De Moivre's Theorem

Prerequisites

You should be familiar with the various ways of representing a complex number in Cartesian form, in polar (trigonometric) form and in exponential form.

$$\begin{aligned} z &= [|z|, \arg z] \\ &= [r, \theta] \\ &= r(\cos\theta + i\sin\theta) \\ &= (x, y) \\ &= x + iy \\ &= re^{i\theta} \\ &= |z|e^{i\arg z} \end{aligned}$$

The first three here are three forms of the polar representation of z; the next two are Cartesian forms, the last two are exponent forms. To understand this chapter you also require knowledge of mathematical induction.

De Moivre's theorem

De Moivre's theorem is a result that enables us to find powers and roots of complex numbers. It tells us how to evaluate powers of a complex number – that is, how to find z^n. It can be expressed in Cartesian and polar (trigonometric) form.

De Moivre's theorem – Cartesian form

$$z^n = r(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

De Moivre's theorem – Polar form

$$z^n = [r, \theta]^n = [r^n, n\theta]$$

Example (1)

Express $\left(2\cos\frac{\pi}{8} + 2i\sin\frac{\pi}{8}\right)^2$ in the form $x + iy$.

Solution

$$\left(2\cos\frac{\pi}{8}+2i\sin\frac{\pi}{8}\right)^2=\left[2,\frac{\pi}{8}\right]^2 \qquad \text{[Putting } z \text{ in polar form]}$$
$$=\left[2^2,2\times\frac{\pi}{8}\right] \qquad \text{[Applying De Moivre's theorem]}$$
$$=\left[4,\frac{\pi}{4}\right]$$
$$=\left(4\cos\frac{\pi}{4}+i4\sin\frac{\pi}{4}\right) \qquad \text{[Returning to Cartesian form]}$$
$$=2\sqrt{2}+2\sqrt{2}\,i$$

Proof of De Moivre's Theorem

The proof of De Moivre's theorem follows by mathematical induction and exploits the property of multiplication of complex numbers. In polar form this is

$[r_1,\theta_1][r_2,\theta_2]=[r_1r_2,\theta_1+\theta_2]$

The proof in polar form is particularly straightforward and elegant.

Proof of De Moivre's Theorem

To prove

$z^n=[r,\theta]^n=[r^n,n\theta]$

Proof by mathematical induction.

For the particular step, when $n=1$ $[r,\theta]^1=[r^1,1\times\theta]$

For the induction step the induction hypothesis is

For $n=k$ $[r,\theta]^k=[r^k,k\theta]$

$[r,\theta]^k=[r^k,k\theta]$

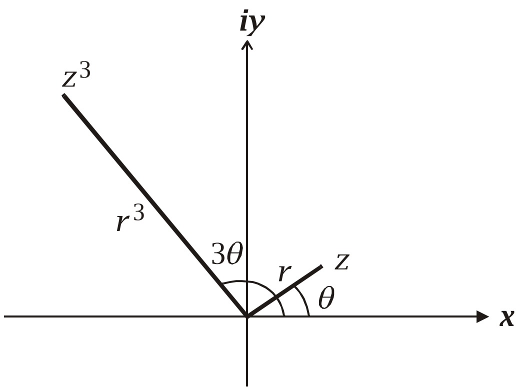
To prove for $n=k+1$ $[r,\theta]^{k+1}=[r^{k+1},(k+1)\theta]$. Now

$$[r,\theta]^{k+1}=[r,\theta][r,\theta]^k$$
$$=[r,\theta][r^k,k\theta] \qquad \text{[By the induction hypothesis]}$$
$$=[r\times r^{k+1},\theta+k\theta] \qquad \text{[Multiplication of complex numbers]}$$
$$=[r^{k+1},(k+1)\theta]$$

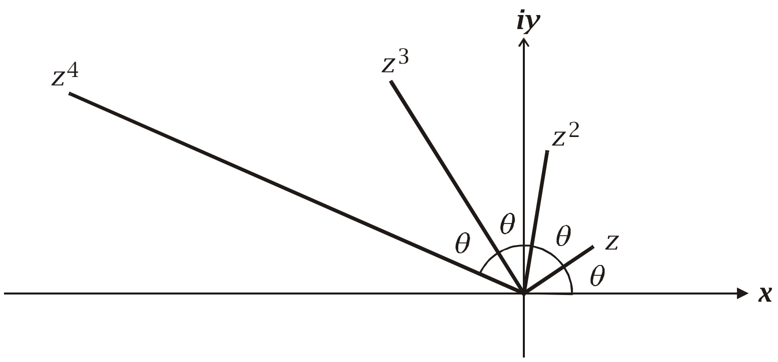
Hence the induction step holds and the result is true for all *n*. Converting into Cartesian form gives: $z^n=r(\cos\theta+i\sin\theta)^n=r^n\left(\cos n\theta+i\sin n\theta\right)$

Interpretation of De Moivre's Theorem and the *n* roots of unity

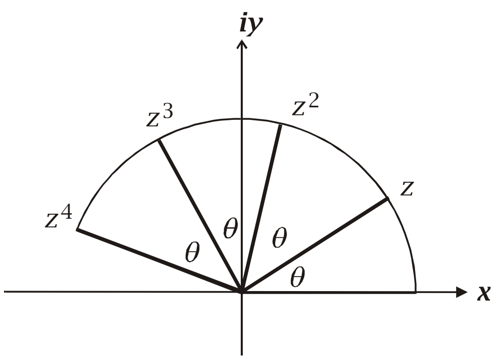
Suppose that $z = [r, \theta]$. For a definite illustration let us consider $z^3 = [r^3, 3\theta]$. Then graphically we plot z^3 by noting (1) that the argument of z^3 is 3 times the argument of *z*; (2) that the modulus of z^3 is the cube of the modulus of *z*.



If $r > 1$ then the values of $z^2, z^3, z^4, \ldots$ "spiral outwards".



If $r < 1$ then the values $z^2, z^3, z^4, \ldots$ "spiral inwards". Whilst if $r = 1$ then the values of $z^2, z^3, z^4, \ldots$ all lie on the unit circle.



The previous illustration suggests that we can apply De Moivre's theorem in reverse to find solutions to the equation $z^n = 1$. This is indeed the case. We observe that the equation $x^2 = 1$ has two solutions, $x = i$ and $x = -i$. Likewise, we expect the equation

$$z^n = 1$$

to have n solutions, and this is the case. In polar form the equation $z^n = 1$ takes the form

$$[r,\theta]^n = [1,0]$$

Applying De Moivre's theorem we get

$$[r^n, n\theta] = [1,0]$$

Hence $r^n = 1$ and $n = 1$ and $n\theta = 0$. One solution to the equation $n\theta = 0$ is $\theta = 0$. However, we should recall that the angle 0 is given modulo 2π and that

$$0 \equiv 2\pi = 4\pi \equiv ... \equiv 2n\pi \equiv ... \quad (\text{mod}\, 2\pi)$$

Hence the n roots of unity – that is the n roots to the equation $z^n = 1$ are given by the n distinct solutions to the equation $n\theta \equiv 0(\text{mod}\, 2\pi)$

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, ...$$

The solutions in polar form are the n distinct complex numbers

$$[1,0], \left[1, \frac{2\pi}{n}\right], \left[1, \frac{4\pi}{n}\right], \left[1, \frac{6\pi}{n}\right], ...$$

Example (2)

Solve $z^3 = 1$

Solution

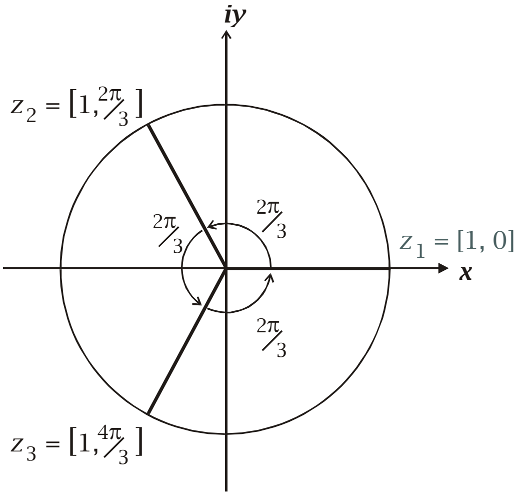
By substitution of $n = 3$ into the formula

$$[1,0], \left[1, \frac{2\pi}{n}\right], \left[1, \frac{4\pi}{n}\right], \left[1, \frac{6\pi}{n}\right], ...$$

the solutions are

$$[1,0], \left[1, \frac{2\pi}{3}\right], \left[1, \frac{4\pi}{3}\right]$$

Graphically, these solutions are represented as follows.



In Cartesian form

$z_1 = \cos 0 + i \sin 0 = (1,0)$ $\qquad$ $z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$z_3 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

We can also use De Moivre's theorem to find solutions to equations such as $z^4 = -1$.

Example (3)

Solve $z^4 = -1$.

Solution

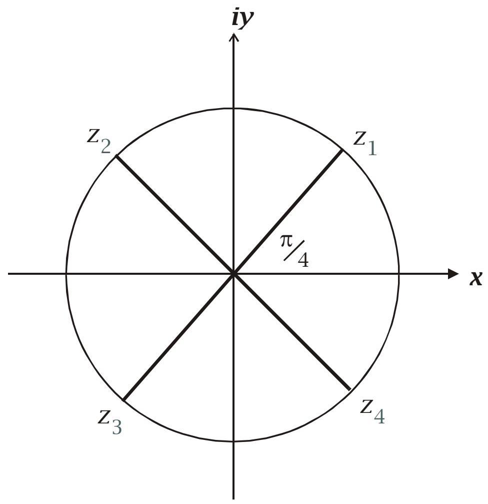
$[1,\theta]^4 = [1,\pi]$

$[1,4\theta] = [1,\pi]$

$\therefore 4\theta = \pi \quad (\mathrm{mod}\, 2\pi)$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$z_1 = \left[1, \frac{\pi}{4}\right] \qquad z_2 = \left[1, \frac{3\pi}{4}\right] \qquad z_3 = \left[1, \frac{5\pi}{4}\right] \qquad z_4 = \left[1, \frac{7\pi}{4}\right]$



In Cartesian form

$z_1 = \cos \pi/4 + i \sin \pi/4 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

$z_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ $z_3 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$ $z_4 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$

Applications of De Moivre's theorem to trigonometric identities

By expanding $(\cos\theta + i\sin\theta)^n$ using the Binomial theorem (or Pascal's triangle) and equating with $\cos n\theta + i\sin n\theta$ we can obtain further trigonometric identities. Recall that De Moivre's theorem is $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Since the real and imaginary parts of both sides of this equation are independent of each other, we can equate real and imaginary parts to obtain trigonometric identities. The whole process is best grasped through illustration.

Example (4)

Prove $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

Solution

By De Moivre's theorem

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

Pascal's triangle up to $n = 5$ gives

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Hence

$$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta$$
$$+10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$

Since $i^2 = -1$ we have

$$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$$
$$-10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

On equating real parts and using the identity $\cos^2\theta + \sin^2\theta \equiv 1$ we get

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta) \\ &= 11\cos^5\theta - 10\cos^3\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

By equating imaginary parts we can also show

$$\begin{aligned}\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\ &= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta \\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\end{aligned}$$

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)	12. $\int \tan x\,dx = \ln\lvert\sec x\rvert + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$	13. $\int \cot x\,dx = \ln\lvert\sin x\rvert + C$
3. $\int \frac{dx}{x} = \ln\lvert x\rvert + C$	14. $\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + C$
4. $\int e^x\,dx = e^x + C$	15. $\int \csc x\,dx = -\ln\lvert\csc x + \cot x\rvert + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$	16. $\int \sinh x\,dx = \cosh x + C$
6. $\int \sin x\,dx = -\cos x + C$	17. $\int \cosh x\,dx = \sinh x + C$
7. $\int \cos x\,dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
8. $\int \sec^2 x\,dx = \tan x + C$	19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
9. $\int \csc^2 x\,dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left\lvert\frac{x}{a}\right\rvert + C$
10. $\int \sec x \tan x\,dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
11. $\int \csc x \cot x\,dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x > a > 0)$

EXAMPLE 2 Complete the square to evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{8x - x^2}} &= \int \frac{dx}{\sqrt{16 - (x - 4)^2}} \\ &= \int \frac{du}{\sqrt{a^2 - u^2}} && a = 4,\ u = (x - 4),\ du = dx \\ &= \sin^{-1}\left(\frac{u}{a}\right) + C && \text{Table 8.1, Formula 18} \\ &= \sin^{-1}\left(\frac{x - 4}{4}\right) + C. \end{aligned}$$

■

EXAMPLE 3 Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x)\,dx.$$

Solution Here we can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\begin{aligned}
\int (\cos x \sin 2x + \sin x \cos 2x)\,dx &= \int (\sin(x + 2x))\,dx \\
&= \int \sin 3x\,dx \\
&= \int \frac{1}{3}\sin u\,du && u = 3x,\ du = 3\,dx \\
&= -\frac{1}{3}\cos 3x + C. && \text{Table 8.1, Formula 6}
\end{aligned}$$

■

In Section 5.5 we found the indefinite integral of the secant function by multiplying it by a fractional form identically equal to one, and then integrating the equivalent result. We can use that same procedure in other instances as well, which we illustrate next.

EXAMPLE 4 Find $\displaystyle\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

Solution We multiply the numerator and denominator of the integrand by $1 + \sin x$, which is simply a multiplication by a form of the number one. This procedure transforms the integral into one we can evaluate:

$$\begin{aligned}
\int_0^{\pi/4} \frac{dx}{1 - \sin x} &= \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\,dx \\
&= \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x}\,dx \\
&= \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\,dx \\
&= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\,dx && \text{Use Table 8.1, Formulas 8 and 10} \\
&= \Big[\tan x + \sec x\Big]_0^{\pi/4} = \big(1 + \sqrt{2} - (0 + 1)\big) = \sqrt{2}.
\end{aligned}$$

■

EXAMPLE 5 Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx.$$

Solution The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

$$\begin{array}{r}
x - 3 \\
3x + 2\,\overline{)\,3x^2 - 7x} \\
\underline{3x^2 + 2x} \\
-9x \\
\underline{-9x - 6} \\
+\,6
\end{array}$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx = \int\left(x - 3 + \frac{6}{3x + 2}\right)dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C. \qquad \blacksquare$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\,dx, \qquad \text{so} \qquad x\,dx = -\frac{1}{2}\,du.$$

Then we obtain

$$\begin{aligned} 3\int \frac{x\,dx}{\sqrt{1 - x^2}} &= 3\int \frac{(-1/2)\,du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\,du \\ &= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1. \end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2\sin^{-1}x + C_2. \qquad \text{Table 8.1, Formula 18}$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = -3\sqrt{1 - x^2} + 2\sin^{-1}x + C. \qquad \blacksquare$$

The question of what to substitute for in an integrand is not always quite so clear. Sometimes we simply proceed by trial-and-error, and if nothing works out, we then try another method altogether. The next several sections of the text present some of these new methods, but substitution works in the next example.

EXAMPLE 7 Evaluate

$$\int \frac{dx}{(1 + \sqrt{x})^3}.$$

Solution We might try substituting for the term $\sqrt{x}$, but we quickly realize the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1 + \sqrt{x})$, and it turns out this works:

$$\begin{aligned} \int \frac{dx}{(1 + \sqrt{x})^3} &= \int \frac{2(u - 1)\,du}{u^3} \qquad u = 1 + \sqrt{x},\ du = \frac{1}{2\sqrt{x}}\,dx;\ \ dx = 2\sqrt{x}\,du = 2(u - 1)\,du \\ &= \int\left(\frac{2}{u^2} - \frac{2}{u^3}\right)du \end{aligned}$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C$$

$$= \frac{1 - 2u}{u^2} + C$$

$$= \frac{1 - 2\left(1 + \sqrt{x}\right)}{\left(1 + \sqrt{x}\right)^2} + C$$

$$= C - \frac{1 + 2\sqrt{x}}{\left(1 + \sqrt{x}\right)^2}.$$

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

EXAMPLE 8 Evaluate $\int_{-\pi/2}^{\pi/2} x^3 \cos x\, dx$.

Solution No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\pi/2, \pi/2]$. Moreover, the factor x^3 is an odd function, and $\cos x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 \cos x\, dx = 0. \qquad \text{Theorem 8, Section 5.6}$$

Exercises 8.1

Assorted Integrations

The integrals in Exercises 1–40 are in no particular order. Evaluate each integral using any algebraic method or trigonometric identity you think is appropriate, and then use a substitution to reduce it to a standard form.

1. $\int_0^1 \frac{16x}{8x^2 + 2}\, dx$
2. $\int \frac{x^2}{x^2 + 1}\, dx$
3. $\int (\sec x - \tan x)^2\, dx$
4. $\int_{\pi/4}^{\pi/3} \frac{dx}{\cos^2 x \tan x}$
5. $\int \frac{1 - x}{\sqrt{1 - x^2}}\, dx$
6. $\int \frac{dx}{x - \sqrt{x}}$
7. $\int \frac{e^{-\cot z}}{\sin^2 z}\, dz$
8. $\int \frac{2^{\ln z^3}}{16z}\, dz$
9. $\int \frac{dz}{e^z + e^{-z}}$
10. $\int_1^2 \frac{8\, dx}{x^2 - 2x + 2}$
11. $\int_{-1}^0 \frac{4\, dx}{1 + (2x + 1)^2}$
12. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3}\, dx$
13. $\int \frac{dt}{1 - \sec t}$
14. $\int \csc t \sin 3t\, dt$
15. $\int_0^{\pi/4} \frac{1 + \sin\theta}{\cos^2\theta}\, d\theta$
16. $\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$
17. $\int \frac{\ln y}{y + 4y \ln^2 y}\, dy$
18. $\int \frac{2^{\sqrt{y}}\, dy}{2\sqrt{y}}$
19. $\int \frac{d\theta}{\sec\theta + \tan\theta}$
20. $\int \frac{dt}{t\sqrt{3 + t^2}}$
21. $\int \frac{4t^3 - t^2 + 16t}{t^2 + 4}\, dt$
22. $\int \frac{x + 2\sqrt{x - 1}}{2x\sqrt{x - 1}}\, dx$
23. $\int_0^{\pi/2} \sqrt{1 - \cos\theta}\, d\theta$
24. $\int (\sec t + \cot t)^2\, dt$
25. $\int \frac{dy}{\sqrt{e^{2y} - 1}}$
26. $\int \frac{6\, dy}{\sqrt{y}\,(1 + y)}$
27. $\int \frac{2\, dx}{x\sqrt{1 - 4\ln^2 x}}$
28. $\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$
29. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$
30. $\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx$
31. $\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1}\, dx$
32. $\int_{-1}^1 \sqrt{1 + x^2}\, \sin x\, dx$
33. $\int_{-1}^0 \sqrt{\frac{1 + y}{1 - y}}\, dy$
34. $\int e^{z + e^z}\, dz$

35. $\displaystyle\int \frac{7\,dx}{(x-1)\sqrt{x^2-2x-48}}$

36. $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$

37. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

38. $\displaystyle\int \frac{d\theta}{\cos\theta-1}$

39. $\displaystyle\int \frac{dx}{1+e^x}$

Hint: Use long division.

40. $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$

Hint: Let $u = x^{3/2}$.

Theory and Examples

41. **Area** Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

42. **Volume** Find the volume of the solid generated by revolving the region in Exercise 41 about the x-axis.

43. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

44. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

45. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

46. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

47. The functions $y = e^{x^3}$ and $y = x^3e^{x^3}$ do not have elementary antiderivatives, but $y = (1+3x^3)e^{x^3}$ does. Evaluate

$$\int (1+3x^3)e^{x^3}\,dx.$$

48. Use the substitution $u = \tan x$ to evaluate the integral

$$\int \frac{dx}{1+\sin^2 x}.$$

49. Use the substitution $u = x^4 + 1$ to evaluate the integral

$$\int x^7\sqrt{x^4+1}\,dx.$$

50. **Using different substitutions** Show that the integral

$$\int ((x^2-1)(x+1))^{-2/3}dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x+1)$

b. $u = ((x-1)/(x+1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$

e. $u = \tan^{-1}((x-1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral?

8.2 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x\cos x\,dx \qquad \text{and} \qquad \int x^2e^x\,dx$$

are such integrals because $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly to become zero, and $g(x) = \cos x$ or $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x\,dx \qquad \text{and} \qquad \int e^x\cos x\,dx.$$

In the first case, $f(x) = \ln x$ is easy to differentiate and $g(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}\left[f(x)g(x)\right]dx = \int \left[f'(x)g(x) + f(x)g'(x)\right]dx$$

or

$$\int \frac{d}{dx}\left[f(x)g(x)\right]dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}\left[f(x)g(x)\right]dx - \int f'(x)g(x)\,dx,$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique. To avoid mistakes, we always list our choices for u and dv, then we add to the list our calculated new terms du and v, and finally we apply the formula in Equation (2).

EXAMPLE 1 Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C. \qquad \blacksquare$$

There are four apparent choices available for u and dv in Example 1:

1. Let $u = 1$ and $dv = x\cos x\,dx$.
2. Let $u = x$ and $dv = \cos x\,dx$.
3. Let $u = x\cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x\,dx$.

Choice 2 was used in Example 1. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3, with $du = (\cos x - x \sin x)\,dx$, leads to the integral

$$\int (x \cos x - x^2 \sin x)\, dx.$$

The goal of integration by parts is to go from an integral $\int u\, dv$ that we don't see how to evaluate to an integral $\int v\, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. When finding v from dv, any antiderivative will work and we usually pick the simplest one; no arbitrary constant of integration is needed in v because it would simply cancel out of the right-hand side of Equation (2).

EXAMPLE 2 Find

$$\int \ln x\, dx.$$

Solution Since $\int \ln x\, dx$ can be written as $\int \ln x \cdot 1\, dx$, we use the formula $\int u\, dv = uv - \int v\, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x}\, dx,$ $\qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x\, dx = x \ln x - \int x \cdot \frac{1}{x}\, dx = x \ln x - \int dx = x \ln x - x + C.$$ ■

Sometimes we have to use integration by parts more than once.

EXAMPLE 3 Evaluate

$$\int x^2 e^x\, dx.$$

Solution With $u = x^2$, $dv = e^x\, dx$, $du = 2x\, dx$, and $v = e^x$, we have

$$\int x^2 e^x\, dx = x^2 e^x - 2\int x e^x\, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x\, dx = x e^x - \int e^x\, dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2 e^x\, dx &= x^2 e^x - 2\int x e^x\, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C,\end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. ■

The technique of Example 3 works for any integral $\int x^n e^x\, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 4 Evaluate

$$\int e^x \cos x\, dx.$$

Solution Let $u = e^x$ and $dv = \cos x\, dx$. Then $du = e^x\, dx$, $v = \sin x$, and

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\, dx, \qquad v = -\cos x, \qquad du = e^x\, dx.$$

Then

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\, dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

■

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x\, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\, dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1 - \cos^2 x)\cos^{n-2} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.$$

If we add

$$(n-1)\int \cos^n x\, dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in Example 5 tells us that

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

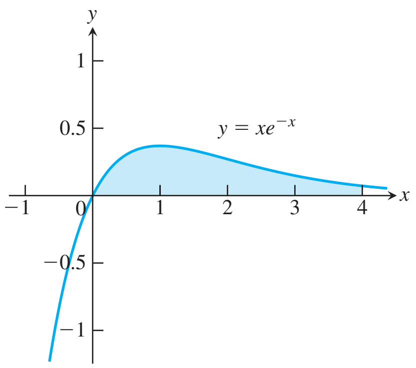


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx \\ &= \left[-4e^{-4} - (-0e^{-0})\right] + \int_0^4 e^{-x}\,dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

Tabular Integration Can Simplify Repeated Integrations

We have seen that integrals of the form $\int f(x)g(x)\,dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the notation and calculations can be cumbersome; or, you choose substitutions for a repeated integration by parts that just ends up giving back the original integral you were trying to find. In situations like these, there is a nice way to organize the calculations that prevents these pitfalls and simplifies the work. It is called **tabular integration** and is illustrated in the next examples.

