Ministry of higher Education and Scientific Research
Foundation of Technical Education
North Technical University
Technical Institute of Kirkuk
Computer System Department

# Mathematics

First Class Students

Subject teacher
Amal N. Khalaf 2024/2025

# Matrixes

In mathematics, a matrix (plural matrices) is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. Matrices are commonly written in box brackets. The horizontal and vertical lines of entries in a matrix are called rows and columns, respectively. The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m -by- n matrix, while m and n are called its dimensions.The dimensions of the following matrix are 2 × 3 up(read “two by three”), because there are two rows and three columns.

For example,

$$\begin{bmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{bmatrix}$$

$A = ai, j$ to represent a matrix where a $i, j$ refers to the element found in the $ith$ row and the $jth$ column.

$$A = [ai, j] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad m * n$$

المصفوفة     مدخلات المصفوفة

حيث: $m$ = صف، $n$= عمود

الصفRow $i = 1,2,3 \ldots \ldots m$

العمود Column $j = 1,2,3 \ldots \ldots n$

## Types of Matrices:

There are several types of matrices, but the most commonly used are:

### 1. Row Matrix:

A Matrix is said to be a row matrix if it has only one row.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

### 2. Column Matrix:

A matrix is said to be a column matrix if it has only one column.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

### 3. **Square matrix** :

If the number of rows and the number of columns in a matrix are equal, then it is called a square matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 3 & 0 & 0 \end{bmatrix}_{3*3}$$

## • Rectangular Matrix:

A matrix is said to be rectangular if the number of rows is not equal to the number of columns.

$$A=\begin{bmatrix} 2 & 1 & 7 \\ 3 & 9 & 4 \end{bmatrix}$$

## • Diagonal Matrix:

A diagonal matrix is a square matrix in which all entries are zero, except for those on the leading diagonal. It is also called the scaling matrix because multiplication with the diagonal matrix scales an object in a corresponding vector space.

القطر الرئيس

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

القطر الثانوي

## • Null Matrix or Zero Matrix:

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by 0.

For example $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 x 4.

## • Transposition of Matrix :

Suppose A is a given matrix, then the matrix obtained by interchanging its rows into columns is called the transpose of A. It is denoted by $A^T$.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}_{2*3} \quad ; \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}_{3*2}$$

## • Identity Matrix:

if all of non-zero elements of a diagonal matrix are equal to (1) .

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2*2} \quad ; \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3*3}$$

## • Triangular Matrix:

A square matrix is said to be triangular if all of its elements above the principal diagonal are zero (lower triangular matrix) or all of its elements below the principal diagonal are zero (upper triangular matrix).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 6 & 0 \\ 2 & 5 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 6 & 9 \\ 0 & 0 & 8 \end{bmatrix}$$

Equality Matrices

Lower triangular matrix upper triangular matrix

Two matrices A and B are said to be equal if A and B have the same order and their corresponding elements be equal. Thus if $A = (aij)_{m,n}$ and $B = (bij)_{m,n}$ then A = B if and only if $a_{ij} = b_{ij}\ for\ i = 1,2,3,\ldots\ldots,m;\ j = 1,2,3,\ldots\ldots\ldots,n.$

The number of rows in matrix A = The number of rows in matrix B and The number of columns in matrix A = The number of columns in matrix B .

**Examples of Equal Matrices:**

**1.** The matrices A = [5] and B = [5] are equal, because both matrices are of the same order 1 × 1 and their corresponding entries are equal.

**2.** The matrices A=$\begin{bmatrix} 2 & 7 \\ 3 & 1 \end{bmatrix}$ $and\ B = \begin{bmatrix} 2 & 7 \\ 3 & 1 \end{bmatrix}$ $are$ equal, because both matrices are of the same order 2 × 2 and their corresponding entries are equal.

**3.** The matrices

$$A = \begin{bmatrix} 6 & \sqrt{4} \\ \frac{1}{2} & -1 \\ 0 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 2 \\ 0.5 & \frac{-2}{2} \\ 0 & 3^2 \end{bmatrix}$$

are equal, because both matrices are of the same order 3 × 3 and their corresponding entries are equal.

**4.** The matrices

$$A = \begin{bmatrix} 2 & -1 & 6 & 5 \\ 5 & 4 & 3 & -3 \\ 7 & -7 & 9 & 5 \\ 2 & 3 & 8 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 6 & 5 \\ 5 & 4 & 3 & -3 \\ 7 & -7 & 9 & 5 \\ 2 & 3 & 8 & 4 \end{bmatrix}$$

are equal, because both matrices are of the same order 4 × 4 and their corresponding entries are equal.

**Operations on Matrices:**

Addition, subtraction and multiplication are the basic operations on the matrix. To add or subtract matrices, these must be of identical order and for multiplication; the number of columns in the first matrix equals the number of rows in the second matrix.

Addition of Matrices

Subtraction of Matrices

Multiplication of Matrices

**Addition:**

The sum A+B of two m-by-n matrices A and B.

