Mathematics
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Lecture # 1

Vectors

Introduction

In single-variable calculus, the functions that one encounters are functions of a variable (usually x or t) that varies over some subset of the real number line (which we denote by $\mathbb{R}$). For such a function, say, $y = f(x)$, the **graph** of the function f consists of the points $(x, y) = (x, f(x))$. These points lie in the **Euclidean plane**, which, in the **Cartesian** or **rectangular** coordinate system, consists of all ordered pairs of real numbers (a, b). We use the word "Euclidean" to denote a system in which all the usual rules of Euclidean geometry hold. We denote the Euclidean plane by $\mathbb{R}^2$; the "2" represents the number of *dimensions* of the plane. The Euclidean plane has two perpendicular **coordinate axes**: the x-axis and the y-axis.

In vector (or multivariable) calculus, we will deal with functions of two or three variables (usually x, y or x, y, z, respectively). The graph of a function of two variables, say, $z = f(x, y)$, lies in **Euclidean space**, which in the Cartesian coordinate system consists of all ordered triples of real numbers (a, b, c). Since Euclidean space is 3-dimensional, we denote it by $\mathbb{R}^3$. The graph of f consists of the points $(x, y, z) = (x, y, f(x, y))$. The 3-dimensional coordinate system of Euclidean space can be represented on a flat surface, such as this page or a blackboard, only by giving the illusion of three dimensions, in the manner shown in Figure 1.1.1. Euclidean space has three mutually perpendicular coordinate axes (x, y and z), and three mutually perpendicular coordinate planes: the xy-plane, yz-plane and xz-plane (see Figure 1.1.2).

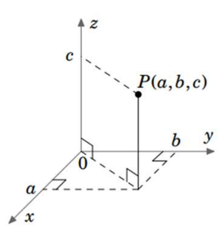


Figure 1.1.1

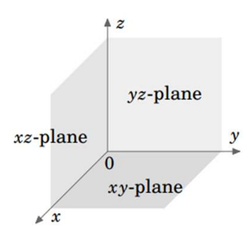


Figure 1.1.2

The coordinate system shown in Figure 1.1.1 is known as a **right-handed coordinate system**, because it is possible, using the right hand, to point the index finger in the positive direction of the x-axis, the middle finger in the positive direction of the y-axis, and the thumb in the positive direction of the z-axis, as in Figure 1.1.3.

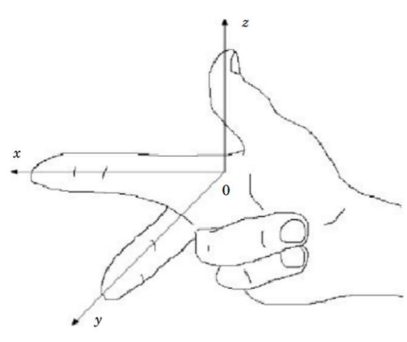


Figure 1.1.3 Right-handed coordinate system

An equivalent way of defining a right-handed system is if you can point your thumb upwards in the positive z-axis direction while using the remaining four fingers to rotate the x-axis towards the y-axis. Doing the same thing with the left hand is what defines a **left-handed coordinate system**. Notice that switching the x- and y-axes in a right-handed system results in a left-handed system, and that rotating either type of system does not change its "handedness". Throughout the book we will use a right-handed system.

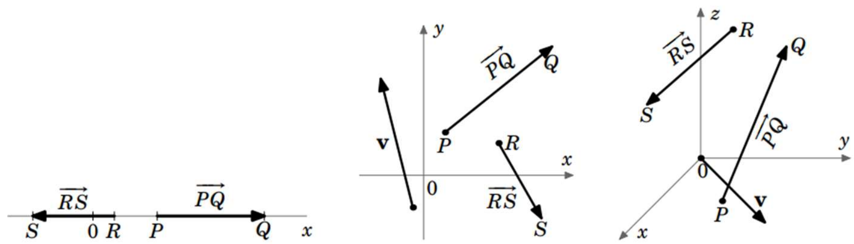
For functions of three variables, the graphs exist in 4-dimensional space (i.e. $\mathbb{R}^4$), which we can not see in our 3-dimensional space, let alone simulate in 2-dimensional space. So we can only think of 4-dimensional space abstractly. For an entertaining discussion of this subject, see the book by ABBOTT.[1]

So far, we have discussed the *position* of an object in 2-dimensional or 3-dimensional space. But what about something such as the velocity of the object, or its acceleration? Or the gravitational force acting on the object? These phenomena all seem to involve motion and *direction* in some way. This is where the idea of a *vector* comes in.

You have already dealt with velocity and acceleration in single-variable calculus. For example, for motion along a straight line, if $y = f(t)$ gives the displacement of an object after time t, then $dy/dt = f'(t)$ is the velocity of the object at time t. The derivative $f'(t)$ is just a number, which is positive if the object is moving in an agreed-upon "positive" direction, and negative if it moves in the opposite of that direction. So you can think of that number, which was called the velocity of the object, as having two components: a *magnitude*, indicated by a nonnegative number, preceded by a *direction*, indicated by a plus or minus symbol (representing motion in the positive direction or the negative direction, respectively), i.e. $f'(t) = \pm a$ for some number $a \geq 0$. Then a is the magnitude of the velocity (normally called the *speed* of the object), and the $\pm$ represents the direction of the velocity (though the + is usually omitted for the positive direction).

For motion along a straight line, i.e. in a 1-dimensional space, the velocities are also contained in that 1-dimensional space, since they are just numbers. For general motion along a curve in 2- or 3-dimensional space, however, velocity will need to be represented by a multi-dimensional object which should have both a magnitude and a direction. A geometric object which has those features is an arrow, which in elementary geometry is called a "directed line segment". This is the motivation for how we will define a vector.

To indicate the direction of a vector, we draw an arrow from its initial point to its terminal point. We will often denote a vector by a single bold-faced letter (e.g. $\mathbf{v}$) and use the terms "magnitude" and "length" interchangeably. Note that our definition could apply to systems with any number of dimensions (see Figure 1.1.4 (a)-(c)).



(a) One dimension **(b)** Two dimensions **(c)** Three dimensions

Figure 1.1.4 Vectors in different dimensions

Definition 1.2. Two nonzero vectors are **equal** if they have the same magnitude and the same direction. Any vector with zero magnitude is equal to the zero vector.

By this definition, vectors with the same magnitude and direction but with different initial points would be equal. For example, in Figure 1.1.5 the vectors **u**, **v** and **w** all have the same magnitude $\sqrt{5}$ (by the Pythagorean Theorem). And we see that **u** and **w** are parallel, since they lie on lines having the same slope $\frac{1}{2}$, and they point in the same direction. So $\mathbf{u} = \mathbf{w}$, even though they have different initial points. We also see that **v** is parallel to **u** but points in the opposite direction. So $\mathbf{u} \neq \mathbf{v}$.

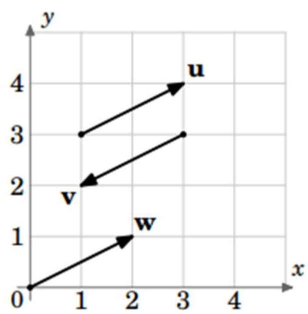


Figure 1.1.5

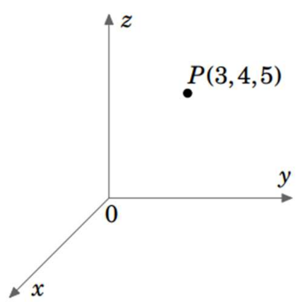
So we can see that there are an infinite number of vectors for a given magnitude and direction, those vectors all being equal and differing only by their initial and terminal points. Is there a single vector which we can choose to represent all those equal vectors? The answer is yes, and is suggested by the vector **w** in Figure 1.1.5.

Unless otherwise indicated, when speaking of "the vector" with a given magnitude and direction, we will mean the one whose initial point is at the origin of the coordinate system.

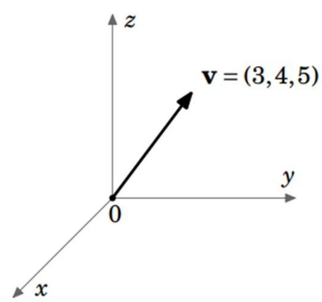
Thinking of vectors as starting from the origin provides a way of dealing with vectors in a standard way, since every coordinate system has an origin. But there will be times when it is convenient to consider a different initial point for a vector (for example, when adding vectors, which we will do in the next section).

Another advantage of using the origin as the initial point is that it provides an easy correspondence between a vector and its terminal point.

Example 1.1. Let $\mathbf{v}$ be the vector in $\mathbb{R}^3$ whose initial point is at the origin and whose terminal point is $(3,4,5)$. Though the *point* $(3,4,5)$ and the vector $\mathbf{v}$ are different objects, it is convenient to write $\mathbf{v} = (3,4,5)$. When doing this, it is understood that the initial point of $\mathbf{v}$ is at the origin $(0,0,0)$ and the terminal point is $(3,4,5)$.



(a) The point (3,4,5)



(b) The vector (3,4,5)

Figure 1.1.6 Correspondence between points and vectors

Example 1.2. Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in $\mathbb{R}^3$, where $P = (2,1,5), Q = (3,5,7), R = (1,-3,-2)$ and $S = (2,1,0)$. Does $\overrightarrow{PQ} = \overrightarrow{RS}$?
Solution: The vector $\overrightarrow{PQ}$ is equal to the vector $\mathbf{v}$ with initial point $(0,0,0)$ and terminal point $Q - P = (3,5,7) - (2,1,5) = (3-2, 5-1, 7-5) = (1,4,2)$.
Similarly, $\overrightarrow{RS}$ is equal to the vector $\mathbf{w}$ with initial point $(0,0,0)$ and terminal point $S - R = (2,1,0) - (1,-3,-2) = (2-1, 1-(-3), 0-(-2)) = (1,4,2)$.
So $\overrightarrow{PQ} = \mathbf{v} = (1,4,2)$ and $\overrightarrow{RS} = \mathbf{w} = (1,4,2)$.
$\therefore \overrightarrow{PQ} = \overrightarrow{RS}$

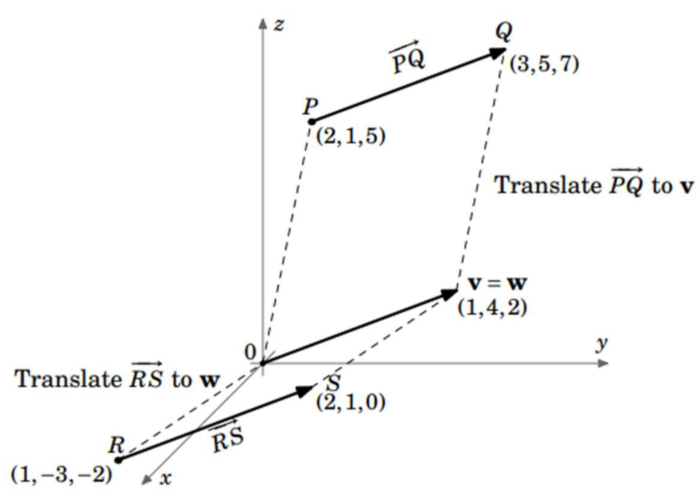


Figure 1.1.7

Recall the distance formula for points in the Euclidean plane:

For points $P=(x_1,y_1)$, $Q=(x_2,y_2)$ in $\mathbb{R}^2$, the distance d between P and Q is:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \tag{1.1}$$

By this formula, we have the following result:

For a vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, the magnitude of $\overrightarrow{PQ}$ is:

$$\|\overrightarrow{PQ}\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \tag{1.2}$$

Finding the magnitude of a vector $\mathbf{v}=(a,b)$ in $\mathbb{R}^2$ is a special case of formula (1.2) with $P=(0,0)$ and $Q=(a,b)$:

For a vector $\mathbf{v}=(a,b)$ in $\mathbb{R}^2$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\|=\sqrt{a^2+b^2} \tag{1.3}$$

To calculate the magnitude of vectors in $\mathbb{R}^3$, we need a distance formula for points in Euclidean space (we will postpone the proof until the next section):

Theorem 1.1. The distance d between points $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$ in $\mathbb{R}^3$ is:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \tag{1.4}$$

The proof will use the following result:

Theorem 1.2. For a vector $\mathbf{v}=(a,b,c)$ in $\mathbb{R}^3$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\|=\sqrt{a^2+b^2+c^2} \tag{1.5}$$

Example 1.3. Calculate the following:

(a) The magnitude of the vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with $P = (-1,2)$ and $Q = (5,5)$.
Solution: By formula (1.2), $\|\overrightarrow{PQ}\| = \sqrt{(5-(-1))^2 + (5-2)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$.

(b) The magnitude of the vector $\mathbf{v} = (8,3)$ in $\mathbb{R}^2$.
Solution: By formula (1.3), $\|\mathbf{v}\| = \sqrt{8^2 + 3^2} = \sqrt{73}$.

(c) The distance between the points $P = (2,-1,4)$ and $Q = (4,2,-3)$ in $\mathbb{R}^3$.
Solution: By formula (1.4), the distance $d = \sqrt{(4-2)^2 + (2-(-1))^2 + (-3-4)^2} = \sqrt{4+9+49} = \sqrt{62}$.

(d) The magnitude of the vector $\mathbf{v} = (5,8,-2)$ in $\mathbb{R}^3$.
Solution: By formula (1.5), $\|\mathbf{v}\| = \sqrt{5^2 + 8^2 + (-2)^2} = \sqrt{25+64+4} = \sqrt{93}$.

Exercises

A

1. Calculate the magnitudes of the following vectors:
(a) $\mathbf{v} = (2,-1)$ (b) $\mathbf{v} = (2,-1,0)$ (c) $\mathbf{v} = (3,2,-2)$ (d) $\mathbf{v} = (0,0,1)$ (e) $\mathbf{v} = (6,4,-4)$

2. For the points $P = (1,-1,1)$, $Q = (2,-2,2)$, $R = (2,0,1)$, $S = (3,-1,2)$, does $\overrightarrow{PQ} = \overrightarrow{RS}$?

3. For the points $P = (0,0,0)$, $Q = (1,3,2)$, $R = (1,0,1)$, $S = (2,3,4)$, does $\overrightarrow{PQ} = \overrightarrow{RS}$?

1.2 Vector Algebra

Now that we know what vectors are, we can start to perform some of the usual algebraic operations on them (e.g. addition, subtraction). Before doing that, we will introduce the notion of a *scalar*.

Definition 1.3. A **scalar** is a quantity that can be represented by a single number.

For our purposes, scalars will always be real numbers.[3] Examples of scalar quantities are mass, electric charge, and speed (not velocity).[4] We can now define *scalar multiplication* of a vector.

Definition 1.4. For a scalar k and a nonzero vector $\mathbf{v}$, the **scalar multiple** of $\mathbf{v}$ by k, denoted by $k\mathbf{v}$, is the vector whose magnitude is $|k|\,\|\mathbf{v}\|$, points in the same direction as $\mathbf{v}$ if $k > 0$, points in the opposite direction as $\mathbf{v}$ if $k < 0$, and is the zero vector $\mathbf{0}$ if $k = 0$. For the zero vector $\mathbf{0}$, we define $k\mathbf{0} = \mathbf{0}$ for any scalar k.

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **parallel** (denoted by $\mathbf{v} \parallel \mathbf{w}$) if one is a scalar multiple of the other. You can think of scalar multiplication of a vector as stretching or shrinking the vector, and as flipping the vector in the opposite direction if the scalar is a negative number (see Figure 1.2.1).

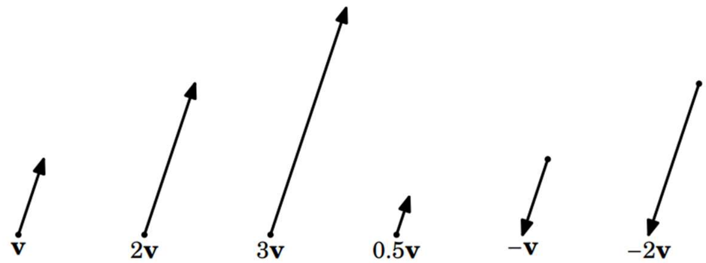


Figure 1.2.1

Recall that **translating** a nonzero vector means that the initial point of the vector is changed but the magnitude and direction are preserved. We are now ready to define the *sum* of two vectors.

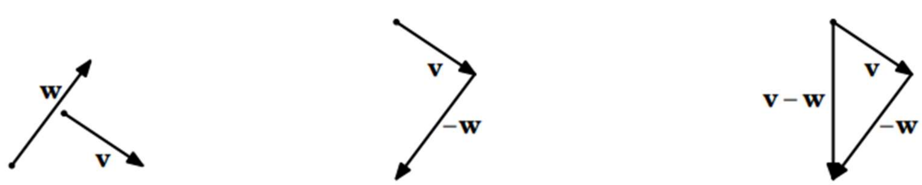
Definition 1.5. The **sum** of vectors **v** and **w**, denoted by $\mathbf{v}+\mathbf{w}$, is obtained by translating **w** so that its initial point is at the terminal point of **v**; the initial point of $\mathbf{v}+\mathbf{w}$ is the initial point of **v**, and its terminal point is the new terminal point of **w**.

Intuitively, adding **w** to **v** means tacking on **w** to the end of **v** (see Figure 1.2.2).

(a) Vectors **v** and **w**   **(b)** Translate **w** to the end of **v**   **(c)** The sum $\mathbf{v}+\mathbf{w}$

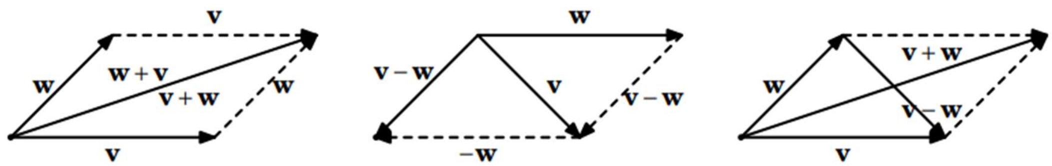
Figure 1.2.2 Adding vectors **v** and **w**

Notice that our definition is valid for the zero vector (which is just a point, and hence can be translated), and so we see that $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$ for any vector **v**. In particular, $\mathbf{0}+\mathbf{0}=\mathbf{0}$. Also, it is easy to see that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$, as we would expect. In general, since the scalar multiple $-\mathbf{v}=-1\,\mathbf{v}$ is a well-defined vector, we can define **vector subtraction** as follows: $\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$. See Figure 1.2.3.



(a) Vectors **v** and **w**   **(b)** Translate −**w** to the end of **v**   **(c)** The difference $\mathbf{v}-\mathbf{w}$

Figure 1.2.4 shows the use of "geometric proofs" of various laws of vector algebra, that is, it uses laws from elementary geometry to prove statements about vectors. For example, (a) shows that $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ for any vectors $\mathbf{v}$, $\mathbf{w}$. And (c) shows how you can think of $\mathbf{v}-\mathbf{w}$ as the vector that is tacked on to the end of $\mathbf{w}$ to add up to $\mathbf{v}$.



(a) Add vectors **(b)** Subtract vectors **(c)** Combined add/subtract

Figure 1.2.4 "Geometric" vector algebra

Notice that we have temporarily abandoned the practice of starting vectors at the origin. In fact, we have not even mentioned coordinates in this section so far. Since we will deal mostly with Cartesian coordinates in this book, the following two theorems are useful for performing vector algebra on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ starting at the origin.

Theorem 1.3. Let $\mathbf{v}=(v_1,v_2)$, $\mathbf{w}=(w_1,w_2)$ be vectors in $\mathbb{R}^2$, and let k be a scalar. Then
(a) $k\mathbf{v}=(kv_1,kv_2)$
(b) $\mathbf{v}+\mathbf{w}=(v_1+w_1,v_2+w_2)$

Theorem 1.4. Let $\mathbf{v}=(v_1,v_2,v_3)$, $\mathbf{w}=(w_1,w_2,w_3)$ be vectors in $\mathbb{R}^3$, let k be a scalar. Then
(a) $k\mathbf{v}=(kv_1,kv_2,kv_3)$
(b) $\mathbf{v}+\mathbf{w}=(v_1+w_1,v_2+w_2,v_3+w_3)$

The following theorem summarizes the basic laws of vector algebra.

Theorem 1.5. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and scalars k,l, we have

(a) $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$	Commutative Law
(b) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$	Associative Law
(c) $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$	Additive Identity
(d) $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$	Additive Inverse
(e) $k(l\mathbf{v})=(kl)\mathbf{v}$	Associative Law
(f) $k(\mathbf{v}+\mathbf{w})=k\mathbf{v}+k\mathbf{w}$	Distributive Law
(g) $(k+l)\mathbf{v}=k\mathbf{v}+l\mathbf{v}$	Distributive Law

Example 1.4. Let $\mathbf{v} = (2, 1, -1)$ and $\mathbf{w} = (3, -4, 2)$ in $\mathbb{R}^3$.

