</u>**, and **can apply** the **<u>Gauss-Seidel method</u>** to **approximate** the **solution**.

Rewriting the **given system** in the **form**

$$\boldsymbol{x_1} = \frac{\boldsymbol{x_2 + 6}}{\boldsymbol{7}}$$
$$\boldsymbol{x_2} = \frac{\boldsymbol{x_1 + 4}}{\boldsymbol{5}}$$

Using the **initial approximation** $\boldsymbol{x_1 = 0}$ and $\boldsymbol{x_2 = 0}$**, yields**

$$x_1^1 = \frac{x_2^0 + 6}{7} = \frac{0 + 6}{7} = \frac{6}{7} = 0.8571$$

$$x_2^1 = \frac{x_1^1 + 4}{5} = \frac{0.8571 + 4}{5} = \frac{4.8571}{5} = 0.9714$$

$$x_1^2 = \frac{x_2^1 + 6}{5} = \frac{0.9714 + 6}{7} = \frac{6}{7} = 0.9959$$

$$x_2^2 = \frac{x_1^2 + 4}{5} = \frac{0.9959 + 4}{5} = \frac{4.8571}{5} = 0.9992$$

Continuing iterations, a **sequence** of **approximations can** be **obtain** as **shown** in the **following table**:

n	0	1	2	3	4	5
x_1	0	0.8571	0.9959	0.9999	1.0000	1.0000
x_2	0	0.9714	0.9992	1.0000	1.0000	1.0000

So, the **solution** is:

$\boldsymbol{x_1 = 1}$ and $\boldsymbol{x_2 = 1}$.

Application of Linear systems

There is a **wide range** of **applications** of **linear systems**, a **few** of **them** are:

1. **Fitting polynomials**,
2. **Network analysis**,
3. **Kirchhoff 's Laws for electrical networks**

1. <u>Polynomial curve fitting</u>

Recall the facts:

i. There is **exactly** **<u>one line</u>** $\boldsymbol{y = mx + c}$ that **passes through** **<u>two</u>** **given points**;

ii. There is **exactly** **<u>one parabola</u>** $\boldsymbol{y = ax^2 + bx + c}$ that **passes through** **<u>three</u>** **given points** (**barring some exceptions**).

iii. More generally, **given** $\boldsymbol{n}$ **number** of **points** in the **plane** and there is **exactly one polynomials** $\boldsymbol{p(x)}$ of **degree** $\boldsymbol{n-1}$, so that the **graph** of $\boldsymbol{y = p(x)}$ will **pass through** these $\boldsymbol{n}$ **points** **describing** it **as follows**:

Suppose a **collection** of **data** is **represented** by $\boldsymbol{n}$ **points**:

$$(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$$

If the $\boldsymbol{x}$ – **coordinates** $\boldsymbol{x_1, x_2, , \cdots, x_n}$ are **distinct**, then there is a **UNIQUE polynomial**

$$p(x) = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1}$$

of degree $\boldsymbol{n-1}$ (or **less**) so that the **graph** of $\boldsymbol{y = p(x)}$ **passes through these points**. Given $\boldsymbol{n}$ **such points**, to **determine** $\boldsymbol{p(x)}$ it needs to **find** the **coefficients** $\boldsymbol{a_0, a_1, a_2, \cdots, a_{n-1}}$. Since the **points** $\boldsymbol{(x_i, y_i)}$ **pass through** the **graph** of $\boldsymbol{y = p(x)}$, we have $\boldsymbol{y_i = p(x_i)}$. **More explicitly**,

$$\begin{array}{ccccccccc}
a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + \cdots + & a_{n-1} x_1^{n-1} & = & y_1 \\
a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + \cdots + & a_{n-1} x_2^{n-1} & = & y_2 \\
a_0 & + & a_1 x_3 & + & a_2 x_3^2 & + \cdots + & a_{n-1} x_3^{n-1} & = & y_3 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_0 & + & a_1 x_n & + & a_2 x_n^2 & + \cdots + & a_{n-1} x_n^{n-1} & = & y_n
\end{array}$$

This is a **linear system** of $\boldsymbol{n}$ **equations**, with $\boldsymbol{n}$ **unknowns** (**variables**) $\boldsymbol{a_0, a_1, a_2, \cdots, a_{n-1}}$.. It is known that, **under our condition** that $\boldsymbol{x_1, x_2, , \cdots, x_n}$ are **distinct**, the **system has** a **unique solution**.

The **augmented matrix** of this **linear system** is:

$$\begin{pmatrix} \mathbf{1} & \boldsymbol{x_1} & \boldsymbol{x_1^2} \cdots \boldsymbol{x_1^{n-1}} & \boldsymbol{y_1} \\ \mathbf{1} & \boldsymbol{x_2} & \boldsymbol{x_2^2} \cdots \boldsymbol{x_2^{n-1}} & \boldsymbol{y_2} \\ \mathbf{1} & \boldsymbol{x_3} & \boldsymbol{x_3^2} \cdots \boldsymbol{x_3^{n-1}} & \boldsymbol{y_3} \\ \vdots & \vdots & \vdots \quad\quad \vdots & \vdots \\ \mathbf{1} & \boldsymbol{x_n} & \boldsymbol{x_n^2} \cdots \boldsymbol{x_n^{n-1}} & \boldsymbol{y_n} \end{pmatrix}$$

and the **coefficients matrix** is

$$\begin{pmatrix} \mathbf{1} & \boldsymbol{x_1} & \boldsymbol{x_1^2} & \cdots & \boldsymbol{x_1^{n-1}} \\ \mathbf{1} & \boldsymbol{x_2} & \boldsymbol{x_2^2} & \cdots & \boldsymbol{x_2^{n-1}} \\ \mathbf{1} & \boldsymbol{x_3} & \boldsymbol{x_3^2} & \cdots & \boldsymbol{x_3^{n-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{1} & \boldsymbol{x_n} & \boldsymbol{x_n^2} & \cdots & \boldsymbol{x_n^{n-1}} \end{pmatrix}$$

The **coefficients matrix** is called **Vandermonde-matrix** in$\boldsymbol{x_1, x_2, , \cdots, x_n}$.

<u>**Example**</u> **1: Determine** the **polynomial function** of **degree 2** that **passes through** the **points** $(\mathbf{2}, \mathbf{4}), (\mathbf{3}, \mathbf{6}), (\mathbf{4}, \mathbf{10})$.

<u>**Solution**</u>**:**

Let the **polynomial function** of **degree 2** be:

$$\boldsymbol{p(x) = a + bx + cx^2}$$

Since **these points pass through** the **graph** of

$$\boldsymbol{y = p(x) = a + bx + cx^2}$$

we have

$$\begin{aligned} \boldsymbol{a + b2 + c2^2} &= \mathbf{4} & & & \boldsymbol{a + 2b + 4c} &= \mathbf{4} \\ \boldsymbol{a + b3 + c3^2} &= \mathbf{6} & &\text{or} & \boldsymbol{a + 3b + 9c} &= \mathbf{6} \\ \boldsymbol{a + b4 + c4^2} &= \mathbf{10} & & & \boldsymbol{a + 4b + 16c} &= \mathbf{10} \end{aligned}$$

The **augmented matrix** of this system is:

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{4} \\ \mathbf{1} & \mathbf{3} & \mathbf{9} & \mathbf{6} \\ \mathbf{1} & \mathbf{4} & \mathbf{16} & \mathbf{10} \end{pmatrix}$$

Now reducing the **matrix** to the **row-echelon form**. To do this **subtract** **row-1** from **row-2** and **row-3**:

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{5} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{12} & \mathbf{6} \end{pmatrix}$$

Multiply **row-2** by **2** and **Subtract** from **row-3**, yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{5} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{2} \end{pmatrix}$$

Divide **row-3** by **2,** yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{5} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

The **matrix** is in **row-echelon form**. They **linear system corresponding to** this **matrix** is:

$$\boldsymbol{a + 2b + 4c = 4}$$

$$\boldsymbol{b + 5c = 2}$$

$$\boldsymbol{c = 1}$$

By **back substitution**, yields

$$\boldsymbol{c = 1 \quad \Rightarrow \quad b = 2 - 5 = -3 \quad \Rightarrow \quad a = 4 - 4 + 6 = 6}$$

So,

$$\boldsymbol{p(x) = a + bx + cx^2 = 6 - 3x + x^2}$$

<u>Example</u> 2: Some **US census population data** is given in the following table

Year t	**1980**	**1990**	**2000**
Population y	**227**	**249**	**281**

Here **population** is given in **millions**.

1. **Fit** a **second degree polynomial passing through these points**.
2. **Use it** to **predict population** in **year 2010** and **2020**.

<u>Solution</u>:

Let $\boldsymbol{t}$ be the **variable time** and **set** $\boldsymbol{t = 0}$ for the **year 1980**. The **table reduces to**

$\boldsymbol{t}$	**0**	**10**	**20**
$\boldsymbol{y}$	**227**	**249**	**281**

Let the **polynomial function** of **degree 2 that fits this data** be:

$$\boldsymbol{p(x) = a + bt + ct^2}$$

Since the **data points pass through** the **graph** of

$$\boldsymbol{p(x) = a + bt + ct^2}$$

we have

$$\begin{aligned} \boldsymbol{a + b0 + c0^2} &= \boldsymbol{227} & & & \boldsymbol{a} &= \boldsymbol{227} \\ \boldsymbol{a + b10 + c(10)^2} &= \boldsymbol{249} & \quad \text{or} \quad & & \boldsymbol{a + 10b + 100c} &= \boldsymbol{249} \\ \boldsymbol{a + b20 + c(20)^2} &= \boldsymbol{281} & & & \boldsymbol{a + 20b + 400c} &= \boldsymbol{281} \end{aligned}$$

The **augmented matrix** is

$$\begin{pmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{227} \\ \boldsymbol{1} & \boldsymbol{10} & \boldsymbol{100} & \boldsymbol{249} \\ \boldsymbol{1} & \boldsymbol{20} & \boldsymbol{400} & \boldsymbol{281} \end{pmatrix}$$

Subtract **row-1** from **row-2 and row-3**, yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{227} \\ \mathbf{0} & \mathbf{10} & \mathbf{100} & \mathbf{22} \\ \mathbf{0} & \mathbf{20} & \mathbf{400} & \mathbf{54} \end{pmatrix}$$

Multiply **row-2** by **2** and **subtract** from **row-3**, yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{227} \\ \mathbf{0} & \mathbf{10} & \mathbf{100} & \mathbf{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{200} & \mathbf{10} \end{pmatrix}$$

Divide **row-2** and **row-3** by **10**, yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{227} \\ \mathbf{0} & \mathbf{1} & \mathbf{10} & \mathbf{2.2} \\ \mathbf{0} & \mathbf{0} & \mathbf{20} & \mathbf{1} \end{pmatrix}$$

Divide **row-3** by **2**, yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{227} \\ \mathbf{0} & \mathbf{1} & \mathbf{10} & \mathbf{2.2} \\ \mathbf{0} & \mathbf{0} & \mathbf{10} & \mathbf{0.5} \end{pmatrix}$$

Subtract **row-3** from **row-2**, yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{227} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1.7} \\ \mathbf{0} & \mathbf{0} & \mathbf{10} & \mathbf{0.5} \end{pmatrix}$$

Divide **row-3** by **10**, yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{227} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1.7} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0.05} \end{pmatrix}$$

So, $\boldsymbol{a = 227}$, $\boldsymbol{b = 1.7}$, $\boldsymbol{c = 0.05}$ and

$$\boldsymbol{y = p(t) = 227 + 1.7\,t + 0.05\,t^2}$$

For **year 2010**, $\boldsymbol{t = 30}$ and **predicated population** is:

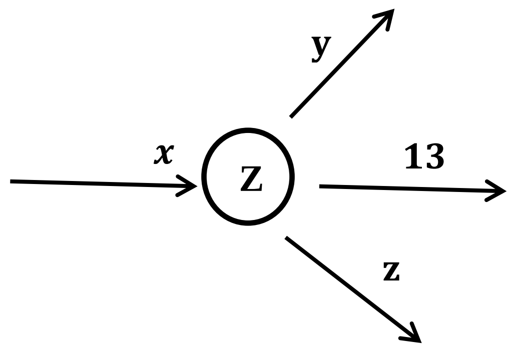
$$\boldsymbol{p(t) = 227 + 1.7 \times 30 + 0.05 \times (30)^2 = 323\ mi}$$

Similarly, for **year 2020**, $\boldsymbol{t = 40}$ and **predicated population** is:

$$\boldsymbol{p(t) = 227 + 1.7 \times 40 + 0.05 \times (40)^2 = 375\ mi}$$

2. Network Analysis

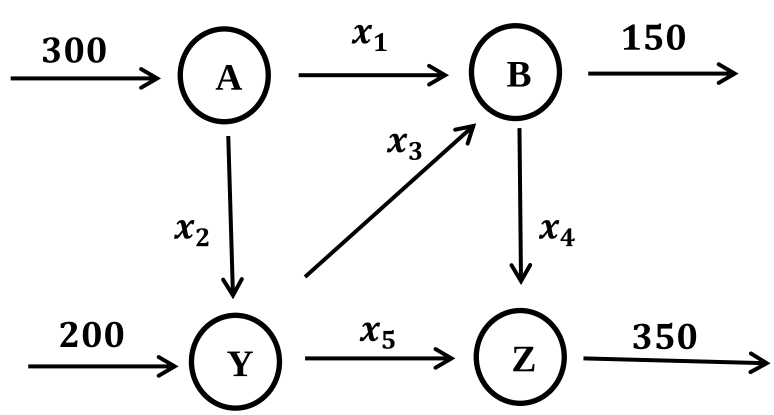
A **network system** consists of **junctions** and **branches**. They are used to **model diverse situations**, including in **economics**, **traffic**, **telephone signal** and **electrical engineering**. Such **models assume**, the **total flow into** a **junction** is **equal** to **total flow out** of the **junction**.