When to add two matrix to be equal of order:

$$A= \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2*2} \qquad B = \begin{bmatrix} a` & b` \\ c` & d` \end{bmatrix}_{2*2}$$

$$A+B=\begin{bmatrix} a+a` & b+b` \\ c+c` & d+d` \end{bmatrix}_{2*2}$$

**Example\ 1**

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix}_{2*3}, B = \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix}_{2*3}$$

A+B=

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}_{2*3}$$

**Example\ 2**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 4 \\ 1 & 0 & 5 \end{bmatrix}_{3*3}, B = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & -6 \\ 3 & 0 & 4 \end{bmatrix}_{3*3}$$

$$A+B=\begin{bmatrix}2 & 1 & 0\\3 & 2 & 4\\1 & 0 & 5\end{bmatrix}_{3*3}+\begin{bmatrix}0 & 2 & 4\\1 & 2 & -6\\3 & 0 & 4\end{bmatrix}_{3*3}=\begin{bmatrix}2 & 3 & 4\\4 & 4 & -2\\4 & 0 & 9\end{bmatrix}_{3*3}$$

## Subtraction:

Two matrices A and B are said to be conformable for subtraction if they have the same order (i.e. same number of rows and columns) and their difference A - B is defined to be the addition of A and (-B).

A= $\begin{bmatrix}a & b\\c & d\end{bmatrix}_{2*2}$ $\qquad$ $B=\begin{bmatrix}a` & b`\\c` & d`\end{bmatrix}_{2*2}$

A-B=$\begin{bmatrix}a-a` & b-b`\\c-c` & d-d`\end{bmatrix}_{2*2}$

**Example**

$$A=\begin{bmatrix}2 & 1 & 0\\3 & 2 & 4\\1 & 0 & 5\end{bmatrix}_{3*3}\quad , \quad B=\begin{bmatrix}0 & 2 & 4\\1 & 2 & -6\\3 & 0 & 4\end{bmatrix}_{3*3}$$

$$A-B=\begin{bmatrix}2 & 1 & 0\\3 & 2 & 4\\1 & 0 & 5\end{bmatrix}_{3*3}-\begin{bmatrix}0 & 2 & 4\\1 & 2 & -6\\3 & 0 & 4\end{bmatrix}_{3*3}=\begin{bmatrix}2 & -1 & -4\\2 & 0 & 10\\-2 & 0 & 1\end{bmatrix}_{3*3}$$

**Multiplication:**

If A and B be any two matrices, then their product AB will be defined only when the number of columns in A is equal to the number of rows in B.

The following will show how to multiply two 2x2 matrices:

**Example / 1**

$$A=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}_{2*2}\quad ,B=\begin{bmatrix}-8 & 4\\2 & 5\end{bmatrix}_{2*2}$$

**Solution**

$$A.B=\begin{bmatrix}1*-8\ +\ 0*2 & 1*4\ +\ 0*5\\0*-8\ +\ 1*2 & 0*4\ +\ 1*5\end{bmatrix}_{2*2}=\begin{bmatrix}-8 & 4\\2 & 5\end{bmatrix}_{2*2}$$

**Example / 2**

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -6 & 2 \end{bmatrix}_{2*3} \qquad , B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 4 & 0 & 5 \end{bmatrix}_{3*3}$$

$A.B=$

$$\begin{bmatrix} 2*1 + 1*2 + 3*4 & 2*3 + 1*1 + 3*0 & 2*1 + 1*2 + 3*5 \\ \mathbf{4*1 + -6*2 + 2*4} & \mathbf{4*3 + -6*1 + 2*0} & \mathbf{4*1 + -6*2 + 2*5} \end{bmatrix}_{2*3}$$

$$A.B = \begin{bmatrix} 16 & 7 & 19 \\ 0 & 9 & 2 \end{bmatrix}_{2*3}$$

**Determinant of a Matrix saris method :**

In mathematics, the determinant is a scalar value that is a function of the entries of a square matrix. ... The determinant of a product of matrices is the product of their determinants (the preceding property is a corollary of this one). The determinant of a matrix A is denoted det(A), det A, or |A|.

The matrix has to be square (same number of rows and columns) Like shown below:

**For a 2×2 Matrix**

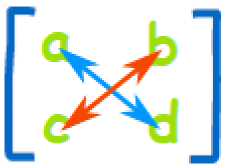
For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

|A| = ad − bc *"The determinant of A equals a times d minus b times c"* It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)




$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix
(This one has 2 Rows and 2 Columns)

Let us calculate the determinant of that matrix:

$$3\times6 - 8\times4$$
$$= 18 - 32$$
$$= \mathbf{-14}$$

Example: find the determinant of

$$C = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

Answer:

$$|C| = 4\times8 - 6\times3$$
$$= 32 - 18$$
$$= 14$$

**For a 3×3 Matrix sarus method**

**Example**

Calculate the determinant of the following 3x3 matrix using Sarrus' rule.