(a) Find $\mathbf{v} - \mathbf{w}$.
Solution: $\mathbf{v} - \mathbf{w} = (2-3, 1-(-4), -1-2) = (-1, 5, -3)$

(b) Find $3\mathbf{v} + 2\mathbf{w}$.
Solution: $3\mathbf{v} + 2\mathbf{w} = (6, 3, -3) + (6, -8, 4) = (12, -5, 1)$

(c) Write $\mathbf{v}$ and $\mathbf{w}$ in component form.
Solution: $\mathbf{v} = 2\,\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{w} = 3\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$

(d) Find the vector $\mathbf{u}$ such that $\mathbf{u} + \mathbf{v} = \mathbf{w}$.
Solution: By Theorem 1.5, $\mathbf{u} = \mathbf{w} - \mathbf{v} = -(\mathbf{v} - \mathbf{w}) = -(-1, 5, -3) = (1, -5, 3)$, by part(a).

(e) Find the vector $\mathbf{u}$ such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.
Solution: By Theorem 1.5, $\mathbf{u} = -\mathbf{w} - \mathbf{v} = -(3, -4, 2) - (2, 1, -1) = (-5, 3, -1)$.

(f) Find the vector $\mathbf{u}$ such that $2\mathbf{u} + \mathbf{i} - 2\mathbf{j} = \mathbf{k}$.
Solution: $2\mathbf{u} = -\mathbf{i} + 2\,\mathbf{j} + \mathbf{k} \Longrightarrow \mathbf{u} = -\frac{1}{2}\,\mathbf{i} + \mathbf{j} + \frac{1}{2}\,\mathbf{k}$

(g) Find the unit vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$.
Solution: $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{2^2+1^2+(-1)^2}}(2, 1, -1) = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

Exercises

A

1. Let $\mathbf{v} = (-1, 5, -2)$ and $\mathbf{w} = (3, 1, 1)$.

(a) Find $\mathbf{v} - \mathbf{w}$. (b) Find $\mathbf{v} + \mathbf{w}$. (c) Find $\frac{\mathbf{v}}{\|\mathbf{v}\|}$. (d) Find $\left\|\frac{1}{2}(\mathbf{v} - \mathbf{w})\right\|$.

(e) Find $\left\|\frac{1}{2}(\mathbf{v} + \mathbf{w})\right\|$. (f) Find $-2\,\mathbf{v} + 4\,\mathbf{w}$. (g) Find $\mathbf{v} - 2\,\mathbf{w}$.

(h) Find the vector $\mathbf{u}$ such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{i}$.

(i) Find the vector $\mathbf{u}$ such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = 2\,\mathbf{j} + \mathbf{k}$.

(j) Is there a scalar m such that $m(\mathbf{v} + 2\,\mathbf{w}) = \mathbf{k}$? If so, find it.

1.3 Dot Product

You may have noticed that while we did define multiplication of a vector by a scalar in the previous section on vector algebra, we did not define multiplication of a vector by a vector. We will now see one type of multiplication of vectors, called the *dot product*.

Definition 1.6. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$. The **dot product** of **v** and **w**, denoted by $\mathbf{v} \cdot \mathbf{w}$, is given by:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \tag{1.6}$$

Similarly, for vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ in $\mathbb{R}^2$, the dot product is:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 \tag{1.7}$$

Definition 1.7. The **angle** between two nonzero vectors with the same initial point is the smallest angle between them.

We do not define the angle between the zero vector and any other vector. Any two nonzero vectors with the same initial point have two angles between them: θ and $360° - \theta$. We will always choose the smallest nonnegative angle θ between them, so that $0° \le \theta \le 180°$. See Figure 1.3.1.

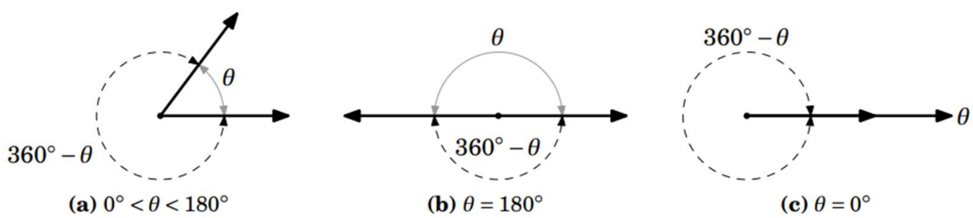


Figure 1.3.1 Angle between vectors

We can now take a more geometric view of the dot product by establishing a relationship between the dot product of two vectors and the angle between them.

Theorem 1.6. Let **v**, **w** be nonzero vectors, and let θ be the angle between them. Then

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|} \tag{1.8}$$

Example 1.5. Find the angle θ between the vectors $\mathbf{v} = (2,1,-1)$ and $\mathbf{w} = (3,-4,1)$.

Solution: Since $\mathbf{v}\cdot\mathbf{w} = (2)(3)+(1)(-4)+(-1)(1) = 1$, $\|\mathbf{v}\| = \sqrt{6}$, and $\|\mathbf{w}\| = \sqrt{26}$, then

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{1}{\sqrt{6}\sqrt{26}} = \frac{1}{2\sqrt{39}} \approx 0.08 \implies \theta = 85.41^\circ$$

Two nonzero vectors are **perpendicular** if the angle between them is 90°. Since $\cos 90^\circ = 0$, we have the following important corollary to Theorem 1.6:

Corollary 1.7. Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

We will write $\mathbf{v}\perp\mathbf{w}$ to indicate that $\mathbf{v}$ and $\mathbf{w}$ are perpendicular.

Corollary 1.8. If θ is the angle between nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{v}\cdot\mathbf{w} \text{ is } \begin{cases} >0 & \text{for } 0^\circ \le \theta < 90^\circ \\ 0 & \text{for } \theta = 90^\circ \\ <0 & \text{for } 90^\circ < \theta \le 180^\circ \end{cases}$$

By Corollary 1.8, the dot product can be thought of as a way of telling if the angle between two vectors is acute, obtuse, or a right angle, depending on whether the dot product is positive, negative, or zero, respectively. See Figure 1.3.3.

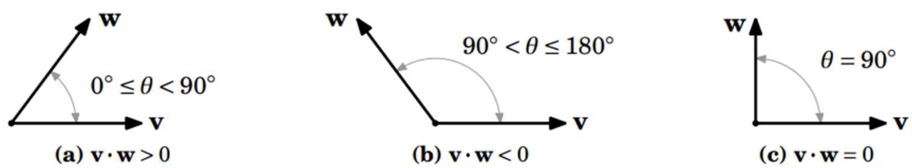


Figure 1.3.3 Sign of the dot product & angle between vectors

Example 1.6. Are the vectors $\mathbf{v} = (-1,5,-2)$ and $\mathbf{w} = (3,1,1)$ perpendicular?

Solution: Yes, $\mathbf{v}\perp\mathbf{w}$ since $\mathbf{v}\cdot\mathbf{w} = (-1)(3)+(5)(1)+(-2)(1) = 0$.

Exercises

A

1. Let $\mathbf{v} = (5,1,-2)$ and $\mathbf{w} = (4,-4,3)$. Calculate $\mathbf{v}\cdot\mathbf{w}$.

2. Let $\mathbf{v} = -3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ and $\mathbf{w} = 6\mathbf{i}+4\mathbf{j}+2\mathbf{k}$. Calculate $\mathbf{v}\cdot\mathbf{w}$.

For Exercises 3-8, find the angle θ between the vectors $\mathbf{v}$ and $\mathbf{w}$.

3. $\mathbf{v} = (5,1,-2)$, $\mathbf{w} = (4,-4,3)$

4. $\mathbf{v} = (7,2,-10)$, $\mathbf{w} = (2,6,4)$

5. $\mathbf{v} = (2,1,4)$, $\mathbf{w} = (1,-2,0)$

6. $\mathbf{v} = (4,2,-1)$, $\mathbf{w} = (8,4,-2)$

7. $\mathbf{v} = -\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $\mathbf{w} = -3\mathbf{i}+6\mathbf{j}+3\mathbf{k}$

8. $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = 3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$

9. Let $\mathbf{v} = (8,4,3)$ and $\mathbf{w} = (-2,1,4)$. Is $\mathbf{v} \perp \mathbf{w}$? Justify your answer.

10. Let $\mathbf{v} = (6,0,4)$ and $\mathbf{w} = (0,2,-1)$. Is $\mathbf{v} \perp \mathbf{w}$? Justify your answer.

1.4 Cross Product

In Section 1.3 we defined the dot product, which gave a way of multiplying two vectors. The resulting product, however, was a scalar, not a vector. In this section we will define a product of two vectors that does result in another vector. This product, called the *cross product*, is only defined for vectors in $\mathbb{R}^3$. The definition may appear strange and lacking motivation, but we will see the geometric basis for it shortly.

Definition 1.8. Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be vectors in $\mathbb{R}^3$. The **cross product** of $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \times \mathbf{w}$, is the vector in $\mathbb{R}^3$ given by:

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \tag{1.10}$$

Example 1.7. Find $\mathbf{i} \times \mathbf{j}$.

Solution: Since $\mathbf{i} = (1,0,0)$ and $\mathbf{j} = (0,1,0)$, then

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= ((0)(0) - (0)(1), (0)(0) - (1)(0), (1)(1) - (0)(0)) \\ &= (0,0,1) \\ &= \mathbf{k}\end{aligned}$$

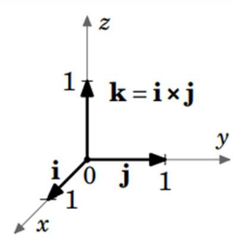


Figure 1.4.1

Similarly it can be shown that $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Theorem 1.11. If the cross product $\mathbf{v} \times \mathbf{w}$ of two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ is also a nonzero vector, then it is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

Theorem 1.13. Area of triangles and parallelograms

(a) The area A of a triangle with adjacent sides $\mathbf{v}$, $\mathbf{w}$ (as vectors in $\mathbb{R}^3$) is:

$$A = \frac{1}{2}\|\mathbf{v} \times \mathbf{w}\|$$

(b) The area A of a parallelogram with adjacent sides $\mathbf{v}$, $\mathbf{w}$ (as vectors in $\mathbb{R}^3$) is:

$$A = \|\mathbf{v} \times \mathbf{w}\|$$

Example 1.9. Calculate the area of the triangle $\triangle PQR$, where $P = (2,4,-7)$, $Q = (3,7,18)$, and $R = (-5,12,8)$.