Accordingly, **above network** is **represented** by

$$x = y + 13 + z$$

Example 1: The **flow** of **traffic** (in **vehicles per hour**) through a **network** of **street** is **shown** in the **following figure**:



1. **Solve** this **system** for $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, $\boldsymbol{x_3}$, $\boldsymbol{x_4}$, and $\boldsymbol{x_5}$.
2. **Find** the **traffic flow** when $\boldsymbol{x_2 = 200}$ and $\boldsymbol{x_3 = 50}$.
3. **Find** the **traffic flow** when $\boldsymbol{x_2 = 150}$ and $\boldsymbol{x_3 = 0}$.

Solution:

From **junction A**, we get $\boldsymbol{x_1 + x_2 = 300}$

From **junction B**, we get $\boldsymbol{x_1 + x_3 = 150 + x_4 \implies x_1 + x_3 - x_4 = 150}$

From **junction Y**, we get $\boldsymbol{x_2 + 200 = x_3 + x_5 \implies x_2 - x_3 - x_5 = -200}$

From **junction Z**, we get $\boldsymbol{x_4 + x_5 = 350}$

Thus, the **system** of **linear equations** is:

$$\boldsymbol{x_1 + x_2 = 300}$$

$$\boldsymbol{x_1 + x_3 - x_4 = 150}$$

$$\boldsymbol{x_2 - x_3 - x_5 = -200}$$

$$\boldsymbol{x_4 + x_5 = 350}$$

To **solve** this **linear system**, we **first of all write** the **augmented matrix**:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 1 & 0 & 1 & -1 & 0 & 150 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 1 & 1 & 350 \end{pmatrix}$$

Then, reduce this **matrix** to **row-echelon form**:

<u>**Subtract**</u> **row-1** from **row-2,** yields

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 0 & -1 & 1 & -1 & 0 & -150 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 1 & 1 & 350 \end{pmatrix}$$

<u>**Add**</u> **row-2** to **row-3**, yields

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 0 & -1 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & -1 & -1 & -350 \\ 0 & 0 & 0 & 1 & 1 & 350 \end{pmatrix}$$

<u>Add</u> **row-3** to **row-4,** yields

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{300} \\ \mathbf{0} & \mathbf{1} & -\mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{150} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{350} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

The **matrix** is in **<u>row-echelon form</u>**. The **corresponding linear system** is **given** by:

$$\boldsymbol{x_1 + x_2 = 300}$$
$$\boldsymbol{x_2 - x_3 + x_4 = 150}$$
$$\boldsymbol{x_4 + x_5 = 350}$$
$$\mathbf{0 = 0}$$

Parametrically, with $\boldsymbol{x_2 = t}$, $\boldsymbol{x_3 = s}$, yields

$$\boldsymbol{x_1 = 300 - t}\,,\quad \boldsymbol{x_2 = t}\,,\quad \boldsymbol{x_3 = s}\,,$$
$$\boldsymbol{x_4 = 150 - t + s}\,,\qquad \boldsymbol{x_5 = 350 - x_4 = 150 + t - s}$$

These **answers** for **part (1)**.

For **part (2)** $\boldsymbol{t = X_2 = 200}$. $\boldsymbol{s = x_3 = 50}$.

So, $\boldsymbol{x_1 = 100}$, $\boldsymbol{x_2 = 200}$, $\boldsymbol{x_3 = 50}$, $\boldsymbol{x_4 = 0}$, $\boldsymbol{x_5 = 300}$

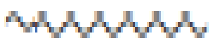
For **part (3)** $\boldsymbol{t = X_2 = 150}$. $\boldsymbol{s = x_3 = 0}$.

So, $\boldsymbol{x_1 = 150}$, $\boldsymbol{x_2 = 150}$, $\boldsymbol{x_3 = 0}$, $\boldsymbol{x_4 = 0}$, $\boldsymbol{x_5 = 350}$

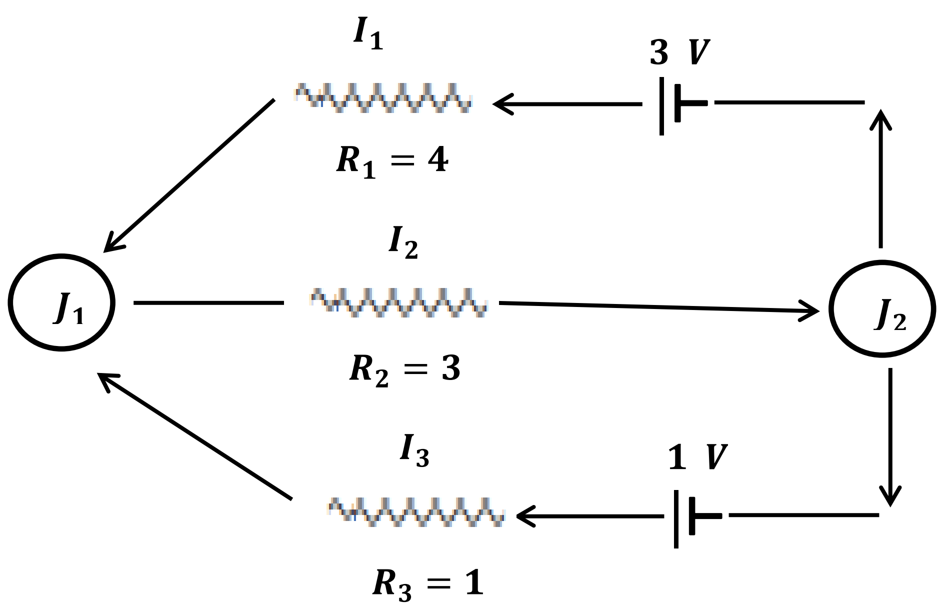
3. <u>Kirchhoff 's Laws</u>

System of **Linear equations** is also **applicable** in **electrical network**. **Analysis** of **electrical network** is **guided** by **<u>two properties</u>** **known** as **Kirchhoff 's Laws**:

1. **All** the **current flowing into a junction must flow out of it**.
2. The **sum** of the **products** ***IR*** (***I*** is **<u>current</u>** and ***R*** is **<u>resistance</u>**) **around** a **closed path** is **<u>equal</u>** to the **<u>total voltage</u>**.

A **battery** or **voltage source** is **denoted** by |⊢ or ⊣| and the **resistance** is **denoted** by 

<u>Example</u> 2: Consider the **electrical circuit drawn** in **figure below**



Use Kirchhoff-Law to **determine** I_1 , I_2 , I_3 .

<u>Solution</u>:

- **Applying (1)** of **<u>Kirchhoff-Law</u>** to **junction** J_1, we have

$$I_1 + I_3 = I_2 \quad \Rightarrow \quad I_1 - I_2 + I_3 = 0 \qquad Eq.(1)$$

- **Applying (2)** of **Kirchhoff-Law:**

$$R_1I_1 + R_2I_2 = 3 \quad \Longrightarrow \quad 4I_1 + 3I_2 = 3 \qquad Eq.(2)$$

$$R_2I_2 + R_3I_3 = 1 \quad \Longrightarrow \quad 3I_2 + I_3 = 1 \qquad Eq.(3)$$

The **network system** is **given** by

$$I_1 - I_2 + I_3 = 0 \qquad Eq.(1)$$

$$4I_1 + 3I_2 = 3 \qquad Eq.(2)$$

$$3I_2 + I_3 = 1 \qquad Eq.(3)$$

First, **writing** the **augmented matrix** is:

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 4 & 3 & 0 & 3 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Then, **reducing this matrix** to **row-echelon form** as follows:

Multiply **row-1** by **4** and **subtract** from **row-2**, yields

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Divide **row-2** by **7,** yields

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

Multiply **row-2** by **3** and **subtract** from **row-3**, yields

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & \frac{19}{7} & -\frac{2}{7} \end{pmatrix}$$

Multiply **row-3** by $\frac{7}{19}$, yields

$$\begin{pmatrix} 1-1 & 1 & 0 \\ 0 \quad 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 \quad 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

Further reducing the **matrix** to **Gauss-Jordan form**:

Add **row-2** to **row-1**, yields

$$\begin{pmatrix} 1 & 0 & \frac{3}{7} & \frac{3}{7} \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

Multiply **row-3** by $\frac{3}{7}$ and **subtract** from **row-1**, yields

$$\begin{pmatrix} 1 & 0 & 0 & \frac{9}{19} \\ 0 & 1 & -\frac{4}{7} & \frac{3}{7} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

Multiply **row-3** by $\frac{4}{7}$ and **add** to **row-2**, yields

$$\begin{pmatrix} 1 & 0 & 0 & \frac{9}{19} \\ 0 & 1 & 0 & \frac{7}{19} \\ 0 & 0 & 1 & -\frac{2}{19} \end{pmatrix}$$

The **corresponding linear system** is:

$$I_1 = \frac{9}{19}\ , \quad I_2 = \frac{7}{19}\ , \quad I_3 = -\frac{2}{19}$$

Gaussian Quadrature

The Gaussian quadrature method is an approximate method of calculation of a certain integral

$$\boldsymbol{I = \int_{a}^{b} y(x)dx}$$

by replacing the variables

$$\boldsymbol{x = (b-a)\frac{t}{2} + (a+b)\frac{t}{2}} \quad , \quad \boldsymbol{f(t) = (b-a)\frac{y(x)}{2}}$$

the desired integral is reduced to the form

$$\boldsymbol{I = \int_{-1}^{1} f(t)dt}$$

The **Gaussian quadrature formula** is:

$$\boldsymbol{I = \int_{-1}^{1} f(t)dt = \sum_{i=1}^{n} A_i f(t_i)}$$

The cusps $\boldsymbol{t_i}$ of the Gaussian quadrature formula are the roots of a Legendre polynomial of degree $\boldsymbol{n}$, $\boldsymbol{P_n(t)}$**.** The Legendre polynomial has exactly $\boldsymbol{n}$ real and various roots in the **interval (–1, 1).** The weights $\boldsymbol{A_i}$ of the Gaussian quadrature formula are defined by

$$\boldsymbol{A_i = \frac{2}{(1-t_i^2)[P'_n(t_i)]^2}}$$

Given in the table are the cusps and weights of the Gaussian quadrature formula for the first five values $\boldsymbol{n}$.

n	**Cusps**	**Weight**
2	± 0.577350	**1**
3	**0**	$^{8}/_{9}$
	± 0.774597	$^{5}/_{9}$
4	± 0.339981	**0.652145**
	± 0.861136	**0.347855**
5	**0**	**0.568889**
	± 0.538469	**0.478629**
	± 0.906180	**0.236927**

Gaussian quadrature formula is **exact** for an **arbitrary polynomial** of **degree not higher than** $(2n-1)$. The **remainder term** of **Gaussian formula** R_n for the integral $\int_a^b y(x)dx$ is expressed as follows:

$$R_n = \frac{(b-a)^{2n-1} n!}{(2n+1)[(2n)!]^3} y^{2n}(\xi); a \le \xi \le b$$

The Gaussian quadrature method is applied when a subintegral function is smooth enough and a gain in the number of cusps is essential (for instance, in calculating multiple integrals as iterated integrals).

The **Gaussian quadrature formula** is widely used in **solving problems** of **radiation heat transfer** in **direct integration** of the **equation** of **transfer of radiation over space**. The application of **Gaussian formula** in this case **works very well** especially when the **number of intervals** of **spectrum decomposition** is **great**.

Gaussian quadrature 1-point formula:

$$I = \int_a^b f(x)dx$$

$$x = \left|\frac{b-a}{2}\right| A + \frac{b+a}{2}$$

$$I = \int_{-1}^{1} f(x)dx = 2f(0)$$

Gaussian quadrature 2-point formula:

$$I = \int_a^b f(x)dx$$

$$I = \int_{-1}^{1} f(x)dx = \frac{b-a}{2}\ [A_1 f(x_1) + A_2 f(x_2)]$$

where,

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2} \quad , \quad x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2}$$

$$A_1 = A_2 = 1 \quad , \quad z_1 = \frac{-1}{\sqrt{3}} = -0.5774 \quad , \quad z_2 = \frac{1}{\sqrt{3}} = 0.5774$$

$$I = \int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gaussian quadrature 3-point formula:

$$I = \int_{-1}^{1} f(x)dx = \frac{b-a}{2}\ [A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)]$$

where,

$$A_1 = \frac{5}{9} = 0.5555 \ , \qquad A_2 = \frac{8}{9} = 0.8888 \ , \qquad A_3 = \frac{5}{9} = 0.5555$$

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2},$$

$$x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2},$$

$$x_3 = \frac{b-a}{2} z_3 + \frac{b+a}{2}$$

$$z_1 = -\sqrt{\frac{3}{5}} = -0.7746 \quad , \quad z_2 = 0 \quad , \quad z_3 = \sqrt{\frac{3}{5}} = 0.7746$$

$$I = \int_{-1}^{1} f(x)dx = \left(\frac{5}{9}\right) f\left(-\sqrt{\frac{3}{5}}\right) + \left(\frac{8}{9}\right) f(0) + \left(\frac{5}{9}\right) f\left(\sqrt{\frac{3}{5}}\right)$$

<u>Example</u> 1: Use Gauss-Legendre two – point and three – point formula to evaluate

$$I = \int_{-2}^{2} e^{-\frac{x}{2}} dx$$

<u>Solution</u>:

Given data are:

$$a = -2 \quad , \quad b = 2 \quad \text{and} \quad f(x) = e^{-\frac{x}{2}}$$

<u>Gaussian quadrature 2 – point formula</u>:

$$I = \frac{b-a}{2} \left[A_1 f(x_1) + A_2 f(x_2)\right]$$

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2} \quad , \quad x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2}$$

$$A_1 = A_2 = 1 \quad , \quad z_1 = \frac{-1}{\sqrt{3}} \quad , \quad z_2 = \frac{1}{\sqrt{3}}$$

$$x_1 = \frac{2-(-2)}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{2-2}{2} = \frac{-2}{\sqrt{3}} \quad ,$$

$$x_2 = \frac{2-(-2)}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{2-2}{2} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad I = \frac{2-(-2)}{2}\left[(1)f\left(\frac{-2}{\sqrt{3}}\right) + (1)f\left(\frac{2}{\sqrt{3}}\right)\right]$$

$$= 2\left[e^{-\frac{x_1}{2}} + e^{-\frac{x_2}{2}}\right] = 2\left[e^{-\frac{\left(\frac{-2}{\sqrt{3}}\right)}{2}} + e^{-\frac{\left(\frac{2}{\sqrt{3}}\right)}{2}}\right] = 4.6854$$

<u>Gaussian quadrature 3-point formula</u>:

$$I = \frac{b-a}{2}\left[A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)\right]$$

where, $A_1 = \frac{5}{9}$, $A_2 = \frac{8}{9}$, $A_3 = \frac{5}{9}$,

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2} = \frac{2-(-2)}{2}(-0.7746) + \frac{2-2}{2} = -1.5492$$

$$x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2} = \frac{2-(-2)}{2}(0) + \frac{2-2}{2} = 0$$

$$x_3 = \frac{b-a}{2}z_3 + \frac{b+a}{2} = \frac{2-(-2)}{2}(0.7746) + \frac{2-2}{2} = 1.5492$$

$$z_1 = -\sqrt{\frac{3}{5}} = -0.7746 \quad , \quad z_2 = 0 \quad , \quad z_3 = \sqrt{\frac{3}{5}} = 0.7746$$

$$\Rightarrow \quad I = \frac{2-(-2)}{2}\left[\left(\frac{5}{9}\right)f\left(-2\sqrt{\frac{3}{5}}\right) + \left(\frac{8}{9}\right)f(0) + \left(\frac{5}{9}\right)f\left(2\sqrt{\frac{3}{5}}\right)\right]$$

$$= 2\left[\left(\frac{5}{9}\right)e^{-\frac{x_1}{2}} + \left(\frac{8}{9}\right)e^{-\frac{x_2}{2}} + \left(\frac{5}{9}\right)e^{-\frac{x_3}{2}}\right]$$

$$= 2\left[\left(\frac{5}{9}\right)e^{-\frac{\left(-2\sqrt{\frac{3}{5}}\right)}{2}} + \left(\frac{8}{9}\right)e^{-\frac{0}{2}} + \left(\frac{5}{9}\right)e^{-\frac{\left(2\sqrt{\frac{3}{5}}\right)}{2}}\right] = 4.7008$$

Applications of Integrals

- **Arc Length** – determining the length of a curve over a given interval.
- **Surface Area** – determining the surface area of a solid of revolution, i.e. a solid obtained by rotating a region bounded by two curves about a vertical or horizontal axis.
- **Center of Mass** – determining the centre of mass or centroid of a thin plate where the plate can be described as a region bounded by two curves (one of which may the x or y-axis).
- **Hydrostatic Pressure and Force** – determining the hydrostatic pressure and force on a vertical plate submerged in water. The plates used in the examples can all be described as regions bounded by one or more curves/lines.
- **Probability** – Many quantities can be described with probability density functions. For example, the length of time a person waits in line at a checkout counter or the life span of a light bulb. None of these quantities is fixed values and will depend on a variety of factors. probability density functions and computing the mean (think average wait in line or average life span of a light blub) of a probability density function.

Fourier Series

➔ Even and Odd Functions

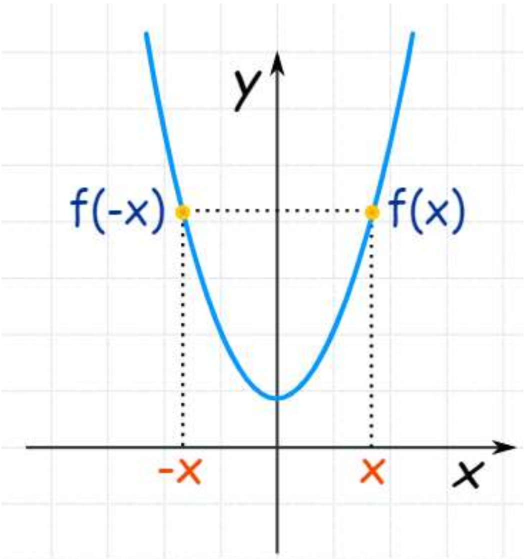
- **Even Functions**

A **function** is "**even**" when:

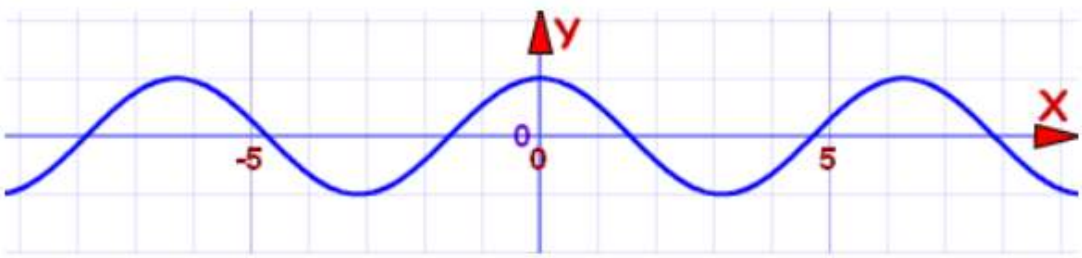
$$f\,(x) = f(-x) \quad \text{for all } x$$

In other words, there is **symmetry** about the **y - axis** (like a reflection).

for example, the curve below of function $f\,(x) = x^2 + 1$



They **got called "even"** functions **because** the **functions** $x^2, x^4, x^6, x^8, \cdots$ etc **behave like that**, but there are **other functions** that **behave like that too**, such as $\cos x$:



cosine function: $f(x) = \cos x$ is an **even function**

But an **even exponent does not always make an even function**, for example $(x+1)^2$ is **not** an **even function**.

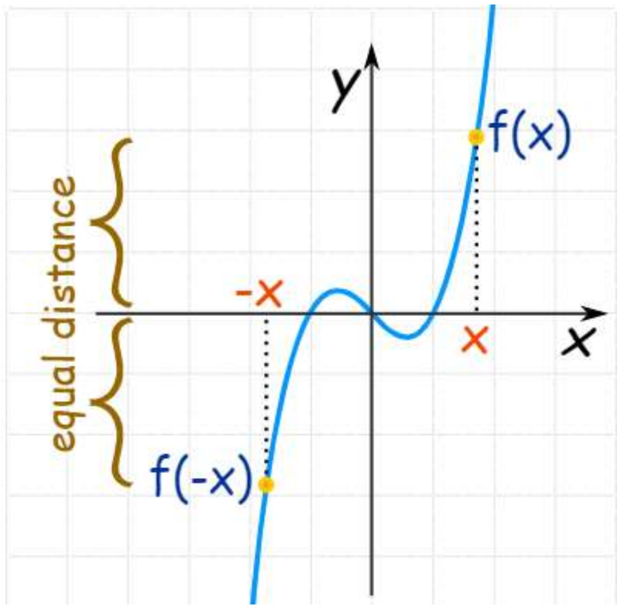
- **<u>Odd Functions</u>**

A **function** is "**odd**" when:

$$-\boldsymbol{f(x) = f(-x)} \quad \text{for all } \boldsymbol{x}$$

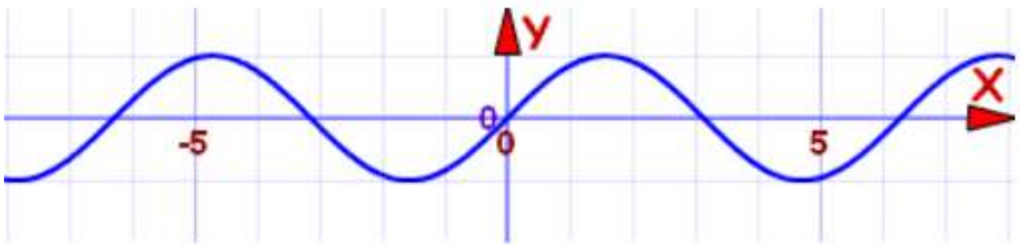
Note the **minus** is in the **front of** $\boldsymbol{f(x)}$: $-\boldsymbol{f(x)}$

The **symmetry** is **about** the **origin** as shown in the **figure below** of the **curve** of the **function** $\boldsymbol{f(x) = x^3 - x}$.



Cosine function: $\boldsymbol{f(x) = x^3 - x}$ is an **even function**

They got called **"odd" functions** because the functions $\boldsymbol{x}, \boldsymbol{x^3}, \boldsymbol{x^5}, \boldsymbol{x^7}$, $\boldsymbol{x^9}, \cdots$, etc behave like that, but there are other functions that behave like that too, such as $\boldsymbol{\sin x}$:

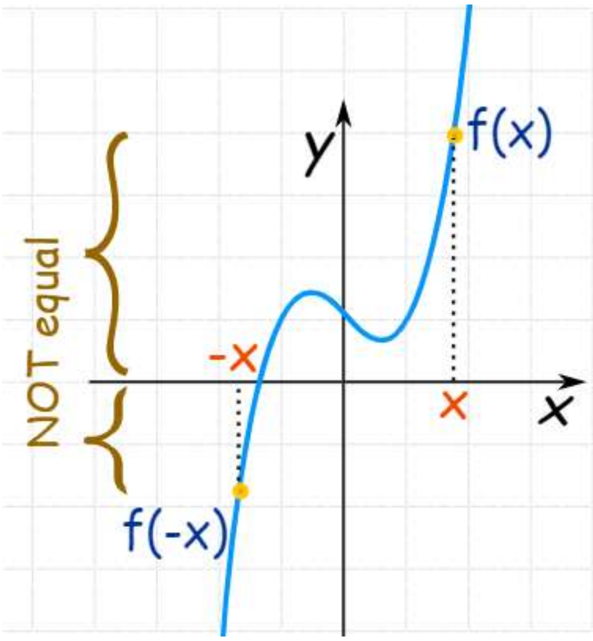


Sine function: $\boldsymbol{f(x) = \sin x}$ is an odd function

But an **odd exponent does not always make an odd function**, for example $\boldsymbol{x^3 + 1}$ is **not** an **odd function**.

- **Neither Odd nor Even Functions**

Don't be misled by the names **"odd"** and **"even"**, they are just **names** and a function **does not have** to be **even** or **odd**. In fact, **most functions** are **neither odd nor even**. For example, just **adding 1** to the function above the curve will be as follows:



Cosine function: $\boldsymbol{f(x) = x^3 - x + 1}$ is **not an odd function**, and it is not an **even function either.**
It is **neither odd nor even function**

Example: is $\boldsymbol{f(x) = \frac{x}{x^2-1}}$ **Even** or **Odd** or **neither**?

Solution:

Substituting $(-\boldsymbol{x})$ in place of $(\boldsymbol{x})$ in the **given function:**

$$f(-x) = \frac{-x}{(-x)^2 - 1} = \frac{-x}{x^2 - 1} = -\frac{x}{x^2 - 1} = -f(x)$$

So, $\boldsymbol{f(-x) = -f(x)}$**,** which **makes** the **function an odd function.**

- **Even and Odd**

The **only function** that is **even and odd** is , $\boldsymbol{f(x) = 0}$

<u>Special Properties</u>

<u>Adding</u>:

- The **sum** of **two even functions** is **even.**
- The **sum** of **two odd functions** is **odd.**
- The **sum** of **an even and odd function** is **neither even nor odd** (**unless one function is zero**).

<u>Multiplying</u>:

- The **product** of **two even functions** is **an even function**.
- The **product** of **two odd functions** is **an even function**.
- The **produc**t of **an even function and an odd function** is **an odd function**.