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3*3}$$

**Solution**

Write the first two columns outside the determinant of the matrix and draw diagonal lines like this:

$$\begin{array}{ccc|cc} 2 & 2 & -2 & 2 & 2 \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 4 & 2 & 3 \end{array}$$

**| A | = (2*2*4 + 2* 3 *2 + (-2)*1*3) – ((-2)*2*2 + 2*3* 3 + 2*1*4)**

**| A | = 16+12-6+8-18-8=4**

**Inverse matrix :**

• Inverse of Matrix for a matrix A is A-1. The inverse of a 2 × 2 matrix can be calculated using a simple formula. Further, to find the inverse of a 3 × 3 matrix, we need to know about the determinant and adjoint of the matrix. The inverse of matrix is another matrix, which on multiplying with the given matrix gives the multiplicative identity.

The inverse of matrix is used of find the solution of linear equations through the matrix inversion method. Here, let us learn about the formula, methods, and terms related to the inverse of matrix.

**We can calculate the Inverse of a Matrix** by:

• Step 1: calculating the Matrix of Minors,

• Step 2: then turn that into the Matrix of Cofactors,

• Step 3: then the Adjugate, and

• Step 4: multiply that by 1/Determinant.

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{vmatrix} a & -b \\ -c & d \end{vmatrix}$$

$$A^{-1} = \frac{1}{|A|} \, adj \, A$$

The inverse of matrix exists only if the determinant of the matrix is a non-zero value. The matrix whose determinant is non-zero and for which the inverse matrix can be calculated is called an invertible matrix.

The inverse matrix formula can be given as,

A-1 = adj(A)/|A|; |A| ≠ 0

**Example 1:** Find the inverse of matrix A = $\begin{pmatrix} -3 & 4 \\ 2 & 5 \end{pmatrix}$.

**Solution:**

The given matrix is A = $\begin{pmatrix} -3 & 4 \\ 2 & 5 \end{pmatrix}$.

The formula to calculate the inverse of matrix for a matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Using this formula we can calculate $A^{-1}$ as follows.

$$A^{-1} = \frac{1}{(-3) \times 5 - 4 \times 2}\begin{pmatrix} 5 & -4 \\ -2 & -3 \end{pmatrix}$$

$$= \frac{1}{-15 - 8}\begin{pmatrix} 5 & -4 \\ -2 & -3 \end{pmatrix}$$

$$= \frac{-1}{23}\begin{pmatrix} 5 & -4 \\ -2 & -3 \end{pmatrix}$$

**Answer:** Therefore $A^{-1} = \frac{-1}{23}\begin{pmatrix} 5 & -4 \\ -2 & -3 \end{pmatrix}$

**Example**

Find the inverse of the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

**Solution**

Using the formula

$$A^{-1} = \frac{1}{(3)(2)-(1)(4)}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$

This could be written as

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

**Example 3:** Find the inverse of $\begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix}$.

**Solution:**

To find: Inverse of matrix $\begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix}$

Using the inverse of a matrix formula,

$$A^{-1} = \frac{adj(A)}{|A|}; A \neq 0$$

$$A^{-1} = \frac{1}{\det\begin{pmatrix} 4 & 2 \\ -1 & 5 \end{pmatrix}}\begin{pmatrix} 5 & -2 \\ 1 & 4 \end{pmatrix}$$

Since, det $\begin{pmatrix} 4 & 2 \\ -1 & 5 \end{pmatrix} = 22$

$$A^{-1} = \frac{1}{22}\begin{pmatrix} 5 & -2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5/22 & -2/22 \\ 1/22 & 4/22 \end{pmatrix}$$

**Answer: Inverse of matrix** $\begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 5/22 & -1/11 \\ 1/22 & 2/11 \end{bmatrix}$

Exe: find the($A^{-1}$) of matrix:

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix}_{3\times 3}$$

$$|A| = \begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ 1 & 0 & -2 \end{vmatrix}\begin{matrix} 1 & -1 \\ 3 & 0 \\ 1 & 0 \end{matrix}\Bigg|$$

**|A|= 1.0. (-2) + (-1).1.1 + (-2).3.0 - 0 + 0 + (-1).3.-2 = -1 + 6 = 5 ≠ 0**

$$adj\ (A) = \begin{bmatrix} A11 & A12 & A13 \\ A21 & A22 & A23 \\ A31 & A32 & A33 \end{bmatrix}_{3\times 3}$$

***First Row***

$$A11 = (-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 0 & -2 \end{vmatrix} = +(\mathbf{0} - \mathbf{0}) = \mathbf{0}$$

$$A12 = (-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -(\mathbf{6} - \mathbf{1}) = -\mathbf{4}$$

$$A13 = (-1)^{1+1}\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = +(0 - 0) = 0$$