EXAMPLE 7 Evaluate

$$\int x^2 e^x\, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 3. ■

EXAMPLE 8 Find the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

for $f(x) = 1$ on $[-\pi, 0)$ and $f(x) = x^3$ on $[0, \pi]$, where n is a positive integer.

Solution The integral is

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx &= \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\, dx \\ &= \frac{1}{n\pi}\sin nx\Big]_{-\pi}^{0} + \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\, dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} x^3\cos nx\, dx.\end{aligned}$$

Using tabular integration to find an antiderivative, we have

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\cos nx$
$3x^2$	$(-)$	$\frac{1}{n}\sin nx$
$6x$	$(+)$	$-\frac{1}{n^2}\cos nx$
6	$(-)$	$-\frac{1}{n^3}\sin nx$
0		$\frac{1}{n^4}\cos nx$

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{x^3}{n}\sin nx + \frac{3x^3}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left(\frac{3\pi^2\cos n\pi}{n^2} - \frac{6\cos n\pi}{n^4} + \frac{6}{n^4}\right)$$
$$= \frac{3}{\pi}\left(\frac{\pi^2 n^2(-1)^n + 2(-1)^{n+1} + 2}{n^4}\right). \qquad \cos n\pi = (-1)^n$$

Integrals like those in Example 8 occur frequently in electrical engineering.

Exercises 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int xe^x\, dx$
8. $\int xe^{3x}\, dx$
9. $\int x^2e^{-x}\, dx$
10. $\int (x^2 - 2x + 1)e^{2x}\, dx$
11. $\int \tan^{-1} y\, dy$
12. $\int \sin^{-1} y\, dy$
13. $\int x \sec^2 x\, dx$
14. $\int 4x \sec^2 2x\, dx$
15. $\int x^3e^x\, dx$
16. $\int p^4e^{-p}\, dp$
17. $\int (x^2 - 5x)e^x\, dx$
18. $\int (r^2 + r + 1)e^r\, dr$
19. $\int x^5e^x\, dx$
20. $\int t^2e^{4t}\, dt$
21. $\int e^{\theta} \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

Using Substitution

Evaluate the integrals in Exercise 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1 - x}\, dx$
27. $\int_0^{\pi/3} x \tan^2 x\, dx$
28. $\int \ln (x + x^2)\, dx$
29. $\int \sin (\ln x)\, dx$
30. $\int z(\ln z)^2\, dz$

Evaluating Integrals

Evaluate the integrals in Exercises 31–52. Some integrals do not require integration by parts.

31. $\int x \sec x^2\, dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$
33. $\int x (\ln x)^2\, dx$
34. $\int \frac{1}{x (\ln x)^2}\, dx$
35. $\int \frac{\ln x}{x^2}\, dx$
36. $\int \frac{(\ln x)^3}{x}\, dx$
37. $\int x^3 e^{x^4}\, dx$
38. $\int x^5 e^{x^3}\, dx$
39. $\int x^3\sqrt{x^2 + 1}\, dx$
40. $\int x^2 \sin x^3\, dx$
41. $\int \sin 3x \cos 2x\, dx$
42. $\int \sin 2x \cos 4x\, dx$
43. $\int \sqrt{x} \ln x\, dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
45. $\int \cos \sqrt{x}\, dx$
46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
49. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$
50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\, dx$
51. $\int x \tan^{-1} x\, dx$
52. $\int x^2 \tan^{-1} \frac{x}{2}\, dx$

Theory and Examples

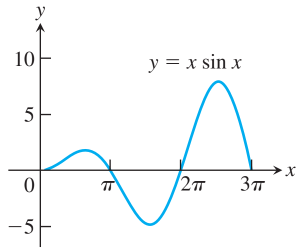
53. Finding area Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$.

b. $\pi \le x \le 2\pi$.

c. $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n + 1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



54. Finding area Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for

a. $\pi/2 \le x \le 3\pi/2$.

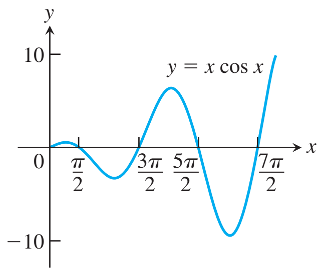
b. $3\pi/2 \le x \le 5\pi/2$.

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n - 1}{2}\right)\pi \le x \le \left(\frac{2n + 1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



55. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

56. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis.

b. about the line $x = 1$.

57. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis.

b. the line $x = \pi/2$.

58. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about

a. the y-axis.

b. the line $x = \pi$.

(See Exercise 53 for a graph.)

59. Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the x-axis.

c. Find the volume of the solid formed by revolving this region about the line $x = -2$.

d. Find the centroid of the region.

60. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.

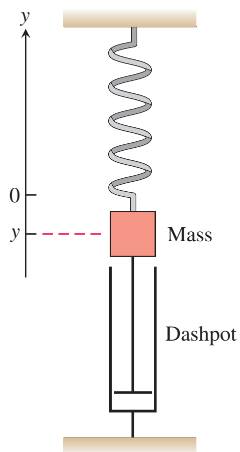
a. Find the area of the region.

b. Find the volume of the solid formed by revolving this region about the y-axis.

61. Average value A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



62. Average value In a mass-spring-dashpot system like the one in Exercise 61, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 63–67, use integration by parts to establish the reduction formula.

63. $\displaystyle\int x^n \cos x\, dx = x^n \sin x - n \int x^{n-1} \sin x\, dx$

64. $\displaystyle\int x^n \sin x\, dx = -x^n \cos x + n \int x^{n-1} \cos x\, dx$

65. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

66. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

67. $\int x^m(\ln x)^n\,dx = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\cdot$

$\int x^m(\ln x)^{n-1}\,dx, \quad m \neq -1$

68. Use Example 5 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \left(\frac{\pi}{2}\right)\frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n \text{ even} \\ \frac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n \text{ odd} \end{cases}$$

69. Show that

$$\int_a^b \left(\int_x^b f(t)\,dt\right) dx = \int_a^b (x-a)f(x)\,dx.$$

70. Use integration by parts to obtain the formula

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x)\,dx &= \int y f'(y)\,dy && y = f^{-1}(x),\quad x = f(y),\ dx = f'(y)\,dy \\ &= yf(y) - \int f(y)\,dy && \text{Integration by parts with } u = y,\ dv = f'(y)\,dy \\ &= xf^{-1}(x) - \int f(y)\,dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x\,dx &= \int ye^y\,dy && y = \ln x,\quad x = e^y,\ dx = e^y\,dy \\ &= ye^y - e^y + C \\ &= x\ln x - x + C. \end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned} \int \cos^{-1} x\,dx &= x\cos^{-1} x - \int \cos y\,dy && y = \cos^{-1} x \\ &= x\cos^{-1} x - \sin y + C \\ &= x\cos^{-1} x - \sin(\cos^{-1} x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 71–74. Express your answers in terms of x.

71. $\int \sin^{-1} x\,dx$

72. $\int \tan^{-1} x\,dx$

73. $\int \sec^{-1} x\,dx$

74. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \tag{5}$$

Exercises 75 and 76 compare the results of using Equations (4) and (5).

75. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ — Eq. (4)

b. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C$ — Eq. (5)

Can both integrations be correct? Explain.

76. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ — Eq. (4)

b. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1+x^2} + C$ — Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 77 and 78 with (**a**) Eq. (4) and (**b**) Eq. (5). In each case, check your work by differentiating your answer with respect to x.

77. $\int \sinh^{-1} x\,dx$

78. $\int \tanh^{-1} x\,dx$

8.3 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\,dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m* is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x\, dx.$$

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \sin x\, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) && \sin x\, dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2)\, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

■

EXAMPLE 2 Evaluate

$$\int \cos^5 x\, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x) \qquad \cos x\, dx = d(\sin x)$$

$$= \int (1 - u^2)^2\, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

■

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x\, dx.$$

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x\, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x\, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). \qquad \text{Omitting the constant of integration until the final result}$$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\, dx = \int (1 - \sin^2 2x)\cos 2x\, dx \qquad u = \sin 2x,\ du = 2\cos 2x\, dx$$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). \qquad \text{Again omitting } C$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

5.6 Definite Integral Substitutions and the Area Between Curves

There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to *definite* integrals by changing the limits of integration. We apply the new formula introduced here to the problem of computing the area between two curves.

The Substitution Formula

The following formula shows how the limits of integration change when the variable of integration is changed by substitution.

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Proof Let F denote any antiderivative of f. Then,

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = F(g(x))\Big]_{x=a}^{x=b}$$

$\frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$

$$= F(g(b)) - F(g(a))$$

$$= F(u)\Big]_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)\, du.$$

Fundamental Theorem, Part 2 ■

To use the formula, make the same u-substitution $u = g(x)$ and $du = g'(x)\, dx$ you would use to evaluate the corresponding indefinite integral. Then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

EXAMPLE 1 Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx$.

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx$$

Let $u = x^3 + 1$, $du = 3x^2\, dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\, du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3+1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Which method is better—evaluating the transformed definite integral with transformed limits using Theorem 7, or transforming the integral, integrating, and transforming back to use the original limits of integration? In Example 1, the first method seems easier, but that is not always the case. Generally, it is best to know both methods and to use whichever one seems better at the time.

EXAMPLE 2 We use the method of transforming the limits of integration.

(a)

$$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta\,d\theta = \int_1^0 u\cdot(-du)$$

Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$,
$-du = \csc^2\theta\,d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b)

$$\int_{-\pi/4}^{\pi/4} \tan x\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\,dx$$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x$, $du = -\sin x\,dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= -\ln|u|\Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0 \qquad \text{Integrate, zero width interval}$$

Definite Integrals of Symmetric Functions

The Substitution Formula in Theorem 7 simplifies the calculation of definite integrals of even and odd functions (Section 1.1) over a symmetric interval $[-a, a]$ (Figure 5.23).

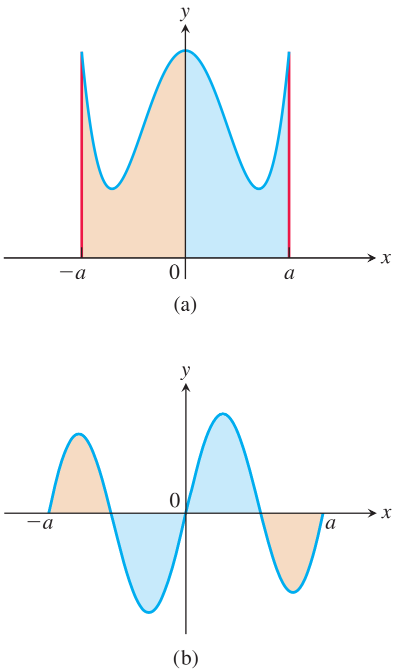


FIGURE 5.23 (a) For f an even function, the integral from $-a$ to a is twice the integral from 0 to a. (b) For f an odd function, the integral from $-a$ to a equals 0.

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof of Part (a)

$$
\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Additivity Rule for Definite Integrals} \\
&= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Order of Integration Rule} \\
&= -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)\,dx && \text{Let } u = -x,\ du = -dx. \text{ When } x = 0,\ u = 0. \text{ When } x = -a,\ u = a. \\
&= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx && f \text{ is even, so } f(-u) = f(u). \\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}
$$

The proof of part (b) is entirely similar and you are asked to give it in Exercise 114. ■

The assertions of Theorem 8 remain true when f is an integrable function (rather than having the stronger property of being continuous).

EXAMPLE 3 Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$
\begin{aligned}
\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx &= 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx \\
&= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2 \\
&= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.
\end{aligned}
$$
■

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Figure 5.24). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area with an integral.

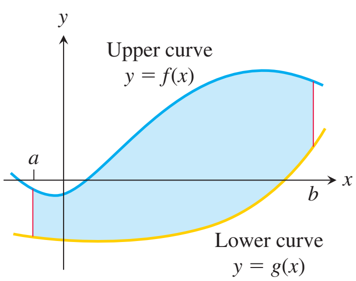


FIGURE 5.24 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

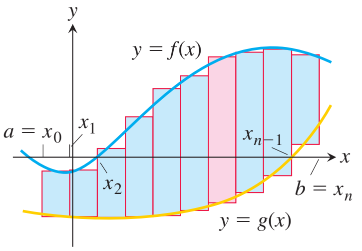


FIGURE 5.25 We approximate the region with rectangles perpendicular to the x-axis.

To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ (Figure 5.25). The area of the kth rectangle (Figure 5.26) is

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)]\,\Delta x_k.$$

We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k. \qquad \text{Riemann sum}$$

As $\|P\| \to 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)]\,dx$ because f and g are continuous. We take the area of the region to be the value of this integral. That is,

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k = \int_a^b [f(x) - g(x)]\,dx.$$

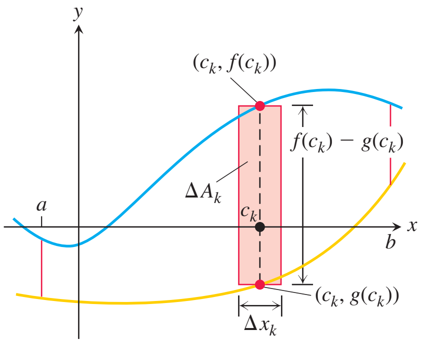


FIGURE 5.26 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

When applying this definition it is helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g. It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x. Then you can integrate the function $f - g$ for the area between the intersections.

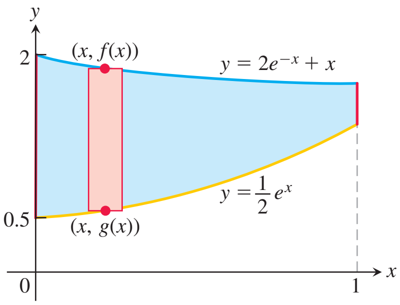


FIGURE 5.27 The region in Example 4 with a typical approximating rectangle.

EXAMPLE 4 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Solution Figure 5.27 displays the graphs of the curves and the region whose area we want to find. The area between the curves over the interval $0 \leq x \leq 1$ is given by

$$A = \int_0^1 \left[(2e^{-x} + x) - \frac{1}{2}e^x\right] dx = \left[-2e^{-x} + \frac{1}{2}x^2 - \frac{1}{2}e^x\right]_0^1$$

$$= \left(-2e^{-1} + \frac{1}{2} - \frac{1}{2}e\right) - \left(-2 + 0 - \frac{1}{2}\right)$$

$$= 3 - \frac{2}{e} - \frac{e}{2} \approx 0.9051.$$

■

EXAMPLE 5 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.28). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$\begin{aligned}
2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x).\\
x^2 - x - 2 &= 0 && \text{Rewrite.}\\
(x + 1)(x - 2) &= 0 && \text{Factor.}\\
x = -1, \qquad x &= 2. && \text{Solve.}
\end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

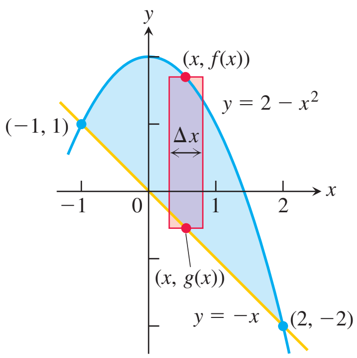


FIGURE 5.28 The region in Example 5 with a typical approximating rectangle.

The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}.$$

If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

EXAMPLE 6 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

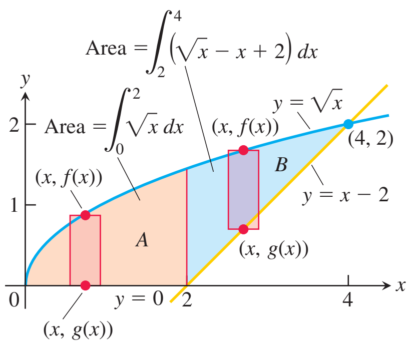


FIGURE 5.29 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

Solution The sketch (Figure 5.29) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.29.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\sqrt{x} = x - 2 \qquad \text{Equate } f(x) \text{ and } g(x).$$

$$x = (x - 2)^2 = x^2 - 4x + 4 \qquad \text{Square both sides.}$$

$$x^2 - 5x + 4 = 0 \qquad \text{Rewrite.}$$

$$(x - 1)(x - 4) = 0 \qquad \text{Factor.}$$

$$x = 1, \quad x = 4. \qquad \text{Solve.}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

We add the areas of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

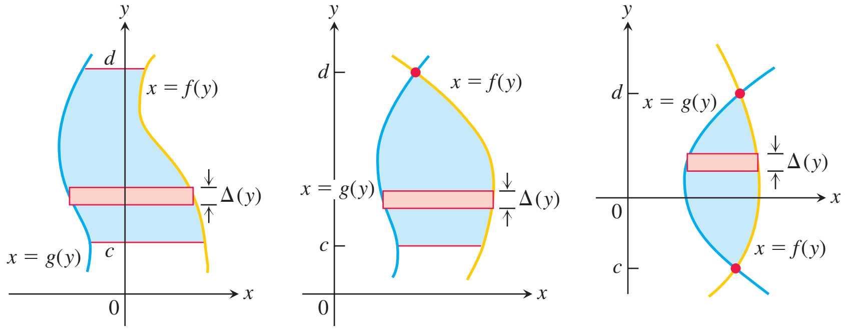
$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these:



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

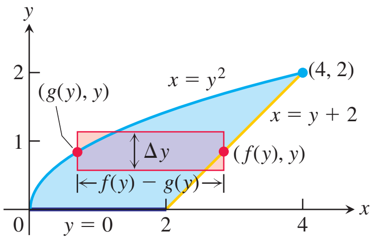


FIGURE 5.30 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 7).

EXAMPLE 7 Find the area of the region in Example 6 by integrating with respect to y.

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 5.30). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$y + 2 = y^2 \qquad \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2.$$
$$y^2 - y - 2 = 0 \qquad \text{Rewrite.}$$
$$(y + 1)(y - 2) = 0 \qquad \text{Factor.}$$
$$y = -1, \quad y = 2 \qquad \text{Solve.}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$$\begin{aligned} A = \int_c^d [f(y) - g(y)]\,dy &= \int_0^2 [y + 2 - y^2]\,dy \\ &= \int_0^2 [2 + y - y^2]\,dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \end{aligned}$$

This is the result of Example 6, found with less work. ■

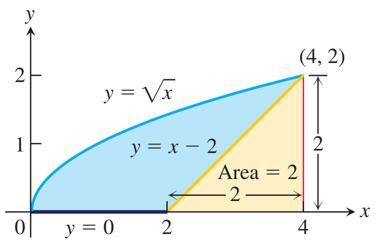


FIGURE 5.31 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Although it was easier to find the area in Example 6 by integrating with respect to y rather than x (just as we did in Example 7), there is an easier way yet. Looking at Figure 5.31, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x-axis for $0 \le x \le 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\begin{aligned}\text{Area} &= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2)\\ &= \frac{2}{3}x^{3/2}\Big]_0^4 - 2\\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.\end{aligned}$$

Exercises 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–46.

1. a. $\int_0^3 \sqrt{y+1}\,dy$ **b.** $\int_{-1}^0 \sqrt{y+1}\,dy$

2. a. $\int_0^1 r\sqrt{1-r^2}\,dr$ **b.** $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ **b.** $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\,dx$ **b.** $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a. $\int_0^1 t^3(1+t^4)^3\,dt$ **b.** $\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ **b.** $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ **b.** $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ **b.** $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ **b.** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ **b.** $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a. $\int_0^1 t\sqrt{4+5t}\,dt$ **b.** $\int_1^9 t\sqrt{4+5t}\,dt$

12. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$ **b.** $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ **b.** $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$ **b.** $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$

16. $\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

18. $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$

20. $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$

24. $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

25. $\int_0^{\pi/4} (1+e^{\tan\theta})\sec^2\theta\,d\theta$

26. $\int_{\pi/4}^{\pi/2} (1+e^{\cot\theta})\csc^2\theta\,d\theta$

27. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$

28. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

29. $\int_1^2 \frac{2\ln x}{x}\,dx$

30. $\int_2^4 \frac{dx}{x\ln x}$

31. $\int_2^4 \frac{dx}{x(\ln x)^2}$

32. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

33. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

34. $\int_{\pi/4}^{\pi/2} \cot t\,dt$

35. $\int_0^{\pi/3} \tan^2\theta\cos\theta\,d\theta$

36. $\int_0^{\pi/12} 6\tan 3x\,dx$

37. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1 + (\sin\theta)^2}$

38. $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1 + (\cot x)^2}$

39. $\displaystyle\int_{0}^{\ln\sqrt{3}} \frac{e^x\, dx}{1 + e^{2x}}$

40. $\displaystyle\int_{1}^{e^{\pi/4}} \frac{4\, dt}{t(1 + \ln^2 t)}$

41. $\displaystyle\int_{0}^{1} \frac{4\, ds}{\sqrt{4 - s^2}}$

42. $\displaystyle\int_{0}^{\sqrt[3]{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

43. $\displaystyle\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$

44. $\displaystyle\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$

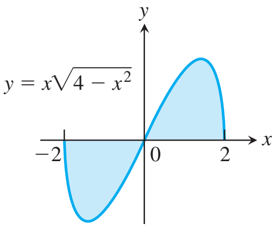
45. $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$

46. $\displaystyle\int_{0}^{3} \frac{y\, dy}{\sqrt{5y + 1}}$

Area

Find the total areas of the shaded regions in Exercises 47–62.

47.