➔ <u>Periodic Functions</u>

Periodic function is a **function** that **repeats itself** at **equal intervals** in the **abscissa**, then the **function** $\boldsymbol{f(t)}$ is said to be a **periodic function** if

$$\boldsymbol{f(t) = f(t+T) = f(t+2T) = \cdots}$$

where $\boldsymbol{T}$ is called the **period** of the **function** $\boldsymbol{f(t)}$. It is a **positive constant** value $(\boldsymbol{T > 0})$**.** For example, **all trigonometric function** have **period** of $\boldsymbol{T = 2\pi}$. So, the **sine and cosine function** is a **periodic function** with a **period** of $\boldsymbol{2\pi}$:

$$\boldsymbol{sin\ t = sin\ (t + 2\pi)} \quad \text{or} \quad \boldsymbol{sin\ t = sin\ (t + 2n\pi)}$$

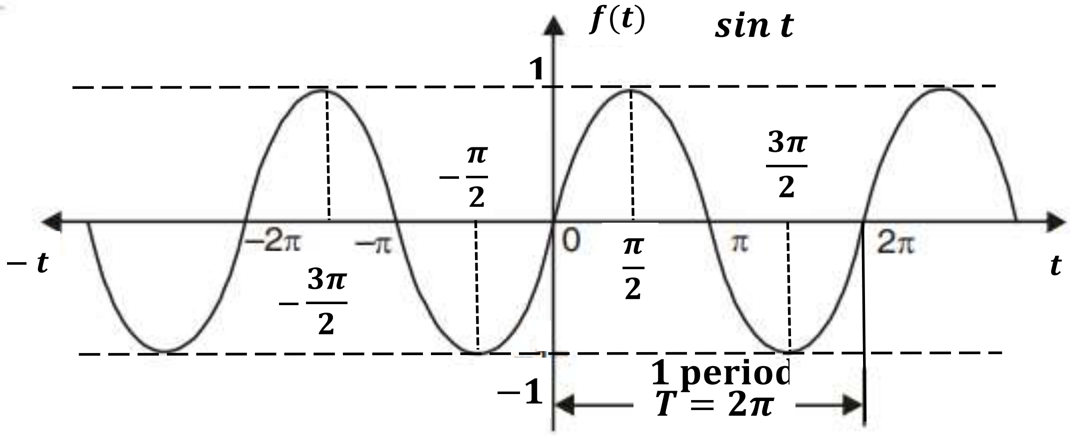
$$\boldsymbol{cos\ t = cos\ (t + 2\pi)} \quad \text{or} \quad \boldsymbol{cos\ t = cos\ (t + 2n\pi)}$$

where $\quad \boldsymbol{n = 1, 2, 3, 4, \cdots}$

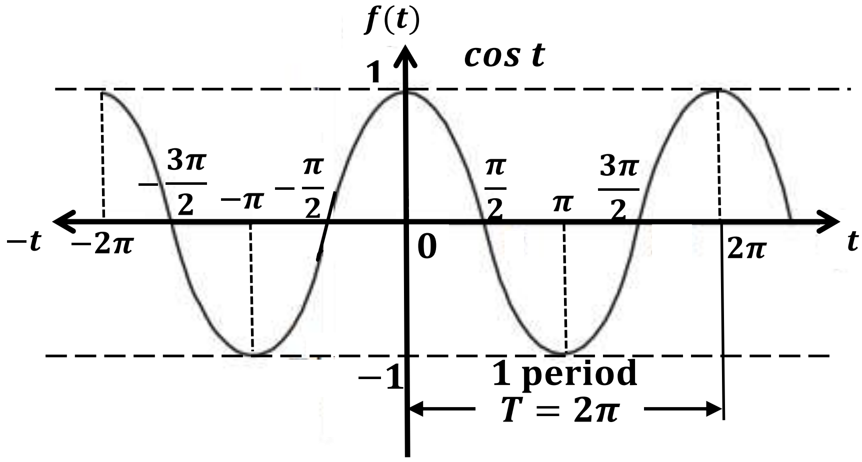
For example:

$$\boldsymbol{sin\ t = sin\ (t + 2\pi) = sin\ (t + 4\pi) = sin\ (t + 6\pi) = \cdots}$$

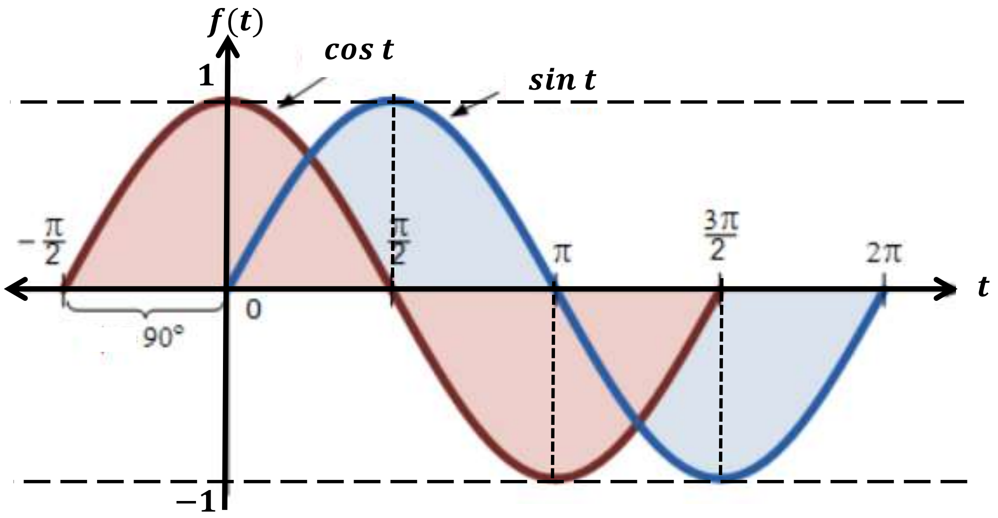
The **Sine Function** has **up-down curve** which **repeats** every **2π radians** (**360°)**. It **starts** at **0**, **heads up** to **1** by $\frac{\pi}{2}$ **radians (90°)** and then **heads down** to **-1**.



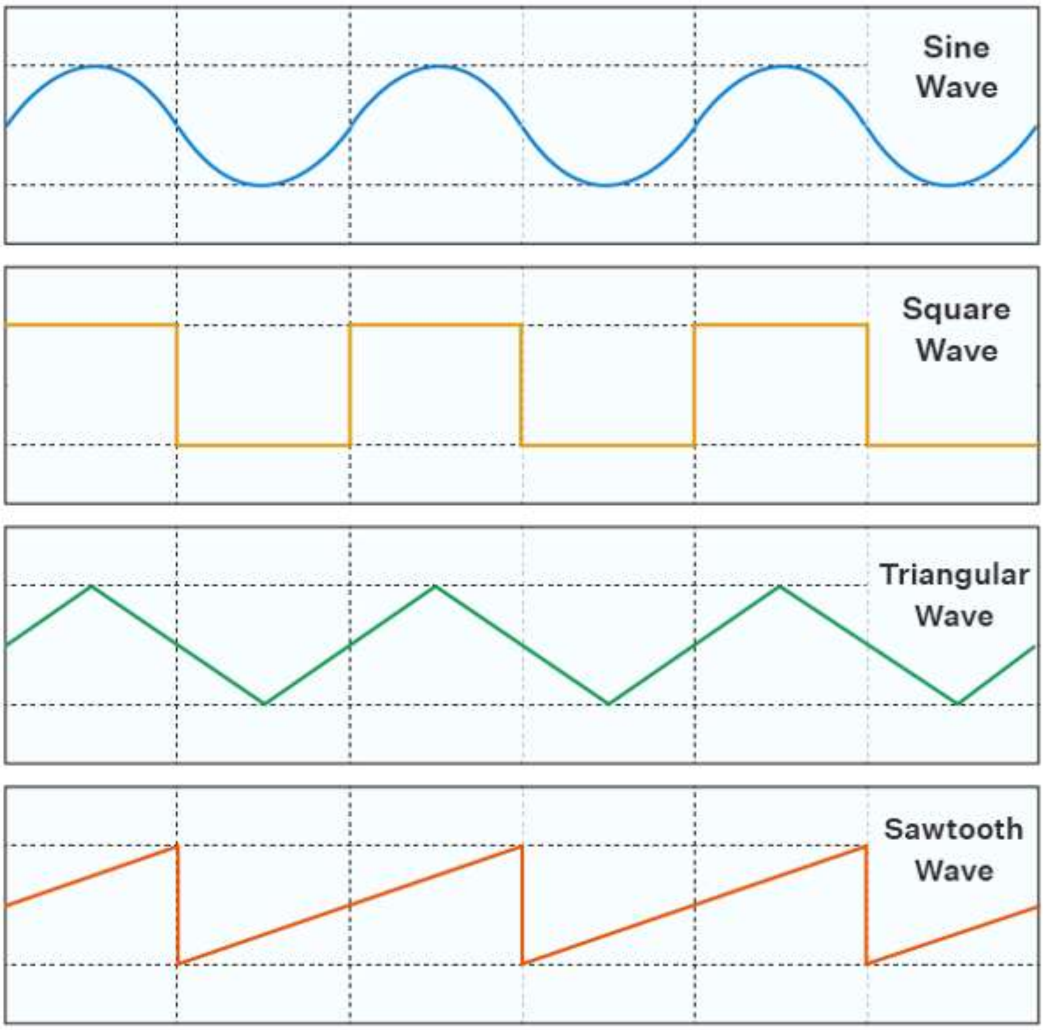
Cosine is just **like Sine**, but it **starts at 1** and **heads down until π radians (180°)** and then heads up again.



In fact, **Sine** and **Cosine** are like **good friends follow each other**, exactly $\frac{\pi}{2}$ **radians (90°) separated** by a **distance**; at a **specified distance** from each other in **time** or **space**.



The following are the graphs of some of the periodic functions. The graph of each of the below periodic functions has translational symmetry.



The **period** of some of the important periodic functions are as follows.

- The period of $\boldsymbol{sin}\ (\boldsymbol{x})$ and $cos\ (\boldsymbol{x})$ is $\boldsymbol{2\pi}$.
- The period of $\boldsymbol{tan}\ (\boldsymbol{x})$and $\boldsymbol{cotan}\ (\boldsymbol{x})$ is $\boldsymbol{\pi}$.
- The period of $\boldsymbol{sec}\ (\boldsymbol{x})$and $\boldsymbol{cosec}\ (\boldsymbol{x})$is $\boldsymbol{2\pi}$.

The **graph** of a **periodic function** is **symmetric** and **repeats itself along** the **horizontal axis**.

Example 1: Find the **period** of the **periodic function** $\boldsymbol{sin}\ (\boldsymbol{4x+5})$.

Solution:

The period of $\boldsymbol{sin}\ (\boldsymbol{x})$ is $\boldsymbol{2\pi}$, and the period of $\boldsymbol{sin}\ (\boldsymbol{4x+5})$ is :

$$\boldsymbol{2\pi = 4x \quad \Rightarrow \quad x = \frac{2\pi}{4} = \frac{\pi}{2}}$$

Therefore, the **period** of $\boldsymbol{sin}\ (\boldsymbol{4x+5})$ is $\boldsymbol{\frac{\pi}{2}}$.

Example 2: Find the **period** of the function

$$\boldsymbol{f(x) = tan\ 3x + sin\frac{5x}{2}}$$

Solution:

The **period** of $\boldsymbol{tan\ x}$ is $\boldsymbol{\pi}$, and the **period** of $\boldsymbol{tan\ x}$ is:

$$\boldsymbol{3x = \pi \quad \Rightarrow \quad x = \frac{\pi}{3}}$$

The period of $\boldsymbol{sin\ x}$ is $\boldsymbol{2\pi}$, and the period of $\boldsymbol{sin\frac{5x}{2}}$ is:

$$\boldsymbol{\frac{5x}{2} = 2\pi \quad \Rightarrow \quad x = \frac{4\pi}{5}}$$

Thus, the **period** of the function $\boldsymbol{f(x) = tan\ 3x + sin\frac{5x}{2}}$, is as follows:
Period of $\boldsymbol{f(x)}$ is:

$$\boldsymbol{f(x) = \frac{\text{LCM of } \pi \text{ and } 4\pi}{\text{HCF of } 3 \text{ and } 5} = \frac{4\pi}{1} = 4\pi}$$

Therefore, the **period** of the **function**

$$\boldsymbol{f(x) = tan\ 3x + sin\frac{5x}{2}}$$

is $\boldsymbol{4\pi}$.

The **Fourier series** is a **superposition** of **various periodic wave function series** to **form** a **complex periodic function**. It is usually **composed** of **sine** and **cosine functions**, and the **summation** of these **wave functions** is taken by **assigning respective weight components** to these series. The **Fourier series** has applications in the **representation** of **heatwaves**, **vibration analysis**, **quantum mechanics**, **electrical engineering**, **signal processing**, **image processing**.

➔ <u>Fourier Series</u>

A **series** of **sines** and **cosines** of an **angle** and its **multiples** of the **form**

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx + \cdots + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin nx + \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + + \sum_{n=1}^{\infty} b_n \sin nx$$

is **called** the **<u>Fourier series</u>**, where a_1 , a_2 , ... , a_n, and b_1 , b_1 , b_1 , … , b_n… are **constants**.

A **periodic function** $f(x)$ can be **expanded** in a **Fourier Series**.

Useful Integrals

The **following integrals** are **useful** in **Fourier Series**.

(1)	$\int_0^{2\pi} \sin nx\, dx = 0$	**(5)**	$\int_0^{2\pi} \sin nx \cdot \sin mx\, dx = 0$
(2)	$\int_0^{2\pi} \cos nx\, dx = 0$	**(6)**	$\int_0^{2\pi} \cos nx \cdot \cos mx\, dx = 0$
(3)	$\int_0^{2\pi} \sin^2 nx\, dx = \pi$	**(7)**	$\int_0^{2\pi} \sin nx \cdot \cos mx\, dx = 0$
(4)	$\int_0^{2\pi} \cos^2 nx\, dx = \pi$	**(8)**	$\int_0^{2\pi} \sin nx \cdot \cos nx\, dx = 0$
(9)	$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$ where $v_1 = \int v\, dx, v_2 = \int v_1 dx$ and so on $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on, and $(x) \sin n\pi = 0$, $\cos n\pi = (-1)^n$ *where* $n \epsilon I$		

Determination of Fourier Coefficients (Euler's Formulae)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx$$
$$+ \cdots \qquad (1)$$

➔ **To find a_0, integrating both sides of the Equation (1)** from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x\, dx + a_2 \int_0^{2\pi} \cos 2x\, dx$$

$$+\cdots+a_n\int_0^{2\pi}\cos nx\,dx+b_1\int_0^{2\pi}\sin x\,dx$$
$$+b_2\int_0^{2\pi}\sin 2x\,dx+\cdots+b_n\int_0^{2\pi}\sin nx\,dx+\cdots$$

$\int_0^{2\pi} f(x)dx = \frac{a_0}{2}\int_0^{2\pi} dx$ [**other integrals** $= \mathbf{0}$ by **formula** $(\mathbf{1})$and $(\mathbf{2})$]

$$\Rightarrow \int_0^{2\pi} f(x)dx = \frac{a_0}{2}2\pi \quad \Rightarrow \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

➔ **<u>To find a_n</u>, Multiplying each side of the Equation (1)** by $\boldsymbol{cos\ nx}$ and **integrating** from $x = 0$ to $x = 2\pi$.