***Second Row***

$$A21 = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 0 & 2 \end{vmatrix} = -(-2 - 0) = 2$$

$$A22 = (-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = +(\mathbf{2} - \mathbf{2}) = \mathbf{0}$$

$$A23 = (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0 + 1) = 1$$

***Third Row***

$$A31 = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = +(-1 - 0) = -1$$

$$A32 = (-1)^{3+1} \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(\mathbf{1} - \mathbf{6}) = \mathbf{5}$$

$$A33 = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = +(0 + 3) = 3$$

$$adj(A) = \begin{matrix} 0 & -5 & 0 \\ 2 & 0 & -1 \\ -1 & 5 & 3 \end{matrix}$$

$$adj(A) = \begin{bmatrix} 0 & 2 & -1 \\ -5 & 0 & 5 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{\begin{bmatrix} 0 & 2 & -1 \\ -5 & 0 & 5 \\ 0 & -1 & 3 \end{bmatrix}}{5}$$

$$\mathrm{A}^{-1} = \begin{bmatrix} \frac{0}{5} & \frac{2}{5} & -\frac{1}{5} \\ -\frac{5}{5} & \frac{0}{5} & \frac{5}{5} \\ \frac{0}{5} & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

**Cofactor Matrix**

The co-factor matrix is useful to find the adjoint of the matrix and the inverse of the given matrix.

Co-factor matrix is a matrix having the co-factors as the elements of the matrix. First, let us understand more about the co-factor of an element within the matrix. Co-factor of an element within the matrix is obtained when the minor $Mij$ of the element is multiplied with $(-1)i + j$. Here $i$ and $j$ are the positional values of the element and refers to the row and the column to which the given element belongs. The co-factor of the element is denoted as $ij$ . If the minor of the element is Mij, then the co-factor of element would be:

$$\boldsymbol{Cij = (-1)\, i + j \,|\, Mij \,|}$$

Here first we need to find the minor of the element of the matrix and then the co-factor, to obtain the co-factor matrix .

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The minor of the element a12 is as follows.

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

**For a 3×3 Matrix cofactor method**

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

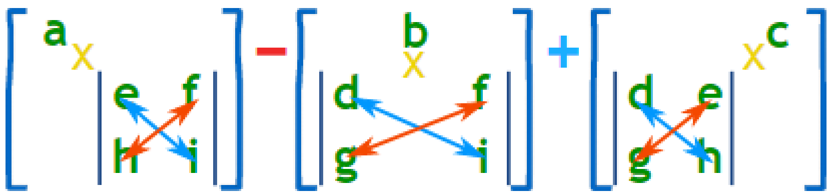
For a 3×3 matrix (3 rows and 3 columns):

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

*"The determinant of A equals ... etc"*

It may look complicated, but **there is a pattern**:



To work out the determinant of a **3×3** matrix:

- Multiply **a** by the **determinant of the 2×2 matrix** that is **not in a**'s row or column.
- Likewise for **b**, and for **c**
- Sum them up, but remember the minus in front of the **b**

As a formula *(remember the vertical bars || mean "determinant of")*:

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

*"The determinant of A equals a times the determinant of ... etc"*

**Ex: Find the determinant of method A by using cofactor matrix.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$\boldsymbol{A\,ij\; = (-1)i + j\,|\,Mij\,|}$

***First row:***

$$A11 = (-1)^{1+1} \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = +\,(\mathbf{4.2 - 1.1}) = \mathbf{7}$$

$$A12 = (-1)^{1+1} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = -\,(\mathbf{2.2 - 1.0}) = \mathbf{-4}$$

$$A13 = (-1)^{1+1} \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = +\,(2.1 - 4.0) = 2$$

$$\boldsymbol{|\,A\,| = a11\,.\,A11 + a12\,.\,A12 + a13\,.\,A13}$$

$$= \mathbf{1.7 + 2.(-4) + 3\,.\,2}$$

$$= \mathbf{7 - 8 + 6 = 5}$$

**Linear Equation:**

**Solving a System of Linear Equations Using the Inverse of a Matrix**

**Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: X is the matrix representing the variables of the system, and B is the matrix representing the constants. Using matrix multiplication, we may define a system of equation with the same number of equations as variables as**

**AX=B**

**To solve a system of linear equations using an inverse matrix let display style AX be the coefficient matrix, let X be the variable matrix, and let B be the constant matrix. Thus, we want to solve a system AX = B . For example, look at the following system of equations.**

$$\boldsymbol{a_1 x + b_1 y = c_1}$$

$$\boldsymbol{a_2 x + b_2 y = c_2}$$

**From this system, the coefficient matrix is**

$$A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{b_1} \\ \boldsymbol{a_2} & \boldsymbol{b_2} \end{bmatrix}$$

The variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

In addition, the constant matrix is

$$B = \begin{bmatrix} \boldsymbol{c_1} \\ \boldsymbol{c_2} \end{bmatrix}$$

**Solve the system of the linear equation by using inverse matrix**

$ax_1 + ax_2 + ax_3 + ax_4 \; ....... = b$

$X = A^{-1}B$

**Ex1 : Solve the system of linear equation by using inverse matrix**

$X + 2Y = 5$

$3X - Y = 1$

**Sol :**

$$X = \begin{bmatrix} X \\ Y \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$X = A^{-1} B$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$|A| = 1 . (-1) \; - 2 . 3 = -7$