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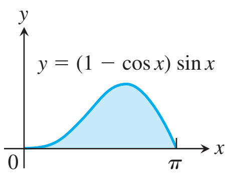

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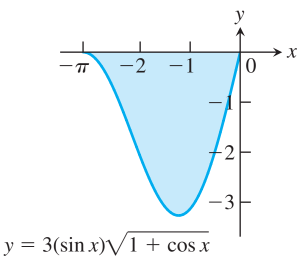

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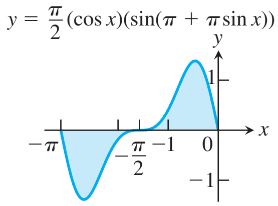

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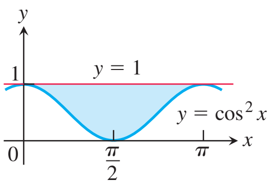

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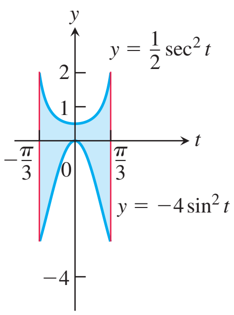

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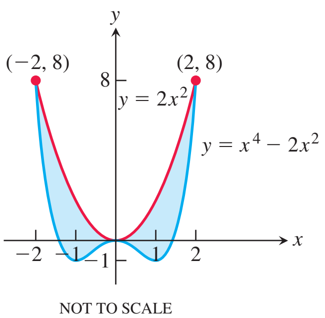

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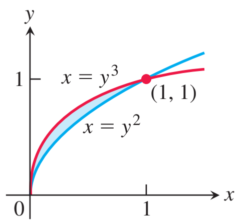

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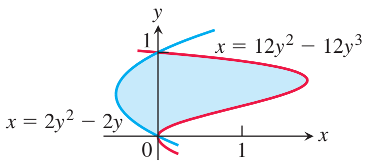

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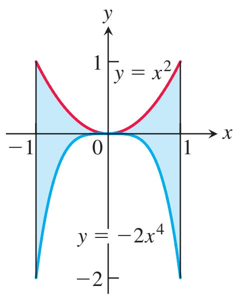

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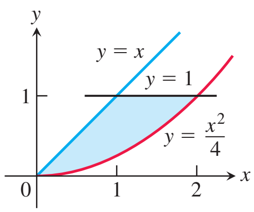

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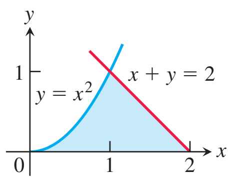

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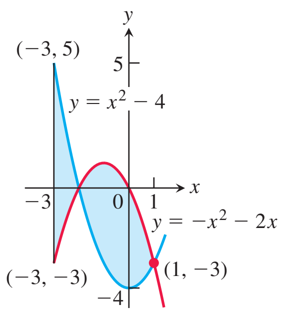

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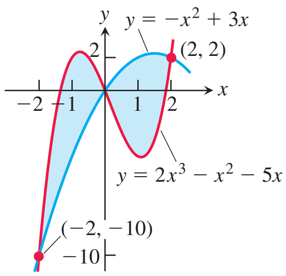

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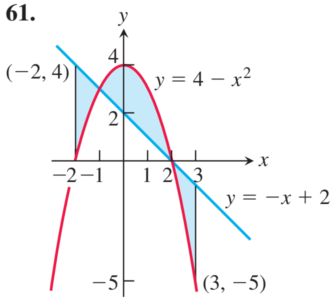

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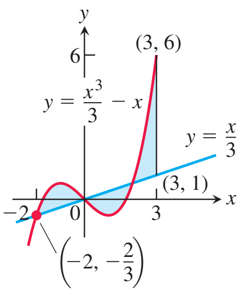


Find the areas of the regions enclosed by the lines and curves in Exercises 63–72.

63. $y = x^2 - 2$ and $y = 2$

64. $y = 2x - x^2$ and $y = -3$

65. $y = x^4$ and $y = 8x$

66. $y = x^2 - 2x$ and $y = x$

67. $y = x^2$ and $y = -x^2 + 4x$

68. $y = 7 - 2x^2$ and $y = x^2 + 4$

69. $y = x^4 - 4x^2 + 4$ and $y = x^2$

70. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

71. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

72. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 73–80.

73. $x = 2y^2$, $x = 0$, and $y = 3$

74. $x = y^2$ and $x = y + 2$

75. $y^2 - 4x = 4$ and $4x - y = 16$

76. $x - y^2 = 0$ and $x + 2y^2 = 3$

77. $x + y^2 = 0$ and $x + 3y^2 = 2$

78. $x - y^{2/3} = 0$ and $x + y^4 = 2$

79. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

80. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 81–84.

81. $4x^2 + y = 4$ and $x^4 - y = 1$

82. $x^3 - y = 0$ and $3x^2 - y = 4$

83. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

84. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 85–92.

85. $y = 2 \sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

86. $y = 8 \cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

87. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

88. $y = \sin(\pi x/2)$ and $y = x$

89. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

90. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

91. $x = 3 \sin y \sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

92. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

Area Between Curves

93. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

94. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

95. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

96. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

97. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

98. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

99. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

100. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2 \ln 2$.

101. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \leq x \leq 2$ of the x-axis.

102. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \leq x \leq 1$ of the x-axis.

103. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

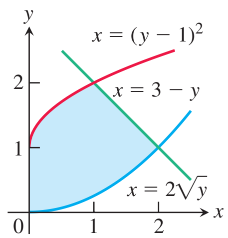
b. Find c by integrating with respect to y. (This puts c in the limits of integration.)

c. Find c by integrating with respect to x. (This puts c into the integrand as well.)

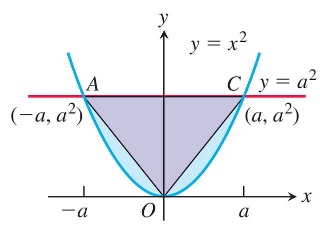
104. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

105. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

106. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



107. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

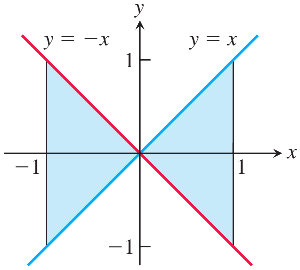


108. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

109. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\int_{-1}^{1} (x - (-x))\, dx = \int_{-1}^{1} 2x\, dx$

b. $\int_{-1}^{1} (-x - (x))\, dx = \int_{-1}^{1} -2x\, dx$



110. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_{a}^{b} [f(x) - g(x)]\, dx.$$

Give reasons for your answer.

Theory and Examples

111. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_{1}^{3} \frac{\sin 2x}{x}\, dx$$

in terms of F.

112. Show that if f is continuous, then

$$\int_{0}^{1} f(x)\, dx = \int_{0}^{1} f(1 - x)\, dx.$$

113. Suppose that

$$\int_{0}^{1} f(x)\, dx = 3.$$

Find

$$\int_{-1}^{0} f(x)\, dx$$

if **a.** f is odd, **b.** f is even.

114. a. Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\, dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

115. If f is a continuous function, find the value of the integral

$$I = \int_{0}^{a} \frac{f(x)\, dx}{f(x) + f(a - x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

116. By using a substitution, prove that for all positive numbers x and y,

$$\int_{x}^{xy} \frac{1}{t}\, dt = \int_{1}^{y} \frac{1}{t}\, dt.$$

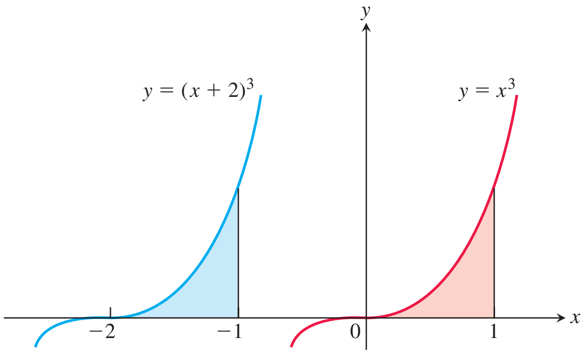
The Shift Property for Definite Integrals A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_{a}^{b} f(x)\, dx = \int_{a-c}^{b-c} f(x + c)\, dx. \qquad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x + 2)^3\, dx = \int_{0}^{1} x^3\, dx$$

because the areas of the shaded regions are congruent.



117. Use a substitution to verify Equation (1).

118. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x + c)$ over $[a - c, b - c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x) = x^2, \quad a = 0, \quad b = 1, \quad c = 1$

b. $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad c = \pi/2$

c. $f(x) = \sqrt{x - 4}, \quad a = 4, \quad b = 8, \quad c = 5$

COMPUTER EXPLORATIONS

In Exercises 119–122, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a. Plot the curves together to see what they look like and how many points of intersection they have.

b. Use the numerical equation solver in your CAS to find all the points of intersection.

c. Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

d. Sum together the integrals found in part (c).

119. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}, \quad g(x) = x - 1$

120. $f(x) = \frac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

121. $f(x) = x + \sin(2x), \quad g(x) = x^3$

122. $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$

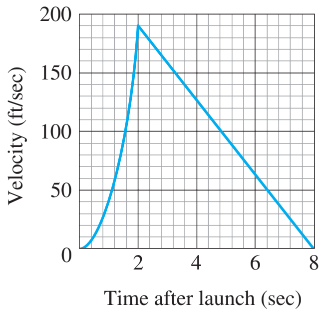
Chapter 5 Questions to Guide Your Review

1. How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
2. What is sigma notation? What advantage does it offer? Give examples.
3. What is a Riemann sum? Why might you want to consider such a sum?
4. What is the norm of a partition of a closed interval?
5. What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
6. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
7. What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
8. Describe the rules for working with definite integrals (Table 5.6). Give examples.
9. What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
10. What is the Net Change Theorem? What does it say about the integral of velocity? The integral of marginal cost?
11. Discuss how the processes of integration and differentiation can be considered as "inverses" of each other.
12. How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
13. How is integration by substitution related to the Chain Rule?
14. How can you sometimes evaluate indefinite integrals by substitution? Give examples.
15. How does the method of substitution work for definite integrals? Give examples.
16. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

Chapter 5 Practice Exercises

Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.

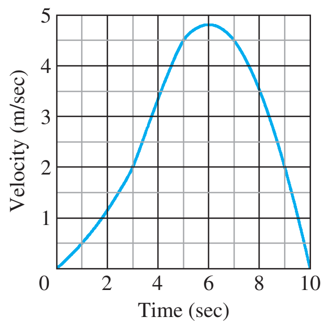


   a. Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.3, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)

   b. Sketch a graph of the rocket's height above ground as a function of time for $0 \le t \le 8$.

2. a. The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?

   b. Sketch a graph of s as a function of t for $0 \le t \le 10$, assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

   a. $\sum_{k=1}^{10} \frac{a_k}{4}$ b. $\sum_{k=1}^{10} (b_k - 3a_k)$

   c. $\sum_{k=1}^{10} (a_k + b_k - 1)$ d. $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

   a. $\sum_{k=1}^{20} 3a_k$ b. $\sum_{k=1}^{20} (a_k + b_k)$

   c. $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$ d. $\sum_{k=1}^{20} (a_k - 2)$

Improper Integral :

The definition of the definite integral as :

$$\int_a^b f(x)dx$$

This integration includes the requirements that the interval [a,b] be an finite and that f be continous on [a,b] , in this lecture we will study integrals that do not satisfy these requirements because of one of the conditions below :

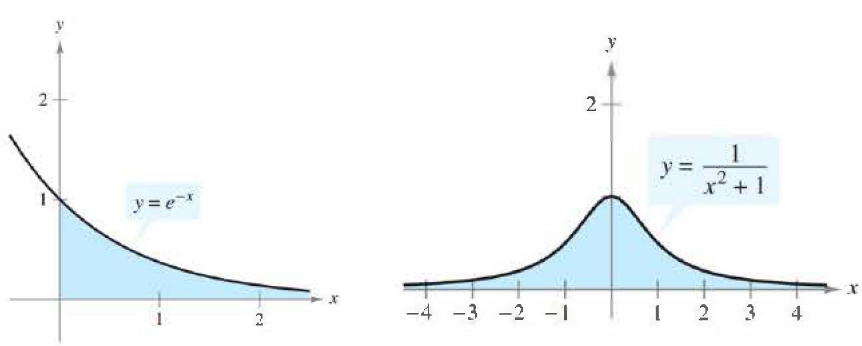
- One or both of limit of integration are finite .
- f(x) has an infinite discontinuity in the interval [a,b] .

Integral having either of these charctristics are called Improper Integral

For instance the integrals :

$$\int_0^\infty e^{-x}dx \quad , \quad \int_{-\infty}^\infty \frac{1}{x^2+1}dx$$

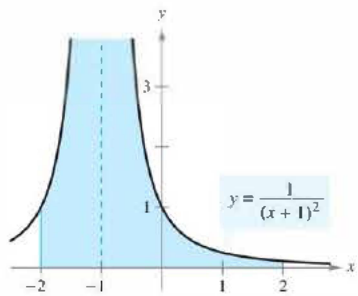
are improper integral because one or both limits of integration are infinite as indicated figures:

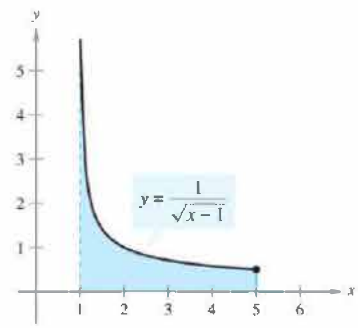


Similary :

$$\int_{1}^{5} \frac{1}{\sqrt{x-1}}\,dx \quad , \quad \int_{-2}^{2} \frac{1}{(x+1)^2}\,dx$$

are improper integral because their integrands has infinite discontinuity that is they approach infinity some when in the interval of integration as indicated figures :





CASE 1

Improper Integrals with infinite limits

1) if f is continous function on the interval $[a,\infty)$ then :

$$\int_{a}^{\infty} f(x)dx = \lim_{b\to\infty} \int_{a}^{b} f(x)\,dx$$

2) if f is continous function on the interval $(-\infty,b]$ then :

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$$

3) if f is continous function on the interval $(-\infty,\infty)$ then :

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

where c is any real number .

in the first two cases if the limit exists then the improper integral convergence , otherwisethe improper integral divergent ,
in the third case the integral on the left side will diverges if either one of the integral on the right side diverges .

Example 1 : Determine the convergence or divergence of :

$$\int_{1}^{\infty} \frac{1}{x}dx$$

Solution :

$$\int_{1}^{\infty} \frac{1}{x}dx = \lim_{b\to\infty}\int_{1}^{b} \frac{1}{x}dx$$

$$\int_{1}^{\infty} \frac{1}{x}dx = \lim_{b\to\infty} Lnx|_1^b = \lim_{b\to\infty} Lnb - Ln1 = Ln\infty = \infty$$

Because the limit is infinite , then the improper integral divergent .

Example 2 : Determine the convergence or divergence of :

$$\int_{1}^{\infty} \frac{1}{x^2} dx$$

Solution :

$$\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{x^2} dx$$

$$\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{b\to\infty} \frac{-1}{x}\Big|_{1}^{b} = \lim_{b\to\infty} -(\frac{1}{b} - 1) = 1$$

Because the limit is finite , then the improper integral convergent to 1 .

Example 3 : Evaluate the following integral if available :

$$\int_{-\infty}^{0} \frac{1}{(1-2x)^{\frac{3}{2}}} dx$$

Solution :

$$\int_{-\infty}^{0} \frac{1}{(1-2x)^{\frac{3}{2}}} dx = \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{(1-2x)^{\frac{3}{2}}} dx$$

$$\int_{-\infty}^{0} \frac{1}{(1-2x)^{\frac{3}{2}}} dx = \lim_{a\to-\infty} \int_{a}^{0} (1-2x)^{\frac{-3}{2}} dx =$$

$$\int_{-\infty}^{0} \frac{1}{(1-2x)^{\frac{3}{2}}} dx = \lim_{a \to -\infty} (1-2x)^{\frac{-1}{2}} \Big|_a^0 = \lim_{a \to -\infty} \frac{1}{\sqrt{1-2x}} \Big|_a^0$$

$$\int_{-\infty}^{0} \frac{1}{(1-2x)^{\frac{3}{2}}} dx = \lim_{a \to -\infty} \left(1 - \frac{1}{\sqrt{1-2a}}\right) = 1 - 0 = 1$$

Because the limit is finite , then the improper integral convergent to 1 .

Example 4 : Find the following integral if available :

$$\int_{0}^{\infty} x e^{-x^2} dx$$

Solution :

$$\int_{0}^{\infty} x e^{-x^2} dx = \lim_{b \to \infty} \int_{0}^{b} x e^{-x^2} dx$$

$$\int_{0}^{\infty} x e^{-x^2} dx = \lim_{b \to \infty} \frac{-1}{2} e^{-x^2} \Big|_0^b = \lim_{b \to \infty} \frac{-1}{2} (e^{-b^2} - 1)$$

$$\int_{0}^{\infty} x e^{-x^2} dx = \frac{-1}{2} (e^{-\infty} - 1) = \frac{-1}{2} (0 - 1) = \frac{1}{2}$$

Because the limit is finite , then the improper integral convergent to 0.5

Example 5: Determine the convergence or divergence of :

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx$$

Solution :

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_{-b}^{b} \frac{1}{x^2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_{-b}^{b} x^{-2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \frac{-1}{x}\Big|_{-b}^{b} = \lim_{b\to\infty} -(\frac{1}{b}+\frac{1}{b}) = \lim_{b\to\infty} -\frac{2}{b} = 0$$

Because the limit is finite , then the improper integral convergent to 1 .

CHECK POINT

Example H·W :

Find the following integrals if available :

1) $\int_{1}^{\infty} \frac{1}{x^3} dx$

2) $\int_{1}^{\infty} \frac{1}{\sqrt{x}} dx$

3) $\int_{-\infty}^{0} \frac{1}{(x-1)^2} dx$

4) $\int_{-\infty}^{\infty} \frac{1}{4+x^2} dx$

5) $\int_{2}^{\infty} \frac{1}{(x-1)(x^2+1)} dx$

CASE 2

Improper Integrals with infinite integrands

1) if f is continous function on the interval $[a,b)$ and approaches to infinity at b then :

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

2) if f is continous function on the interval $(a,b]$ and approaches to infinity at a then :

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

3) if f is continous function on the interval $[a,b]$ except some values of c in (a,b) at which f approaches infinity then :

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where c is any real number lies between a and b ·

in the first two cases if the limit exists then the improper integral convergence , otherwisethe improper integral divergent ,
in the third case the integral on the left side will diverges if either one of the integral on the right side diverges ·

Example 6 : Determine the convergence or divergence of :

$$\int_1^2 \frac{1}{\sqrt[3]{x-1}}dx$$

Solution :

Singular points is ' 1 ' (makes the function not continous)

$$\int_1^2 \frac{1}{\sqrt[3]{x-1}}dx = \lim_{c\to 1^+}\int_c^2 \frac{1}{\sqrt[3]{x-1}}dx$$

$$\int_1^2 \frac{1}{\sqrt[3]{x-1}}dx = \lim_{c\to 1^+}\int_c^2 (x-1)^{\frac{-1}{3}}dx = \lim_{c\to 1^+}\frac{3}{2}(x-1)^{\frac{2}{3}}\Big|_c^2$$

$$\int_1^2 \frac{1}{\sqrt[3]{x-1}}dx = \frac{3}{2}\lim_{c\to 1^+}\left(1-(c-1)^{\frac{2}{3}}\right) = \frac{3}{2}(1-0) = \frac{3}{2}$$

Because the limit is finite , then the improper integral convergent to 1.5

Example 7 : Evaluate the integral :

$$\int_1^2 \frac{2}{x^2-2x}dx$$

Solution :

Singular points is ' 0 , 2 ' (2 makes the function not continous)

By using partial fraction we produce :

$$\int_1^2 \frac{2}{x^2-2x}dx = \lim_{c\to 2^-}\int_1^c \frac{1}{x-2}-\frac{1}{x}dx$$

$$\int_1^2 \frac{2}{x^2-2x}dx = \lim_{c\to 2^-}\left[\,Ln|x-2| - Ln|x|\right]_1^c$$

$$\int_1^2 \frac{2}{x^2-2x}dx = \lim_{c\to 2^-} Ln\left|\frac{x-2}{x}\right|_1^c = \lim_{c\to 2^-} Ln\left|\frac{c-2}{c}\right| - Ln1$$

$$\int_1^2 \frac{2}{x^2-2x}dx = \lim_{c\to 2^-} Ln\left|1-\frac{2}{c}\right| = Ln0 = -\infty$$

Because the limit is infinite , then the improper integral divergent.

Example 8 : Evaluate the integral :

$$\int_{-1}^2 \frac{2}{x^3}dx$$

Solution :

Singular points is ' 0 ' (makes the function not continous)

This integral is improper because the integrand has an infinite discontinuity at the interior value x=0 , so we can write :

$$\int_{-1}^{2} \frac{2}{x^3} dx = \underbrace{\int_{-1}^{0} \frac{2}{x^3} dx}_{I.I.1} + \underbrace{\int_{0}^{2} \frac{2}{x^3} dx}_{I.I.2}$$

1st Integral

$$\int_{-1}^{0} \frac{2}{x^3} dx = \lim_{b \to 0^-} \int_{-1}^{b} 2x^{-3} dx = \lim_{b \to 0^-} \frac{-1}{x^2}\Big|_{-1}^{b}$$

$$\int_{-1}^{0} \frac{2}{x^3} dx = \lim_{b \to 0^-} -\left(\frac{1}{b^2} - 1\right) = -\infty$$

The 1st integral is divergent .

2nd Integral

$$\int_{0}^{2} \frac{2}{x^3} dx = \lim_{a \to 0^+} \int_{a}^{2} 2x^{-3} dx = \lim_{a \to 0^+} \frac{-1}{x^2}\Big|_{a}^{2}$$

$$\int_{-1}^{0} \frac{2}{x^3} dx = \lim_{a \to 0^+} -\left(\frac{1}{4} - \frac{1}{a^2}\right) = \infty$$

The 1st integral is divergent too .

So the whole integral is divergent .

Note :

in the previous example had you not recognized that the integral was improper you would have obtained the incorrected result

$$\int_{-1}^{2} \frac{2}{x^3} dx = \int_{a}^{2} 2x^{-3} dx = \frac{-1}{x^2}\bigg|_{-1}^{2} = -\left(\frac{1}{4} - 1\right) = \frac{3}{4} \quad (incorrect\ result)$$

CHECK POINT

Example H.W :

Find the following integrals if available :

1) $\int_{1}^{2} \frac{1}{\sqrt{x-1}} dx$

2) $\int_{1}^{3} \frac{3}{x^3 - 3x} dx$

3) $\int_{0}^{2} \frac{x+2}{(x-1)^2} dx$

4) $\int^{4} \frac{x^2}{\sqrt[3]{x^3 - x^5}} dx$

water's temperature was 39°C; 10 min after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

44. **Silver cooling in air** The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be
   a. 15 min from now?
   b. 2 hours from now?
   c. When will the silver be 10°C above room temperature?

45. **The age of Crater Lake** The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

46. **The sensitivity of carbon-14 dating to measurement** To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:
   a. A bone fragment found in central Illinois in the year 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.
   b. Repeat part (a), assuming 18% instead of 17%.
   c. Repeat part (a), assuming 16% instead of 17%.

47. **Carbon-14** The oldest known frozen human mummy, discovered in the Schnalstal glacier of the Italian Alps in 1991 and called *Otzi*, was found wearing straw shoes and a leather coat with goat fur, and holding a copper ax and stone dagger. It was estimated that Otzi died 5000 years before he was discovered in the melting glacier. How much of the original carbon-14 remained in Otzi at the time of his discovery?

48. **Art forgery** A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

49. **Lascaux Cave paintings** Prehistoric cave paintings of animals were found in the Lascaux Cave in France in 1940. Scientific analysis revealed that only 15% of the original carbon-14 in the paintings remained. What is an estimate of the age of the paintings?