➔

$$\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^{2\pi}\cos nx\,dx + a_1\int_0^{2\pi}\cos x\cos nx\,dx$$
$$+a_2\int_0^{2\pi}\cos 2x\cos nx\,dx+\cdots+\ a_n\int_0^{2\pi}\cos^2 nx\,dx$$
$$+b_1\int_0^{2\pi}\sin x\cos nx\,dx+b_2\int_0^{2\pi}\sin 2x\cos nx\,dx$$
$$+\cdots+b_n\int_0^{2\pi}\sin nx\cos nx\,dx+\cdots$$
$$= a_n\int_0^{2\pi}\cos^2 nx\,dx = a_n\pi$$

(All **other integrals** $= \mathbf{0}$ by **formula** $(\mathbf{1})$ to $(\mathbf{8})$

$$\Rightarrow \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

By **taking** $\boldsymbol{n} = \mathbf{1, 2, 3}$, ... the **values** of a_1, a_2, a_3, ... **can be found.**

➔ **To find b_n, Multiplying each side** of **Equation (1)** by $\boldsymbol{\sin nx}$ and **integrate** from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x)\sin nx\, dx = \frac{a_0}{2}\int_0^{2\pi} \sin nx\, dx + a_1\int_0^{2\pi} \cos x \sin nx\, dx$$

$$+a_2\int_0^{2\pi} \cos 2x \sin nx\, dx + \cdots + a_n\int_0^{2\pi} \cos nx \sin nx\, dx$$

$$+b_1\int_0^{2\pi} \sin x \sin nx\, dx + b_2\int_0^{2\pi} \sin 2x \sin nx\, dx$$

$$+\cdots + b_n\int_0^{2\pi} \sin^2 nx\, dx + \cdots$$

$$= b_n\int_0^{2\pi} \sin^2 nx\, dx = b_n\pi$$

(**All other integrals** $= 0$ by **formula** (1) to (8))

$$\Longrightarrow \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

Note: To **get similar formula** of a_0, $\frac{1}{2}$ has been **written** with a_0 in **Fourier series**.

Example 1: Find the **Fourier Series representing**

$$f(x) = x \qquad 0 < x < 2\pi$$

and **sketch its graph** from $x = -4\pi$ to $x = 4\pi$.

Solution:

Let

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots + b_1\sin x + b_2\sin 2x + \cdots$$

Hence,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{4\pi^2}{2} - 0\right] = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

By **integrating** by **parts**, yields

$$\frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[uv - \int_0^{2\pi} vdu\right]$$

$$= \left|\begin{matrix} u = x \quad \Rightarrow \quad du = dx \\ dv = \cos nx\,dx \Rightarrow v = \dfrac{\sin nx}{n} \end{matrix}\right|$$

$$= \frac{1}{\pi}\left[x\,\frac{\sin nx}{n} - \int_0^{2\pi} \frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[x\,\frac{\sin nx}{n} - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi\,\frac{\sin 2n\pi}{n} + \frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

By **integrating** by **parts**, yields

$$\frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[uv - \int_0^{2\pi} vdu\right]$$

$$= \left|\begin{matrix} u = x \quad \Rightarrow \quad du = dx \\ dv = \sin nx = dx \;\Rightarrow\; v = -\dfrac{\cos nx}{n} \end{matrix}\right|$$

$$\frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \int_0^{2\pi}\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \int_0^{2\pi}\frac{\cos nx}{n}\,dx\right] = \frac{1}{\pi}\left[-x\,\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\ \frac{\cos 2n\pi}{n} + \frac{\sin 2n\pi}{n^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[-2\pi \frac{\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

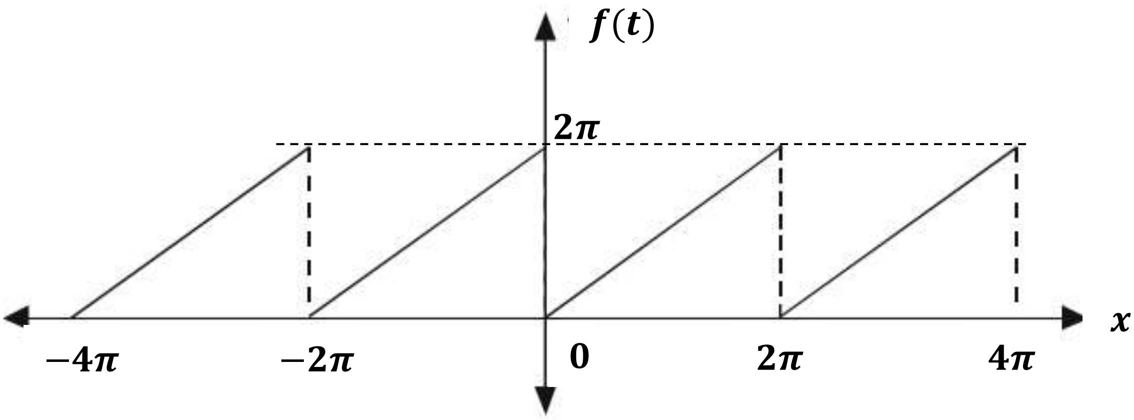
Substituting $a_0 = 2\pi$, $a_n = 0$, $b_n = -\frac{2}{n}$, into **Equation** of $f(x)$**,** yields

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

$$\Rightarrow\ f(x) = x = \frac{2\pi}{2} - \frac{2}{1}\ \sin x - \frac{2}{2}\ \sin 2x - \frac{2}{3} \sin 3x - \cdots$$

$$\Rightarrow\quad x = \pi - 2\left(\sin x + \frac{1}{2}\ \sin 2x + \frac{1}{3} \sin 3x + \cdots\right)$$

Now, drawing the **graph** from $x = -4\pi$ to $x = 4\pi$.



<u>Example</u> 2: Given that $\boldsymbol{f(x) = x + x^2}$ for $\boldsymbol{-\pi < x < \pi}$, find the **Fourier expression** of $f(x)$. **Deduce** that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

<u>Solution</u>:

Let

$$x + x^2 = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$$

$\boldsymbol{f(x) = x + x^2 = -x + (-x)^2 = -(x - x^2) = -f(x)}$,

so $\boldsymbol{f(x)}$is **an odd function**.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$\Rightarrow \quad a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$\boldsymbol{f(x) = x \cos nx = -x \cos(-nx) = -\cos(nx) = -f(x)}$,

so $\boldsymbol{f(x)}$ is **an odd function.**

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - (2x)\frac{(-\cos nx)}{n^2} + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 \frac{\sin n\pi}{n} - (2x)\frac{(-\cos n\pi)}{n^2} + 2\left(-\frac{\sin n\pi}{n^3}\right)\right] = \frac{4(-1)^n}{n^2}$$

$$\Rightarrow a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\, dx$$

$\boldsymbol{f(x) = x^2 \sin nx = (-x)^2 \sin(-nx) = -x^2 \sin(nx) = -f(x)}$, so $\boldsymbol{f(x)}$ is **an odd function.**

$$\boldsymbol{b_n = \frac{2}{\pi}\int_0^{\pi} x\, sin\, nx\, dx = \frac{2}{\pi}\left[x\left(-\frac{cos\, nx}{n}\right) - (1)\left(-\frac{sin\, nx}{n^2}\right)\right]_0^{\pi}}$$

$$\boldsymbol{= \frac{2}{\pi}\left[(-\pi)\frac{cos\, n\pi}{n} + 2\frac{sin\, n\pi}{n^2}\right] = \frac{2}{\pi}\left[\frac{\pi}{n} cos\, n\pi\right] = -\frac{2}{n}(-1)^n}$$

$$\boldsymbol{\Rightarrow b_n = -\frac{2}{n}(-1)^{n+1}}$$

Substituting the values of $\boldsymbol{a_0}$, $\boldsymbol{a_n}$, $\boldsymbol{b_n}$ in **Fourier series** of the function $\boldsymbol{f(x)}$ yields

$$\boldsymbol{x + x^2 = \frac{\pi^2}{3} + 4\left[-cos\, x + \frac{1}{2^2}\, cos\, 2x - \frac{1}{3^2}\, cos\, 3x + \cdots\right]}$$

$$\boldsymbol{+2\left[sin\, x - \frac{1}{2}\, sin\, 2x + \frac{1}{3}\, sin\, 3x - \cdots\right]}$$

Putting $\boldsymbol{x = \pi}$

$$\boldsymbol{\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]} \qquad \boldsymbol{(1)}$$

Putting $\boldsymbol{x = -\pi}$

$$\boldsymbol{-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]} \qquad \boldsymbol{(2)}$$

Adding Equations (3) and (4)

$$\boldsymbol{\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]}$$

$$\boldsymbol{\Longrightarrow \quad \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

Laplace Transform

The **Laplace transform** is an **integral transform** named **after** its **inventor Pierre-Simon Laplace**. It **transforms a function** of a **real variable** t (**often time**) to a **function** of a **complex variable** s (**complex frequency**). The **transformation** was **named** by this **name** in **relation to** the **French scientist Laplac**e who **lived** in the **nineteenth century.** The **transform** has **many applications** in **science** and **engineering**.

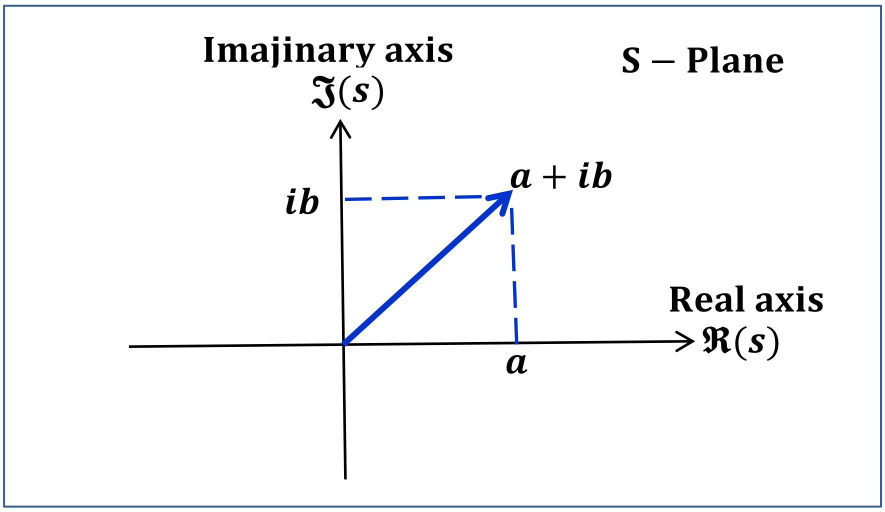
By **Laplace transform**, a **real function** f in **time** t is **translated into** the **s-plane** by taking the **integral** of the **function multiplied** by e^{-st} **from 0** to ∞,

$$F(s) = \int_0^{\infty} f(t)\ e^{-st} dt$$

where s is a **complex number** with the **form**

$$s = a + bi$$

where a and b are **real numbers**, and i is **called** an **imaginary number** which is a **solution** of the **equation** $x^2 = -1$, because **no real number satisfies** this **equation**. Therefore, a is **called** the **real part**, and ib is **called** the **imaginary part**.



This **transformation from** the ***t*-domain** into the **s-domain** is known as a **Laplace transform** and the **function** $\boldsymbol{F(s)}$ is called the **Laplace transform** of $\boldsymbol{f}$, and **denoted** by $\boldsymbol{\mathcal{L}[f(t)]}$ and **defined** as

$$\boldsymbol{F(s) = \mathcal{L}[f(t)] = \int_{0}^{\infty} f(t)\, e^{-st} \quad dt}$$

With this **alternate notation**, **note** that the **transform** is **really a function** of a **new variable**, ***s***, and that all the ***t's*** **will drop out** in the **integration process**.

Now, the **integral** in the **definition** of the **transform** is **called** an **improper integral** and it would probably **be best** to **recall how these kinds** of **integrals work** before **jump** into **computing** some **transforms**.

<u>Example</u> 1: If $\boldsymbol{k \neq 0}$, evaluate the following integral.

$$\boldsymbol{\int_{0}^{\infty} e^{kt}\, dt}$$

<u>Solution</u>:

Convert improper integrals to **limits** as follows,

$$\boldsymbol{\int_{0}^{\infty} e^{kt}\, dt = \lim_{n\to\infty} \int_{0}^{n} e^{kt}\, dt = \lim_{n\to\infty} \left[\frac{1}{k} e^{kt}\right]_{0}^{n}}$$

$$\boldsymbol{= \lim_{n\to\infty} \left[\frac{1}{k} e^{kn} - \frac{1}{k} e^{0}\right]}$$

$$\boldsymbol{= \lim_{n\to\infty} \left[\frac{1}{k} e^{kn} - \frac{1}{k}\right]}$$

At this point, careful has to be got. The value of $\boldsymbol{k}$ will affect the answer. $\boldsymbol{k}$ has already **assumed** to be **non-zero**, now need to be worry about the sign of $\boldsymbol{k}$.

- If $\boldsymbol{k}$ is **<u>positive</u>** the **exponential** $\boldsymbol{(e^{kn})}$ will **<u>go to infinity</u>**.
- if $\boldsymbol{k}$ is **<u>negative</u>** the **exponential** $\boldsymbol{(e^{kn})}$ will **<u>go to zero</u>**.