First Row

A11= -1

A12 = - 3

Second Row

A21 = -2

A22 = 1

$$Adj(A) = \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}$$

$X = A^{-1} B$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$|A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} \begin{matrix} -2 & 2 \\ 2 & 1 \\ -1 & -2 \end{matrix}$$

$$|A| = [(-2) . 1 . 0 + 2 . (-6) . (-1) + (-3) . 2 . (-2)] - [(-3) . 1 . (-1) + (-2) . (-6) . (-2) + 2 . 2 . 0]$$

$$= (0 + 12 + 12) - (3 - 24 + 0) = 45 \neq 0$$

$$Adj(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{ij} = (-1)^{1+j} |M_{ij}|$$

## First Row

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} = +(0 - 12) = -12$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} = -(0 - 6) = 6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} = +(-4 + 1) = -3$$

## Second Row

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -3 \\ -2 & 0 \end{vmatrix} = -(0 - 6) = 6$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} = +(0 - 3) = -3$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} -2 & 2 \\ -1 & -2 \end{vmatrix} = -(4 - (-2)) = -6$$

## Third Row

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -3 \\ 1 & -6 \end{vmatrix} = +(-12 + 3) = -9$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} -2 & -3 \\ 2 & -6 \end{vmatrix} = -(12 + 6) = -18$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = +(-2 - 4) = -6$$

$$adj(A) = \begin{bmatrix} -12 & 6 & -3 \\ 6 & -3 & -6 \\ -9 & -18 & -6 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} -12 & 6 & -9 \\ 6 & -3 & -18 \\ -3 & -6 & -6 \end{bmatrix}$$

$$X = A^{-1} B$$

$$X = \frac{Adj(A)}{|A|} \cdot B$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{\begin{bmatrix} -12 & 6 & -9 \\ 6 & -3 & -18 \\ -3 & -6 & -6 \end{bmatrix}}{45} \cdot \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$$

$$X_1 = \frac{-12 . 5 + 6 . 10 + (-9) . (-5)}{45} \quad \frac{-60 + 60 + 45}{45} \quad \frac{45}{45} = 1$$

$$X_2 = \frac{6 . 5 + (-3) . 10 + (-18) . (-5)}{45} \quad \frac{30 + (-30) + 90}{45} \quad \frac{90}{45} = 2$$

$$X_3 = \frac{(-3) . 5 + (-6) . 10 + (-6) . (-5)}{45} \quad \frac{-15 - 60 + 30}{45} \quad \frac{-45}{45} = -1$$

## Derivative :

Derivatives are defined as the varying rate of change of a function with respect to an independent variable. The derivative is primarily used when there is some varying quantity, and the rate of change is not constant.

If an infinitesimal change in x is denoted as dx, then the derivative of y with respect to x is written as **dy/dx**.

$\bar{y} \cdot \frac{dy}{dx}$

$y = x \Rightarrow \bar{y} = 1$

$y = ax \Rightarrow \bar{y} = a$

$y = x^n \Rightarrow \bar{y} = n.x^{n-1}$

ملاحظة : اذا كان عدنا حاصل ضرب دالتين :

الأولى x مشتقة الثانية + الثانية x مشتقة الأولى .

حاصل القسمة دالتين :

مشتقة البسط x المقام - مشتقة المقام x البسط

(المقام )$^2$

**ex : find the derivative for every flowing function :**

1- $y = x^3 - 5x^2 + 2x^3 + 7$

$\bar{y} = 3x^2 - 10x + 6x^2$

2- $y = (2x + 4)^2$

$\bar{y} = 2\,(2x + 4)\,.\,2$

$\bar{y} = 4\;(2x + 4)$

3- $y = (x - 4)\,(x + 5)$ ✓

$\bar{y} = (x - 4)\,.\,1 + (x + 5)\,.\,1$

$\bar{y} = (x - 4) + (x + 5)$

4- $y = \dfrac{(4x - 1)}{(5x + 3)}$

$\bar{y} = \dfrac{4 . (5x + 3) - 5 . (4x - 1)}{(5x + 3)^2}$

$y = \dfrac{20x + 12 - 20x + 5}{(5x + 3)^2}$

$\bar{y} = \dfrac{17}{(5x + 3)^2}$

تكامل

**Integration :**

This process is the reverse of finding a derivative. Integrations are the anti-derivatives. Integrations are the way of adding the parts to find the whole. Integration is the whole pizza and the slices are the differentiable functions which can be integrated. If f(x) is any function and f'(x) is its derivatives. The integration of f'(x) with respect to dx is given as