50. **Incan mummy** The frozen remains of a young Incan woman were discovered by archeologist Johan Reinhard on Mt. Ampato in Peru during an expedition in 1995.
   a. How much of the original carbon-14 was present if the estimated age of the "Ice Maiden" was 500 years?
   b. If a 1% error can occur in the carbon-14 measurement, what is the oldest possible age for the Ice Maiden?

7.3 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical and engineering applications. In this section we give a brief introduction to these functions, their graphs, their derivatives, their integrals, and their inverse functions.

Definitions and Identities

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

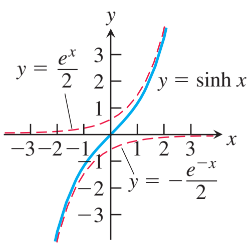
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{and} \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

We pronounce $\sinh x$ as "cinch x," rhyming with "pinch x," and $\cosh x$ as "kosh x," rhyming with "gosh x." From this basic pair, we define the hyperbolic tangent, cotangent, secant, and cosecant functions. The defining equations and graphs of these functions are shown in Table 7.4. We will see that the hyperbolic functions bear many similarities to the trigonometric functions after which they are named.

Hyperbolic functions satisfy the identities in Table 7.5. Except for differences in sign, these resemble identities we know for the trigonometric functions. The identities are proved directly from the definitions, as we show here for the second one:

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^{2x} - e^{-2x}}{2} \\ &= \sinh 2x. \end{aligned}$$

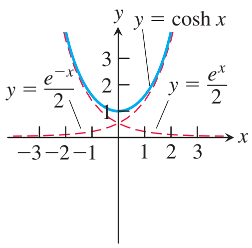
TABLE 7.4 The six basic hyperbolic functions



(a)

Hyperbolic sine:

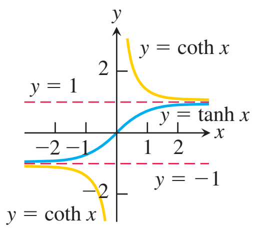
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



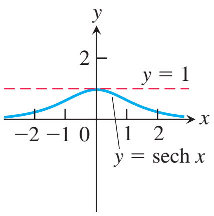
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

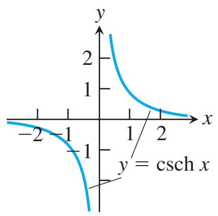
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

TABLE 7.5 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

The other identities are obtained similarly, by substituting in the definitions of the hyperbolic functions and using algebra. Like many standard functions, hyperbolic functions and their inverses are easily evaluated with calculators, which often have special keys for that purpose.

For any real number u, we know the point with coordinates $(\cos u, \sin u)$ lies on the unit circle $x^2 + y^2 = 1$. So the trigonometric functions are sometimes called the *circular* functions. Because of the first identity

$$\cosh^2 u - \sinh^2 u = 1,$$

with u substituted for x in Table 7.5, the point having coordinates $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$. This is where the *hyperbolic* functions get their names (see Exercise 86).

Hyperbolic functions are useful in finding integrals, which we will see in Chapter 8. They play an important role in science and engineering as well. The hyperbolic cosine describes the shape of a hanging cable or wire that is strung between two points at the same height and hanging freely (see Exercise 83). The shape of the St. Louis Arch is an inverted hyperbolic cosine. The hyperbolic tangent occurs in the formula for the velocity of an ocean wave moving over water having a constant depth, and the inverse hyperbolic tangent describes how relative velocities sum according to Einstein's Law in the Special Theory of Relativity.

TABLE 7.6 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Derivatives and Integrals of Hyperbolic Functions

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 7.6). Again, there are similarities with trigonometric functions.

TABLE 7.7 Integral formulas for hyperbolic functions

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

The derivative formulas are derived from the derivative of e^u:

$$\begin{aligned}\frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of } \sinh u\\ &= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u\\ &= \cosh u \frac{du}{dx}. && \text{Definition of } \cosh u\end{aligned}$$

This gives the first derivative formula. From the definition, we can calculate the derivative of the hyperbolic cosecant function, as follows:

$$\begin{aligned}\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of } \operatorname{csch} u\\ &= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule for derivatives}\\ &= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.}\\ &= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of } \operatorname{csch} u \text{ and } \coth u\end{aligned}$$

The other formulas in Table 7.6 are obtained similarly.

The derivative formulas lead to the integral formulas in Table 7.7.

EXAMPLE 1 We illustrate the derivative and integral formulas.

(a)
$$\begin{aligned}\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) &= \operatorname{sech}^2\sqrt{1+t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)\\ &= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}\end{aligned}$$

(b)
$$\begin{aligned}\int \coth 5x\, dx &= \int \frac{\cosh 5x}{\sinh 5x}dx = \frac{1}{5}\int\frac{du}{u} && u = \sinh 5x,\ du = 5\cosh 5x\, dx\\ &= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C\end{aligned}$$

(c)
$$\begin{aligned}\int_0^1 \sinh^2 x\, dx &= \int_0^1 \frac{\cosh 2x - 1}{2}dx && \text{Table 7.5}\\ &= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\, dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1\\ &= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672 && \text{Evaluate with a calculator.}\end{aligned}$$

(d)
$$\begin{aligned}\int_0^{\ln 2} 4e^x \sinh x\, dx &= \int_0^{\ln 2} 4e^x\frac{e^x - e^{-x}}{2}dx = \int_0^{\ln 2}(2e^{2x} - 2)\, dx\\ &= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)\\ &= 4 - 2\ln 2 - 1 \approx 1.6137\end{aligned}$$

Inverse Hyperbolic Functions

The inverses of the six basic hyperbolic functions are very useful in integration (see Chapter 8). Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by

$$y = \sinh^{-1} x.$$

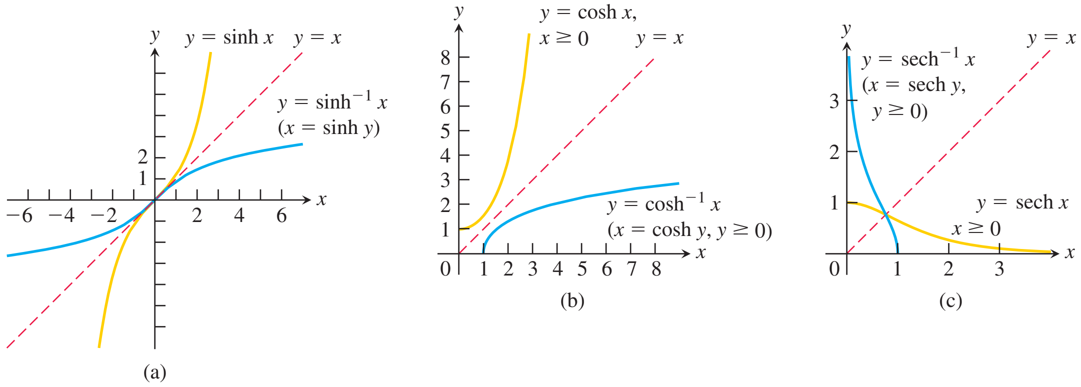


FIGURE 7.8 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Figure 7.8a.

The function $y = \cosh x$ is not one-to-one because its graph in Table 7.4 does not pass the horizontal line test. The restricted function $y = \cosh x$, $x \geq 0$, however, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x. The graphs of $y = \cosh x$, $x \geq 0$, and $y = \cosh^{-1} x$ are shown in Figure 7.8b.

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x. The graphs of $y = \operatorname{sech} x$, $x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Figure 7.8c.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Figure 7.9.

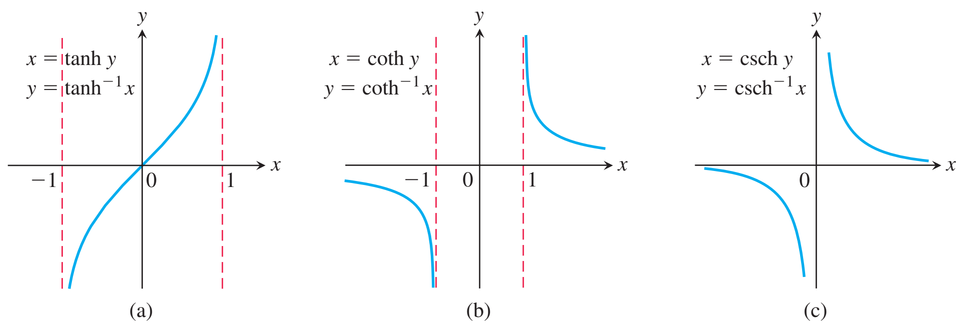


FIGURE 7.9 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

TABLE 7.8 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

Useful Identities

We use the identities in Table 7.8 to calculate the values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ on calculators that give only $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$. These identities are direct consequences of the definitions. For example, if $0 < x \le 1$, then

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x.$$

We also know that $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$, so because the hyperbolic secant is one-to-one on $(0, 1]$, we have

$$\cosh^{-1}\left(\frac{1}{x}\right) = \operatorname{sech}^{-1} x.$$

Derivatives of Inverse Hyperbolic Functions

An important use of inverse hyperbolic functions lies in antiderivatives that reverse the derivative formulas in Table 7.9.

TABLE 7.9 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1 - u^2}}\frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1 + u^2}}\frac{du}{dx}, \qquad u \ne 0$$

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $\tanh^{-1} u$ and $\coth^{-1} u$ come from the natural restrictions on the values of these functions. (See Figure 7.9a and b.) The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas.

We illustrate how the derivatives of the inverse hyperbolic functions are found in Example 2, where we calculate $d(\cosh^{-1} u)/dx$. The other derivatives are obtained by similar calculations.

EXAMPLE 2 Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}.$$

Solution First we find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem 3 of Section 3.8 with $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem 3 can be applied because the derivative of $\cosh x$ is positive for $0 < x$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3, Section 3.8} \\
&= \frac{1}{\sinh(\cosh^{-1} x)} && f'(u) = \sinh u \\
&= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} && \cosh^2 u - \sinh^2 u = 1, \ \sinh u = \sqrt{\cosh^2 u - 1} \\
&= \frac{1}{\sqrt{x^2 - 1}} && \cosh(\cosh^{-1} x) = x
\end{aligned}$$

HISTORICAL BIOGRAPHY
Sonya Kovalevsky
(1850–1891)

The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

With appropriate substitutions, the derivative formulas in Table 7.9 lead to the integration formulas in Table 7.10. Each of the formulas in Table 7.10 can be verified by differentiating the expression on the right-hand side.

TABLE 7.10 Integrals leading to inverse hyperbolic functions

1.	$\displaystyle\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,$	$a > 0$
2.	$\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,$	$u > a > 0$
3.	$\displaystyle\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$	
4.	$\displaystyle\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C,$	$0 < u < a$
5.	$\displaystyle\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C,$	$u \neq 0$ and $a > 0$

EXAMPLE 3 Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3 + 4x^2}}.$$

Solution The indefinite integral is

$$\begin{aligned}
\int \frac{2\,dx}{\sqrt{3 + 4x^2}} &= \int \frac{du}{\sqrt{a^2 + u^2}} && u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3} \\
&= \sinh^{-1}\left(\frac{u}{a}\right) + C && \text{Formula from Table 7.10} \\
&= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.
\end{aligned}$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

Exercises 7.3

Values and Identities

Each of Exercises 1–4 gives a value of sinh x or cosh x. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$

2. $\sinh x = \frac{4}{3}$

3. $\cosh x = \frac{17}{15},\quad x > 0$

4. $\cosh x = \frac{13}{5},\quad x > 0$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

5. $2\cosh(\ln x)$

6. $\sinh(2\ln x)$

7. $\cosh 5x + \sinh 5x$

8. $\cosh 3x - \sinh 3x$

9. $(\sinh x + \cosh x)^4$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

11. Prove the identities

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.$$

Then use them to show that

a. $\sinh 2x = 2\sinh x \cosh x.$

b. $\cosh 2x = \cosh^2 x + \sinh^2 x.$

12. Use the definitions of cosh x and sinh x to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Finding Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6\sinh\frac{x}{3}$

14. $y = \frac{1}{2}\sinh(2x + 1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t}$

16. $y = t^2\tanh\frac{1}{t}$

17. $y = \ln(\sinh z)$

18. $y = \ln(\cosh z)$

19. $y = \operatorname{sech}\theta(1 - \ln\operatorname{sech}\theta)$

20. $y = \operatorname{csch}\theta(1 - \ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \frac{1}{2}\tanh^2 v$

22. $y = \ln\sinh v - \frac{1}{2}\coth^2 v$

23. $y = (x^2 + 1)\operatorname{sech}(\ln x)$

(*Hint:* Before differentiating, express in terms of exponentials and simplify.)

24. $y = (4x^2 - 1)\operatorname{csch}(\ln 2x)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1}\sqrt{x}$

26. $y = \cosh^{-1}2\sqrt{x + 1}$

27. $y = (1 - \theta)\tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta + 1)$

29. $y = (1 - t)\coth^{-1}\sqrt{t}$

30. $y = (1 - t^2)\coth^{-1}t$

31. $y = \cos^{-1}x - x\operatorname{sech}^{-1}x$

32. $y = \ln x + \sqrt{1 - x^2}\operatorname{sech}^{-1}x$

33. $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^\theta$

34. $y = \operatorname{csch}^{-1}2^\theta$

35. $y = \sinh^{-1}(\tan x)$

36. $y = \cosh^{-1}(\sec x),\quad 0 < x < \pi/2$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a. $\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x) + C$

b. $\int \operatorname{sech} x\,dx = \sin^{-1}(\tanh x) + C$

38. $\int x\operatorname{sech}^{-1}x\,dx = \frac{x^2}{2}\operatorname{sech}^{-1}x - \frac{1}{2}\sqrt{1 - x^2} + C$

39. $\int x\coth^{-1}x\,dx = \frac{x^2 - 1}{2}\coth^{-1}x + \frac{x}{2} + C$

40. $\int \tanh^{-1}x\,dx = x\tanh^{-1}x + \frac{1}{2}\ln(1 - x^2) + C$

Evaluating Integrals

Evaluate the integrals in Exercises 41–60.

41. $\int \sinh 2x\,dx$

42. $\int \sinh\frac{x}{5}\,dx$

43. $\int 6\cosh\left(\frac{x}{2} - \ln 3\right)dx$

44. $\int 4\cosh(3x - \ln 2)\,dx$

45. $\int \tanh\frac{x}{7}\,dx$

46. $\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

47. $\int \operatorname{sech}^2\left(x - \frac{1}{2}\right)dx$

48. $\int \operatorname{csch}^2(5 - x)\,dx$

49. $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}\,dt}{\sqrt{t}}$

50. $\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)\,dt}{t}$

51. $\int_{\ln 2}^{\ln 4} \coth x\,dx$

52. $\int_0^{\ln 2} \tanh 2x\,dx$

53. $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\, d\theta$ **54.** $\int_{0}^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta$

55. $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\, d\theta$ **56.** $\int_{0}^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta$

57. $\int_{1}^{2} \frac{\cosh(\ln t)}{t}\, dt$ **58.** $\int_{1}^{4} \frac{8\cosh\sqrt{x}}{\sqrt{x}}\, dx$

59. $\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right) dx$ **60.** $\int_{0}^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right) dx$

Inverse Hyperbolic Functions and Integrals

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms, as shown here.

$$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right), \quad -\infty < x < \infty$$
$$\cosh^{-1}x = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1$$
$$\tanh^{-1}x = \frac{1}{2}\ln\frac{1 + x}{1 - x}, \quad |x| < 1$$
$$\operatorname{sech}^{-1}x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), \quad 0 < x \leq 1$$
$$\operatorname{csch}^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right), \quad x \neq 0$$
$$\coth^{-1}x = \frac{1}{2}\ln\frac{x + 1}{x - 1}, \quad |x| > 1$$

Use the formulas in the box here to express the numbers in Exercises 61–66 in terms of natural logarithms.

61. $\sinh^{-1}(-5/12)$ **62.** $\cosh^{-1}(5/3)$

63. $\tanh^{-1}(-1/2)$ **64.** $\coth^{-1}(5/4)$

65. $\operatorname{sech}^{-1}(3/5)$ **66.** $\operatorname{csch}^{-1}\left(-1/\sqrt{3}\right)$

Evaluate the integrals in Exercises 67–74 in terms of

a. inverse hyperbolic functions.
b. natural logarithms.

67. $\int_{0}^{2\sqrt{3}} \frac{dx}{\sqrt{4 + x^2}}$ **68.** $\int_{0}^{1/3} \frac{6\, dx}{\sqrt{1 + 9x^2}}$

69. $\int_{5/4}^{2} \frac{dx}{1 - x^2}$ **70.** $\int_{0}^{1/2} \frac{dx}{1 - x^2}$

71. $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1 - 16x^2}}$ **72.** $\int_{1}^{2} \frac{dx}{x\sqrt{4 + x^2}}$

73. $\int_{0}^{\pi} \frac{\cos x\, dx}{\sqrt{1 + \sin^2 x}}$ **74.** $\int_{1}^{e} \frac{dx}{x\sqrt{1 + (\ln x)^2}}$

Applications and Examples

75. Show that if a function f is defined on an interval symmetric about the origin (so that f is defined at $-x$ whenever it is defined at x), then

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}. \quad (1)$$

Then show that $(f(x) + f(-x))/2$ is even and that $(f(x) - f(-x))/2$ is odd.

76. Derive the formula $\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right)$ for all real x. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

77. Skydiving If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity t sec into the fall satisfies the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that the variation in the air's density will not affect the outcome significantly.)

a. Show that

$$v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}\,t\right)$$

satisfies the differential equation and the initial condition that $v = 0$ when $t = 0$.

b. Find the body's *limiting velocity*, $\lim_{t\to\infty} v$.

c. For a 160-lb skydiver ($mg = 160$), with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity?

78. Accelerations whose magnitudes are proportional to displacement Suppose that the position of a body moving along a coordinate line at time t is

a. $s = a\cos kt + b\sin kt$.

b. $s = a\cosh kt + b\sinh kt$.

Show in both cases that the acceleration d^2s/dt^2 is proportional to s but that in the first case it is directed toward the origin, whereas in the second case it is directed away from the origin.

79. Volume A region in the first quadrant is bounded above by the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

80. Volume The region enclosed by the curve $y = \operatorname{sech} x$, the x-axis, and the lines $x = \pm\ln\sqrt{3}$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

81. Arc length Find the length of the graph of $y = (1/2)\cosh 2x$ from $x = 0$ to $x = \ln\sqrt{5}$.

82. Use the definitions of the hyperbolic functions to find each of the following limits.

a. $\lim_{x\to\infty}\tanh x$ **b.** $\lim_{x\to-\infty}\tanh x$

c. $\lim_{x\to\infty}\sinh x$ **d.** $\lim_{x\to-\infty}\sinh x$

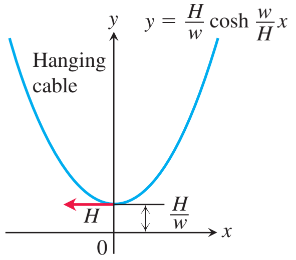
e. $\lim_{x\to\infty}\operatorname{sech} x$ **f.** $\lim_{x\to\infty}\coth x$

g. $\lim_{x\to0^+}\coth x$ **h.** $\lim_{x\to0^-}\coth x$

i. $\lim_{x\to-\infty}\operatorname{csch} x$

83. Hanging cables Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is a constant w and the horizontal tension at its lowest point is a *vector* of length H. If we choose a coordinate system for the plane of the cable in which the x-axis is horizontal, the force of gravity is straight down, the positive y-axis points straight up, and the lowest point of the cable lies at the point $y = H/w$ on the y-axis (see accompanying figure), then it can be shown that the cable lies along the graph of the hyperbolic cosine

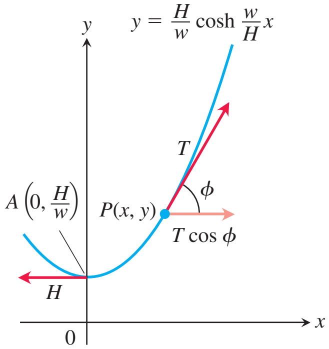
$$y = \frac{H}{w}\cosh\frac{w}{H}x.$$



Such a curve is sometimes called a **chain curve** or a **catenary**, the latter deriving from the Latin *catena*, meaning "chain."

a. Let $P(x, y)$ denote an arbitrary point on the cable. The next accompanying figure displays the tension at P as a vector of length (magnitude) T, as well as the tension H at the lowest point A. Show that the cable's slope at P is

$$\tan\phi = \frac{dy}{dx} = \sinh\frac{w}{H}x.$$



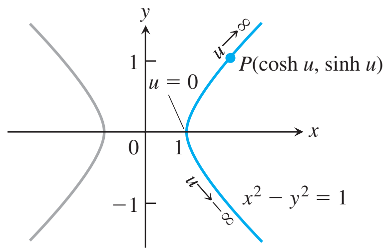
b. Using the result from part (a) and the fact that the horizontal tension at P must equal H (the cable is not moving), show that $T = wy$. Hence, the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.

84. (*Continuation of Exercise 83.*) The length of arc AP in the Exercise 83 figure is $s = (1/a)\sinh ax$, where $a = w/H$. Show that the coordinates of P may be expressed in terms of s as

$$x = \frac{1}{a}\sinh^{-1} as, \qquad y = \sqrt{s^2 + \frac{1}{a^2}}.$$

85. Area Show that the area of the region in the first quadrant enclosed by the curve $y = (1/a)\cosh ax$, the coordinate axes, and the line $x = b$ is the same as the area of a rectangle of height $1/a$ and length s, where s is the length of the curve from $x = 0$ to $x = b$. Draw a figure illustrating this result.

86. The hyperbolic in hyperbolic functions Just as $x = \cos u$ and $y = \sin u$ are identified with points (x, y) on the unit circle, the functions $x = \cosh u$ and $y = \sinh u$ are identified with points (x, y) on the right-hand branch of the unit hyperbola, $x^2 - y^2 = 1$.



Since $\cosh^2 u - \sinh^2 u = 1$, the point $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$ for every value of u.

Another analogy between hyperbolic and circular functions is that the variable u in the coordinates $(\cosh u, \sinh u)$ for the points of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ is twice the area of the sector AOP pictured in the accompanying figure. To see why this is so, carry out the following steps.

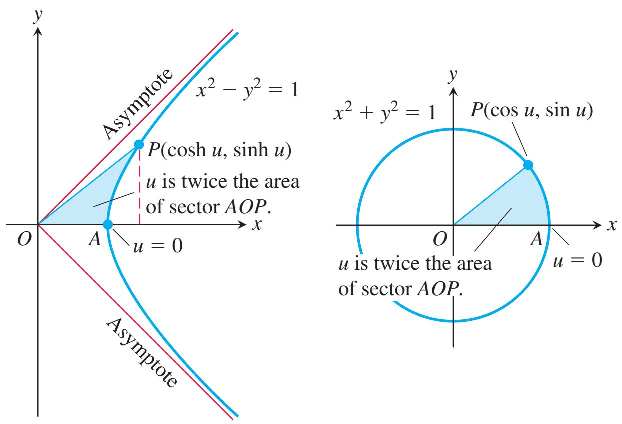
a. Show that the area $A(u)$ of sector AOP is

$$A(u) = \frac{1}{2}\cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx.$$

b. Differentiate both sides of the equation in part (a) with respect to u to show that

$$A'(u) = \frac{1}{2}.$$

c. Solve this last equation for $A(u)$. What is the value of $A(0)$? What is the value of the constant of integration C in your solution? With C determined, what does your solution say about the relationship of u to $A(u)$?