So, the **integral** is only **convergent** (i.e. the **limit exists** and is **finite**) **provided $k < 0$**. In this case yields,

$$\int_0^\infty e^{kt}\,dt = \lim_{n\to\infty}\left[\underbrace{\frac{1}{k}e^{kn}}_{=0} - \frac{1}{k}\right] = -\frac{1}{k} \quad \text{provided } k < 0$$

Example 2: Find $\mathcal{L}\,(1)$

Solution:

$$F(s) = \mathcal{L}\,[f(t)] = \int_0^\infty f(t)\,e^{-st}\;\;dt$$

$$f(t) = 1 \quad \Rightarrow \quad F(s) = \mathcal{L}\,(1) = \int_0^\infty (1)\,e^{-st}\,dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

$$\Rightarrow \quad \mathcal{L}\,(1) = -\frac{1}{s}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{s}\left(\frac{1}{e^{\infty}} - 1\right) = -\frac{1}{s}\,(0-1) = \frac{1}{s}$$

$$\Rightarrow \quad \mathcal{L}\,(1) = \frac{1}{s}$$

Thus, the **Laplace transform** of the **constant** is:

$$\mathcal{L}\,(k) = \frac{k}{s} \quad , \quad k = 1, 2, 3, \cdots, \infty$$

Examples are:

$$\mathcal{L}\,(2) = \frac{2}{s}\,, \qquad \mathcal{L}\,(10) = \frac{10}{s}\,, \qquad \mathcal{L}\,(23) = \frac{23}{s}\,, \qquad \ldots\ldots$$

Example 3: Find $\mathcal{L}\,(e^{at})$

Solution:

$$F(s) = \mathcal{L}\,[f(t)] = \int_0^\infty f(t)\,e^{-st}\;\;dt$$

$$f(t) = e^{at} \quad \Rightarrow \quad \mathcal{L}\,(e^{at}) = \int_0^\infty (e^{at})\,e^{-st}\,dt = \int_0^\infty e^{at-st}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

$$\Rightarrow \mathcal{L}\,(e^{at}) = \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^{\infty} = \frac{1}{a-s}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{a-s}\left(\frac{1}{e^{\infty}} - 1\right) = \frac{1}{s-a}(0-1) = \frac{1}{s-a}$$

Thus, the **Laplace transform** of $f(t) = e^{\pm at}$ is:

$$\mathcal{L}\,(e^{\pm at}) = \frac{1}{s \mp a} \qquad a = 1, 2, 3, \cdots, \infty$$

$$\mathcal{L}\,(e^{-2t}) = \frac{1}{s+2}\ , \qquad \mathcal{L}\,(10) = \frac{10}{s}\ , \qquad \mathcal{L}\,(23) = \frac{23}{s}\ , \qquad \ldots..$$

<u>Examples are</u>:

$$\mathcal{L}\,(e^{2t}) = \frac{1}{s-2} \quad , \quad \mathcal{L}\,(e^{-3t}) = \frac{1}{s+3} \quad , \qquad \ldots\ldots$$

<u>Example</u> 4: Find $\mathcal{L}\,[sin(at)]$

<u>Solution</u>:

$$F(s) = \mathcal{L}\,[f(t)] = \int_0^{\infty} f(t)\, e^{-st} \; dt$$

$$f(t) = sin(at) \Rightarrow \mathcal{L}\,[sin(at)] = \int_0^{\infty} sin(at)\, e^{-st}\, dt$$

$$= \lim_{n \to \infty} \int_0^{n} e^{-st} sin(at)\, dt$$

Now, if **integrate by parts** yields,

$$\int u\, dv = uv - \int v\, du$$

$$\int_0^{n} e^{-st} sin(at) dt$$

$$= \left| \begin{array}{c} u = e^{-st} \qquad \Rightarrow \qquad du = -s\, e^{-st}\, dt \\ dv = sin\,(at)dt \;\Rightarrow\; v = \displaystyle\int_0^{n} sin\,(at)dt \\ \Rightarrow \; v = -\left.\dfrac{cos\,(at)}{a}\right|_0^{n} = -\left[\dfrac{cos\,(an)}{a} - \dfrac{cos\,0}{a}\right] \\ \Rightarrow \; v = -\dfrac{cos\,(an)}{a} + \dfrac{1}{a} = \dfrac{1}{a}[1 - cos\,(an)] \end{array} \right|$$

$$= e^{-sn}\frac{1}{a}[1-\cos(an)] - \int_{0}^{n} \frac{1}{a}[1-\cos(an)](-s\,e^{-st}\,dt)$$

$$= \frac{1}{a}\,e^{-sn}\,[1-\cos(an)] + \frac{s}{a}\int_{0}^{n} e^{-st}[1-\cos(an)]\,dt$$

$$= -\frac{1}{a}\,e^{-sn}\,[1-\cos(an)] + \frac{s}{a}\int_{0}^{n} e^{-st}\,dt - \frac{s}{a}\int_{0}^{n} e^{-st}\cos(an)\,dt$$

$$\frac{s}{a}\int_{0}^{n} e^{-st}\,dt = \frac{s}{a}\left[-\frac{1}{s}e^{-st}\right]_{0}^{n} = -\frac{1}{a}(e^{-sn}-e^{0}) = \frac{1}{a}(e^{-sn}-1)$$

$$\int_{0}^{n} e^{-st}\cos(an)\,dt = \left|\begin{array}{c} u = e^{-st} \implies du = -s\,e^{-st}\,dt \\ dv = \cos(an)dt \implies v = \int_{0}^{n}\cos(at)dt \\ \implies v = \left.\dfrac{\sin(at)}{a}\right|_{0}^{n} = \dfrac{\sin(an)}{a} + \dfrac{\sin 0}{a} \\ \implies v = \dfrac{\sin(an)}{a} \end{array}\right|$$

$$= e^{-st}\,\frac{\sin(an)}{a} - \int_{0}^{n}\frac{\sin(an)}{a}(-s\,e^{-st}\,dt)$$

$$= \frac{1}{a}e^{-st}\sin(an) + \frac{s}{a}\int_{0}^{n} e^{-st}\sin(an)\,dt$$

$$\implies \int_{0}^{n} e^{-st}\sin(at)dt =$$

$$= -\frac{1}{a}\,e^{-sn}\,[1-\cos(an)] + \frac{1}{a}(e^{-sn}-1) - \frac{s}{a}e^{-st}\sin(an)$$

$$-\frac{s^2}{a^2}\int_{0}^{n} e^{-st}\sin(an)\,dt$$

$$= -\frac{1}{a}\,e^{-sn} + \frac{1}{a}\,e^{-sn}\cos(an) + \frac{1}{a}e^{-sn} - \frac{1}{a} - \frac{s}{a}e^{-st}\sin(an)$$

$$-\frac{s^2}{a^2}\int_{0}^{n} e^{-st}\sin(an)\,dt$$

$$= \frac{1}{a}\,e^{-sn}\cos(an) - \frac{1}{a} - \frac{s}{a}e^{-st}\sin(an) - \int_{0}^{n} e^{-st}\cos(an)\,dt$$

$\Rightarrow \mathcal{L}\,[sin(at)] =$

$$= \lim_{n\to\infty}\left[\frac{1}{a}\, e^{-sn} cos\,(an) - \frac{1}{a} - \frac{s}{a} e^{-st} sin\,(an) - \int_0^n e^{-st} cos\,(an)\, dt\right]$$

Now, notice that in the **limits** it **had to assume** that $s > 0$ in order **to do** the following **two limits:**

$$\lim_{n\to\infty} e^{-sn}\, cos(an) = 0$$

$$\lim_{n\to\infty} e^{-sn}\, sin(an) = 0$$

Without **this assumption**, we get a **divergent integral again**. Also, note that when **got back** to the **integral** it just **converted** the **upper limit back to infinity**. The **reason** for this is that, if you think about it, **this integral** is **nothing more than** the **integral** that **started with**. Therefore, now get,

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}\, F(s)$$

Now, **simply solve** for $F(s)$ to get,

$$\mathcal{L}\,[sin(at)] = F(s) = \frac{a}{s^2 + a^2}$$

<u>Example</u> 5: Find $\mathcal{L}\,(sinhat)$

<u>Solution</u>:

$$sinhx = \frac{e^x - e^{-x}}{2} \quad , \qquad coshx = \frac{e^x + e^{-x}}{2}$$

substituting, yields

$$\mathcal{L}\,(sinhat) = \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2}\mathcal{L}(e^{at} - e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a-s+a}{(s-a)(s+a)}\right)$$

$$= \frac{1}{2}\left(\frac{2a}{s^2 - a^2}\right) = \frac{a}{s^2 - a^2}$$

Thus,

$$\mathcal{L}\,(sinhat) = \frac{a}{s^2 - a^2} \quad , \quad \mathcal{L}\,(coshat) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}\,(sinat) = \frac{a}{s^2 + a^2} \quad , \quad \mathcal{L}\,(cosat) = \frac{s}{s^2 + a^2}$$

<u>Examples</u> are:

$$\mathcal{L}\,(sinh2t) = \frac{2}{s^2 - 2^2} = \frac{2}{s^2 - 4} \ ,$$

$$\mathcal{L}\,(cosh3t) = \frac{s}{s^2 - 3^2} = \frac{s}{s^2 - 9}$$

$$\mathcal{L}\,(sin4t) = \frac{4}{s^2 + 4^2} = \frac{4}{s^2 + 16} \ ,$$

$$\mathcal{L}\,(cos6t) = \frac{s}{s^2 + 6^2} = \frac{s}{s^2 + 36}$$

<u>Example</u> 6: Find $\mathcal{L}\,(sin^2 t)$

<u>Solution</u>:

$$sin^2 t = \frac{1 - cos2x}{2} \quad , \quad cos^2 t = \frac{1 + cos2x}{2}$$

$$\mathcal{L}\,(sin^2 t) = \mathcal{L}\left(\frac{1 - cos2x}{2}\right) = \frac{1}{2}\,\mathcal{L}\,(1 - cos2t)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 2^2}\right)$$

$$\Longrightarrow \quad \mathcal{L}\,(sin^2 t) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 4}\right)$$

<u>Example</u> 7: Find $\mathcal{L}\,(t^n)$

<u>Solution</u>:

$$\mathcal{L}\,(t^n) = \frac{n!}{s^{n+1}} \quad , \qquad n = 1, 2, 3, \cdots, \infty$$

Examples are:

$$\mathcal{L}\,(t) = \mathcal{L}\,(t^1) = \frac{1!}{s^2} = \frac{1}{s^2} \ ,$$

$$\mathcal{L}\,(t^2) = \frac{2!}{s^3} = \frac{2 \times 1}{s^3} = \frac{2}{s^3} \, ,$$

$$\mathcal{L}\,(t^3) = \frac{3!}{s^4} = \frac{3 \times 2 \times 1}{s^4} = \frac{6}{s^4}$$

Example 8: Find $\mathcal{L}\,(t^2+2)^2$

Solution:

$$\mathcal{L}(t^4+4t^2+4)=\frac{4!}{s^5}+4\frac{2!}{s^3}+\frac{4}{s}$$

$\mathcal{L}\,(k)$	$=\dfrac{k}{s}$, $k=1,2,3,\cdots,\infty$
$\mathcal{L}\,(e^{\pm at})$	$=\dfrac{1}{s\mp a}$, $a=1,2,3,\cdots,\infty$
$\mathcal{L}\,(sinat)$	$=\dfrac{a}{s^2+a^2}$, $a=1,2,3,\cdots,\infty$
$\mathcal{L}\,(cosat)$	$=\dfrac{s}{s^2+a^2}$, $a=1,2,3,\cdots,\infty$
$\mathcal{L}\,(sinhat)$	$=\dfrac{a}{s^2-a^2}$, $a=1,2,3,\cdots,\infty$
$\mathcal{L}\,(coshat)$	$=\dfrac{s}{s^2-a^2}$, $a=1,2,3,\cdots,\infty$
$\mathcal{L}\,(t^n)$	$=\dfrac{n!}{s^{n+1}}$, $n=1,2,3,\cdots,\infty$

Example 9:

$$\mathcal{L}\,[e^{\pm at}\,f(t)]=\mathcal{L}\,f(t)]_{s\to s\mp a}$$

$$\mathcal{L}\,[e^{3t}\,sin(2t)]=\mathcal{L}\,sin(2t)]_{s\to s-3}=\left.\frac{2}{s^2+2^2}\right]_{s\to s-3}=\frac{2}{(s-3)^2+4}$$

$$\Longrightarrow\quad \mathcal{L}\,[e^{3t}\,sin(2t)]=\frac{2}{s^2-6s+9+4}=\frac{2}{s^2-6s+13}$$

Example 10:

$$\mathcal{L}\,[e^{-at}\,f(t)]=\mathcal{L}\,f(t)]_{s\to s+a}$$

1. **Find** $\mathcal{L}\,[e^{-t}\,cos^2(2t)]$

Solution:

$$\mathcal{L}\,[e^{-t}\,cos^2(2t)] = \mathcal{L}\,cos^2(2t)]_{s\to s+1} = \mathcal{L}\,\frac{1+cos4t}{2}\Big]_{s\to s+1}$$

$$= \frac{1}{2}\mathcal{L}\,[1+cos4t]_{s\to s+1} = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4^2}\right]_{s\to s+1}$$

$$= \frac{1}{2}\left[\frac{1}{s+1}+\frac{s}{(s+1)^2+16}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+1}+\frac{s}{s^2+2s+1+16}\right]$$

$$\Rightarrow \quad \mathcal{L}\,[e^{-t}\,cos^2(2t)] = \frac{1}{2}\left[\frac{1}{s+1}+\frac{s}{s^2+2s+17}\right]$$

2. **Find** $\mathcal{L}\,[sinht\;cos(2t)]$

Solution:

$$sinht = \frac{e^x - e^{-x}}{2}$$

$$\Rightarrow \mathcal{L}\,[sinht\;cos(2t)] = \mathcal{L}\left[\frac{e^t - e^{-t}}{2}\;cos(2t)\right]$$

$$= \frac{1}{2}\mathcal{L}\,[(e^t - e^{-t})\;cos(2t)] = \frac{1}{2}\mathcal{L}\,[e^t cos(2t) - e^{-t}cos(2t)]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+2^2}\right)_{s\to s-1} - \left(\frac{s}{s^2+2^2}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{s-1}{(s-1)^2+2^2}\right) - \left(\frac{s+1}{(s+1)^2+2^2}\right)\right]$$

Homework:

1. $\mathcal{L}\,(cosht\;cos^2t)$
2. $\mathcal{L}\,(sinh3t\;cosh2t)$
3. $\mathcal{L}\,(sin^2t\;cos^2t)$