$\int f'(x)\, dx = f(x) + C$

$\int x^n\, dx$

$= \dfrac{x^{n+1}}{n+1} + C$

$\int dx \Longrightarrow = X + C$

$\int 2\, dx = 2X + C$

Ex1 : Calculator the flowing Integration

$$\int (6x^2 + 4x + 2)\,dx$$

Sol : $2+1 \quad 1+1 \Rightarrow 6x^{2+1}$

$$6\frac{x^3}{3} + 4\frac{x^2}{2} + 2x + c$$

$$2x^3 + 2x^2 + 2x + c$$

Ex 2:

$$\int \frac{x}{(x^2+1)^2}\,dx$$

Sol:

$$= \int x(x^2+1)^{-2}\,dx$$

$$= \frac{1}{2}\int 2x(x^2+1)^{-2}\,dx$$

$$= \frac{1}{2}\frac{(x^2+1)^{-2+1}}{-2+1} + C$$

$$= \frac{1}{2}\frac{(x^2+1)^{-1}}{-1} + C$$

$$= -\frac{1}{2}(x^2+1)^{-1} + C$$

Ex3 :

$$\int \frac{x}{\sqrt{2x^2+4}}\,dx$$

Sol:

$$= \int x\sqrt{2x^2+4}\;dx$$

$$= \int x(2x^2+4)^{-\frac{1}{2}}\,dx$$

اخراج مشتقة داخل القوس

تعديل

$$= \frac{1}{4}\int 4x(2x^2+4)^{-\frac{1}{2}}\,dx$$

$$\frac{1}{4}\frac{(2x^2+4)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$$

$$= \frac{1}{4}\frac{(2x^2+4)^{\frac{1}{2}}}{\frac{1}{2}}+C$$

$$= \frac{1}{4}\cdot\frac{2}{1}(2x^2+4)^{\frac{1}{2}}+c$$

$$= \frac{1}{2}(2x^2+4)^{\frac{1}{2}}+c$$

باعتبار الجذر التربيعي

$$= \frac{1}{2}\sqrt{2x^2+4}+C$$

Ex4 : $\int \frac{x}{(x^2+1)^2} dx$

Sol:

$=\int x(x^2+1)^{-2}\,dx$

$=\frac{1}{2}\int 2^x(x^2+1)^{-2}\,dx$

$$= \frac{1}{2}\frac{(x^2+1)^{-2+1}}{-2+1} + C$$

Ex 5: $\int \frac{x}{(x^2+1)^2} dx$

Sol:

$=\int x(x^2+1)^{-2}\,dx$

$=\frac{1}{2}\int 2^x(x^2+1)^{-2}\,dx$

$$= \frac{1}{2}\frac{(x^2+1)^{-2+1}}{-2+1} + C$$

$=\frac{1}{2}\frac{(x^2+1)^{-1}}{-1} + C$

$$= -\frac{1}{2}(x^2+1)^{-1} + C$$

logarithmtic functions :

Logarithmic functions are the inverses of exponential functions. The inverse of the exponential function $y = a^x$ is $x = a^y$. The logarithmic function $y = \log_a x$ is defined to be equivalent to the exponential equation $x = a^y$. $y = \log_a x$ only under the following conditions: $x = a^y$, $a > 0$, and $a \neq 1$. It is called the logarithmic function with base a.

Consider what the inverse of the exponential function means: $x = a^y$. Given a number x and a base a, to what power y must a be raised to equal x? This unknown exponent, y, equals $\log_a x$. So you see a logarithm is nothing more than an exponent. By definition, $a^{\log_a x} = x$, for every real $x > 0$.

صيغة عامة /

$y = \ln(x)$

$\ln 1 = 0$ → صفر

$\ln(ax) = \ln(a) + \ln(x)$

$\ln\frac{x}{a} = \ln(x) - \ln(a)$

$\ln x^n = n \ln(x)$

**Derivative of the Logarithm Function $y = \ln x$**

The derivative of the logarithmic function $y = \ln x$ is given by:

دالة

$y = \ln(x) \Rightarrow \bar{y} = \frac{1}{x} \cdot 1$

ملاحظة :

$\frac{1}{\text{دالة}}$ X ( مشتقة دالة )

ex: Find the derivative of

$$y = \ln(x^2)$$

sol : $\bar{y} = \frac{1}{x^2} \cdot (2x)$

$$\bar{y} = \frac{2x}{x^2}$$

find the derivative of the function :

ex 1: $y = \ln(2x + 5)$

$$\bar{y} = \frac{1}{2x+5} \cdot 2$$

$$\bar{y} = \frac{2}{2x+5}$$

Ex2:

$$y = \ln \frac{(x^3+1)^6}{(3x+7)^{\frac{1}{3}}}$$

:- $\ln \frac{x}{a} = \ln(x) - \ln(a)$ حسب الخاصية

$= \ln(x^3 + 1)^6 - \ln(3x+7)^{1/3}$

:- $\ln x^n = n \ln(x)$ حسب الخاصية

$y = 6\ln(x^3 + 1) - 1/3 \ln(3x+7)$

$$\bar{y} = 6\left[\frac{1}{x^3+1} \cdot (3x^2)\right] - \frac{1}{3}\left[\frac{1}{3x+7} \cdot 3\right]$$

$$\bar{y} = \frac{18x^2}{x^3+1} - \frac{1}{3x+7}$$

Ex3:

$$y = [ln(5x+1)]^3$$

Sol

$$\bar{y} = 3(ln(5x+1))^2 \cdot \frac{1}{5x+1} \cdot 5$$

$$\frac{15}{5x+1}[ln(5x+1)]^2$$

Ex4: y = ln (ln (x))

$$\bar{y} = \frac{1}{ln(x)} \cdot \frac{1}{x} \cdot 1$$

واحدها Ln x
تحت واحدها
الـ x

Ex5: y= ln (ln $x^2$))

$$\bar{y} = \frac{1}{ln(x^2)} \cdot \frac{1}{x^2} \cdot 2x$$

$$\bar{y} = \frac{2}{x\, ln(x^2)}$$

| |
|---|
| ∫ 1/دالة dx<br>اذا توفرت مشتقة الدالة من ضمن المقدار<br>= ln (دالة) + c |