One of the analogies between hyperbolic and circular functions is revealed by these two diagrams (Exercise 86).

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left side of the equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

On the right side of the equation:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

The graph of the function is shown in Figure 9.1. ■

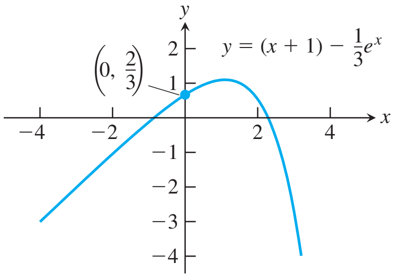


FIGURE 9.1 Graph of the solution to the initial value problem in Example 2.

Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 9.2a shows a slope field, with a particular solution sketched into it in Figure 9.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.

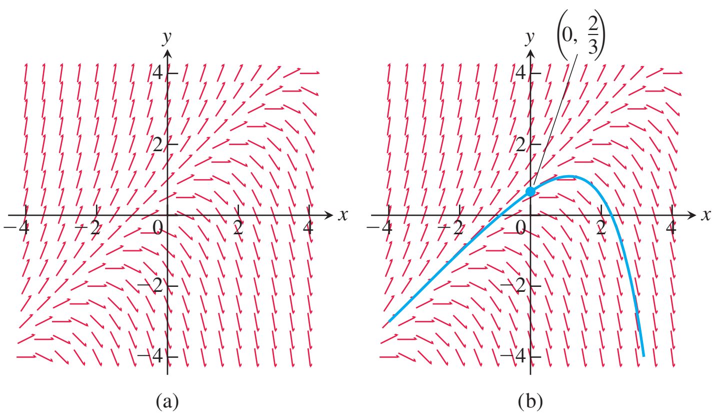


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

Figure 9.3 shows three slope fields and we see how the solution curves behave by following the tangent line segments in these fields. Slope fields are useful because they display the overall behavior of the family of solution curves for a given differential equation.

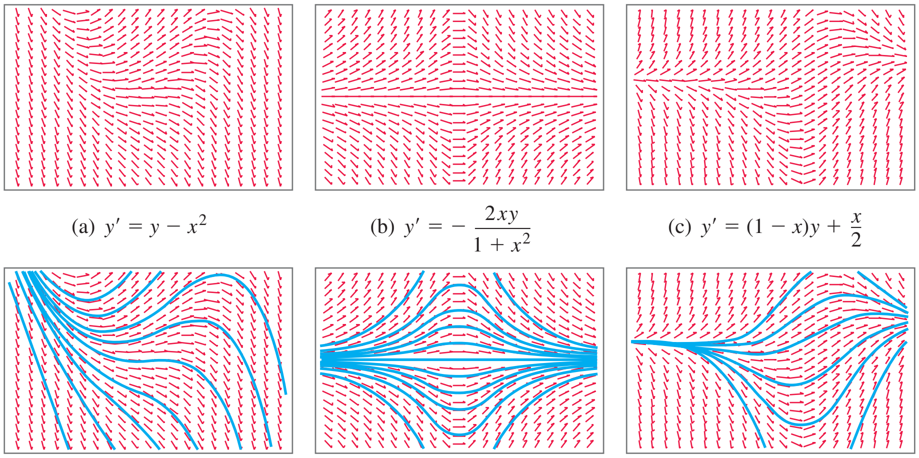


FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here, but they should be considered as just tangent line segments.

For instance, the slope field in Figure 9.3b reveals that every solution $y(x)$ to the differential equation specified in the figure satisfies $\lim_{x \to \pm\infty} y(x) = 0$. We will see that knowing the overall behavior of the solution curves is often critical to understanding and predicting outcomes in a real-world system modeled by a differential equation.

Constructing a slope field with pencil and paper can be quite tedious. All our examples were generated by computer software.

Euler's Method

If we do not require or cannot immediately find an *exact* solution giving an explicit formula for an initial value problem $y' = f(x, y), y(x_0) = y_0$, we can often use a computer to generate a table of approximate numerical values of y for values of x in an appropriate interval. Such a table is called a **numerical solution** of the problem, and the method by which we generate the table is called a **numerical method**.

Given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \qquad \text{or} \qquad L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

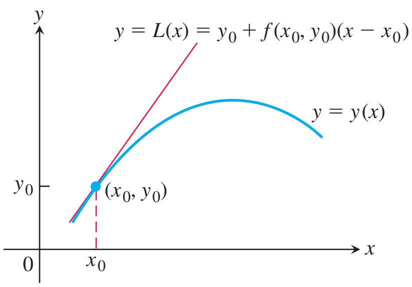


FIGURE 9.4 The linearization $L(x)$ of $y = y(x)$ at $x = x_0$.

The function $L(x)$ gives a good approximation to the solution $y(x)$ in a short interval about x_0 (Figure 9.4). The basis of Euler's method is to patch together a string of linearizations to approximate the curve over a longer stretch. Here is how the method works.

We know the point (x_0, y_0) lies on the solution curve. Suppose that we specify a new value for the independent variable to be $x_1 = x_0 + dx$. (Recall that $dx = \Delta x$ in the definition of differentials.) If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)\, dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0), which lies *exactly* on the solution curve, we have obtained the point (x_1, y_1), which lies very close to the point $(x_1, y(x_1))$ on the solution curve (Figure 9.5).

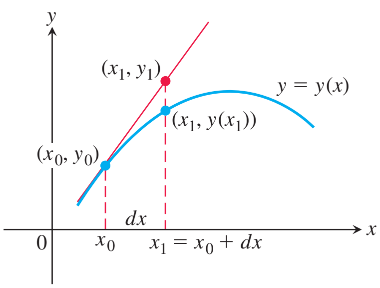


FIGURE 9.5 The first Euler step approximates $y(x_1)$ with $y_1 = L(x_1)$.

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1), we take a second step. Setting $x_2 = x_1 + dx$, we use the linearization of the solution curve through (x_1, y_1) to calculate

$$y_2 = y_1 + f(x_1, y_1)\, dx.$$

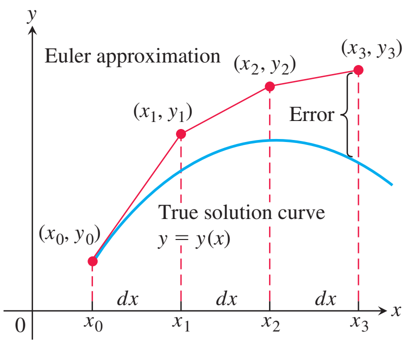


FIGURE 9.6 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

This gives the next approximation (x_2, y_2) to values along the solution curve $y = y(x)$ (Figure 9.6). Continuing in this fashion, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the third approximation

$$y_3 = y_2 + f(x_2, y_2)\,dx,$$

and so on. We are literally building an approximation to one of the solutions by following the direction of the slope field of the differential equation.

The steps in Figure 9.6 are drawn large to illustrate the construction process, so the approximation looks crude. In practice, dx would be small enough to make the red curve hug the blue one and give a good approximation throughout.

EXAMPLE 3 Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1,$$

starting at $x_0 = 0$ with $dx = 0.1$.

Solution We have the starting values $x_0 = 0$ and $y_0 = 1$. Next we determine the values of x at which the Euler approximations will take place: $x_1 = x_0 + dx = 0.1$, $x_2 = x_0 + 2\,dx = 0.2$, and $x_3 = x_0 + 3\,dx = 0.3$. Then we find

$$\begin{aligned}
\textit{First:}\quad y_1 &= y_0 + f(x_0, y_0)\,dx \\
&= y_0 + (1 + y_0)\,dx \\
&= 1 + (1 + 1)(0.1) = 1.2 \\
\textit{Second:}\quad y_2 &= y_1 + f(x_1, y_1)\,dx \\
&= y_1 + (1 + y_1)\,dx \\
&= 1.2 + (1 + 1.2)(0.1) = 1.42 \\
\textit{Third:}\quad y_3 &= y_2 + f(x_2, y_2)\,dx \\
&= y_2 + (1 + y_2)\,dx \\
&= 1.42 + (1 + 1.42)(0.1) = 1.662
\end{aligned}$$

The step-by-step process used in Example 3 can be continued easily. Using equally spaced values for the independent variable in the table for the numerical solution, and generating n of them, set

$$\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx.
\end{aligned}$$

Then calculate the approximations to the solution,

$$\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\,dx \\
y_2 &= y_1 + f(x_1, y_1)\,dx \\
&\vdots \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\,dx.
\end{aligned}$$

The number of steps n can be as large as we like, but errors can accumulate if n is too large.

Euler's method is easy to implement on a computer or calculator. The software program generates a table of numerical solutions to an initial value problem, allowing us to input x_0 and y_0, the number of steps n, and the step size dx. It then calculates the approximate solution values $y_1, y_2, \ldots, y_n$ in iterative fashion, as just described.

HISTORICAL BIOGRAPHY
Leonhard Euler
(1703–1783)

Solving the separable equation in Example 3, we find that the exact solution to the initial value problem is $y = 2e^x - 1$. We use this information in Example 4.

EXAMPLE 4 Use Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking **(a)** $dx = 0.1$ and **(b)** $dx = 0.05$. Compare the approximations with the values of the exact solution $y = 2e^x - 1$.

Solution

(a) We used a computer to generate the approximate values in Table 9.1. The "error" column is obtained by subtracting the unrounded Euler values from the unrounded values found using the exact solution. All entries are then rounded to four decimal places.

TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

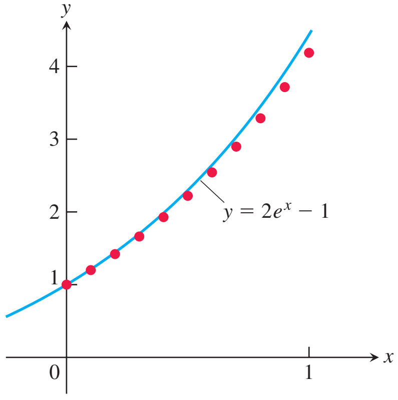


FIGURE 9.7 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1 (Example 4).

By the time we reach $x = 1$ (after 10 steps), the error is about 5.6% of the exact solution. A plot of the exact solution curve with the scatterplot of Euler solution points from Table 9.1 is shown in Figure 9.7.

(b) One way to try to reduce the error is to decrease the step size. Table 9.2 shows the results and their comparisons with the exact solutions when we decrease the step size to 0.05, doubling the number of steps to 20. As in Table 9.1, all computations are performed before rounding. This time when we reach $x = 1$, the relative error is only about 2.9%. ■

It might be tempting to reduce the step size even further in Example 4 to obtain greater accuracy. Each additional calculation, however, not only requires additional computer time but more importantly adds to the buildup of round-off errors due to the approximate representations of numbers inside the computer.

The analysis of error and the investigation of methods to reduce it when making numerical calculations are important but are appropriate for a more advanced course. There are numerical methods more accurate than Euler's method, usually presented in a further study of differential equations or in a numerical analysis course.

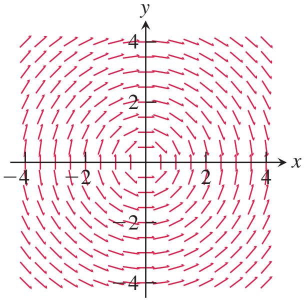
TABLE 9.2 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.05$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287
0.45	2.1027	2.1366	0.0340
0.50	2.2578	2.2974	0.0397
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0853
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

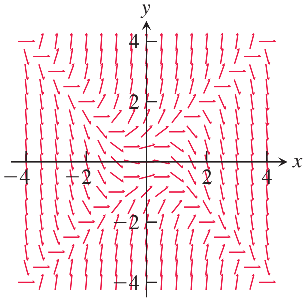
Exercises 9.1

Slope Fields

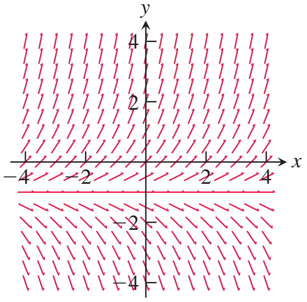
In Exercises 1–4, match the differential equations with their slope fields, graphed here.



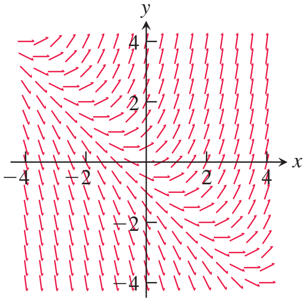
(a)



(b)



(c)



(d)

1. $y' = x + y$

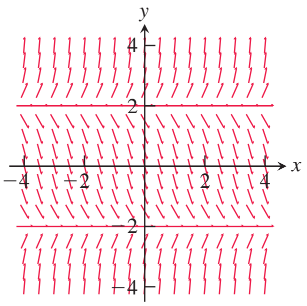
2. $y' = y + 1$

3. $y' = -\frac{x}{y}$

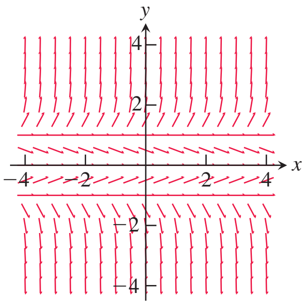
4. $y' = y^2 - x^2$

In Exercises 5 and 6, copy the slope fields and sketch in some of the solution curves.

5. $y' = (y + 2)(y - 2)$



6. $y' = y(y + 1)(y - 1)$



Integral Equations

In Exercises 7–10, write an equivalent first-order differential equation and initial condition for y.

7. $y = -1 + \int_1^x (t - y(t))\, dt$

8. $y = \int_1^x \frac{1}{t}\, dt$

9. $y = 2 - \int_0^x (1 + y(t)) \sin t\, dt$

10. $y = 1 + \int_0^x y(t)\, dt$

Using Euler's Method

In Exercises 11–16, use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round your results to four decimal places.

11. $y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$

12. $y' = x(1 - y), \quad y(1) = 0, \quad dx = 0.2$

13. $y' = 2xy + 2y, \quad y(0) = 3, \quad dx = 0.2$

14. $y' = y^2(1 + 2x), \quad y(-1) = 1, \quad dx = 0.5$

T **15.** $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1$

T **16.** $y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

17. Use the Euler method with $dx = 0.2$ to estimate $y(1)$ if $y' = y$ and $y(0) = 1$. What is the exact value of $y(1)$?

18. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = y/x$ and $y(1) = 2$. What is the exact value of $y(2)$?

19. Use the Euler method with $dx = 0.5$ to estimate $y(5)$ if $y' = y^2/\sqrt{x}$ and $y(1) = -1$. What is the exact value of $y(5)$?

20. Use the Euler method with $dx = 1/3$ to estimate $y(2)$ if $y' = x \sin y$ and $y(0) = 1$. What is the exact value of $y(2)$?

21. Show that the solution of the initial value problem

$$y' = x + y, \qquad y(x_0) = y_0$$

is

$$y = -1 - x + (1 + x_0 + y_0)\, e^{x - x_0}.$$

22. What integral equation is equivalent to the initial value problem $y' = f(x), y(x_0) = y_0$?

COMPUTER EXPLORATIONS

In Exercises 23–28, obtain a slope field and add to it graphs of the solution curves passing through the given points.

23. $y' = y$ with

a. (0, 1) **b.** (0, 2) **c.** (0, −1)

24. $y' = 2(y - 4)$ with

a. (0, 1) **b.** (0, 4) **c.** (0, 5)

25. $y' = y(x + y)$ with

a. (0, 1) **b.** (0, −2) **c.** (0, 1/4) **d.** (−1, −1)

26. $y' = y^2$ with

a. (0, 1) **b.** (0, 2) **c.** (0, −1) **d.** (0, 0)

27. $y' = (y - 1)(x + 2)$ with

a. (0, −1) **b.** (0, 1) **c.** (0, 3) **d.** (1, − 1)

28. $y' = \frac{xy}{x^2 + 4}$ with

a. (0, 2) **b.** (0, −6) **c.** $\left(-2\sqrt{3}, -4\right)$

In Exercises 29 and 30, obtain a slope field and graph the particular solution over the specified interval. Use your CAS DE solver to find the general solution of the differential equation.

29. A logistic equation $y' = y(2 - y), \; y(0) = 1/2; \; 0 \le x \le 4, \; 0 \le y \le 3$

30. $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \le x \le 6, \quad -6 \le y \le 6$

Exercises 31 and 32 have no explicit solution in terms of elementary functions. Use a CAS to explore graphically each of the differential equations.

31. $y' = \cos(2x - y), \quad y(0) = 2; \quad 0 \le x \le 5, \quad 0 \le y \le 5$

32. A Gompertz equation $y' = y(1/2 - \ln y), \quad y(0) = 1/3; \; 0 \le x \le 4, \quad 0 \le y \le 3$

33. Use a CAS to find the solutions of $y' + y = f(x)$ subject to the initial condition $y(0) = 0$, if $f(x)$ is

a. $2x$ **b.** $\sin 2x$ **c.** $3e^{x/2}$ **d.** $2e^{-x/2} \cos 2x$.

Graph all four solutions over the interval $-2 \le x \le 6$ to compare the results.

34. a. Use a CAS to plot the slope field of the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

over the region $-3 \le x \le 3$ and $-3 \le y \le 3$.

b. Separate the variables and use a CAS integrator to find the general solution in implicit form.

c. Using a CAS implicit function grapher, plot solution curves for the arbitrary constant values $C = -6, -4, -2, 0, 2, 4, 6$.

d. Find and graph the solution that satisfies the initial condition $y(0) = -1$.

In Exercises 35–38, use Euler's method with the specified step size to estimate the value of the solution at the given point x^*. Find the value of the exact solution at x^*.

35. $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1, \quad x^* = 1$

36. $y' = 2y^2(x - 1), \quad y(2) = -1/2, \quad dx = 0.1, \quad x^* = 3$

37. $y' = \sqrt{x}/y, \quad y > 0, \quad y(0) = 1, \quad dx = 0.1, \quad x^* = 1$

38. $y' = 1 + y^2, \quad y(0) = 0, \quad dx = 0.1, \quad x^* = 1$

Use a CAS to explore graphically each of the differential equations in Exercises 39–42. Perform the following steps to help with your explorations.

a. Plot a slope field for the differential equation in the given xy-window.

b. Find the general solution of the differential equation using your CAS DE solver.

c. Graph the solutions for the values of the arbitrary constant $C = -2, -1, 0, 1, 2$ superimposed on your slope field plot.

d. Find and graph the solution that satisfies the specified initial condition over the interval $[0, b]$.

e. Find the Euler numerical approximation to the solution of the initial value problem with 4 subintervals of the x-interval and plot the Euler approximation superimposed on the graph produced in part (d).

f. Repeat part (e) for 8, 16, and 32 subintervals. Plot these three Euler approximations superimposed on the graph from part (e).

g. Find the error (y(exact) $-$ y(Euler)) at the specified point $x = b$ for each of your four Euler approximations. Discuss the improvement in the percentage error.

39. $y' = x + y, \quad y(0) = -7/10; \quad -4 \le x \le 4, \quad -4 \le y \le 4; \quad b = 1$

40. $y' = -x/y, \quad y(0) = 2; \quad -3 \le x \le 3, \quad -3 \le y \le 3; \; b = 2$

41. $y' = y(2 - y), \quad y(0) = 1/2; \quad 0 \le x \le 4, \; 0 \le y \le 3; \; b = 3$

42. $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \le x \le 6, \quad -6 \le y \le 6; \quad b = 3\pi/2$

9.2 First-Order Linear Equations

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**. Since the exponential growth/decay equation $dy/dx = ky$ (Section 7.2) can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function $\left(\text{such as } \sin y,\, e^y, \text{ or } \sqrt{dy/dx}\right)$.

EXAMPLE 1 Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x.$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ■

Ordinary Differential Equation

Differential equation. A differential equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

$$F(x,y,y',y'',\ldots,y^{(n)})=0$$

Where x is called the independent variable and y is the dependent.

Here are a few more examples of differential equations.

$$ay''+by'+cy=g(t) \quad (5)$$

$$\sin(y)\frac{d^2y}{dx^2}=(1-y)\frac{dy}{dx}+y^2e^{-5y} \quad (6)$$

$$y^{(4)}+10y'''-4y'+2y=\cos(t) \quad (7)$$

$$\alpha^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \quad (8)$$

$$a^2u_{xx}=u_{tt} \quad (9)$$

$$\frac{\partial^3 u}{\partial^2 x\partial t}=1+\frac{\partial u}{\partial y} \quad (10)$$

Order

The order of a differential equation is the largest derivative present in the differential equation.

Examples: In the differential equations listed above(5), (6), (8), and (9) are second order differential equations, (10) is a third order differential equation and (7) is a fourth order differential equation.

Ordinary and Partial Differential Equations

Definition A differential equation is called an **ordinary differential equation**, abbreviated by **ode,** if it has ordinary derivatives in it

$$F(x,y,y',y'',\ldots,y^{(n)})=0$$

Definition a differential equation is called a **partial differential equation**, abbreviated by **pde,** if it has differential derivatives in it. In the differential

Example: equations above (5) - (7) are ode's and (8) - (10) are pde's.

A linear differential equation is any differential equation that can be written in the following form.

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=g(t) \quad \text{...(11)}$$

The important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives occur to any power other than the first power. The coefficients $a_0(t)$, ... ,$a_n(t)$ and $g(t)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, $y(t)$, and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, (11) then it is called a **non-linear** differential equation.

Examples In (5) - (7) above only (6) is non-linear, the other two are linear differential equations.

Definition A **solution** to a differential equation on an interval $\alpha < t < \beta$ is any function y=y(t)

which satisfies the differential equation in question on the interval

Example Show that $y(x)=x^{-\frac{3}{2}}$

is a solution to $4x^2y''+12xy'+3y=0$ for $x>0$.

Solution We'll need the first and second derivative to do this.

$$y'(x)=-\frac{3}{2}x^{-\frac{5}{2}} \qquad y''(x)=\frac{15}{4}x^{-\frac{7}{2}}$$

Put these function into the differential equation.

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right)+12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right)+3\left(x^{-\frac{3}{2}}\right)=0$$

$$15x^{-\frac{3}{2}}-18x^{-\frac{3}{2}}+3x^{-\frac{3}{2}}=0$$

$$0=0$$

So, $y(x)=x^{-\frac{5}{2}}$ does satisfy the differential equation and hence is a solution.

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions (often abbreviated i.c.'s) are of the form,

$$y(t_0)=y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0)=y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points.

Note The ***number*** of initial conditions that are required for a given differential equation will depend upon the ***order*** of the differential equation as we will see.

Example $y(x)=x^{-\frac{3}{2}}$ is a solution to

$4x^2y''+12xy'+3y=0$, $y(4)=\frac{1}{8}$, and $y'(4)=-\frac{3}{64}$.

Solution As we saw in previous example the function is a solution and we can then note that

$$y(4)=4^{-\frac{3}{2}}=\frac{1}{\left(\sqrt{4}\right)^3}=\frac{1}{8}$$

$$y'(4)=-\frac{3}{2}4^{-\frac{5}{2}}=-\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5}=-\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4)=\frac{1}{8}$ and $y'(4)=-\frac{3}{64}$

Definition An **Initial Value Problem** (or **IVP**) is a differential equation along with an appropriate number of initial conditions.