Solution: Let $\mathbf{v} = \overrightarrow{PQ}$ and $\mathbf{w} = \overrightarrow{PR}$, as in Figure 1.4.4. Then $\mathbf{v} = (3,7,18)-(2,4,-7) = (1,3,25)$ and $\mathbf{w} = (-5,12,8)-(2,4,-7) = (-7,8,15)$, so the area A of the triangle $\triangle PQR$ is

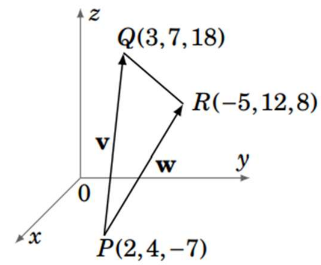


Figure 1.4.4

$$
\begin{aligned}
A &= \frac{1}{2}\|\mathbf{v}\times\mathbf{w}\| = \frac{1}{2}\|(1,3,25)\times(-7,8,15)\| \\
&= \frac{1}{2}\left\|((3)(15)-(25)(8),(25)(-7)-(1)(15),(1)(8)-(3)(-7))\right\| \\
&= \frac{1}{2}\left\|(-155,-190,29)\right\| \\
&= \frac{1}{2}\sqrt{(-155)^2+(-190)^2+29^2} = \frac{1}{2}\sqrt{60966} \\
A &\approx 123.46
\end{aligned}
$$

Example 1.10. Calculate the area of the parallelogram $PQRS$, where $P = (1,1)$, $Q = (2,3)$, $R = (5,4)$, and $S = (4,2)$.

Solution: Let $\mathbf{v} = \overrightarrow{SP}$ and $\mathbf{w} = \overrightarrow{SR}$, as in Figure 1.4.5. Then $\mathbf{v} = (1,1)-(4,2) = (-3,-1)$ and $\mathbf{w} = (5,4)-(4,2) = (1,2)$. But these are vectors in $\mathbb{R}^2$, and the cross product is only defined for vectors in $\mathbb{R}^3$. However, $\mathbb{R}^2$ can be thought of as the subset of $\mathbb{R}^3$ such that the z-coordinate is always 0. So we can write $\mathbf{v} = (-3,-1,0)$ and $\mathbf{w} = (1,2,0)$. Then the area A of $PQRS$ is

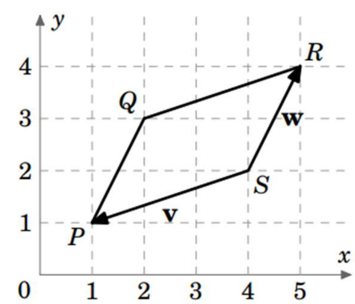


Figure 1.4.5

$$
\begin{aligned}
A &= \|\mathbf{v}\times\mathbf{w}\| = \left\|(-3,-1,0)\times(1,2,0)\right\| \\
&= \left\|((-1)(0)-(0)(2),(0)(1)-(-3)(0),(-3)(2)-(-1)(1))\right\| \\
&= \left\|(0,0,-5)\right\| \\
A &= 5
\end{aligned}
$$

The following theorem summarizes the basic properties of the cross product.

Theorem 1.14. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$, and scalar k, we have

(a) $\mathbf{v}\times\mathbf{w} = -\mathbf{w}\times\mathbf{v}$ — Anticommutative Law

(b) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$ — Distributive Law

(c) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$ — Distributive Law

(d) $(k\mathbf{v})\times\mathbf{w} = \mathbf{v}\times(k\mathbf{w}) = k(\mathbf{v}\times\mathbf{w})$ — Associative Law

(e) $\mathbf{v}\times\mathbf{0} = \mathbf{0} = \mathbf{0}\times\mathbf{v}$

(f) $\mathbf{v}\times\mathbf{v} = \mathbf{0}$

(g) $\mathbf{v}\times\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v}\parallel\mathbf{w}$

Example 1.11. Adding to Example 1.7, we have

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$
$$\mathbf{j}\times\mathbf{i}=-\mathbf{k} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$
$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$$

The scalar triple product can also be written as a determinant. In fact, by Example 1.12, the following theorem provides an alternate definition of the determinant of a 3×3 matrix as the volume of a parallelepiped whose adjacent sides are the rows of the matrix and form a right-handed system (a left-handed system would give the negative volume).

Theorem 1.17. For any vectors $\mathbf{u}=(u_1,u_2,u_3)$, $\mathbf{v}=(v_1,v_2,v_3)$, $\mathbf{w}=(w_1,w_2,w_3)$ in $\mathbb{R}^3$:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \tag{1.15}$$

Example 1.17. Find the volume of the parallelepiped with adjacent sides $\mathbf{u}=(2,1,3)$, $\mathbf{v}=(-1,3,2)$, $\mathbf{w}=(1,1,-2)$ (see Figure 1.4.9).

Solution: By Theorem 1.15, the volume $\mathrm{vol}(P)$ of the parallelepiped P is the absolute value of the scalar triple product of the three adjacent sides (in any order). By Theorem 1.17,

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & 1 & -2 \end{vmatrix}$$
$$=2\begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}-1\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix}+3\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix}$$
$$=2(-8)-1(0)+3(-4)=-28, \text{ so}$$
$$\mathrm{vol}(P)=|-28|=28.$$

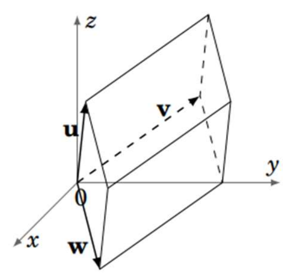


Figure 1.4.9 P

Exercises

A

For Exercises 1-6, calculate $\mathbf{v} \times \mathbf{w}$.

1. $\mathbf{v} = (5, 1, -2)$, $\mathbf{w} = (4, -4, 3)$

2. $\mathbf{v} = (7, 2, -10)$, $\mathbf{w} = (2, 6, 4)$

3. $\mathbf{v} = (2, 1, 4)$, $\mathbf{w} = (1, -2, 0)$

4. $\mathbf{v} = (1, 3, 2)$, $\mathbf{w} = (7, 2, -10)$

5. $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{w} = -3\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$

6. $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

For Exercises 7-8, calculate the area of the triangle $\triangle PQR$.

7. $P = (5, 1, -2)$, $Q = (4, -4, 3)$, $R = (2, 4, 0)$

8. $P = (4, 0, 2)$, $Q = (2, 1, 5)$, $R = (-1, 0, -1)$

For Exercises 9-10, calculate the area of the parallelogram $PQRS$.

9. $P = (2, 1, 3)$, $Q = (1, 4, 5)$, $R = (2, 5, 3)$, $S = (3, 2, 1)$

10. $P = (-2, -2)$, $Q = (1, 4)$, $R = (6, 6)$, $S = (3, 0)$

For Exercises 11-12, find the volume of the parallelepiped with adjacent sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

11. $\mathbf{u} = (1, 1, 3)$, $\mathbf{v} = (2, 1, 4)$, $\mathbf{w} = (5, 1, -2)$

12. $\mathbf{u} = (1, 3, 2)$, $\mathbf{v} = (7, 2, -10)$, $\mathbf{w} = (1, 0, 1)$

For Exercises 13-14, calculate $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

13. $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (3, 0, 2)$, $\mathbf{w} = (2, 2, 2)$

14. $\mathbf{u} = (1, 0, 2)$, $\mathbf{v} = (-1, 0, 3)$, $\mathbf{w} = (2, 0, -2)$

15. Calculate $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z})$ for $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (3, 0, 2)$, $\mathbf{w} = (2, 2, 2)$, $\mathbf{z} = (2, 1, 4)$.

1.5 Lines and Planes

Now that we know how to perform some operations on vectors, we can start to deal with some familiar geometric objects, like lines and planes, in the language of vectors. The reason for doing this is simple: using vectors makes it easier to study objects in 3-dimensional Euclidean space. We will first consider lines.

Line through a point, parallel to a vector

Let $P = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, let $\mathbf{v} = (a, b, c)$ be a nonzero vector, and let L be the line through P which is parallel to $\mathbf{v}$ (see Figure 1.5.1).

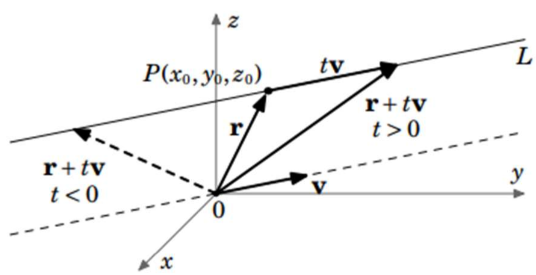


Figure 1.5.1

Let $\mathbf{r} = (x_0, y_0, z_0)$ be the *vector* pointing from the origin to P. Since multiplying the vector $\mathbf{v}$ by a scalar t lengthens or shrinks $\mathbf{v}$ while preserving its direction if $t > 0$, and reversing its direction if $t < 0$, then we see from Figure 1.5.1 that every point on the line L can be obtained by adding the vector $t\mathbf{v}$ to the vector $\mathbf{r}$ for some scalar t. That is, as t varies over all real numbers, the vector $\mathbf{r} + t\mathbf{v}$ will point to every point on L. We can summarize the *vector representation of* L as follows:

For a point $P = (x_0, y_0, z_0)$ and nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$, the line L through P parallel to $\mathbf{v}$ is given by

$$\mathbf{r} + t\mathbf{v}, \quad \text{for } -\infty < t < \infty \tag{1.16}$$

where $\mathbf{r} = (x_0, y_0, z_0)$ is the vector pointing to P.

Note that we used the correspondence between a vector and its terminal point. Since $\mathbf{v} = (a, b, c)$, then the terminal point of the vector $\mathbf{r} + t\mathbf{v}$ is $(x_0 + at, y_0 + bt, z_0 + ct)$. We then get the *parametric representation of* L with the *parameter* t:

For a point $P = (x_0, y_0, z_0)$ and nonzero vector $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$, the line L through P parallel to $\mathbf{v}$ consists of all points (x, y, z) given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for } -\infty < t < \infty \tag{1.17}$$

Note that in both representations we get the point P on L by letting $t = 0$.

For a point $P=(x_0,y_0,z_0)$ and vector $\mathbf{v}=(a,b,c)$ in $\mathbb{R}^3$ with a, b and c all nonzero, the line L through P parallel to $\mathbf{v}$ consists of all points (x,y,z) given by the equations

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c} \tag{1.18}$$

Example 1.19. Write the line L through the point $P=(2,3,5)$ and parallel to the vector $\mathbf{v}=(4,-1,6)$, in the following forms: (a) vector, (b) parametric, (c) symmetric. Lastly: (d) find two points on L distinct from P.

Solution: (a) Let $\mathbf{r}=(2,3,5)$. Then by formula (1.16), L is given by:

$$\mathbf{r}+t\mathbf{v}=(2,3,5)+t(4,-1,6),\ \text{ for } -\infty<t<\infty$$

(b) L consists of the points (x,y,z) such that

$$x=2+4t,\quad y=3-t,\quad z=5+6t,\ \text{ for } -\infty<t<\infty$$

(c) L consists of the points (x,y,z) such that

$$\frac{x-2}{4}=\frac{y-3}{-1}=\frac{z-5}{6}$$

(d) Letting $t=1$ and $t=2$ in part(b) yields the points $(6,2,11)$ and $(10,1,17)$ on L.