التكامل

Integration logarithmtic :

Ex1:

$$\int \frac{1}{x} dx$$

=ln (x) + C

Ex2:

$$\int \frac{2x}{(x^2+1)}dx$$

مشتقة المقام متوفرة بالبسط

تهمل

$x^2 = 2x$

تكامل مقام ...

$$= \ln(x^2+1) + C$$

Ex3:

$$\int \frac{x}{(x^2+1)}dx$$

لا نكامل لعدم وجود مشتقة المقام في البسط

$$= \frac{1}{2}\int \frac{2x}{(x^2+1)}dx$$

$$= \frac{1}{2}\ln(x^2+1) + c$$

Ex4:

$$\int \frac{2}{(2x+1)[ln(2x+1)]^3}\,dx$$

$$ln(2x+1) = \frac{1}{2x+1}\cdot 2$$

Sol:

$$\int \frac{2}{(2x+1)}ln(2x+1)^{-3}dx$$

$$\frac{ln(2x+1)^{-2}}{-2} + c$$

Ex5:

$$\int \frac{x}{(x^2-9)}dx$$

Sol:

$$=\frac{1}{2}\int \frac{2x}{x^2-9}dx$$

$$=\frac{1}{2}ln(x^2-9)+c$$

$x^2 = 2x$

نقسم على ② ونضرب في ②

نهمل 2x لوجود المشتقة في المقام

---

Exponential function:

exponential function, in mathematics, a relation of the form y = $a^x$, with the independent variable x ranging over the entire real number line as the exponent of a positive number a. Probably the most important of the exponential functions is y = $e^x$, sometimes written y = exp (x), in which e (2.7182818…) is the base of the natural system of logarithms (ln).

$y = e^x$ , x > 0

$e^0 = 1$

$e^x \cdot e^y = e^{xy}$

$(e^x)^y = e^{xy}$

$\frac{e^x}{e^y} = e^{x-y}$

$e\, ln(x) = x$

$ln\, e^x = x\, ln\, e^0 = x$

$y = e^u$

$\bar{y} = e^u \cdot \frac{dy}{dx}$

find the derivative of the function :

ex 1:

$y = e^{x^3}$

Sol:

$\bar{y} = e^{x^3} \cdot 3x^2$

$\bar{y} = 3x^2 e^{x^3}$

Ex2:

$y = e^{-2x}$

$\bar{y} = e^{-2x} \cdot (-2)$

$\bar{y} = -2e^{-2x}$

2022

رياضيات

(...ة فقط)

Ex3:

$y = x^2 \cdot e^{-x^2}$

$\bar{y} = x^2 \cdot [e^{-x^2} \cdot (-2x)] + e^{-x^2} \cdot 2x$

$= -2x^3 e^{-x^2} + 2x e^{-x^2}$

$= 2e^{-x^2}[-x^3 + x]$

ترتيب (تبسيط) (المعادلة)

Ex4:

$y = e^x \cdot ln(x)$

$\bar{y} = e^x \cdot \frac{1}{x} \cdot 1 + ln(x) \cdot e^x \cdot 1$

$\bar{y} = \frac{e^x}{x} + ln(x)e^x$

تبسيط معادلة

## Integration Exponential :

find the integration of the function :

Ex1:

$\int e^x \, dx$

Sol:

$= e^x + c$

Ex2:

$\int x e^{x^2} \, dx$

ملاحظة :

$e^{x^2} = e^{x^2} \cdot 2x$

$= \frac{1}{2} \int 2x \, e^{x^2} \, dx$ → نقسم على 2 ونضرب في 2

يهمل

$= \frac{1}{2} \left[ e^{x^2} + C \right]$

Ex3:

تهمل

$\int e^{3x} \, dx$

$e^{3x} \cdot 3$

$= \frac{1}{3} \int 3 e^{3x} \, dx$

$\frac{1}{3} \left[ e^{3x} + c \right]$

Ex4:

$\int x^2 e^{(x^3+1)} \, dx$

$e^{x^3+1} = e^{x^3+1} \cdot 3x^2$

$= \frac{1}{3} \int 3x^2 e^{(x^3+1)} \, dx$

$= \frac{1}{3} \left[ e^{(x^3+1)} + c \right]$

Ex5:

$$\int \frac{e^x - e^{-x}}{2}\, dx$$

$$= \frac{1}{2} \int e^x - e^{-x}\, dx \longrightarrow$$ تجزئة

$$= \frac{1}{2} \int e^x\, dx - \int e^{-x}\, dx$$

$$= \frac{1}{2} [e^x - [-\int -e^{-x}\, dx]$$

$$= \frac{1}{2} [e^x + e^{-x}] + c$$

الى هنا امتحان شهري

---

trigonometric functions :

trigonometry, the branch of mathematics concerned with specific functions of angles and their application to calculations. There are six functions of an angle commonly used in trigonometry. Their names and abbreviations are sine (sin), cosine (cos), tangent (tan), cotangent (cot), secant (sec), and cosecant (csc).