Example The following is an IVP.

$$4x^2y''+12xy'+3y=0 \qquad y(4)=\frac{1}{8}, \quad y'(4)=-\frac{3}{64}$$

Example Here's another IVP.

$$2ty' + 4y = 3 \qquad y(1) = -4$$

Definition The **general solution** to a differential equation is the most general form that the solution can take and doesn't take any initial conditions into account i.e contains **a constants** same as the order of DE.

Example $y(t) = (3/4) + (c/t^2)$ is the general solution to

$$2ty' + 4y = 3$$

Definition The **particular solution** to a differential equation is the specific solution that not only satisfies the differential equation, but also satisfies the given initial condition(s).

Example 6 What is the particular solution to the following IVP?

$$2ty' + 4y = 3 \qquad y(1) = -4$$

Solution This is actually easier to do than it might at first appear. From the previous example we already know (well that is provided you believe my solution to this example...) that all solutions to the differential equation are of the form.

$$y(t) = \frac{3}{4} + \frac{c}{t^2}$$

All that we need to do is determine the value of c that will give us the solution that we're after. To find this all we need do is use our initial condition as follows.

$$-4 = y(1) = \frac{3}{4} + \frac{c}{1^2} \qquad \Rightarrow \qquad c = -4 - \frac{3}{4} = -\frac{19}{4}$$

So, the actual solution to the IVP is.

$$y(t) = \frac{3}{4} - \frac{19}{4t^2}$$

Separable Differential Equations

A separable differential equation is any differential equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x) \qquad (1)$$

Note that in order for a differential equation to be separable all the y's in the differential equation must be multiplied by the derivative and all the x's in the differential equation must be on the other side of the equal sign.

Solving separable differential equation is fairly easy. We first rewrite the differential equation as the following

$$N(y)\,dy = M(x)\,dx$$

Then you integrate both sides.

$$\int N(y)\,dy = \int M(x)\,dx \qquad (2)$$

The solution in the form $y = y(x)$

Example 1 Solve the following differential equation

$$\frac{dy}{dx} = 6y^2x \qquad y(1) = \frac{1}{25}$$

Solution

$$y^{-2}dy = 6x\,dx$$

$$\int y^{-2}dy = \int 6x\,dx$$

$$-\frac{1}{y} = 3x^2 + c$$

So apply the initial condition and find the value of c.

$$-\frac{1}{1/25} = 3(1)^2 + c \qquad c = -28$$

Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 - 28$$

$$y(x) = \frac{1}{28 - 3x^2}$$

Example 2 Solve the following.

$$y' = \frac{3x^2 + 4x - 4}{2y - 4} \qquad y(1) = 3$$

Solution
This differential equation is clearly separable, so let's put it in the proper form and then integrate both sides.

$$(2y-4)\,dy = (3x^2 + 4x - 4)\,dx$$
$$\int (2y-4)\,dy = \int (3x^2 + 4x - 4)\,dx$$
$$y^2 - 4y = x^3 + 2x^2 - 4x + c$$

We now have our implicit solution, so as with the first example let's apply the initial condition at this point to determine the value of c.

$$(3)^2 - 4(3) = (1)^3 + 2(1)^2 - 4(1) + c \qquad c = -2$$

The solution is then

$$y^2 - 4y = x^3 + 2x^2 - 4x - 2$$

We now need to find the explicit solution. This is actually easier than it might look and you already know how to do it. First we need to rewrite the solution a little

$$y^2 - 4y - (x^3 + 2x^2 - 4x - 2) = 0$$

To solve this all we need to recognize is that this is quadratic in y and so we can use the quadratic formula to solve it. However, unlike quadratics you are used to, at least some of the "constants" will not actually be constant, but will in fact involve x's.

So, upon using the quadratic formula on this we get.

$$y(x) = \frac{4 \pm \sqrt{16 - 4(1)\left(-(x^3 + 2x^2 - 4x - 2)\right)}}{2}$$
$$= \frac{4 \pm \sqrt{16 + 4(x^3 + 2x^2 - 4x - 2)}}{2}$$

Next, notice that we can factor a 4 out from under the square root (it will come out as a 2…) and then simplify a little.

$$y(x) = \frac{4 \pm 2\sqrt{4 + (x^3 + 2x^2 - 4x - 2)}}{2}$$
$$= 2 \pm \sqrt{x^3 + 2x^2 - 4x + 2}$$

We are almost there. Notice that we've actually got two solutions here (the " $\pm$ ") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x = 1$ into the solution gives.

$$3 = y(1) = 2 \pm \sqrt{1+2-4+2} = 2 \pm 1 = 3,1$$

In this case it looks like the "+" is the correct sign for our solution. Note that it is completely possible that the " — " could be the solution so don't always expect it to be one or the other.

Example 4 Solve the following IVP.

$$y' = \mathrm{e}^{-y}(2x-4) \qquad y(5) = 0$$

Solution
This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$\mathrm{e}^{y}\,dy = (2x-4)\,dx$$
$$\int \mathrm{e}^{y}\,dy = \int (2x-4)\,dx$$
$$\mathrm{e}^{y} = x^2 - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad c = -4$$

This then gives an implicit solution of.

$$\mathrm{e}^{y} = x^2 - 4x - 4$$

Example 5 Solve the following IVP.

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} \qquad r(1) = 2$$

Solution
This is actually a fairly simple differential equation to solve. I'm doing this one mostly because of the interval of validity.

So, get things separated out and then integrate.

$$\frac{1}{r^2}\,dr = \frac{1}{\theta}\,d\theta$$

$$\int \frac{1}{r^2}\,dr = \int \frac{1}{\theta}\,d\theta$$

$$-\frac{1}{r} = \ln|\theta| + c$$

Now, apply the initial condition to find c.

$$-\frac{1}{2} = \ln(1) + c \qquad c = -\frac{1}{2}$$

So, the implicit solution is then,

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

Solving for r gets us our explicit solution.

$$r = \frac{1}{\frac{1}{2} - \ln|\theta|}$$

Example 6 Solve the following IVP.

$$\frac{dy}{dt} = \mathbf{e}^{y-t}\sec(y)\left(1+t^2\right) \qquad y(0) = 0$$

Solution

This problem will require a little work to get it separated and in a form that we can integrate, so let's do that first.

$$\frac{dy}{dt} = \frac{\mathbf{e}^{y}\mathbf{e}^{-t}}{\cos(y)}\left(1+t^2\right)$$

$$\mathbf{e}^{-y}\cos(y)\,dy = \mathbf{e}^{-t}\left(1+t^2\right)dt$$

Now, with a little integration by parts on both sides we can get an implicit solution.

$$\int \mathbf{e}^{-y}\cos(y)\,dy = \int \mathbf{e}^{-t}\left(1+t^2\right)dt$$

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y) - \cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+c$$

Applying the initial condition gives.

$$\frac{1}{2}(-1) = -(3)+c \qquad c = \frac{5}{2}$$

Therefore, the implicit solution is.

$$\frac{\mathbf{e}^{-y}}{2}\left(\sin(y) - \cos(y)\right) = -\mathbf{e}^{-t}\left(t^2+2t+3\right)+\frac{5}{2}$$

Solving Linear Equations

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by multiplying both sides by a *positive* function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by $v(x)$ works:

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{Original equation is in standard form.}$$

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x) \qquad \text{Multiply by positive } v(x).$$

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x) \qquad v(x) \text{ is chosen to make } v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y).$$

$$v(x) \cdot y = \int v(x)Q(x)\,dx \qquad \text{Integrate with respect to } x.$$

$$y = \frac{1}{v(x)}\int v(x)Q(x)\,dx \tag{2}$$

Equation (2) expresses the solution of Equation (1) in terms of the functions $v(x)$ and $Q(x)$. We call $v(x)$ an **integrating factor** for Equation (1) because its presence makes the equation integrable.

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly, in the construction of the positive function $v(x)$. We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy \qquad \text{Condition imposed on } v$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy \qquad \text{Derivative Product Rule}$$

$$y\frac{dv}{dx} = Pvy \qquad \text{The terms } v\frac{dy}{dx} \text{ cancel.}$$

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\,dx \qquad \text{Variables separated, } v > 0$$

$$\int \frac{dv}{v} = \int P\,dx \qquad \text{Integrate both sides.}$$

$$\ln v = \int P\,dx \qquad \text{Since } v > 0, \text{ we do not need absolute value signs in } \ln v.$$

$$e^{\ln v} = e^{\int P\,dx} \qquad \text{Exponentiate both sides to solve for } v.$$

$$v = e^{\int P\,dx} \tag{3}$$

Thus a formula for the general solution to Equation (1) is given by Equation (2), where $v(x)$ is given by Equation (3). However, rather than memorizing the formula, just remember how to find the integrating factor once you have the standard form so $P(x)$ is correctly identified. Any antiderivative of P works for Equation (3).

> To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

When you integrate the product on the left-hand side in this procedure, you always obtain the product $v(x)y$ of the integrating factor and solution function y because of the way v is defined.

HISTORICAL BIOGRAPHY
Adrien Marie Legendre
(1752–1833)

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int (-3/x)\,dx} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln|x|} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\begin{aligned} \frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) &= \frac{1}{x^3}\cdot x \\ \frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y &= \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} && \text{Left-hand side is } \frac{d}{dx}(v\cdot y). \\ \frac{1}{x^3}y &= \int \frac{1}{x^2}\,dx && \text{Integrate both sides.} \\ \frac{1}{x^3}y &= -\frac{1}{x} + C. \end{aligned}$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$

■

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \qquad \text{Left-hand side is } vy.$$

Integration by parts of the right-hand side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

or, solving for y,

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C,$$

so

$$C = 2.$$

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4.$$

In solving the linear equation in Example 2, we integrated both sides of the equation after multiplying each side by the integrating factor. However, we can shorten the amount of work, as in Example 3, by remembering that the left-hand side *always* integrates into the product $v(x) \cdot y$ of the integrating factor times the solution function. From Equation (2) this means that

$$v(x)y = \int v(x)Q(x)\,dx. \tag{4}$$

We need only integrate the product of the integrating factor $v(x)$ with $Q(x)$ on the right-hand side of Equation (1) and then equate the result with $v(x)y$ to obtain the general solution. Nevertheless, to emphasize the role of $v(x)$ in the solution process, we sometimes follow the complete procedure as illustrated in Example 2.

Observe that if the function $Q(x)$ is identically zero in the standard form given by Equation (1), the linear equation is separable and can be solved by the method of Section 7.2:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = 0 \qquad Q(x) = 0$$

$$\frac{dy}{y} = -P(x)\,dx \qquad \text{Separating the variables}$$

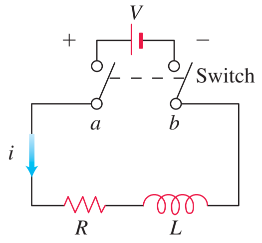


FIGURE 9.8 The *RL* circuit in Example 4.

RL Circuits

The diagram in Figure 9.8 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's Law, $V = RI$, has to be augmented for such a circuit. The correct equation accounting for both resistance and inductance is

$$L\frac{di}{dt} + Ri = V, \tag{5}$$

where i is the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.

EXAMPLE 4 The switch in the *RL* circuit in Figure 9.8 is closed at time $t = 0$. How will the current flow as a function of time?

Solution Equation (5) is a first-order linear differential equation for i as a function of t. Its standard form is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}, \tag{6}$$

and the corresponding solution, given that $i = 0$ when $t = 0$, is

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t}. \tag{7}$$

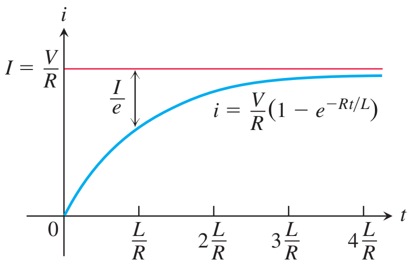


FIGURE 9.9 The growth of the current in the *RL* circuit in Example 4. I is the current's steady-state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 time constants (Exercise 27).

(We leave the calculation of the solution for you to do in Exercise 28.) Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t} \to 0$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

At any given time, the current is theoretically less than V/R, but as time passes, the current approaches the **steady-state value** V/R. According to the equation

$$L\frac{di}{dt} + Ri = V,$$

$I = V/R$ is the current that will flow in the circuit if either $L = 0$ (no inductance) or $di/dt = 0$ (steady current, $i =$ constant) (Figure 9.9).

Equation (7) expresses the solution of Equation (6) as the sum of two terms: a steady-state solution V/R and a transient solution $-(V/R)e^{-(R/L)t}$ that tends to zero as $t \to \infty$. ■

Exercises 9.2

First-Order Linear Equations

Solve the differential equations in Exercises 1–14.

1. $x\frac{dy}{dx} + y = e^x, \quad x > 0$
2. $e^x\frac{dy}{dx} + 2e^x y = 1$
3. $xy' + 3y = \frac{\sin x}{x^2}, \quad x > 0$
4. $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$
5. $x\frac{dy}{dx} + 2y = 1 - \frac{1}{x}, \quad x > 0$
6. $(1 + x)y' + y = \sqrt{x}$
7. $2y' = e^{x/2} + y$
8. $e^{2x}y' + 2e^{2x}y = 2x$
9. $xy' - y = 2x \ln x$
10. $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y, \quad x > 0$

11. $(t-1)^3\dfrac{ds}{dt} + 4(t-1)^2 s = t + 1, \quad t > 1$

12. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

13. $\sin\theta\dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

14. $\tan\theta\dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

Solving Initial Value Problems

Solve the initial value problems in Exercises 15–20.

15. $\dfrac{dy}{dt} + 2y = 3, \quad y(0) = 1$

16. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

17. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y(\pi/2) = 1$

18. $\theta\dfrac{dy}{d\theta} - 2y = \theta^3\sec\theta\tan\theta, \quad \theta > 0, \quad y(\pi/3) = 2$

19. $(x+1)\dfrac{dy}{dx} - 2(x^2+x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1, \quad y(0) = 5$

20. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

21. Solve the exponential growth/decay initial value problem for y as a function of t by thinking of the differential equation as a first-order linear equation with $P(x) = -k$ and $Q(x) = 0$:

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

22. Solve the following initial value problem for u as a function of t:

$$\frac{du}{dt} + \frac{k}{m}u = 0 \quad (k \text{ and } m \text{ positive constants}), \quad u(0) = u_0$$

a. as a first-order linear equation.

b. as a separable equation.

Theory and Examples

23. Is either of the following equations correct? Give reasons for your answers.

a. $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + C$ **b.** $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + Cx$

24. Is either of the following equations correct? Give reasons for your answers.

a. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + C$

b. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + \dfrac{C}{\cos x}$

25. Current in a closed *RL* circuit How many seconds after the switch in an *RL* circuit is closed will it take the current i to reach half of its steady-state value? Notice that the time depends on R and L and not on how much voltage is applied.

26. Current in an open *RL* circuit If the switch is thrown open after the current in an *RL* circuit has built up to its steady-state value $I = V/R$, the decaying current (see accompanying figure) obeys the equation

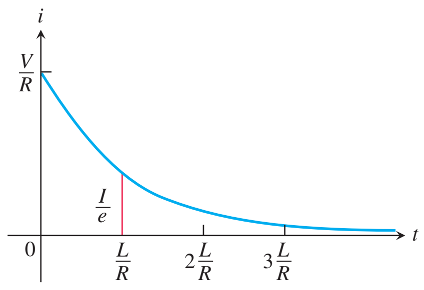
$$L\frac{di}{dt} + Ri = 0,$$

which is Equation (5) with $V = 0$.

a. Solve the equation to express i as a function of t.

b. How long after the switch is thrown will it take the current to fall to half its original value?

c. Show that the value of the current when $t = L/R$ is I/e. (The significance of this time is explained in the next exercise.)



27. Time constants Engineers call the number L/R the *time constant* of the *RL* circuit in Figure 9.9. The significance of the time constant is that the current will reach 95% of its final value within 3 time constants of the time the switch is closed (Figure 9.9). Thus, the time constant gives a built-in measure of how rapidly an individual circuit will reach equilibrium.

a. Find the value of i in Equation (7) that corresponds to $t = 3L/R$ and show that it is about 95% of the steady-state value $I = V/R$.

b. Approximately what percentage of the steady-state current will be flowing in the circuit 2 time constants after the switch is closed (i.e., when $t = 2L/R$)?

28. Derivation of Equation (7) in Example 4

a. Show that the solution of the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

is

$$i = \frac{V}{R} + Ce^{-(R/L)t}.$$

b. Then use the initial condition $i(0) = 0$ to determine the value of C. This will complete the derivation of Equation (7).

c. Show that $i = V/R$ is a solution of Equation (6) and that $i = Ce^{-(R/L)t}$ satisfies the equation

$$\frac{di}{dt} + \frac{R}{L}i = 0.$$

HISTORICAL BIOGRAPHY
James Bernoulli
(1654–1705)

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

For example, in the equation

$$\frac{dy}{dx} - y = e^{-x}y^2$$

we have $n = 2$, so that $u = y^{1-2} = y^{-1}$ and $du/dx = -y^{-2}\,dy/dx$. Then $dy/dx = -y^2\,du/dx = -u^{-2}\,du/dx$. Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x}u^{-2}$$

or, equivalently,

$$\frac{du}{dx} + u = -e^{-x}.$$

This last equation is linear in the (unknown) dependent variable u.

Solve the Bernoulli equations in Exercises 29–32.

29. $y' - y = -y^2$

30. $y' - y = xy^2$

31. $xy' + y = y^{-2}$

32. $x^2y' + 2xy = y^3$

9.3 Applications

We now look at four applications of first-order differential equations. The first application analyzes an object moving along a straight line while subject to a force opposing its motion. The second is a model of population growth. The third application considers a curve or curves intersecting each curve in a second family of curves *orthogonally* (that is, at right angles). The final application analyzes chemical concentrations entering and leaving a container. The various models involve separable or linear first-order equations.

Motion with Resistance Proportional to Velocity

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. Picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

If the resisting force is proportional to velocity, we have

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 7.2)

$$v = v_0e^{-(k/m)t}. \tag{1}$$

What can we learn from Equation (1)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero (because t must be large in the exponent of the equation in order to make kt/m large enough for v to be small). We can learn even more if we integrate Equation (1) to find the position s as a function of time t.

Suppose that an object is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \quad \text{and} \quad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}(1 - e^{-(k/m)t}). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\lim_{t\to\infty} s(t) = \lim_{t\to\infty} \frac{v_0 m}{k}(1 - e^{-(k/m)t})$$
$$= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

The number $v_0 m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least: If m is large, the body will coast a long way.

In the English system, in which weight is measured in pounds, mass is measured in **slugs**. Thus,

$$\text{Pounds} = \text{slugs} \times 32,$$

assuming the gravitational constant is 32 ft/sec^2.

EXAMPLE 1 For a 192-lb ice skater, the k in Equation (1) is about 1/3 slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution We answer the first question by solving Equation (1) for t:

$$11e^{-t/18} = 1 \qquad \text{Eq. (1) with } k = 1/3, m = 6, v_0 = 11, v = 1$$
$$e^{-t/18} = 1/11$$
$$-t/18 = \ln(1/11) = -\ln 11$$
$$t = 18 \ln 11 \approx 43 \text{ sec.}$$

We answer the second question with Equation (3):

$$\text{Distance coasted} = \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3}$$
$$= 198 \text{ ft.}$$

Inaccuracy of the Exponential Population Growth Model

In Section 7.2 we modeled population growth with the Law of Exponential Change:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

where P is the population at time t, $k > 0$ is a constant growth rate, and P_0 is the size of the population at time $t = 0$. In Section 7.2 we found the solution $P = P_0 e^{kt}$ to this model.

To assess the model, notice that the exponential growth differential equation says that

$$\frac{dP/dt}{P} = k \tag{4}$$

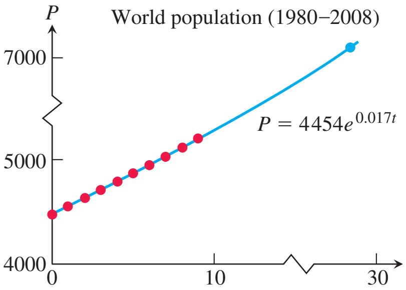


FIGURE 9.10 Notice that the value of the solution $P = 4454e^{0.017t}$ is 7169 when $t = 28$, which is nearly 7% more than the actual population in 2008.

TABLE 9.3 World population (midyear)

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 2007): www.census.gov/ipc/www/idb.

is constant. This rate is called the **relative growth rate**. Now, Table 9.3 gives the world population at midyear for the years 1980 to 1989. Taking $dt = 1$ and $dP \approx \Delta P$, we see from the table that the relative growth rate in Equation (4) is approximately the constant 0.017. Thus, based on the tabled data with $t = 0$ representing 1980, $t = 1$ representing 1981, and so forth, the world population could be modeled by the initial value problem,

$$\frac{dP}{dt} = 0.017P, \qquad P(0) = 4454.$$

The solution to this initial value problem gives the population function $P = 4454e^{0.017t}$. In year 2008 (so $t = 28$), the solution predicts the world population in midyear to be about 7169 million, or 7.2 billion (Figure 9.10), which is more than the actual population of 6707 million from the U.S. Bureau of the Census. A more realistic model would consider environmental and other factors affecting the growth rate, which has been steadily declining to about 0.012 since 1987. We consider one such model in Section 9.4.

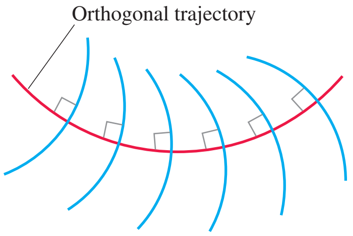


FIGURE 9.11 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally* (Figure 9.11). For instance, each straight line through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$, centered at the origin (Figure 9.12). Such mutually orthogonal systems of curves are of particular importance in physical problems related to electrical potential, where the curves in one family correspond to strength of an electric field and those in the other family correspond to constant electric potential. They also occur in hydrodynamics and heat-flow problems.

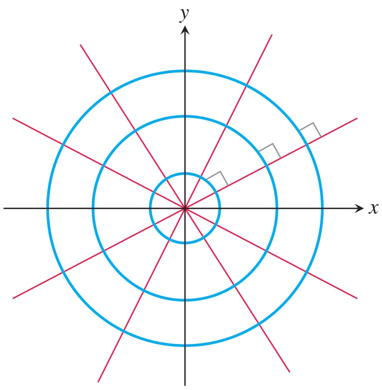


FIGURE 9.12 Every straight line through the origin is orthogonal to the family of circles centered at the origin.

EXAMPLE 2 Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas having the coordinate axes as asymptotes. First we find the slopes of each curve in this family, or their dy/dx values. Differentiating $xy = a$ implicitly gives

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

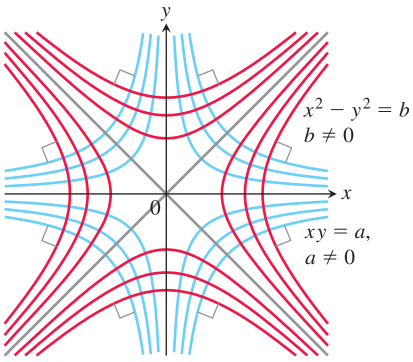


FIGURE 9.13 Each curve is orthogonal to every curve it meets in the other family (Example 2).