Example 11:

$$\mathcal{L}\,[t^n\,f(t)] = (-1)^n\frac{d^n}{ds^n}\;\mathcal{L}\,f(t)$$

$$\mathcal{L}\,[t\;sin(2t)] = (-1)^1\frac{d^1}{ds^1}\;\mathcal{L}(sin2t)$$

$$= -\frac{d}{ds}\,\frac{2}{s^2+2^2} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -\,\frac{0-2s\times s}{(s^2+4)^2} = \frac{4s^2}{(s^2+4)^2}$$

<u>Example</u> 12:

$$\mathcal{L}\,[t^3\,e^{-2t})] = (-1)^3 \frac{d^3}{ds^3}\,\mathcal{L}(e^{-2t}) = -\frac{d^3}{ds^3}\,\frac{1}{s+2} = -\frac{d^3}{ds^3}\,(s+2)^{-1}$$

$$= -\frac{d^2}{ds^2}\,(-1)\,(s+2)^{-2}$$

$$= -\frac{d}{ds}\,(2)\,(s+2)^{-3} = -(-6)\,(s+2)^{-4}$$

$$= \frac{6}{(s+4)^4}$$

or

$$\mathcal{L}\,[t^3\,e^{-2t})] = \mathcal{L}\,t^3]_{s\to s+2} = \left.\frac{3!}{s^4}\right]_{s\to s+2} = \frac{3\times 2\times 1}{(s+2)^4} = \frac{6}{(s+2)^4}$$

<u>Example</u> 13:

$$\mathcal{L}\,[t^2\,cos2t)] = (-1)^2 \frac{d^2}{ds^2}\,\mathcal{L}(cos2t) = -\frac{d^2}{ds^2}\left(\frac{s}{s^2+2^2}\right)$$

$$= -\frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right) = -\frac{d}{ds}\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{(-2s)(s^2+4)^2 - 2(s^2+4)(2s)(4-s^2)}{(s^2+4)^4}$$

$$= \frac{(-2s)(s^2+4)[(s^2+4)+2(4-s^2)]}{(s^2+4)^4}$$

$$= \frac{(-2s)[s^2+4+8-2s^2]}{(s^2+4)^3} = \frac{(-2s)(12-s^2)}{(s^2+4)^3}$$

<u>Example</u> 14:

$$\mathcal{L}\,[4t\,e^{3t}\,sin2t)] = 4\mathcal{L}\,[t\,e^{3t}\,sin2t)] = 4(-1)^1 \frac{d^1}{ds^1}\,\mathcal{L}(e^{3t}\,sin2t)$$

$$= -\,4\frac{d}{ds}\,\mathcal{L}[\,sin2t]_{s\to s-3} = -4\frac{d}{ds}\left[\frac{2}{s^2-2^2}\right]_{s\to s-3}$$

$$= -4\frac{d}{ds}\left[\frac{2}{(s-3)^2-4}\right] = -4\frac{d}{ds}\left[\frac{2}{s^2-6s+9-4}\right]$$

$$= -4\frac{d}{ds}\left[\frac{2}{s^2-6s+5}\right] = -4\left[\frac{0-2(2s-6)}{(s^2-6s+5)^2}\right]$$

$$= \frac{16(s-3)}{(s^2-6s+5)^2}$$

<u>Example</u> 15:

$$\mathcal{L}\frac{f(t)}{t} = \int_s^\infty \mathcal{L}f(t)\ ds$$

$$\mathcal{L}\frac{sint}{t} = \int_s^\infty \mathcal{L}sint\ ds = \int_s^\infty \frac{1}{s^2+1^2}\ ds = [tan^{-1}]_s^\infty$$

$$= tan^{-1}\infty - tan^{-1}s = \frac{\pi}{2} - tan^{-1}s$$

<u>Example</u> 16:

$$\mathcal{L}\left(\frac{e^{3t}-e^{2t}}{t}\right) = \int_s^\infty \mathcal{L}(e^{3t}-e^{2t})\ ds$$

$$= \left(\int_s^\infty \frac{1}{s-3} - \frac{1}{s-2}\right) ds = [ln(s-3) - l(s-2)]_s^\infty$$

$$= \left[ln\frac{(s-3)}{(s-2)}\right]_s^\infty$$

$$= \ ln\frac{(\infty-3)}{\infty-2} - ln\frac{(s-3)}{s-2} = ln1 - ln\frac{(s-3)}{s-2}$$

$$= -ln\frac{(s-3)}{s-2} \quad or \ \ ln\frac{(s-2)}{(s-3)}$$

<u>Properties of Laplace Transform</u>

i. <u>Linearity</u>

$$ax + by \quad \Leftrightarrow \quad aX + bY$$

<u>proof</u>:

$$\mathcal{L}\,[ax(t)+by(t)] = \int_0^\infty [ax(t)+by(t)]\, e^{-st}\, dt$$

$$= a\int_0^\infty x(t)\, e^{-st}\, dt + b\int_0^\infty y(t)\, e^{-st}\, dt = aX(s)+bY(s)$$

ii. Scaling

$$x(at) \;\Leftrightarrow\; \frac{1}{|a|}X\left(\frac{s}{a}\right) \quad , \quad x(at) \;\Leftrightarrow\; \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

proof:

$$\mathcal{L}\,[x(at)] = \int_0^\infty [x(at)]\, e^{-st}\, dt$$

$$= \int_0^\infty x(\tau)\, e^{-s\frac{\tau}{a}}\, \frac{d\tau}{|a|} = \int_0^\infty x(\tau)\, e^{-\frac{s}{a}\tau}\frac{d\tau}{|a|} = \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

iii. Modulation in Time / Time – Shift

$$\mathcal{L}\,[x(t-t_0)\, u(t-t_0)] = X(s)\, e^{-st_0}$$

proof:

$$\mathcal{L}\,[x(t-t_0)\, u(t-t_0)] = \int_0^\infty \left[\mathcal{L}\,[x(t-t_0)]\right] e^{-st}\, dt$$

$$= \int_0^\infty x(\tau)\, e^{-s(\tau+t_0)}\, d\tau$$

$$= e^{-st_0}\int_0^\infty x(\tau)\, e^{-s\tau}d\tau = e^{-st_0}X(s)$$

iv. Modulation in Frequency / Frequency – Shift

$$e^{at}x(t) \;\Leftrightarrow\; X(s-a)$$

proof:

$$\mathcal{L}\,[x(t)\,e^{at}] = \int_0^\infty [x(t)\,e^{at}]\,e^{-st}\,dt = \int_0^\infty x(t)\,e^{at}\,e^{-st}\,dt$$

$$= X(s-a)$$

v. Time – Reverse

$$x(-t) \;\Leftrightarrow\; X(-s)$$

proof:

$$\mathcal{L}\,[x(-t)] = \int_0^\infty [x(-t)]\,e^{-st}\,dt = \int_0^\infty x(\tau)\,e^{st}\,d\tau$$

$$= \int_0^\infty x(\tau)\,e^{-(-st)}\,d\tau = X(-s)$$

vi. Time Differentiation

$$\frac{d}{dt}x(t) \;\Leftrightarrow\; s\,X(s) - X(0)$$

proof:

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = \int_0^\infty \frac{dx(t)}{dt}\,e^{-st}\,dt = \int_0^\infty e^{st}\,dx(t)$$

$$= \underbrace{[e^{st}\,x(t)]_0^\infty}_{x(0)} - \underbrace{\int_0^\infty x(t)\,de^{-st}}_{sX(s)} = sX(s) - x(0)$$

Remark. Repeat,

$$\mathcal{L}\left[\frac{d}{dt}\left(\frac{dx(t)}{dt}\right)\right] = s\,\mathcal{L}\left(\frac{dx(t)}{dt}\right) - \left.\frac{dx(t)}{dt}\right|_{t=0}$$

$$= s\,[s\,X(s) - x(0)] - x(0)$$

$$= s^2X(s) - s\,x(0) - x(0)$$

The **general form**, which can be **prove** by **Mathematical Induction**, is:

$$\mathcal{L}\left[\frac{d^n x(t)}{dt^n}\right] = s^n\, X(s) - s^{n-1}x(0) - \cdots - x^{n-1}(0)$$

vii. Time Integration

$$\int_0^t x(\tau)\, d\tau \;\Leftrightarrow\; \frac{X(s)}{s}$$

proof:

$$\mathcal{L}\left[\int_0^t x(\tau)\, d\tau\right] = \int_0^\infty \left[\int_0^t x(\tau)\, d\tau\right] e^{-st}\, dt$$

$$= \int_0^\infty \left[\int_0^t x(\tau)\, d\tau\right] \frac{de^{-st}}{-s} dx(t)$$

Integrating by parts:

$$\underbrace{\left[\frac{e^{-st}\int_0^t x(\tau)\, d\tau}{-s}\right]_0^\infty}_{=0} + \frac{1}{s}\int_0^\infty e^{-st}\, d\left[\int_0^t x(\tau)\, d\tau\right]$$

By fundamental Theorem of Calculus

$$\frac{d}{dt}\left[\int_0^t x(\tau)\, d\tau\right] = x(t) \quad \Leftrightarrow \quad d\left[\int_0^t x(\tau)\, d\tau\right] = x(t)dt$$

The Laplace Theorem then becomes

$$\Rightarrow \quad \mathcal{L}\left[\int_0^t x(\tau)\, d\tau\right] = \frac{1}{s}\int_0^\infty e^{-st}\, x(t)\, dt = \frac{X(s)}{s}$$

The Inverse Laplace Transform

Given the **Laplace Transform** $\boldsymbol{F(s)}$**,** it can **be obtained** the **corresponding time domain function** $\boldsymbol{f(t)}$ **via the Inverse Laplace Transform**:

$$\boldsymbol{f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\ e^{st} ds}$$

This is **quite tedious** to apply **in practice**. Instead, it can be simplified by using the Laplace Transform tables to obtain the corresponding functions.

Example: what is

$$\boldsymbol{\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right]} = ?$$

Solution:

$$\boldsymbol{\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{(s+1)}\right]}$$

$$\boldsymbol{\frac{1}{s(s+1)} = \quad \frac{A}{s} + \frac{B}{s+1} \qquad \times s(s+1)}$$

$$\boldsymbol{1 = \quad A(s+1) + Bs \quad \Rightarrow \quad 1 = A + s(A+B)}$$

$$\boldsymbol{\Rightarrow \quad A = 1 \ , A + B = 0 \quad \Rightarrow \quad B = -A = -1}$$

$$\boldsymbol{\Rightarrow \quad \frac{1}{s(s+1)} = \quad \frac{1}{s} - \frac{1}{s+1}}$$

$$\boldsymbol{\Rightarrow \quad \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = 1(t) - e^{-t}}$$

In this example, we "**broke up**" the **function** $\boldsymbol{\frac{1}{s(s+1)}}$ into a **sum** of **simpler functions**, and **then applied** the **inverse Laplace Transform** (by **consulting** the **Laplace Transform table**) to **each of them**. This is a **general technique** for **inverting Laplace Transforms**.

<u>Note</u>: If the **denominator** is a **second degree** or **more**, **provided that** it is **more than a term**, then the **fraction partition method** has to be **used.**

This is **illustrated** in the **next example**

<u>Example</u>: Evaluate

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right]$$

<u>Solution</u>:

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{A(s+3)+Bs}{s+3} = \frac{3A+s(A+B)}{s(s+3)}$$

$$A = \frac{1}{3} \quad \text{and} \quad B = -\frac{1}{3}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3}\mathcal{L}^{-1}\frac{1}{s} - \frac{1}{3}\mathcal{L}^{-1}\frac{1}{s+3} = \frac{1}{3}(1) - \frac{1}{3}e^{-3t}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3}(1-e^{-3t})$$

Some **<u>examples</u>** are:

1. $\mathcal{L}^{-1}\left(\frac{3}{s^2-9}\right) = sinh3t$

2. $\mathcal{L}^{-1}\left(\frac{5!}{s^6}\right) = t^5$

3. $\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{3!}\mathcal{L}^{-1}\frac{3!}{s^4} = \frac{1}{3!}t^3$