sin(x) = cos(x)

cos (x) = - sin (x)

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

$$\mathrm{Cos}^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin^2(x) + \cos^2(x) = 1$$

$1 + \tan^2(x) = \sec^2(x)$

$1 + \cot^2(x) = \csc^2(x)$

| | |
|---|---|
| $y = \sin(x)$ | $\bar{y} = \cos(x) \cdot \frac{dy}{dx}$ |
| $y = \cos(x)$ | $\bar{y} = -\sin(x) \cdot \frac{dy}{dx}$ |
| $y = \tan(x)$ | $\bar{y} = \sec^2(x) \cdot \frac{dy}{dx}$ |
| $y = \sec(x)$ | $\bar{y} = \sec(x)\tan(x) \cdot \frac{dy}{dx}$ |
| $y = \cot(x)$ | $\bar{y} = \csc^2(x) \cdot \frac{dy}{dx}$ |
| $y = \csc(x)$ | $\bar{y} = -\csc(x)\cot(x) \cdot \frac{dy}{dx}$ |

find the derivative of the trigonometric function :

ex1:

$y = \cos(4x)$

$\bar{y} = -\sin(4x) \cdot 4$

$\bar{y} = -4\sin(4x)$

ex2:

$y = \sqrt{\sin(x)}$

$\bar{y} = \frac{1}{2}(\sin(x))^{-\frac{1}{2}} \cdot \cos(x)$

$\bar{y} = \frac{\cos(x)}{2(\sin(x))^{\frac{1}{2}}}$

$\bar{y} = \frac{\cos(x)}{2\sqrt{\sin(x)}}$

*Ex3:*

$y = \tan(\sqrt{x})$

sol:

$$\bar{y} = sec^2(x)^{\frac{1}{2}} \cdot \frac{1}{2} \cdot x^{--\frac{1}{2}}$$

$$\bar{y} = \frac{1}{2}\frac{sec\left(x^{\frac{1}{2}}\right)}{x^{1/2}}$$

$$\bar{y} = \frac{1\,sec(\sqrt{x})}{2\sqrt{x}}$$

Ex4:

$y = \cot(x)\csc(x)$

sol :

$$\bar{y} = \cot(x)\,.[-\csc(x)\cot(x)] + \csc(x).(-csc^2(x)]$$

$$\bar{y} = -\csc(x)\ cot^2(x) - csc^3(x)$$

Ex3:

$y = \tan(\sqrt{x})$

sol:

$$\bar{y} = sec^2(x)^{\frac{1}{2}} \cdot \frac{1}{2} \cdot x^{--\frac{1}{2}}$$

$$\bar{y} = \frac{1}{2}\frac{sec\left(x^{\frac{1}{2}}\right)}{x^{1/2}}$$

$$\bar{y} = \frac{1\,sec(\sqrt{x})}{2\sqrt{x}}$$

Ex4:

$y = \cot(x)\csc(x)$

sol :

$$\bar{y} = \cot(x)\,.[-\csc(x)\cot(x)] + \csc(x).(-csc^2(x)]$$

$$\bar{y} = -\csc(x)\ cot^2(x) - csc^3(x)$$

Integration trigonometric:

$\int \sin(x)\ dx = -\cos(x) + c$

$\int \cos(x)\ dx = \sin(x) + c$

$\int sec^2(x)\ dx = \tan(x) + c$

$\int \csc^2(x)\ dx = -\cot(x) + c$

$\int \sec(x) \tan(x)\ dx = \sec(x) + c$

$\int \csc(x) \cot(x)\ dx = -\csc(x) + c$

find the integration of the function :

ex1:

$y = \int \tan(x)\, dx$

sol :

$= \int \frac{\sin(x)}{\cos(x)}\ dx$

$= - \int \frac{-\sin(x)}{\cos(x)}\ dx$

$= - \ln \cos(x) + c$

$= \ln (cos^{-1}(x)) + c$

$= \ln \frac{1}{\cos(x)} + c$

$= \ln \sec(x) + c$

Ex2:

$$\int \cos(5x)\, dx$$

Sol :

$$=\frac{1}{5}\int 5\cos(5x)\, dx$$

$$=\frac{1}{5}[\sin(5x) + c]$$

Ex3 :

$$\int \frac{\cos(x)}{1+\sin(x)}\, dx$$

Sol:

$$= \ln(1+\sin(x)) + c$$

Ex4:

$$\int e^{\tan(x)}\ sec^2(x)$$

Sol:

$$= e^{\tan(x)} + c$$

Ex5:

$$\int cot^2(x)dx$$

$$\int csc^2(x) - 1\,dx$$

$$\int csc^2(x) - 1\,dx$$

$$= -\cot(x) - x + C$$