Thus the slope of the tangent line at any point (x, y) on one of the hyperbolas $xy = a$ is $y' = -y/x$. On an orthogonal trajectory the slope of the tangent line at this same point must be the negative reciprocal, or x/y. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

This differential equation is separable and we solve it as in Section 7.2:

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 - x^2 = b, \tag{5}$$

where $b = 2C$ is an arbitrary constant. The orthogonal trajectories are the family of hyperbolas given by Equation (5) and sketched in Figure 9.13. ■

Mixture Problems

Suppose a chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process, it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{array}{c}\text{Rate of change}\\\text{of amount}\\\text{in container}\end{array} = \left(\begin{array}{c}\text{rate at which}\\\text{chemical}\\\text{arrives}\end{array}\right) - \left(\begin{array}{c}\text{rate at which}\\\text{chemical}\\\text{departs.}\end{array}\right). \tag{6}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)} \cdot (\text{outflow rate}) \\ &= \left(\begin{array}{c}\text{concentration in}\\\text{container at time } t\end{array}\right) \cdot (\text{outflow rate}).\end{aligned} \tag{7}$$

Accordingly, Equation (6) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}). \tag{8}$$

If, say, y is measured in pounds, V in gallons, and t in minutes, the units in Equation (8) are

$$\frac{\text{pounds}}{\text{minutes}} = \frac{\text{pounds}}{\text{minutes}} - \frac{\text{pounds}}{\text{gallons}} \cdot \frac{\text{gallons}}{\text{minutes}}.$$

EXAMPLE 3 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min.

The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 9.14)?

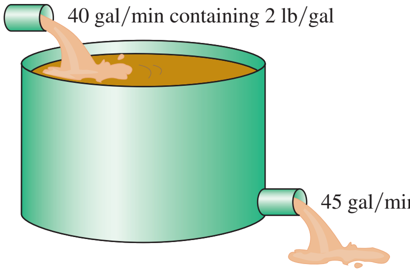


FIGURE 9.14 The storage tank in Example 3 mixes input liquid with stored liquid to produce an output liquid.

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$\begin{aligned} V(t) &= 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min}) \\ &= (2000 - 5t) \text{ gal}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \cdot \text{outflow rate} && \text{Eq. (7)} \\ &= \left(\frac{y}{2000 - 5t}\right) 45 && \text{Outflow rate is 45 gal/min and } V = 2000 - 5t. \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}. \end{aligned}$$

Also,

$$\begin{aligned} \text{Rate in} &= \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right) \\ &= 80 \frac{\text{lb}}{\text{min}}. \end{aligned}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t} \qquad \text{Eq. (8)}$$

in pounds per minute.

To solve this differential equation, we first write it in standard linear form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t} y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$. The integrating factor is

$$\begin{aligned} v(t) &= e^{\int P\, dt} = e^{\int \frac{45}{2000 - 5t} dt} \\ &= e^{-9 \ln (2000 - 5t)} && 2000 - 5t > 0 \\ &= (2000 - 5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives

$$(2000 - 5t)^{-9} \cdot \left(\frac{dy}{dt} + \frac{45}{2000 - 5t}y\right) = 80(2000 - 5t)^{-9}$$

$$(2000 - 5t)^{-9}\frac{dy}{dt} + 45(2000 - 5t)^{-10}y = 80(2000 - 5t)^{-9}$$

$$\frac{d}{dt}\left[(2000 - 5t)^{-9}y\right] = 80(2000 - 5t)^{-9}$$

$$(2000 - 5t)^{-9}y = \int 80(2000 - 5t)^{-9}\,dt$$

$$(2000 - 5t)^{-9}y = 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C.$$

The general solution is

$$y = 2(2000 - 5t) + C(2000 - 5t)^9.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$100 = 2(2000 - 0) + C(2000 - 0)^9$$

$$C = -\frac{3900}{(2000)^9}.$$

The particular solution of the initial value problem is

$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$$

The amount of additive in the tank 20 min after the pumping begins is

$$y(20) = 2\left[2000 - 5(20)\right] - \frac{3900}{(2000)^9}\left[2000 - 5(20)\right]^9 \approx 1342 \text{ lb.}$$ ■

Exercises 9.3

Motion Along a Line

1. **Coasting bicycle** A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The k in Equation (1) is about 3.9 kg/sec.
   a. About how far will the cyclist coast before reaching a complete stop?
   b. How long will it take the cyclist's speed to drop to 1 m/sec?
2. **Coasting battleship** Suppose that an Iowa class battleship has mass around 51,000 metric tons (51,000,000 kg) and a k value in Equation (1) of about 59,000 kg/sec. Assume that the ship loses power when it is moving at a speed of 9 m/sec.
   a. About how far will the ship coast before it is dead in the water?
   b. About how long will it take the ship's speed to drop to 1 m/sec?
3. The data in Table 9.4 were collected with a motion detector and a CBL™ by Valerie Sharritts, then a mathematics teacher at St. Francis DeSales High School in Columbus, Ohio. The table shows the distance s (meters) coasted on inline skates in t sec by her daughter Ashley when she was 10 years old. Find a model for Ashley's

position given by the data in Table 9.4 in the form of Equation (2). Her initial velocity was $v_0 = 2.75$ m/sec, her mass $m = 39.92$ kg (she weighed 88 lb), and her total coasting distance was 4.91 m.

TABLE 9.4 Ashley Sharritts skating data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	2.24	3.05	4.48	4.77
0.16	0.31	2.40	3.22	4.64	4.82
0.32	0.57	2.56	3.38	4.80	4.84
0.48	0.80	2.72	3.52	4.96	4.86
0.64	1.05	2.88	3.67	5.12	4.88
0.80	1.28	3.04	3.82	5.28	4.89
0.96	1.50	3.20	3.96	5.44	4.90
1.12	1.72	3.36	4.08	5.60	4.90
1.28	1.93	3.52	4.18	5.76	4.91
1.44	2.09	3.68	4.31	5.92	4.90
1.60	2.30	3.84	4.41	6.08	4.91
1.76	2.53	4.00	4.52	6.24	4.90
1.92	2.73	4.16	4.63	6.40	4.91
2.08	2.89	4.32	4.69	6.56	4.91

4. **Coasting to a stop** Table 9.5 shows the distance s (meters) coasted on inline skates in terms of time t (seconds) by Kelly Schmitzer. Find a model for her position in the form of Equation (2). Her initial velocity was $v_0 = 0.80$ m/sec, her mass $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m.

TABLE 9.5 Kelly Schmitzer skating data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	1.5	0.89	3.1	1.30
0.1	0.07	1.7	0.97	3.3	1.31
0.3	0.22	1.9	1.05	3.5	1.32
0.5	0.36	2.1	1.11	3.7	1.32
0.7	0.49	2.3	1.17	3.9	1.32
0.9	0.60	2.5	1.22	4.1	1.32
1.1	0.71	2.7	1.25	4.3	1.32
1.3	0.81	2.9	1.28	4.5	1.32

Orthogonal Trajectories

In Exercises 5–10, find the orthogonal trajectories of the family of curves. Sketch several members of each family.

5. $y = mx$
6. $y = cx^2$
7. $kx^2 + y^2 = 1$
8. $2x^2 + y^2 = c^2$
9. $y = ce^{-x}$
10. $y = e^{kx}$
11. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal.
12. Find the family of solutions of the given differential equation and the family of orthogonal trajectories. Sketch both families.
    a. $x\,dx + y\,dy = 0$
    b. $x\,dy - 2y\,dx = 0$

Mixture Problems

13. **Salt mixture** A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.
    a. At what rate (pounds per minute) does salt enter the tank at time t?
    b. What is the volume of brine in the tank at time t?
    c. At what rate (pounds per minute) does salt leave the tank at time t?
    d. Write down and solve the initial value problem describing the mixing process.
    e. Find the concentration of salt in the tank 25 min after the process starts.
14. **Mixture problem** A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.
    a. At what time will the tank be full?
    b. At the time the tank is full, how many pounds of concentrate will it contain?
15. **Fertilizer mixture** A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.
16. **Carbon monoxide pollution** An executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

9.4 Graphical Solutions of Autonomous Equations

In Chapter 4 we learned that the sign of the first derivative tells where the graph of a function is increasing and where it is decreasing. The sign of the second derivative tells the concavity of the graph. We can build on our knowledge of how derivatives determine the shape of a graph to solve differential equations graphically. We will see that the ability to

Exam : Solve the D.E $x^2(1-y^2)dx + y(1+x^2)dy = 0$

Sol:

$$\frac{x^2}{(1+x^2)}dx + \frac{y}{(1-y^2)}dy = 0$$

$$(1-\frac{1}{1+x^2})dx + \frac{y}{(1-y^2)}dy = 0$$

$$x - \tan^{-1}x + \frac{1}{2}Ln\left|1-y^2\right| = c$$

Exam : Solve the D.E $\frac{dy}{dx} = \frac{4y}{x(y-3)}$

Sol:

$$x(y-3)dy = 4y\,dx$$

$$\frac{(y-3)}{y}dy = \frac{4}{x}dx$$

$$(1-\frac{3}{y})dy = \frac{4}{x}dx$$

$$y - 3Ln\left|y\right| = 4Ln\left|x\right| + c$$

2 – Homogeneous D.E.

Def: we said that the function f(x,y) is homo. From degree n if satisfies :

$$f(tx,ty) = t^n f(x,y)$$

Ex:

$$f(x,y) = 4x^2 + 9xy - 8y^2$$

$$f(tx,ty) = 4(tx)^2 + 9txty - 8(ty)^2$$

$$= 4t^2x^2 + 9t^2xy - 8t^2y^2$$

$$= t^2(4x^2 + 9xy - 8y^2) = t^2 f(x,y)$$

$$\therefore\ Homo.\,2$$

Def: we said that the D.E. (M dx + N dy =0) is homo. If M&N are homo. Functions with same degree

Exam : Solve the D.E $(2x-y)dy=(2y-x)dx$

Sol:

$$\underbrace{(2x-y)}_{N}dy=\underbrace{(2y-x)}_{M}dx$$

$$\left.\begin{aligned}M(tx,ty)&=2ty-tx=t(2y-x)\\N(tx,ty)&=2tx-ty=t(2x-y)\end{aligned}\right\}\therefore \text{homo}.$$

$$Let\ \ y=vx\Rightarrow dy=vdx+xdv$$

$$(2x-vx)(vdx+xdv)=(2vx-x)dx$$

$$2xvdx+2x^2dv-v^2xdx-vx^2dv=2vxdx-xdx$$

$$x^2(2-v)dv-x(v^2-1)dx=0$$

$$\frac{(2-v)}{(v^2-1)}dv-\frac{1}{x}dx=0$$

$$\frac{1/2}{v-1}-\frac{3/2}{v+1}-\frac{1}{x}dx=0$$

$$1/2Ln|v-1|-3/2Ln|v+1|-Ln|x|=c$$

$$1/2Ln\left|\frac{y}{x}-1\right|-3/2Ln\left|\frac{y}{x}+1\right|-Ln|x|=c$$

Exam : Solve the D.E $(3x+2y)dx+(2x-4y)dy$

Sol:

$$\left.\begin{aligned}M(tx,ty)&=3tx+2ty=t(3x+2y)=tM\\N(tx,ty)&=2tx-4ty=t(2x-4y)=tN\end{aligned}\right\}\therefore \text{homo}.$$

$$let\ \ y=vx\Rightarrow dy=vdx+xdv$$

$$(3x+2vx)dx+(2x-4vx)(vdx+xdv)$$

$$x(3+4v-4v^2)dx+2x^2(1-2v)dv=0$$

$$\frac{dx}{x}+\frac{2(1-2v)}{(3+4v-4v^2)}dv=0$$

$$Ln|x|+\frac{1}{2}Ln\left|3+4\frac{y}{x}-4(\frac{y}{x})^2\right|=c$$

3 – D.E. of linear cofactors .

المعادلة التفاضلية ذات المعاملات الخطية تكون بالصيغة التالية :

$$(ax + by + c)dx + (\alpha x + \beta y + \gamma)dy = 0$$

Where a, b ,c , α , β and γ are constants

$$m_1 = \frac{-a}{b} , \quad m_2 = \frac{-\alpha}{\beta} \quad , m \ \ is \ \ slope$$

CASE 1 :

اذا كان المستقيمان متقاطعان $m_1 \neq m_2 \ \ (m_1 m_2 = -1)$

طريقة الحل نوجد نقاط التقاطع (h,k) للمستقيمين (المعادلتين) ومن ثم نفرض

$$x = x_1 + h \Rightarrow dx = dx_1 \quad , \ y = y_1 + k \Rightarrow dy = dy_1$$

عندئذ ستتحول المعادلة الى متجانسة .

CASE 2 :

اذا كان المستقيمان متوازيان m1=m2 اي لا توجد نقطة تقاطع

طريقة الحل نفرض ان z=ax+by

<u>Exam :</u> Solve the D.E. $(2x - 3y + 4)dx + (3x + 2y + 1)dy = 0$

Sol :

$$m_1 = \frac{-2}{-3} = \frac{2}{3} \ , \quad m_2 = \frac{-3}{2} \quad , m1.m2 = -1 \text{, Intersecting}$$

$$2x - 3y + 4 = 0$$

$$3x + 2y + 1 = 0$$

============

$$(h,k) = (\frac{-11}{13}, \frac{10}{13})$$

$$x = x_1 - \frac{11}{13} \ \Rightarrow dx = dx_1 \quad , \quad y = y_1 + \frac{10}{13} \ \Rightarrow dy = dy_1$$

$$\left[2(x_1 - \frac{11}{13}) - 3(y_1 + \frac{10}{13}) + 4\right]dx_1 + \left[3(x_1 - \frac{11}{13}) + 2(y_1 + \frac{10}{13}) + 1\right]dy_1 = 0$$

$$(2x_1 - 3y_1)dx_1 + (3x_1 + 2y_1)dy_1 = 0$$

$$Let \ \ y_1 = vx_1 \Rightarrow dy_1 = vdx_1 + x_1 dv$$

$$(2x_1 - 3vx_1)dx_1 + (3x_1 + 2vx_1)(vdx_1 + x_1 dv) = 0$$

$$(2x_1 + 2v^2 x_1)dx_1 + (3x_1^2 + 2vx_1^2)dv = 0$$

$$2x_1(1 + v^2)dx_1 + x_1^2(3 + 2v)dv = 0$$

$$2\frac{dx_1}{x} + \frac{(3 + 2v)}{(1 + v^2)}dv = 0$$

$$2\frac{dx_1}{x_1}+\frac{3}{(1+v^2)}dv+\frac{2v}{(1+v^2)}dv=0$$

$$2Ln|x_1|+3\tan^{-1}v+Ln|1+v^2|=c$$

$$2Ln|x_1|+3\tan^{-1}(\frac{y_1}{x_1})+Ln\left|1+(\frac{y_1}{x_1})^2\right|=c$$

$$2Ln\left|x-\frac{11}{13}\right|+3\tan^{-1}(\frac{y+\frac{10}{13}}{x-\frac{11}{13}})+Ln\left|1+(\frac{y+\frac{10}{13}}{x-\frac{11}{13}})^2\right|=c$$

Exam : **Solve the D.E.** $(2x-3y-1)dx+(12x-18y-6)dy=0$

Sol :

$m_1=\frac{-2}{-3}=\frac{2}{3}$, $m_2=\frac{-12}{-18}=\frac{2}{3}$, $m1=m2$, parallel

$let\ z=2x-3y\Rightarrow y=\frac{-1}{3}(z-2x)\Rightarrow dy=\frac{-1}{3}(dz-2dx)$

$(2x-3y-1)dx+6(2x-3y-1)dy=0$

$(z-1)dx+6(z-1)dy=0$

$(z-1)dx+6(z-1)\left[\frac{-1}{3}(dz-2dx)\right]=0$

$(z-1)dx-2(z-1)(dz-2dx)=0$

$(z-1)dx-2(z-1)dz+4(z-1)dx=0$

$5(z-1)dx-2(z-1)dz=0\Rightarrow 5dx-2dz=0$

$5x-2z=c$

$5x-2(2x-3y)=c$

$x+6y=c$

4-EXACT D.E. **المعادلات التفاضلية التامة**

تكون المعادلة **Mdx + Ndy =0** تامة اذا حققت :

$$\boxed{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}$$

و أن

$$f(x,y) = \int M(x,y)dx + \phi(y)$$

$$f(x,y) = \int N(x,y)dy + G(x)$$

$$\frac{\partial f}{\partial x} = M \quad , \quad \frac{\partial f}{\partial y} = N$$

Exam : **Solve the D.E.** $(2x + 3x^2 y)dx + (x^3 + 3y^2)dy = 0$

Sol :

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2 \\ \frac{\partial N}{\partial x} &= 3x^2 \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \therefore exact$$

$$f(x,y) = \int M(x,y)dx + \phi(y)$$

$$f(x,y) = \int (2x + 3x^2 y)dx + \phi(y)$$

$$f(x,y) = x^2 + x^3 y + \phi(y)$$

$$\frac{\partial f}{\partial y} = x^3 + \phi'(y)$$

$$\cancel{x^3} + 3y^2 = \cancel{x^3} + \phi'(y) \Rightarrow \phi'(y) = 3y^2$$

$$\phi(y) = \int \phi'(y)dy = \int 3y^2 dy = y^3 + c$$

$$f(x,y) = x^2 + x^3 y + y^3 + c$$

الحل بطريقة اخرى :

$$f(x,y) = \int N(x,y)dy + G(x)$$

$$f(x,y) = \int (x^3 + 3y^2)dy + G(x)$$

$$f(x,y) = x^3 y + y^3 + G(x)$$

$$\frac{\partial f}{\partial x} = 3x^2 y + G'(x)$$

$$2x + \cancel{3x^2 y} = \cancel{3x^2 y} + G'(x) \Rightarrow G'(x) = 2x$$

$$G(x) = \int G'(x)dx = \int 2x dx = x^2 + c$$

$$f(x,y) = x^3 y + y^3 + x^2 + c$$

Exam : **Solve the D.E.** $(\cos y + yCosx)dx + (\sin x - xSiny)dy = 0$

Sol :

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -Siny + \cos x \\ \frac{\partial N}{\partial x} &= \cos x - Siny \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \therefore exact$$

$$f(x,y) = \int M(x,y)dx + \phi(y)$$

$$f(x,y) = \int (\cos y + yCosx)dx + \phi(y)$$

$$f(x,y) = x\cos y + ySinx + \phi(y)$$

$$\frac{\partial f}{\partial y} = -xSiny + Sinx + \phi'(y)$$

$$Sinx - xSiny = -xSiny + Sinx + \phi'(y) \Rightarrow \phi'(y) = 0$$

$$\phi(y) = \int \phi'(y)dy = c$$

$$f(x,y) = x\cos y + ySinx + c$$

5 – Integration cofactor . **عامل التكامل**

اذا كانت المعادلة التفاضلية غير تامة اي ان

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

فسوف نحولها الى معادلة تامة وذلك بضرب طرفي المعادلة بمقدر يسمى عامل التكامل u والذي ينتج من :

$$h(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \Rightarrow\Rightarrow u = e^{\int h(x)dx}$$

or

$$g(y) = -\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} \Rightarrow\Rightarrow u = e^{\int g(y)dy}$$

Exam : **Solve the D.E.** $(2y-4x^2)dx+xdy=0$

Sol :

$$\left.\begin{array}{l}\dfrac{\partial M}{\partial y}=2\\[2ex]\dfrac{\partial N}{\partial x}=1\end{array}\right\}\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}\Rightarrow\therefore not\ exact$$

$$h(x)=\frac{2-1}{x}=\frac{1}{x}\Rightarrow\Rightarrow u=e^{\int\frac{1}{x}dx}=e^{Lnx}=x$$

$$(2xy-4x^3)dx+x^2dy=0$$

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$$

$$f(x,y)=\int M(x,y)dx+\phi(y)$$

$$f(x,y)=\int(2xy-4x^3)dx+\phi(y)$$

$$f(x,y)=x^2y-x^4+\phi(y)$$

$$\frac{\partial f}{\partial y}=x^2+\phi'(y)$$

$$\cancel{x^2}=\cancel{x^2}+\phi'(y)\Rightarrow\phi'(y)=0\Rightarrow\phi(y)=c$$

$$f(x,y)=x^2y-x^4+c$$

Exam : **Solve the D.E.**

$$(2x^3y^2+4x^2y+2xy^2+xy^4+2y)dx+2(y^3+x^2y+x)dy=0$$

Sol :

$$\left.\begin{array}{l}\dfrac{\partial M}{\partial y}=4x^3y+4x^2+4xy+4xy^3+2\\[2ex]\dfrac{\partial N}{\partial x}=4xy+2\end{array}\right\}\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}\Rightarrow\therefore not\ exact$$

$$h(x)=\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{4x^3y+4x^2+4xy^3}{2(y^3+x^2y+x)}=\frac{4x\,\cancel{(x^2y+x+y^3)}}{2\,\cancel{(y^3+x^2y+x)}}=2x$$

$$u=e^{\int 2xdx}=e^{x^2}$$

$$(2x^3e^{x^2}y^2 + 4x^2e^{x^2}y + 2xe^{x^2}y^2 + xe^{x^2}y^4 + 2ye^{x^2})dx$$

$$+2(y^3e^{x^2} + x^2e^{x^2}y + xe^{x^2})dy = 0 \quad \Rightarrow exact$$

$$f(x,y) = \int N(x,y)dy + G(x)$$

$$f(x,y) = \int 2y^3e^{x^2} + 2x^2e^{x^2}y + 2xe^{x^2})dy + G(x)$$

$$f(x,y) = \frac{1}{2}y^4e^{x^2} + x^2e^{x^2}y^2 + 2xe^{x^2}y + G(x)$$

$$\frac{\partial f}{\partial x} = xy^4e^{x^2} + x^2e^{x^2}(0) + y^2\left[2x^3e^{x^2} + 2xe^{x^2}\right]$$

$$+y\left[4x^2e^{x^2} + 2e^{x^2}\right] + G'(x)$$

$$\cancel{2x^3e^{x^2}y^2} + \cancel{4x^2e^{x^2}y} + \cancel{2xe^{x^2}y^2} + \cancel{xe^{x^2}y^4} + \cancel{2ye^{x^2}} =$$

$$\cancel{xe^{x^2}y^4} + \cancel{2x^3e^{x^2}y^2} + \cancel{2xe^{x^2}y^2} + \cancel{4x^2e^{x^2}y} + \cancel{2ye^{x^2}} + G'(x)$$

$$G'(x) = 0 \Rightarrow G(x) = c$$

$$f(x,y) = \frac{1}{2}y^4e^{x^2} + x^2e^{x^2}y^2 + 2xe^{x^2}y + c$$

****

HOME WORK

SOLVE

$\langle 1 \rangle\, ydx + (x^2 - 4x)dy = 0$

$\langle 2 \rangle\, Sin^2xCosy\ dx + SinySecx\ dy = 0$

$\langle 3 \rangle\, x(1-y)\dfrac{dy}{dx} + y(1+x) = 0$

$\langle 4 \rangle\, \dfrac{dx}{dy} = \dfrac{x^2}{y^2 + 6y + 9}$

$\langle 5 \rangle\, (x^2+1)(y^2-1)dx + 2xydy = 0$

$\langle 6 \rangle\, (4x + xy^2)dx + (y + x^2y)dy = 0$

$\langle 7 \rangle\, x(y^2+1)dy + (y^3 - 4y)dx = 0$

SOLVE

$\langle 1\rangle xdy - ydx = \sqrt{x^2 + y^2}dx$

$\langle 2\rangle xy^2dy - (x^3 + y^3)dx = 0$

$\langle 3\rangle x(x^2 + 3y^2)dy = y(y^2 + 3x^2)dx$

$\langle 4\rangle \dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}$

SOLVE

$\langle 1\rangle (2x - 3y + 4)dx + (3x - 2y + 1)dy = 0$

$\langle 2\rangle (4x + 2y + 3)dx - (6x + 3y - 2)dy = 0$

$\langle 3\rangle (2x - 3y - 1)dx = 4(x + 1)dy$

$\langle 4\rangle (2y - x - 4)dx = (2x - y + 2)dy$

SOLVE

$\langle 1\rangle 2x(ye^{x^2} - 1)dx + e^{x^2}dy = 0$

$\langle 2\rangle (x^2 + y^2 + x)dx + (xy)dy = 0$

$\langle 3\rangle (2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy$

$\langle 4\rangle yLnydx + (x - Lny)dy = 0$

$\langle 5\rangle ye^{xy}dx + xe^{xy}dy = 0$

SOLVE

$\langle 1\rangle xdy - ydx = x^2y^3dx$

$\langle 2\rangle xdy - 3ydx = x^4y^{-1}dx$

$\langle 3\rangle (2xy^2 - 2y)dx + (3x^2y - 4x)dy$

It is not possible to find an explicit solution for this problem and so we will have to leave the solution in its implicit form

Homogeneous Equation

Definition: A function f(x,y) is said to be *homogeneous equation* of order n if

$$f(tx,ty) = t^n f(x,y)$$

Example:1- $f(x,y)= x^2+y^2\ln\frac{y}{x}$

2- $f(x,y)=\sqrt{y}\,\sin\frac{x}{y}$

3- $x^4-3x^3y+5y^2x^2-2y^4$

4- $e^{\frac{y}{x}}+\tan\left(\frac{y}{x}\right)$

Definition An equation of the form

$$P(x,y)\,dx+Q(x,y)\,dy$$

is said to be *homogeneous equation* the functions P(x,y) and Q(x,y) are homogeneous and of the same order.