4. $\mathcal{L}^{-1}\left(\frac{5}{s^2+25}\right) = sin5t$

5. $\mathcal{L}^{-1}\left(\frac{20}{s^2+25}\right) = \frac{20}{5}\mathcal{L}^{-1}\left(\frac{5}{s^2+25}\right) = 4sin5t$

$$6.\ \mathcal{L}^{-1}\left(\frac{1}{s+5}-\frac{3}{s^3}\right)=e^{-5t}-3\mathcal{L}^{-1}\left(\frac{1}{s^3}\right)=e^{-5t}-\frac{3}{2!}\mathcal{L}^{-1}\left(\frac{2!}{s^3}\right)$$

$$=e^{-5t}-\frac{3}{2!}\mathcal{L}^{-1}(t^2)$$

$$7.\ \mathcal{L}^{-1}\left(\frac{s-6}{s^2+4}\right)=\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)-6\,\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$=cos2t-\frac{6}{2}\,\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)=cos2t-3sin2t$$

$$8.\ \mathcal{L}^{-1}\left[\frac{1}{s^3+6s^2+5s}\right]$$

$$s^3+6s^2+5s=s(s^2+6s+5)=s(s+1)(s+5)$$

$$\frac{1}{s^3+6s^2+5s}=\frac{1}{s(s+1)(s+5)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+5}$$

$$A=\frac{1}{(0+1)(0+5)}=\frac{1}{5}\ ,$$

$$B=\frac{1}{(-1)(-1+5)}=-\frac{1}{4}\ ,$$

$$C=\frac{1}{(-5)(-5+1)}=\frac{1}{20}$$

$$\Rightarrow\mathcal{L}^{-1}\left[\frac{1}{s^3+6s^2+5s}\right]=\frac{1}{5}\mathcal{L}^{-1}\frac{1}{s}-\frac{1}{4}\mathcal{L}^{-1}\frac{1}{s+1}+\frac{1}{20}\mathcal{L}^{-1}\frac{1}{s+5}$$

$$=\frac{1}{5}(1)-\frac{1}{4}e^{-t}+\frac{1}{20}e^{-5t}$$

$$9.\ \mathcal{L}^{-1}\left[\frac{s^2+2s+4}{s(s+3)^3}\right]$$

$$\frac{s^2+2s+4}{s(s+3)^3}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{(s+3)^2}+\frac{D}{(s+3)^3}$$

$$A=\frac{0\pm0+4}{(0+3)^3}=\frac{4}{27}\ ,$$

$$C_k = \frac{1}{(m+k)!}\frac{d^{m-k}}{ds^{m-k}}\left[(s-a)^m F(s)\right] ,$$

where, m − **current derivative** and **k** − **Repetition number**

$$B = \frac{1}{(3-1)!}\frac{d^{3-1}}{ds^{3-1}}\left[(s+3)^3 \times \frac{s^2+2s+4}{s(s+3)^3}\right]$$

$$= \frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{s^2+2s+4}{s}\right] = \frac{1}{2}\frac{s}{ds}\left[\frac{s(2s+2)-s^2-2s-4}{s^2}\right]$$

$$= \frac{1}{2}\frac{s}{ds}\left[\frac{2s^2+2s-s^2-2s-4}{s^2}\right] = \frac{1}{2}\frac{s}{ds}\left[\frac{s^2-4}{s^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s \times s^2 - 2s(s^2-4)}{s^4}\right] = \frac{1}{2}\left[\frac{2s^3-2s^3+8s}{s^4}\right] = \frac{1}{2}\frac{8s}{s^4} = \frac{4}{s^3}$$

$$\Rightarrow \quad B = \frac{4}{s^3} = \frac{4}{(-3)^3} = -\frac{4}{27}$$

$$C = \frac{1}{(3-2)!}\frac{d^{3-2}}{ds^{3-2}}\left[\frac{s^2+2s+4}{s}\right] = \frac{d}{ds}\left[\frac{s^2+2s+4}{s}\right]$$

$$= \left[\frac{(2s+2)s-s^2-2s-4}{s}\right] = \frac{2s^2+2s-s^2-2s-4}{s^2} = \frac{s^2-4}{s^2}$$

$$\Rightarrow \quad C = \frac{(-3)^2-4}{(-3)^2} = \frac{5}{9}$$

$$D = \frac{1}{(3-3)!}\frac{d^{3-3}}{ds^{3-3}}\left[\frac{s^2+2s+4}{s}\right] = \left[\frac{s^2+2s+4}{s}\right]$$

$$0! = 1 \Rightarrow \quad D = \frac{(-3)^2+2(-3)+4}{(-3)} = \frac{9-6+4}{-3} = -\frac{7}{3}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{s^2+2s+4}{s(s+3)^3}\right] = \mathcal{L}^{-1}\left[\frac{4/27}{s} + \frac{-4/27}{s+3} + \frac{5/9}{(s+3)^2} + \frac{-7/3}{(s+3)^3}\right]$$

$$= \frac{4}{27}\mathcal{L}^{-1}\frac{1}{s} - \frac{4}{27}\mathcal{L}^{-1}\frac{1}{s+3} + \frac{5}{9}\mathcal{L}^{-1}\frac{1}{(s+3)^2} - \frac{7}{3}\mathcal{L}^{-1}\frac{1}{(s+3)^3}$$

$$= \frac{4}{27}(1) - \frac{4}{27}e^{-3t} + \frac{5}{9}\mathcal{L}^{-1}\frac{1}{(s+3)^2} - \frac{7}{3}\mathcal{L}^{-1}\frac{1}{(s+3)^3}$$

$\mathcal{L}^{-1}\frac{1}{(s+3)^2}$ and $\mathcal{L}^{-1}\frac{1}{(s+3)^3}$

$$\mathcal{L}^{-1}(s \pm a)^n = e^{\mp at}\,\mathcal{L}^{-1}s^n$$

$$\mathcal{L}^{-1}(s-5)^2 = e^{5t}\,\mathcal{L}^{-1}s^2$$

$$\mathcal{L}^{-1}\frac{1}{(s+3)^2} = e^{-3t}\,\mathcal{L}^{-1}\frac{1}{s^2} = e^{-3t}.t$$

$$\mathcal{L}^{-1}\frac{1}{(s+3)^3} = e^{-3t}\,\mathcal{L}^{-1}\frac{1}{s^3} = e^{-3t}.\frac{1}{2!}t^2$$

$$\Longrightarrow \mathcal{L}^{-1}\left[\frac{s^2+2s+4}{s(s+3)^3}\right] = \frac{4}{27} - \frac{4}{27}e^{-3t} + \frac{5}{9}e^{-3t}.t - \frac{7}{3}e^{-3t}.t^2$$

❖ Applications of Laplace Transform

The **Laplace Transform** has **many applications**. **Two** of the **most important** are the **solution** of:

- **Differential equations** and
- **Convolution**.

1. Solving Differential Equations using Laplace transforms

The **Laplace Transform** can **greatly simplify** the **solution** of **problems involving differential equations**.

The **Laplace transform** can be employed to **solve constant coefficient ordinary differential equations**. In particular **initial value** problems considered, and it shall be found that the **initial conditions** are **automatically included** as **part** of the **solution process**.

The **idea** is **simple**; the **Laplace transform** of **each term** in the **differential equation** is **taken**.

If the **unknown function** is $\boldsymbol{y(t)}$ then, **on taking** the **Laplace transform**, an **algebraic equation** involving $\boldsymbol{Y(s) = \mathcal{L}[y(t)]}$ is **obtained**. This **equation** is **solved** for $\boldsymbol{Y(s)}$ which is **then inverted** to **produce** the **required solution** $\boldsymbol{y(t) = \mathcal{L}^{-1}[Y(s)]}$.

Begin with a **straightforward initial value problem involving** a **<u>first order constant coefficient differential equation</u>**. Let's using the **Laplace transform approach** to **find** the **solution** of:

$$\boldsymbol{\frac{dy}{dt} + 2y = 12e^{3t} \qquad y(0) = 3}$$

Let's write

$$\boldsymbol{\mathcal{L}[y(t)] = Y(s)}$$

Then, **taking** the **Laplace transform** of **every term** in the **differential equation** gives:

$$\boldsymbol{\mathcal{L}\left[\frac{dy}{dt}\right] + \mathcal{L}[2y] = \mathcal{L}[12e^{3t}]}$$

$$\boldsymbol{\mathcal{L}\left[\frac{dy}{dt}\right] = -y(0) + sY(s) = -3 + s\,Y(s)}$$

$\boldsymbol{\mathcal{L}[2y] = 2Y(s)}$ and

$$\boldsymbol{\mathcal{L}[12e^{3t}] = \frac{12}{s-3}}$$

Substituting these **expressions** into the **transformed version** of the **differential equation** gives

$$\boldsymbol{-3 + sY(s) + 2Y(s) = \frac{12}{s-3}}$$

Solving for $\boldsymbol{Y(s)}$ yields

$$\boldsymbol{Y(s)(s+2) = \frac{12}{s-3} + 3 = \frac{3+3s}{s-3}}$$

Therefore, $$\boldsymbol{Y(s) = \frac{3(s+1)}{(s+2)(s-3)}}$$

Now, **using partial fractions**, this **last expression can be written** in a **more convenient form**:

$$\boldsymbol{Y(s) = \frac{3/5}{(s+2)} + \frac{12/5}{(s-3)}}$$

and then, taking inverse of the Laplace transform, yields

$$\boldsymbol{y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{3}{5}\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right] + \frac{12}{5}\mathcal{L}^{-1}\left[\frac{1}{(s-3)}\right]}$$

thus

$$\boldsymbol{y(t) = \frac{3}{5}e^{-2t} + \frac{12}{5}e^{3t}}$$

This is the **solution** to the **given initial value problem**.

<u>Example</u> 1: Solve the **second-order initial-value problem**

$$\boldsymbol{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t} \qquad y(0) = 0\,, \qquad y'(0) = 0}$$

using the **Laplace transform method**.

<u>Solution</u>:

Taking the **Laplace transform** of **each term**:

$$\boldsymbol{\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 2\mathcal{L}\left[\frac{dy}{dt}\right] + 2\mathcal{L}[y] = \mathcal{L}[e^{-t}]}$$

$$\boldsymbol{\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = -y'(0) - s\,y(0) + s^2\,Y(s) = s^2\,Y(s)}$$

$$\boldsymbol{2\mathcal{L}\left[\frac{dy}{dt}\right] = 2[-y(0) + s\,Y(s)] = 2sY(s)}$$

$\boldsymbol{\mathcal{L}[2y] = 2Y(s)}$ and

$$\boldsymbol{\mathcal{L}[e^{-t}] = \frac{1}{s+1}}$$

$$\boldsymbol{\Rightarrow \quad s^2\,Y(s) + 2sY(s) + 2Y(s) = \frac{1}{s+1}}$$

Solving for $\boldsymbol{Y(s)}$ yields

$$\boldsymbol{Y(s)(s^2+2s+2) = \frac{1}{s+1}}$$

$$\Longrightarrow \quad \boldsymbol{Y(s) = \frac{1}{(s+1)(s^2+2s+2)}}$$

Using partial fraction:

$$\boldsymbol{\frac{1}{(s+1)(s^2+2s+2)} \equiv \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}}$$

$$\boldsymbol{\equiv \frac{A(s^2+2s+2)+(Bs+C)(s+1)}{(s+1)(s^2+2s+2)}}$$

$$\Longrightarrow \quad \boldsymbol{1 \equiv As^2+2As+2A+Bs^2+Bs+Cs+C}$$

$$\Longrightarrow \quad \boldsymbol{1 \equiv (A{+}B)s^2+(2A+B+C)s+2A+C}$$

$$\boldsymbol{A+B=0} \ \Longrightarrow \quad \boldsymbol{A=-B}$$

$$\boldsymbol{2A+B+C=0} \quad \Longrightarrow \quad \boldsymbol{C=B}$$

$$\boldsymbol{2A+C=1} \qquad \Longrightarrow \quad \boldsymbol{2A-A=1}$$

$$\Longrightarrow \quad \boldsymbol{A=1} \ \Longrightarrow \quad \boldsymbol{B=-1} \ \Longrightarrow \quad \boldsymbol{C=-1}$$

$$\Longrightarrow \boldsymbol{\frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{s+1} + \frac{(-1)s-1}{s^2+2s+2}}$$

$$\boldsymbol{= \frac{1}{s+1} - \frac{s+1}{s^2+2s+2}}$$

Also, $\boldsymbol{(s^2+2s+2)}$ can be written as:

$$\boldsymbol{s^2+2s+2 \equiv (s+1)^2+1}$$

$$\Longrightarrow \boldsymbol{\frac{1}{(s+1)(s^2+2s+2)} \equiv \frac{1}{s+1} - \frac{(s+1)}{(s+1)^2+1}}$$

Taking the **Inverse Laplace transform**

$$\boldsymbol{y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]}$$

thus

$$\boldsymbol{y(t) = e^{-t} - e^{-t}cost}$$

Solving systems of differential equations

The **Laplace transform method** is also **well suited** to **solving systems** of **differential equations**. A **simple example** will **illustrate** the **technique**. Let $\boldsymbol{x(t)}$, $\boldsymbol{y(t)}$be **two independent functions** which **satisfy** the **coupled differential equation:**

$$\frac{dx}{dt} + y = e^{-t}$$

$$\frac{dy}{dt} - x = 3e^{-t}$$

$$x(0) = 0 \quad , \quad y(0) = 1$$

Taking the **Laplace transform** of **every term** in the **differential equations** and **use** the **notation**:

$$\mathcal{L}[x(t)] = X(s) \quad \text{and} \quad \mathcal{L}[y(t)] = Y(s)$$

and the **initial conditions** and **rearranging** give

$$\mathcal{L}\left[\frac{dx}{dt}\right] = -x(0) + s\,X(s) = s\,X(s)$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -y(0) + sY(s) = -1 + s\,Y(s)$$

$$\mathcal{L}[x(t)] = X(s)$$

$$\mathcal{L}[y] = Y(s)$$

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}$$

and

$$3\mathcal{L}[e^{-t}] = \frac{3}{s+1}$$

$$sX(s) + Y(s) = \frac{1}{s+1}$$

$$-X(s) - 1 + sY(s) = \frac{3}{s+1} \Rightarrow -X(s) + sY(s) = \frac{s+4}{s+1}$$

these algebraic equations can be solved in $\boldsymbol{X(s)}$ and $\boldsymbol{Y(s)}$ **using a variety** of **techniques** (**inverse matrix; Cramer's determinant method** etc.) Here Cramer's method will be used:

$$X(s) = \frac{\begin{vmatrix} \frac{1}{s+1} & 1 \\ \frac{s+4}{s+1} & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{\frac{1}{s+1} - \frac{s+4}{s+1}}{s^2+1} = \frac{-4}{(s^2+1)(s+1)}$$

$$= \frac{2(s-1)}{s^2+1} - \frac{2}{s+1}$$

$$Y(s) = \frac{\begin{vmatrix} s & \frac{1}{s+1} \\ -1 & \frac{s+4}{s+1} \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{\frac{s(s+4)}{s+1} + \frac{1}{s+1}}{s^2+1} = \frac{s^2+4s+1}{(s^2+1)(s+1)}$$

$$= -\frac{1}{s+1} + \frac{2(s+1)}{s^2+1}$$

The **last lines** in each case **having been obtained** using **partial fractions**. **Taking Inverse Laplace Transform** yields

$$x(t) = \mathcal{L}^{-1}[X(s)] = 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - 2\mathcal{L}^{-1}\left[\frac{s}{s+1}\right]$$

$$\Longrightarrow \quad x(t) = 2\,cost - 2\,sint - 2e^{-t}$$

and

$$y(t) = \mathcal{L}^{-1}[Y(s)] = -\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + 2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]$$

$$\Longrightarrow \quad y(t) = 2\,cost + 2\,sint - e^{-t}$$