Distance between a point and a line

Let L be a line in $\mathbb{R}^3$ in vector form as $\mathbf{r}+t\mathbf{v}$ (for $-\infty<t<\infty$), and let P be a point not on L. The distance d from P to L is the length of the line segment from P to L which is perpendicular to L (see Figure 1.5.4). Pick a point Q on L, and let $\mathbf{w}$ be the vector from Q to P. If θ is the angle between $\mathbf{w}$ and $\mathbf{v}$, then $d=\|\mathbf{w}\|\sin\theta$. So since $\|\mathbf{v}\times\mathbf{w}\|=\|\mathbf{v}\|\,\|\mathbf{w}\|\sin\theta$ and $\mathbf{v}\neq\mathbf{0}$, then:

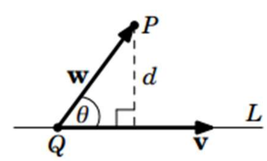


Figure 1.5.4

$$d=\frac{\|\mathbf{v}\times\mathbf{w}\|}{\|\mathbf{v}\|} \tag{1.23}$$

Example 1.21. Find the distance d from the point $P=(1,1,1)$ to the line L in Example 1.20.

Solution: From Example 1.20, we see that we can represent L in vector form as: $\mathbf{r}+t\mathbf{v}$, for $\mathbf{r}=(-3,1,-4)$ and $\mathbf{v}=(7,3,-2)$. Since the point $Q=(-3,1,-4)$ is on L, then for $\mathbf{w}=\overrightarrow{QP}=(1,1,1)-(-3,1,-4)=(4,0,5)$, we have:

$$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\7&3&-2\\4&0&5\end{vmatrix}=\begin{vmatrix}3&-2\\0&5\end{vmatrix}\mathbf{i}-\begin{vmatrix}7&-2\\4&5\end{vmatrix}\mathbf{j}+\begin{vmatrix}7&3\\4&0\end{vmatrix}\mathbf{k}=15\mathbf{i}-43\mathbf{j}-12\mathbf{k}\text{ , so}$$

$$d=\frac{\|\mathbf{v}\times\mathbf{w}\|}{\|\mathbf{v}\|}=\frac{\|15\mathbf{i}-43\mathbf{j}-12\mathbf{k}\|}{\|(7,3,-2)\|}=\frac{\sqrt{15^2+(-43)^2+(-12)^2}}{\sqrt{7^2+3^2+(-2)^2}}=\frac{\sqrt{2218}}{\sqrt{62}}=5.98$$

Plane through a point, perpendicular to a vector

Let P be a plane in $\mathbb{R}^3$, and suppose it contains a point $P_0 = (x_0, y_0, z_0)$. Let $\mathbf{n} = (a, b, c)$ be a nonzero vector which is perpendicular to the plane P. Such a vector is called a **normal vector** (or just a *normal*) to the plane. Now let (x, y, z) be any point in the plane P. Then the vector $\mathbf{r} = (x - x_0, y - y_0, z - z_0)$ lies in the plane P (see Figure 1.5.6). So if $\mathbf{r} \neq \mathbf{0}$, then $\mathbf{r} \perp \mathbf{n}$ and hence $\mathbf{n} \cdot \mathbf{r} = 0$. And if $\mathbf{r} = \mathbf{0}$ then we still have $\mathbf{n} \cdot \mathbf{r} = 0$.

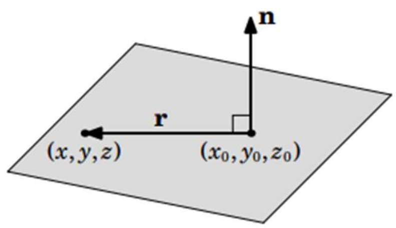


Figure 1.5.6 The plane P

Conversely, if (x, y, z) is any point in $\mathbb{R}^3$ such that $\mathbf{r} = (x - x_0, y - y_0, z - z_0) \neq \mathbf{0}$ and $\mathbf{n} \cdot \mathbf{r} = 0$, then $\mathbf{r} \perp \mathbf{n}$ and so (x, y, z) lies in P. This proves the following theorem:

Theorem 1.18. Let P be a plane in $\mathbb{R}^3$, let (x_0, y_0, z_0) be a point in P, and let $\mathbf{n} = (a, b, c)$ be a nonzero vector which is perpendicular to P. Then P consists of the points (x, y, z) satisfying the vector equation:

$$\mathbf{n} \cdot \mathbf{r} = 0 \qquad (1.24)$$

where $\mathbf{r} = (x - x_0, y - y_0, z - z_0)$, or equivalently:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad (1.25)$$

The above equation is called the **point-normal form** of the plane P.

Example 1.23. Find the equation of the plane P containing the point $(-3, 1, 3)$ and perpendicular to the vector $\mathbf{n} = (2, 4, 8)$.

Solution: By formula (1.25), the plane P consists of all points (x, y, z) such that:

$$2(x + 3) + 4(y - 1) + 8(z - 3) = 0$$

Distance between a point and a plane

The distance between a point in $\mathbb{R}^3$ and a plane is the length of the line segment from that point to the plane which is perpendicular to the plane. The following theorem gives a formula for that distance.

Theorem 1.19. Let $Q = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, and let P be a plane with normal form $ax + by + cz + d = 0$ that does not contain Q. Then the distance D from Q to P is:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{1.27}$$

Example 1.25. Find the distance D from $(2, 4, -5)$ to the plane from Example 1.24.

Solution: Recall that the plane is given by $5x - 3y + z - 10 = 0$. So

$$D = \frac{|5(2) - 3(4) + 1(-5) - 10|}{\sqrt{5^2 + (-3)^2 + 1^2}} = \frac{|-17|}{\sqrt{35}} = \frac{17}{\sqrt{35}} \approx 2.87$$

Exercises

A

For Exercises 1-4, write the line L through the point P and parallel to the vector $\mathbf{v}$ in the following forms: (a) vector, (b) parametric, and (c) symmetric.

1. $P = (2, 3, -2)$, $\mathbf{v} = (5, 4, -3)$

2. $P = (3, -1, 2)$, $\mathbf{v} = (2, 8, 1)$

3. $P = (2, 1, 3)$, $\mathbf{v} = (1, 0, 1)$

4. $P = (0, 0, 0)$, $\mathbf{v} = (7, 2, -10)$

For Exercises 5-6, write the line L through the points P_1 and P_2 in parametric form.

5. $P_1 = (1, -2, -3)$, $P_2 = (3, 5, 5)$

6. $P_1 = (4, 1, 5)$, $P_2 = (-2, 1, 3)$

For Exercises 7-8, find the distance d from the point P to the line L.

7. $P = (1, -1, -1)$, $L : x = -2 - 2t,\ y = 4t,\ z = 7 + t$

8. $P = (0, 0, 0)$, $L : x = 3 + 2t,\ y = 4 + 3t,\ z = 5 + 4t$

For Exercises 9-10, find the point of intersection (if any) of the given lines.

9. $x = 7 + 3s,\ y = -4 - 3s,\ z = -7 - 5s$ and $x = 1 + 6t,\ y = 2 + t,\ z = 3 - 2t$

10. $\dfrac{x-6}{4} = y + 3 = z$ and $\dfrac{x-11}{3} = \dfrac{y-14}{-6} = \dfrac{z+9}{2}$

For Exercises 11-12, write the normal form of the plane P containing the point Q and perpendicular to the vector $\mathbf{n}$.

11. $Q = (5, 1, -2)$, $\mathbf{n} = (4, -4, 3)$

12. $Q = (6, -2, 0)$, $\mathbf{n} = (2, 6, 4)$

For Exercises 13-14, write the normal form of the plane containing the given points.

13. $(1,0,3)$, $(1,2,-1)$, $(6,1,6)$

14. $(-3,1,-3)$, $(4,-4,3)$, $(0,0,1)$

15. Write the normal form of the plane containing the lines from Exercise 9.

16. Write the normal form of the plane containing the lines from Exercise 10.

For Exercises 17-18, find the distance D from the point Q to the plane P.

17. $Q=(4,1,2)$, $P: 3x-y-5z+8=0$

18. $Q=(0,2,0)$, $P: -5x+2y-7z+1=0$

For Exercises 19-20, find the line of intersection (if any) of the given planes.

19. $x+3y+2z-6=0$, $2x-y+z+2=0$

20. $3x+y-5z=0$, $x+2y+z+4=0$

B

21. Find the point(s) of intersection (if any) of the line $\dfrac{x-6}{4}=y+3=z$ with the plane $x+3y+2z-6=0$. (*Hint: Put the equations of the line into the equation of the plane.*)

Introduction:

Learning Outcomes: Complex Numbers:

1- Understand the concept of complex numbers.
2- Represent complex numbers on the Argand diagram.
3- Perform basic arithmetic operations.
4- Find the modulus and argument of a complex number
5- Express complex numbers in polar form
6- Use Euler's formula and exponential form.
7- Apply De Moivre's Theorem.
8- Find the nth roots of a complex number.
9- Solve quadratic and polynomial equations with complex solutions.
10- Apply complex numbers to real world problems.
11- Understand basic properties of complex conjugates and modulus

Mathematics II
Assist. Lecture: Abdul Rahman S. A
@ department of Applied Mechanics
Fall - 2025
Lecture # 3 - 4

COMPLEX NUMBER:

A **complex number** can be represented by an expression of the form $a + bi$, where a and b are real numbers and i is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair (a, b) and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus, the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number a and the **imaginary part** is the real number b. Thus, the real part of $4 - 3i$ is 4 and the imaginary part is -3. Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts:

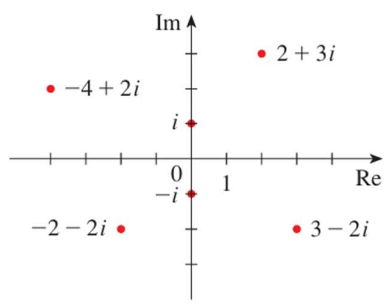


FIGURE 1
Complex numbers as points in the Argand plane

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di)\\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

EXAMPLE 1

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= (-1)(2 - 5i) + 3i(2 - 5i)\\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$

EXAMPLE 2 Express the number $\dfrac{-1 + 3i}{2 + 5i}$ in the form $a + bi$.