Method of solution

By using the substitution

$$y = u\,x$$

$$dy = u\,dx + x\,du$$

the homogeneous equation converted to separable equation.

Example: 1- $2xy\,dx+(x^2+y^2)\,dy=0$

2- $(\sqrt{x^2-y^2}+y)dx-x\,dy=0$ Integration of$(du/(1-u^2)^{1/2})$ is $\sin^{-1}u$

$2xy\,dx + (x^2+y^2)\,dy = 0$

$P(x,y) = 2xy$

$Q(x,y) = x^2+y^2$

$y = ux$, $dy = u\,dx + x\,du$ → $2ux^2dx + (x^2+u^2x^2)(u\,dx + x\,du)$

⇒ $2ux^2dx + ux^2dx + x^3du + u^3x^2dx + u^2x^3du = 0$

$(2ux^2 + ux^2 + u^3x^2)dx + (x^3du + u^2x^3du) = 0$

$\int \frac{x^2\,dx}{x^3} + \int \frac{1+u^2}{3u+u^3}\,du = 0$

Exact Differentials equations

Theorem A necessary and sufficient condition that the DE

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

Be exact is that

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial x}Q(x,y)$$

Method of solution

The solution of the exact DE is given by

$$f(x,y) = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy = c$$

Where (x_0,y_0) is any point at which the functions P(x,y) and Q(x,y) is defined.

Examples Show that the following DE's are exact and find the 1-parameter family of solutions

1- $\cos y\,dx - (x\sin y - y^2)dy = 0$
2- $(x-2xy+e^y)dx+(y-x^2+xe^y)dy = 0$

Integrating Factors A multiplying factor which will convert an inexact DE into exact one is called an *integrating factor.*

Example

$(y^2+y)dx-x\,dy = 0$ is not exact

I.F. $=y^{-2}$

$(1+\frac{1}{y})dx - \frac{x}{y^2}dy = 0 \ , \ y \neq 0$ is exact.

Finding an integrating factor

Simple exact DE

$y\,dx + x\,dy = 0$ then $d(xy)$, $2xydx+x^2dy = 0$ $d(x^2y)$

$y^2dx+2xydy = 0$ $d(xy^2)$, $\frac{ydx-xdy}{y^2} = 0$ $d(x/y)$

Linear First Order DE

These equation are equations of the type:

$$y' + P(x)y = Q(x), y(x_0) = y_0$$

In order to solve this equation

1-multiply both side by $\mu(x) = e^{\int P(x)dx}$ **(Integral factor),** we obtain

$$y' e^{\int P(x)dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx}$$

2-This way the L.H.S. equation is the derivative of some function, which is

$$d\left(y\ e^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

3- By integration

$$y\ e^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx} + c$$

4- use the initial condition to find the constant of integration c.

Example1 Solve the following IVP.

$$y' + \frac{4}{x}y = x^3$$

Example2 Solve the following IVP

$-2v' - 2v = x \quad \Rightarrow \quad v' + v = -\frac{1}{2}x \qquad \mu(x) = e^x$

Bernoulli Differential Equations

In this section we are going to take a look at differential equations in the form,

$$y' + p(x)y = q(x)y^n$$

where $p(x)$ and $q(x)$ are continuous functions on the interval we're working on and n is a real number. Differential equations in this form are called **Bernoulli Equations**.

Note that if $n = 0$ or $n = 1$ then the equation is linear and we already know how to solve it in these cases. Therefore, in this section we're going to be looking at solutions for values of n other than these two.

In order to solve these we'll first divide the differential equation by y^n to get,

$$y^{-n}y' + p(x)\,y^{1-n} = q(x)$$

We are now going to use the substitution

$$v = y^{1-n}$$

So, taking the derivative gives us,

$$v' = (1-n)\,y^{-n}y'$$

Now, plugging this as well as our substitution into the differential equation gives,

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

This is a linear differential equation that we can solve for v and once we have this in hand we can also get the solution to the original differential equation by plugging v back into our substitution and solving for y.

Let's take a look at an example.

Example 1 Solve the following IVP.

$$y' + \frac{4}{x}y = x^3y^2 \qquad y(2) = -1, \qquad x > 0$$

Solution

So, the first thing that we need to do is get this into the "proper" form and that means dividing everything by y^2. Doing this gives,

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3$$

The substitution and derivative that we'll need here is,

$$v = y^{-1} \qquad\qquad v' = -y^{-2}y'$$

With this substitution the differential equation becomes,

$$-v' + \frac{4}{x}v = x^3$$

Here's the solution to this differential equation.

$$v' - \frac{4}{x}v = -x^3 \qquad \Rightarrow \qquad \mu(x) = \mathbf{e}^{\int -\frac{4}{x}dx} = \mathbf{e}^{-4\ln|x|} = x^{-4}$$

$$\int \left(x^{-4}v\right)' dx = \int -x^{-1}\,dx$$

$$x^{-4}v = -\ln|x| + c \qquad \Rightarrow \qquad v(x) = cx^4 - x^4\ln x$$

So, to get the solution in terms of y all we need to do is plug the substitution back in. Doing this gives,

$$y^{-1} = x^4\left(c - \ln x\right)$$

At this point we can solve for y and then apply the initial condition or apply the initial condition and then solve for y. We'll generally do this with the later approach so let's apply the initial condition to get,

$$(-1)^{-1} = c2^4 - 2^4 \ln 2 \qquad \Rightarrow \qquad c = \ln 2 - \frac{1}{16}$$

Plugging in for c and solving for y gives,

$$y(x) = \frac{1}{x^4\left(\ln 2 - \frac{1}{16} - \ln x\right)} = \frac{-16}{x^4\left(1 + 16\ln x - 16\ln 2\right)} = \frac{-16}{x^4\left(1 + 16\ln\frac{x}{2}\right)}$$

Example 2 Solve the following IVP.

$$y' = 5y + \mathbf{e}^{-2x} y^{-2} \qquad y(0) = 2$$

Solution

The first thing we'll need to do here is multiply through by y^2 and we'll also do a little rearranging to get things into the form we'll need for the linear differential equation. This gives,

$$y^2 y' - 5y^3 = \mathbf{e}^{-2x}$$

The substitution here and its derivative is,

$$v = y^3 \qquad v' = 3y^2 y'$$

Plugging the substitution into the differential equation gives,

$$\tfrac{1}{3}v' - 5v = \mathbf{e}^{-2x} \qquad \Rightarrow \qquad v' - 15v = 3\mathbf{e}^{-2x} \qquad \mu(x) = \mathbf{e}^{-15x}$$

$$v(x) = c\mathbf{e}^{15x} - \tfrac{3}{17}\mathbf{e}^{-2x}$$

Now go back to y's.

$$y^3 = c\mathbf{e}^{15x} - \tfrac{3}{17}\mathbf{e}^{-2x}$$

Applying the initial condition and solving for c gives,

$$8 = c - \tfrac{3}{17} \qquad \Rightarrow \qquad c = \tfrac{139}{17}$$

Plugging in c and solving for y gives,

$$y(x) = \left(\frac{139\mathbf{e}^{15x} - 3\mathbf{e}^{-2x}}{17}\right)^{\frac{1}{3}}$$

Example 3 Solve the following IVP.

$$5y' - 2y = x y^4 \qquad y(0) = -2$$

Solution

First get the differential equation in the proper form and then write down the substitution.

$$6y^{-4}y' - 2y^{-3} = x \qquad \Rightarrow \qquad v = y^{-3} \qquad v' = -3y^{-4}y'$$

Plugging the substitution into the differential equation gives,

$$-2v' - 2v = x \qquad \Rightarrow \qquad v' + v = -\tfrac{1}{2}x \qquad \mu(x) = \mathbf{e}^{x}$$

Again, we've rearranged a little and given the integrating factor needed to solve the linear differential equation. Upon solving the linear differential equation we have,

$$v(x) = -\tfrac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now back substitute to get back into y's.

$$y^{-3} = -\tfrac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now we need to apply the initial condition and solve for c.

$$-\tfrac{1}{8} = \tfrac{1}{2} + c \qquad \Rightarrow \qquad c = -\tfrac{5}{8}$$

Plugging in c and solving for y gives,

$$y(x) = -\frac{2}{\left(4x - 4 + 5\mathbf{e}^{-x}\right)^{\frac{1}{3}}}$$

To this point we've only worked examples in which n was an integer (positive and negative) and so we should work a quick example where n is not an integer.

Example 4 Solve the following IVP.

$$y' + \frac{y}{x} - \sqrt{y} = 0 \qquad y(1) = 0$$

Solution

Let's first get the differential equation into proper form.

$$y' + \frac{1}{x}y = y^{\frac{1}{2}} \qquad \Rightarrow \qquad y^{-\frac{1}{2}}y' + \frac{1}{x}y^{\frac{1}{2}} = 1$$

The substitution is then,

$$v = y^{\frac{1}{2}} \qquad v' = \frac{1}{2}y^{-\frac{1}{2}}y'$$

Now plug the substitution into the differential equation to get,

$$2v' + \frac{1}{x}v = 1 \qquad \Rightarrow \qquad v' + \frac{1}{2x}v = \frac{1}{2} \qquad \mu(x) = x^{\frac{1}{2}}$$

As we've done with the previous examples we've done some rearranging and given the integrating factor needed for solving the linear differential equation. Solving this gives us,

$$v(x) = \tfrac{1}{3}x + cx^{-\frac{1}{2}}$$

In terms of y this is,

$$y^{\frac{1}{2}} = \tfrac{1}{3}x + cx^{-\frac{1}{2}}$$

Applying the initial condition and solving for c gives,

$$0 = \tfrac{1}{3} + c \qquad \Rightarrow \qquad c = -\tfrac{1}{3}$$

Plugging in for c and solving for y gives us the solution.

$$y(x) = \left(\tfrac{1}{3}x - \tfrac{1}{3}x^{-\frac{1}{2}}\right)^2 = \frac{x^3 - 2x^{\frac{3}{2}} + 1}{9x}$$

Example 1 :

Solve the D·E : $\frac{dy}{dx} + \frac{y}{x} = x^2$

Solution :

$$p(x) = \frac{1}{x} \quad , \quad Q(x) = x^2$$

$$I = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{Lnx} = x$$

$$yI = \int Q(x)Idx$$

$$yx = \int x^2 x\, dx$$

$$yx = \frac{x^4}{4} + C$$

$$y = \frac{x^3}{4} + \frac{C}{x}$$

Example 2 :

Solve the D·E : $xdy + ydx = xsinx^2\, dx$

Solution :

By dividing on dx

$$x\frac{dy}{dx} + y = xsinx^2$$

$$\frac{dy}{dx}+\frac{y}{x}=sinx^2$$

$$p(x)=\frac{1}{x}\quad,\quad Q(x)=sinx^2$$

$$I=e^{\int p(x)dx}=e^{\int\frac{1}{x}dx}=e^{Lnx}=x$$

$$yI=\int Q(x)Idx$$

$$yx=\int xsinx^2\,dx$$

$$yx=\frac{-1}{2}cosx^2+C$$

$$y=\frac{-1}{2x}cosx^2+\frac{C}{x}$$

Example 3 :

Solve the D·E : $dy+2xydx=xe^{-x^2}\,dx$

Solution :

By dividing on dx

$$\frac{dy}{dx}+2xy=xe^{-x^2}$$

$$p(x)=2x\quad,\quad Q(x)=xe^{-x^2}$$

$$I=e^{\int p(x)dx}=e^{\int 2xdx}=e^{x^2}$$

$$yI = \int Q(x)I\,dx$$

$$ye^{x^2} = \int e^{x^2} x e^{-x^2}\, dx$$

$$ye^{x^2} = \int x\,dx$$

$$ye^{x^2} = \frac{x^2}{2} + C \rightarrow y = \frac{x^2 e^{-x^2}}{2} + C\, e^{-x^2}$$

Example 4 :

Solve the D·E : $y\frac{dx}{dy} + 2x = y^3$

Solution :

By dividing on y

$$\frac{dx}{dy} + \frac{2x}{y} = y^2$$

$$g(y) = \frac{2}{y} \quad , \quad h(y) = y^2$$

$$I = e^{\int g(y)dy} = e^{\int \frac{2}{y} dy} = e^{2Lny} = e^{Lny^2} = y^2$$

$$xI = \int h(y)I\,dy$$

$$xy^2 = \int y^2 y^2\, dy$$

$$xy^2 = \frac{y^5}{5} + C$$

$$x = \frac{y^3}{5} + \frac{C}{y^2}$$

Example 5 :

Solve the D.E : $\frac{dx}{dy} + 2xy = 4y$

Solution :

$$g(y) = 2y \quad , \quad h(y) = 4y$$

$$I = e^{\int g(y)dy} = e^{\int 2ydy} = e^{y^2}$$

$$xI = \int h(y)Idy$$

$$xe^{y^2} = \int 4ye^{y^2}\,dy$$

$$xe^{y^2} = 2e^{y^2} + C$$

$$x = 2 + Ce^{-y^2}$$

CHECK POINT

Example H.W :

Solve the following D.E

1) $y' + y = sinx$

2) $x\frac{dy}{dx} - 2y = x^3 cos4x$

3) $x\frac{dy}{dx} = y + x^3 + 3x^2 - 2x$

4) $\frac{dx}{dy} + x = 4cos2y$

5) $\frac{dy}{dx} + ycotx = 5e^{cosx}$

Bernoli Equation

The general form to bernoli equation is :

$$\frac{dy}{dx} + p(x)y = Q(x)\, y^n \quad ,n \neq 1 \qquad \cdots\cdots(1)$$

Such that p and Q functions for x only .

method of solution :

transform eq(1) to L.D.E by multiplying by y^{-n}

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = Q(x) \qquad \cdots\cdots(2)$$

Let $z = y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} \Rightarrow \frac{1}{1-n}\frac{dz}{dx} = y^{-n}\frac{dy}{dx} \quad \cdots\cdot(3)$

Subistituting eq(3) in eq(2) we get :

$$\frac{1}{1-n}\frac{dz}{dx}+p(x)z=Q(x)$$

The last equation is L.D.E and its solution is :

$$z.I=\int Q(x).I\,dx$$

By the same method ,The general form to bernoli equation is :

$$\frac{dx}{dy}+g(y)x=h(y)\,x^{n} \quad ,n\neq 1 \qquad \cdots\cdots(1)$$

Such that g and h functions for y only .

method of solution :

transform eq(1) to L.D.E by multiplying by x^{-n}

$$x^{-n}\frac{dx}{dy}+g(y)x^{1-n}=h(y) \qquad \cdots\cdots(2)$$

Let $z=x^{1-n}\Rightarrow\frac{dz}{dy}=(1-n)x^{-n}\frac{dx}{dy}\;\Rightarrow\frac{1}{1-n}\frac{dz}{dy}=x^{-n}\frac{dx}{dy} \qquad \cdots\cdots(3)$

Subistituting eq(3) in eq(2) we get :

$$\frac{1}{1-n}\frac{dz}{dy}+g(y)z=h(y)$$

The last equation is L.D.E and its solution is :

$$z.I=\int h(y).I\,dy$$

Example 6 :

Solve the D.E : $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Solution :

multiplying by (-1)

$$\frac{dy}{dx} - xy = -y^3 e^{-x^2} \quad \cdots\cdots(1)$$

Eq(1) is bernoli equation , multiply (1) by y^{-3}

$$y^{-3}\frac{dy}{dx} - xy^{-2} = -e^{-x^2} \quad \cdots\cdots(2)$$

Let $z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow \frac{-1}{2}\frac{dz}{dx} = y^{-3}\frac{dy}{dx} \quad \cdots\cdots(3)$

Subistitute eq(3) in eq(2) we get :

$$\frac{-1}{2}\frac{dz}{dx} - xz = -e^{-x^2}$$

Multiply by (-2)

$\frac{dz}{dx} + 2xz = 2e^{-x^2}$ is L.D.E

$$p(x) = 2x \ , \ Q(x) = 2e^{-x^2}$$

$$I = e^{\int p(x)dx} = e^{\int 2x dx} = e^{x^2}$$

$$zI = \int Q(x) I dx$$

$$ze^{x^2} = \int 2e^{-x^2}e^{x^2}\,dx$$

$$ze^{x^2} = \int 2dx$$

$$ze^{x^2} = 2x + C \Rightarrow z = 2xe^{-x^2} + C\,e^{-x^2}$$

Example 7 :

Solve the D.E : $\frac{dy}{dx} - y = -xy^5$

Solution :

$\frac{dy}{dx} - y = -xy^5$ .....(1)

Eq(1) is bernoli equation , multiply (1) by y^{-5}

$y^{-5}\frac{dy}{dx} - y^{-4} = -x$ .....(2)

Let $z = y^{-4} \Rightarrow \frac{dz}{dx} = -4y^{-5}\frac{dy}{dx} \Rightarrow \frac{-1}{4}\frac{dz}{dx} = y^{-5}\frac{dy}{dx}$ .....(3)

Subistitute eq(3) in eq(2) we get :

$$\frac{-1}{4}\frac{dz}{dx} - z = -x$$

Multiply by (-4)

$\frac{dz}{dx} + 4z = 4x$ is L.D.E

$$p(x) = 4 \quad , \quad Q(x) = 4x$$

$$I = e^{\int p(x)dx} = e^{\int 4dx} = e^{4x}$$

$$zI = \int Q(x)Idx$$

$$ze^{4x} = \int 4x\, e^{4x}\, dx$$

$$ze^{4x} = xe^{4x} - \frac{1}{4}e^{4x} + C$$

$$\frac{e^{4x}}{y^4} = xe^{4x} - \frac{1}{4}e^{4x} + C$$

Example 8 :

Solve the D.E : $dx - xdy = yx^2dy$

Solution :

Dividing by dy

$\frac{dx}{dy} - x = yx^2$ .....(1)

Eq(1) is bernoli equation , multiply (1) by x^{-2}

$x^{-2}\frac{dx}{dy} - x^{-1} = y$ .....(2)

Let $z = x^{-1} \Rightarrow \frac{dz}{dy} = -x^{-2}\frac{dx}{dy} \Rightarrow -\frac{dz}{dy} = x^{-2}\frac{dx}{dy}$ .....(3)

Subistitute eq(3) in eq(2) we get :

$$-\frac{dz}{dy} - z = y$$

Multiply by (-1)

$\frac{dz}{dy} + z = -y$ is L.D.E

$$g(y) = 1 \quad , \quad h(y) = -y$$

$$I = e^{\int g(y)dy} = e^{\int dy} = e^{y}$$

$$zI = \int h(y) I dy$$

$$ze^{y} = \int -y\, e^{y}\, dy$$

$$ze^{y} = -(y\, e^{y} - e^{y}) + C$$

$$\frac{e^{y}}{x} = -y\, e^{y} + e^{y} + C$$

Example 9 :

Solve the D.E : $dx - 2xydy = 6x^{3}y^{2}e^{-2y^{2}}dy$

Solution :

Dividing by dy

$\frac{dx}{dy} - 2xy = 6x^{3}y^{2}e^{-2y^{2}}$ .....(1)

Eq(1) is bernoli equation , multiply (1) by x^{-3}

$x^{-3}\frac{dx}{dy} - 2x^{-2} = 6y^2e^{-2y^2}$ .....(2)

Let $z = x^{-2} \Rightarrow \frac{dz}{dy} = -2x^{-3}\frac{dx}{dy} \Rightarrow \frac{-1}{2}\frac{dz}{dy} = x^{-3}\frac{dx}{dy}$ .....(3)

Subistitute eq(3) in eq(2) we get :

$$\frac{-1}{2}\frac{dz}{dy} - 2zy = 6y^2e^{-2y^2}$$

Multiply by (-2)

$\frac{dz}{dy} + 4zy = -12y^2e^{-2y^2}$ is L.D.E

$$g(y) = 4y \ , \ h(y) = -12y^2e^{-2y^2}$$

$$I = e^{\int 4y dy} = e^{2y^2}$$

$$zI = \int h(y)I dy$$

$$ze^{2y^2} = \int -12y^2e^{-2y^2}e^{2y^2}\, dy$$

$$ze^{2y^2} = \int -12y^2\, dy$$

$$ze^{2y^2} = -4y^3 + C$$

$$x^{-2}e^{2y^2} = -4y^3 + C$$

CHECK POINT

Example H.W :

Solve the following D.E

1) $xdy + ydx = x^3y^6\,dx$

2) $\frac{dy}{dx} + xy = 6x\sqrt{y}$

3) $\frac{dy}{dx} + y = y^3$

4) $dx + xdy = x^2e^y\,dy$

5) $\frac{dx}{dy} - \frac{x}{2y} = \frac{-1}{2}(cosy)x^3$