SOLUTION We multiply numerator and denominator by the complex conjugate of $2 + 5i$, namely $2 - 5i$, and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29}i$$

Properties of Conjugates

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = \bar{z}^n$$

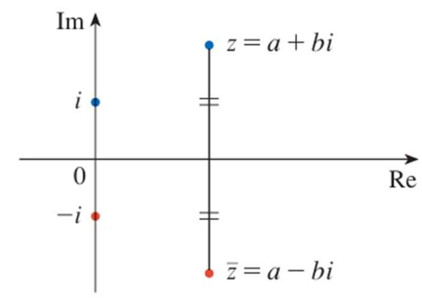


FIGURE 2

The **modulus**, or **absolute value**, $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

and so

$$z\bar{z} = |z|^2$$

This explains why the division procedure in Example 2 works in general:

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

Since $i^2 = -1$, we can think of i as a square root of -1. But notice that we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of -1. We say that i is the **principal square root** of -1 and write $\sqrt{-1} = i$. In general, if c is any positive number, we write

$$\sqrt{-c} = \sqrt{c}\, i$$

With this convention, the usual derivation and formula for the roots of the quadratic equation $ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 3 Find the roots of the equation $x^2 + x + 1 = 0$.

SOLUTION Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

POLAR FORM

We know that any complex number $z = a + bi$ can be considered as a point (a, b) and that any such point can be represented by polar coordinates (r, θ) with $r \geqslant 0$. In fact,

$$a = r\cos\theta \qquad b = r\sin\theta$$

as in Figure 4. Therefore, we have

$$z = a + bi = (r\cos\theta) + (r\sin\theta)i$$

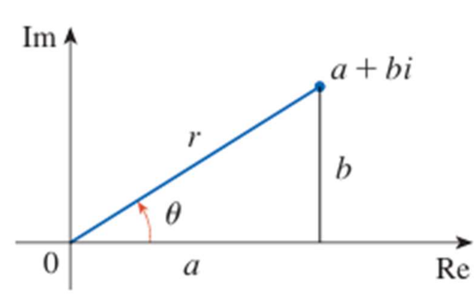


FIGURE 4

Thus, we can write any complex number z in the form

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $\quad r = |z| = \sqrt{a^2 + b^2} \quad$ and $\quad \tan\theta = \dfrac{b}{a}$

The angle θ is called the **argument** of z and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of z differ by an integer multiple of 2π.

EXAMPLE 4 Write the following numbers in polar form.

(a) $z = 1 + i$ (b) $w = \sqrt{3} - i$

SOLUTION

(a) We have $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1$, so we can take $\theta = \pi/4$. Therefore, the polar form is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(b) Here we have $r = |w| = \sqrt{3 + 1} = 2$ and $\tan\theta = -1/\sqrt{3}$. Since w lies in the fourth quadrant, we take $\theta = -\pi/6$ and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

The numbers z and w are shown in Figure 5.

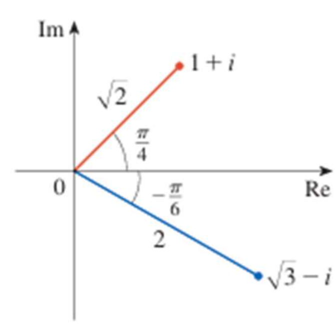


FIGURE 5

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

1

$$\boxed{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]}$$

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments.* (See Figure 6.)

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments.*

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0}$$

In particular, taking $z_1 = 1$ and $z_2 = z$, (and therefore $\theta_1 = 0$ and $\theta_2 = \theta$), we have the following, which is illustrated in Figure 7.

$$\boxed{\text{If} \quad z = r(\cos\theta + i\sin\theta), \quad \text{then} \quad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).}$$

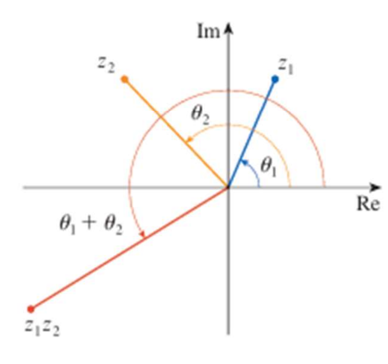


FIGURE 6

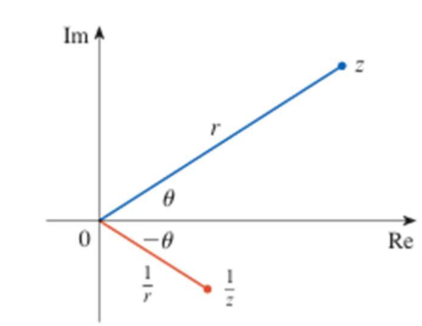


FIGURE 7

EXAMPLE 5 Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in polar form.

SOLUTION From Example 4 we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

and
$$\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

So, by Equation 1,

$$(1 + i)(\sqrt{3} - i) = 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

This is illustrated in Figure 8.

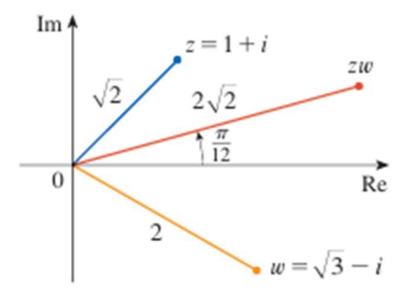


FIGURE 8

2 De Moivre's Theorem If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This says that *to take the nth power of a complex number we take the nth power of the modulus and multiply the argument by n.*

EXAMPLE 6 Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 4(a) that $\frac{1}{2} + \frac{1}{2}i$ has the polar form

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem,

$$\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} = \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$
$$= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i$$

De Moivre's Theorem can also be used to find the nth roots of complex numbers. An n th root of the complex number z is a complex number w such that

$$w^n = z$$

3 Roots of a Complex Number Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

EXAMPLE 7 Find the six sixth roots of $z = -8$ and graph these roots in the complex plane.

SOLUTION In trigonometric form, $z = 8(\cos\pi + i\sin\pi)$. Applying Equation 3 with $n = 6$, we get

$$w_k = 8^{1/6}\left(\cos\frac{\pi + 2k\pi}{6} + i\sin\frac{\pi + 2k\pi}{6}\right)$$

We get the six sixth roots of -8 by taking $k = 0, 1, 2, 3, 4, 5$ in this formula:

$$w_0 = 8^{1/6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_1 = 8^{1/6}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \sqrt{2}\,i$$

$$w_2 = 8^{1/6}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_3 = 8^{1/6}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$w_4 = 8^{1/6}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -\sqrt{2}\,i$$

$$w_5 = 8^{1/6}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

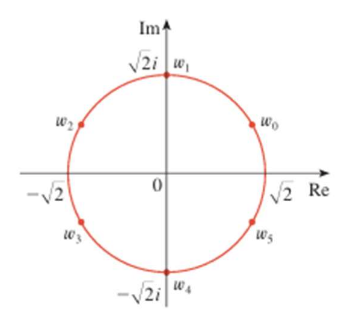


FIGURE 9
The six sixth roots of $z = -8$

All these points lie on the circle of radius $\sqrt{2}$ as shown in Figure 9.

COMPLEX EXPONENTIALS

We also need to give a meaning to the expression e^z when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 8 can be extended to the case where the terms are complex numbers. Using the Taylor series for e^x (8.7.12) as our guide, we define

4 $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

5 $$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

If we put $z = iy$, where y is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \ldots$$

we get

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y \end{aligned}$$

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 8.7.17 and 8.7.16). The result is a famous formula called **Euler's formula**:

6 $$\boxed{e^{iy} = \cos y + i \sin y}$$

Combining Euler's formula with Equation 5, we get

7 $$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

EXAMPLE 8 Evaluate: (a) $e^{i\pi}$ (b) $e^{-1+i\pi/2}$

SOLUTION

(a) From Euler's equation (6) we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e}$$

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos \theta + i \sin \theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

EXERCISES

1–14 ■ Evaluate the expression and write your answer in the form $a + bi$.

1. $(5 - 6i) + (3 + 2i)$

2. $\left(4 - \frac{1}{2}i\right) - \left(9 + \frac{5}{2}i\right)$

3. $(2 + 5i)(4 - i)$

4. $(1 - 2i)(8 - 3i)$

5. $\overline{12 + 7i}$

6. $\overline{2i\left(\frac{1}{2} - i\right)}$

7. $\dfrac{1 + 4i}{3 + 2i}$

8. $\dfrac{3 + 2i}{1 - 4i}$

9. $\dfrac{1}{1 + i}$

10. $\dfrac{3}{4 - 3i}$

11. i^3

12. i^{100}

13. $\sqrt{-25}$

14. $\sqrt{-3}\sqrt{-12}$

15–17 ■ Find the complex conjugate and the modulus of the number.

15. $12 - 5i$

16. $-1 + 2\sqrt{2}\,i$

17. $-4i$

18. Prove the following properties of complex numbers.

(a) $\overline{z + w} = \bar{z} + \bar{w}$

(b) $\overline{zw} = \bar{z}\,\bar{w}$

(c) $\overline{z^n} = \bar{z}^n$, where n is a positive integer

[*Hint:* Write $z = a + bi$, $w = c + di$.]

19–24 ■ Find all solutions of the equation.

19. $4x^2 + 9 = 0$

20. $x^4 = 1$

21. $x^2 + 2x + 5 = 0$

22. $2x^2 - 2x + 1 = 0$

23. $z^2 + z + 2 = 0$

24. $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

25–28 ■ Write the number in polar form with argument between 0 and 2π.

25. $-3 + 3i$

26. $1 - \sqrt{3}\,i$

27. $3 + 4i$

28. $8i$

29–32 ■ Find polar forms for zw, z/w, and $1/z$ by first putting z and w into polar form.

29. $z = \sqrt{3} + i, \quad w = 1 + \sqrt{3}\,i$

30. $z = 4\sqrt{3} - 4i, \quad w = 8i$

31. $z = 2\sqrt{3} - 2i, \quad w = -1 + i$

32. $z = 4(\sqrt{3} + i), \quad w = -3 - 3i$

MECHANICAL EXAMPLE

The following example is for reference only, and students are not required to study it.

Mechanical Example: Analysis of a Mass-Spring-Damper System (Vibration Analysis)

Problem:

A mechanical system consists of a mass m , a spring with stiffness k , and a damper with damping coefficient c . The system is subjected to free vibration. We want to analyze its motion using differential equations — and this is where **complex numbers** become very useful.

The equation of motion is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

This is a second-order linear differential equation used to model vibrations in structures, vehicles, machines, etc.

✅ Step 1: Assume a solution using complex exponentials

To solve this, engineers often assume a solution of the form:

$$x(t) = e^{rt}$$

But since the system oscillates, we use **complex numbers** to represent sinusoidal motion efficiently:

$$x(t) = \mathrm{Re}\left(Ae^{i\omega t}\right)$$

Where:

- A is a complex amplitude (contains magnitude and phase),
- $i = \sqrt{-1}$,
- ω is the angular frequency,
- $\mathrm{Re}()$ means taking the real part (since physical displacement must be real).

Using Euler's formula:
$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

So the solution becomes a **cosine or sine wave** — representing oscillatory motion.

✅ **Step 2: Characteristic Equation and Complex Roots**

When solving the differential equation, we get a characteristic equation:

$$mr^2 + cr + k = 0$$

Solving for r :

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Now, if the damping is **light** ($c^2 < 4mk$), the term under the square root is negative → leads to **complex roots**:

$$r = -\alpha \pm i\omega_d$$

Where:

- $\alpha = \frac{c}{2m}$ (decay rate),
- $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$ (damped natural frequency).

➡️ This means the solution will be:

$$x(t) = e^{-\alpha t}\left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right)$$

Or more compactly using complex notation:

$$x(t) = \mathrm{Re}\left(Be^{(-\alpha + i\omega_d)t}\right)$$

Which represents a **decaying oscillation** — like a car suspe[illegible]on settling after hitting a bump.

Introduction of Matrices

1.1 Definition 1:

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A*,*B*, *C* etc.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

A is a matrix of order $m \times n$. i^{th} row j^{th} column element of the matrix denoted by a_{ij}

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

1.2 Special Types of Matrices:

1. **Square matrix**:

A matrix in which numbers of rows are equal to number of columns is called a square matrix.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}$$

2. **Diagonal matrix**:

A square matrix $A = (a_{ij})_{n \times n}$ is called a diagonal matrix if each of its non-diagonal element is zero.

That is $a_{ij} = 0 \;\; if \;\; i \neq j$ and at least one element $a_{ii} \neq 0$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

3. **Identity Matrix**

A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix and denoted by I_n.

That is $a_{ij} = \begin{cases} 0 & if \; i \neq j \\ 1 & if \; i = j \end{cases}$

Example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. **Upper Triangular matrix**:

A square matrix said to be a Upper triangular matrix if $a_{ij} = 0$ *if* $i > j$.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 8 \\ 0 & -2 & 5 \\ 0 & 0 & 7 \end{pmatrix}$$

5. **Lower Triangular Matrix**:

A square matrix said to be a Lower triangular matrix if $a_{ij} = 0$ *if* $i < j$.

Example:

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 7 & 0 & 0 \\ 9 & 6 & 2 \end{pmatrix}$$

6. **Symmetric Matrix**:

A square matrix $A = (a_{ij})_{n \times n}$ said to be a symmetric if $a_{ij} = a_{ji}$ for all *i* and *j.*

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ -2 & -9 & 3 \\ 7 & 3 & 5 \end{pmatrix}$$

7. Skew- Symmetric Matrix:

A square matrix $A = (a_{ij})_{n\times n}$ said to be a skew-symmetric if $a_{ij} = -a_{ji}$ for all *i* and *j*.

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix} \qquad B = \begin{pmatrix} 8 & -2 & 7 \\ 2 & -9 & 3 \\ -7 & -3 & 5 \end{pmatrix}$$

8. Zero Matrix:

A matrix whose all elements are zero is called as Zero Matrix and order $n \times m$ Zero matrix denoted by $0_{n\times m}$.

Example:

$$0_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

9. Row Vector

A matrix consists a single row is called as a row vector or row matrix.

Example:

$$A = (a_{11} \quad a_{12} \quad a_{13}) \qquad B = (7 \quad 4 \quad -3)$$

10. Column Vector

A matrix consists a single column is called a column vector or column matrix.

Example:

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \qquad B = \begin{pmatrix} 9 \\ -7 \\ 3 \end{pmatrix}$$

Matrix Algebra

2.1. Equality of two matrices:

Two matrices A and B are said to be equal if

(i) They are of same order.
(ii) Their corresponding elements are equal.

That is if $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ then $a_{ij} = b_{ij}$ for all *i* and *j*.

2.2. Scalar multiple of a matrix

Let k be a scalar then scalar product of matrix $A = (a_{ij})_{m\times n}$ given denoted by *kA* and given by $kA = (ka_{ij})_{m\times n}$ or

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ \vdots & & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

2.3. Addition of two matrices:

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ are two matrices with same order then sum of the two matrices are given by

$$A + B = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

Example 2.1: let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 8 \end{pmatrix}.$$

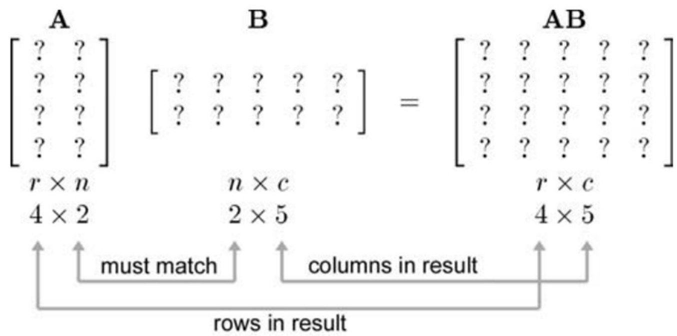
Find (i) 5B (ii) A + B (iii) 4A – 2B (iv) 0 A

2.4. Multiplication of two matrices:

Two matrices *A* and *B* are said to be confirmable for product *AB* if number of columns in *A* equals to the number of rows in matrix *B*. Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times r}$ be two matrices the product matrix *C= AB*, is matrix of order $m\times r$ where

In certain situations, we can take the product of two matrices. The rules that govern when matrix multiplication is allowed, and how the result is computed, may at first seem bizarre. An $r \times n$ matrix **A** may be multiplied by an $n \times c$ matrix **B**. The result, denoted **AB**, is an $r \times c$ matrix.

For example, assume that **A** is a 4×2 matrix, and **B** is a 2×5 matrix. Then **AB** is a 4×5 matrix:



Let's look at a 2×2 example with some real numbers:

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} -3 & 0 \\ 5 & 1/2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -7 & 2 \\ 4 & 6 \end{bmatrix} \\
\mathbf{AB} &= \begin{bmatrix} -3 & 0 \\ 5 & 1/2 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 4 & 6 \end{bmatrix} \\
&= \begin{bmatrix} (-3)(-7) + (0)(4) & (-3)(2) + (0)(6) \\ (5)(-7) + (1/2)(4) & (5)(2) + (1/2)(6) \end{bmatrix} \\
&= \begin{bmatrix} 21 & -6 \\ -33 & 13 \end{bmatrix}
\end{aligned}
$$

And a 3×3 example with some real numbers:

$$
\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 1 & -5 & 3 \\ 0 & -2 & 6 \\ 7 & 2 & -4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -8 & 6 & 1 \\ 7 & 0 & -3 \\ 2 & 4 & 5 \end{bmatrix} \\
\mathbf{AB} &= \begin{bmatrix} 1 & -5 & 3 \\ 0 & -2 & 6 \\ 7 & 2 & -4 \end{bmatrix} \begin{bmatrix} -8 & 6 & 1 \\ 7 & 0 & -3 \\ 2 & 4 & 5 \end{bmatrix} \\
&= \begin{bmatrix} (1)(-8) + (-5)(7) + (3)(2) & (1)(6) + (-5)(0) + (3)(4) & (1)(1) + (-5)(-3) + (3)(5) \\ (0)(-8) + (-2)(7) + (6)(2) & (0)(6) + (-2)(0) + (6)(4) & (0)(1) + (-2)(-3) + (6)(5) \\ (7)(-8) + (2)(7) + (-4)(2) & (7)(6) + (2)(0) + (-4)(4) & (7)(1) + (2)(-3) + (-4)(5) \end{bmatrix} \\
&= \begin{bmatrix} -37 & 18 & 31 \\ -2 & 24 & 36 \\ -50 & 26 & -19 \end{bmatrix}
\end{aligned}
$$

It is easier to see this by looking at a few examples. Let

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}$$

We can now define their product. Here we would say that **B** is pre-multiplied by **A**, or that **A** is post-multiplied by **B**:

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} -2\cdot 3+1\cdot 2+3\cdot 1 & -2\cdot(-2)+1\cdot 4+3\cdot(-3) \\ 4\cdot 3+1\cdot 2+6\cdot 1 & 4\cdot(-2)+1\cdot 4+6\cdot(-3) \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ 20 & -22 \end{bmatrix} \end{aligned}$$

Note that **A** is of order $(2,3)$, and **B** is of order $(3,2)$. Thus, the product **AB** is of order $(2,2)$.

We can similary compute the product **BA** which will be of order $(3,3)$. You can verify that this product is:

$$\mathbf{BA} = \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{bmatrix}$$

This shows that the multiplication of matrices is not commutative. In other words: $\mathbf{AB} \neq \mathbf{BA}$.

Since a vector can be considered a matrix with one row or one column, we can multiply a vector and a matrix using the rules discussed in the previous section. It becomes very important whether we are using row or column vectors. Below we show how 3D row and column vectors may be pre- or post-multiplied by a 3×3 matrix:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} xm_{11}+ym_{21}+zm_{31} & xm_{12}+ym_{22}+zm_{32} & xm_{13}+ym_{23}+zm_{33} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xm_{11}+ym_{12}+zm_{13} \\ xm_{21}+ym_{22}+zm_{23} \\ xm_{31}+ym_{32}+zm_{33} \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = (undefined)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = (undefined)$$

Example 2.2: Let $A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{pmatrix}$

Calculate (i) AB (ii) BA

(iii) is AB = BA ?

Example Find $A^2 - 5A + 6I$ if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Finding A^2

$A^2 = AA$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2)+0(2)+1(1) & 2(0)+0(1)+1(-1) & 2(1)+0(3)+1(0) \\ 2(2)+1(2)+3(1) & 2(0)+1(1)+3(-1) & 2(1)+1(3)+3(0) \\ 1(2)+-1(2)+0(1) & 1(0)+-1(1)+0(-1) & 1(1)+-1(3)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{-1} & \mathbf{2} \\ \mathbf{9} & \mathbf{-2} & \mathbf{5} \\ \mathbf{0} & \mathbf{-1} & \mathbf{-2} \end{bmatrix}$$

Now calculating

$A^2 - 5A + 6I$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 2\times5 & 0\times5 & 1\times5 \\ 2\times5 & 1\times5 & 3\times5 \\ 1\times5 & -1\times5 & 0\times5 \end{bmatrix} + \begin{bmatrix} 1\times6 & 0\times6 & 0\times6 \\ 0\times6 & 1\times6 & 0\times6 \\ 0\times6 & 0\times6 & 1\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2-0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$

First calculating A^2

$A^2 = A.A$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)+1(-1) & 3(1)+1(2) \\ -1(3)+2(-1) & -1(1)+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Now calculating

$\mathbf{A^2 - 5A - 7I}$

$$= \begin{bmatrix} \mathbf{8} & \mathbf{5} \\ \mathbf{-5} & \mathbf{3} \end{bmatrix} - \mathbf{5} \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} + \mathbf{7} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 5(3) & 5(1) \\ 5(-1) & 5(2) \end{bmatrix} + \begin{bmatrix} 7(1) & 7(0) \\ 7(0) & 7(1) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 - (-5) & 3 - 10 + 7 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

= O

= R.H.S.

Since L.H.S = R.H.S

Functions of a Matrix

The Transpose of a Matrix

The transpose of an $m \times n$ matrix $\mathbf{A}$, written $\mathbf{A}^T$, is the $n \times m$ matrix formed by interchanging the rows and columns of A. For example, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

then in terms of the elements of the above matrix, the transpose is

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}.$$

Notice that the transpose of a row vector produces a column vector, and similarly the transpose of a column vector produces a row vector. The transpose of the *product* of two matrices is the *reversed* product of the transpose of the two individual matrices,

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T.$$

The rules of matrix multiplication show that the product of a vector and its transpose is the sum of the squares of all of the elements

$$\mathbf{x}(\mathbf{x}^T) = \sum_{i=1}^{n} (x_i)^2.$$

The transpose of matrix $A = (a_{ij})_{m \times n}$, written A^t $(A' \text{ or } A^T)$ is the matrix obtained by writing the rows of A in order as columns.

That is $A^t = (a_{ji})_{n \times m}$.

Properties of Transpose:

(i) $(A + B)^t = (A^t + B^t)$
(ii) $(A^t)^t = A$
(iii) $(kA)^t = k A^t$ for scalar k.
(iv) $(AB)^t = B^t A^t$

Example 2.3: Using the following matrices A and B, Verify the transpose properties

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

(i) AA^t and A^tA are both symmetric.
(ii) $A + A^t$ is a symmetric matrix.
(iii) $A - A^t$ is a skew-symmetric matrix.
(iv) If A is a symmetric matrix and m is any positive integer then A^m is also symmetric.
(v) If A is skew symmetric matrix then odd integral powers of A is skew symmetric, while positive even integral powers of A is symmetric.

If *A* and *B* are symmetric matrices then

(vi) $(AB + BA)$ is symmetric.
(vii) $(AB - BA)$ is skew-symmetric.

Exercise 2.1: Verify the (i) , (ii) and (iii) using the following matrix A.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ -3 & -5 & 10 \\ 1 & 8 & 9 \end{pmatrix}$$

Exercise: Prove the matrix A is symmetric

$$A = \begin{bmatrix} 9 & 2 & 3 \\ 2 & -1 & -8 \\ 3 & -8 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 9 & 2 & 3 \\ 2 & -1 & -8 \\ 3 & -8 & 0 \end{bmatrix}$$

Since A = A'

∴ A is a symmetric matrix

Exercise: Prove the matrix B is skew – symmetric

$$B = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -9 \\ 3 & 9 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 9 \\ -3 & -9 & 0 \end{bmatrix}$$

Therefore,

B' = −B

So, B is a skew symmetric matrix

2.8 A square matrix A is said to be symmetric if $A = A^t$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$, A is symmetric by the definition of symmetric matrix.

Then

$$A^t = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$

That is $A = A^t$

2.9 A square matrix A is said to be skew- symmetric if $A = -A^t$

Example:

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -5 & 8 \\ 1 & 8 & 9 \end{pmatrix}$$

Determinant, Minor and Adjoint Matrices

Definition 3.1:

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n , then the number $|A|$ called determinant of the matrix A.

(i) Determinant of 2×2 matrix

Let A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $|A|$ = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}a_{22} - a_{12}a_{21}$

(ii) Determinant of 3×3 matrix

Let B = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then $|B|$ = $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$$|B| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) - a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Exercise 3.1: Calculate the determinants of the following matrices

(i) $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 1 & 9 & 5 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

3.1 Properties of the Determinant:

a. The determinant of a matrix A and its transpose A^t are equal.

$$|A| = |A^t|$$

b. Let A be a square matrix

(i) If A has a row (column) of zeros then $|A| = 0$.

(ii) If A has two identical rows (or columns) then $|A| = 0$.

c. If A is triangular matrix then $|A|$ is product of the diagonal elements.

d. If A is a square matrix of order n and k is a scalar then $|kA| = k^n|A|$

3.2 Singular Matrix

If A is square matrix of order n, the A is called singular matrix when $|A| = 0$ and non-singular otherwise.

Example:

Calculate the determinant of the following matrices:

$$a)\begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix} \qquad b)\begin{pmatrix} 1 & 0 & 2 \\ 1 & 3 & 4 \\ 0 & 6 & 0 \end{pmatrix}$$

$$c)\begin{pmatrix} 3 & -2 & 4 \\ 2 & -4 & 5 \\ 1 & 8 & 2 \end{pmatrix} \qquad d)\begin{pmatrix} 8 & -1 & 9 \\ 3 & 1 & 8 \\ 11 & 0 & 17 \end{pmatrix}$$

Minor and Cofactors:

Let A = $(a_{ij})_{n \times n}$ is a square matrix. Then M_{ij} denote a sub matrix of A with order (n-1) $\times$ (n-1) obtained by deleting its i^{th} row and j^{th} column. The determinant $|M_{ij}|$ is called the minor of the element a_{ij} of A.

The cofactor of a_{ij} denoted by A_{ij} and is equal to $(-1)^{i+j}|M_{ij}|$.

Compute the cofactors of the following matrix, $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix}$

Solution

$$\alpha_{11} = (-1)^{1+1}\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} = 20 \qquad \alpha_{12} = (-1)^{1+2}\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} = 10$$

$$\alpha_{13} = (-1)^{1+3}\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 0 \qquad \alpha_{21} = (-1)^{2+1}\begin{bmatrix} 2 & -3 \\ 2 & -6 \end{bmatrix} = 6$$

$$\alpha_{22} = (-1)^{2+2}\begin{bmatrix} 1 & -3 \\ -1 & -6 \end{bmatrix} = -9 \qquad \alpha_{23} = (-1)^{2+3}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = -4$$

$$\alpha_{31} = (-1)^{3+1}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} = -8, \qquad \alpha_{32} = (-1)^{3+2}\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} = -8$$

$$\alpha_{33} = (-1)^{3+3}\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} = -8$$

The Matrix Inverse

Given any non-singular matrix A, its inverse can be found from the formula

$$A^{-1} = \frac{\text{adj}\, A}{|A|}$$

where adj A is the **adjoint matrix** and $|A|$ is the determinant of A. The procedure for finding the adjoint matrix is given below.

Example

Find the adjoint, and hence the inverse, of $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$.

Solution

Follow the stages outlined above. First find the transpose of A by taking the first column of A to be the first row of A^T, and so on:

$$A^T = \begin{pmatrix} 1 & 3 & -1 \\ -2 & 1 & 2 \\ 0 & 5 & 3 \end{pmatrix}$$

$$\text{matrix of minors of } A^T = \begin{pmatrix} -7 & -6 & -10 \\ 14 & 3 & 5 \\ 7 & 0 & 7 \end{pmatrix}$$

Then impose the place sign. This results in the matrix of cofactors, that is, the adjoint of A.

$$\text{adj } A = \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Notice that to complete this last stage, each element in the matrix of minors has been multiplied by 1 or -1 according to its position.

It is a straightforward matter to show that the determinant of A is 21. Finally

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{21}\begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

Compute the $adjA$ given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

Solution

$$adjA = \begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Exercise

1. Show that the inverse of $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ -1 & 3 & 0 \end{pmatrix}$ is $\frac{1}{4}\begin{pmatrix} -3 & 6 & -7 \\ -1 & 2 & -1 \\ 5 & -6 & 5 \end{pmatrix}$.

Solving Systems of Equations Using Matrices

Matrices are particularly useful when solving systems of equations, which, if you remeber, is what we did when we when solved for the least squares estimators. You covered this material in your high school algebra class. Here is an example, with three equations and three unknowns:

$$\begin{aligned} x + 2y + z &= 3 \\ 3x - y - 3z &= -1 \\ 2x + 3y + z &= 4 \end{aligned}$$

There is an easier way, however, and that is to use a matrix. Note that this system of equations can be represented as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \iff \mathbf{Ax} = \mathbf{b}$$

We can solve the problem $\mathbf{Ax} = \mathbf{b}$ by pre-multiplying both sides by $\mathbf{A^{-1}}$ and simplifying. This yields the following:

$$\mathbf{Ax} = \mathbf{b} \rightarrow \mathbf{A^{-1}Ax} = \mathbf{A^{-1}b} \rightarrow \mathbf{x} = \mathbf{A^{-1}b}$$

We can therefore solve a system of equations by computing the inverse of $\mathbf{A}$, and multiplying it by $\mathbf{b}$. Here our matrix $\mathbf{A}$ and its inverse is as follows (using Mathematica to perform the calculation):

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} \qquad \mathbf{A^{-1}} = \begin{bmatrix} 8 & 1 & -5 \\ -9 & -1 & 6 \\ 11 & 1 & -7 \end{bmatrix}$$

We can now solve this system of equations:

$$\mathbf{x} = \mathbf{A^{-1}b} = \begin{bmatrix} 8 & 1 & -5 \\ -9 & -1 & 6 \\ 11 & 1 & -7 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \quad \blacksquare$$

Cramer's Method

Cramer's method is a convenient method for manually solving low-order non-homogeneous sets of linear equations. If the equations are written in matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

then the ith element of the vector $\mathbf{x}$ may be found directly from a ratio of determinants

$$x_i = \frac{\det \mathbf{A}_{(i)}}{\det \mathbf{A}}$$

where $\mathbf{A}_{(i)}$ is the matrix formed by replacing the ith column of $\mathbf{A}$ with the column vector $\mathbf{b}$. For example, solve

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 2x_3 & = & 2 \\ x_1 & + & 10x_2 & - & 3x_3 & = & 5 \\ -x_1 & + & x_2 & + & x_3 & = & -3 \end{array} .$$

Then

$$\det \mathbf{A} = \begin{vmatrix} 2 & -1 & 2 \\ 1 & 10 & -3 \\ -1 & 1 & 1 \end{vmatrix} = 46$$

and

$$\begin{aligned} x_1 &= \frac{1}{46} \begin{vmatrix} 2 & -1 & 2 \\ 5 & 10 & -3 \\ -3 & 1 & 1 \end{vmatrix} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{1}{46} \begin{vmatrix} 2 & 2 & 2 \\ 1 & 5 & -3 \\ -1 & -3 & 1 \end{vmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{1}{46} \begin{vmatrix} 2 & -1 & 2 \\ 1 & 10 & 5 \\ -1 & 1 & -3 \end{vmatrix} \\ &= -1 \end{aligned}$$