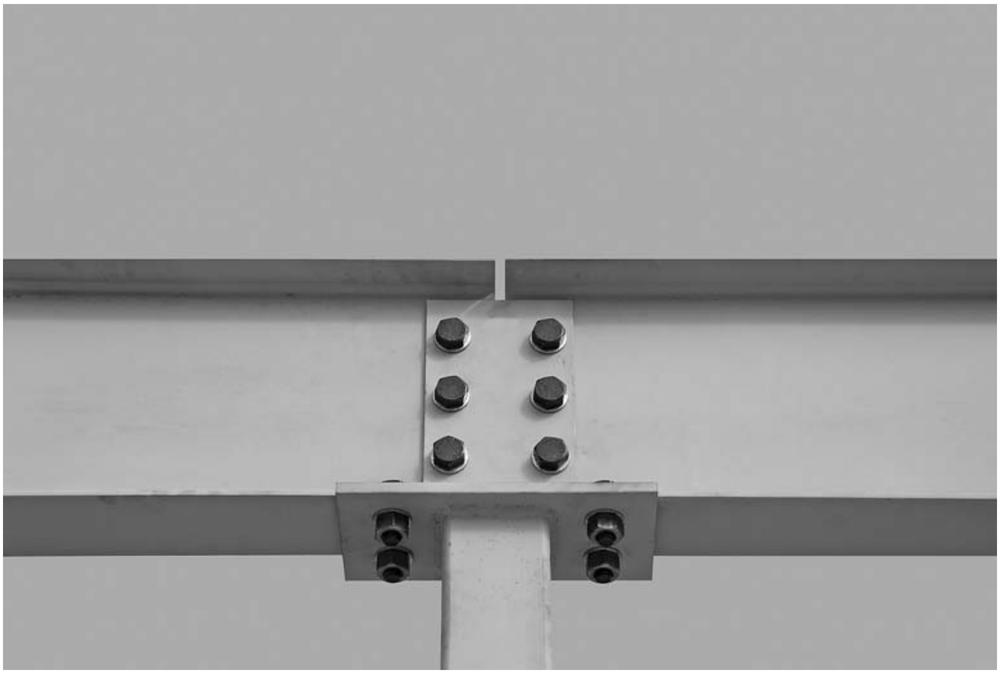
1 Stress



Mark Winfrey/Shutterstock

Bolted connection in a steel frame. The bolts must withstand the shear forces imposed on them by the members of the frame. The stress analysis of bolts and rivets is discussed in this chapter. Courtesy of Mark Winfrey/Shutterstock.

1.1 Introduction

The three fundamental areas of engineering mechanics are statics, dynamics, and mechanics of materials. Statics and dynamics are devoted primarily to the study of the *external effects upon rigid bodies*—that is, bodies for which the change in shape (deformation) can be neglected. In contrast, *mechanics of materials* deals with the *internal effects* and *deformations* that are caused by the applied loads. Both considerations are of paramount importance in design. A machine part or structure must be strong enough to carry the applied load without breaking and, at the same time, the deformations must not be excessive.

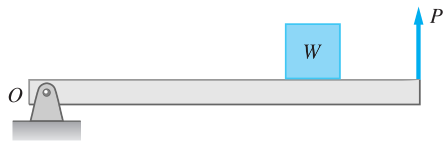


FIG. 1.1 Equilibrium analysis will determine the force P, but not the strength or the rigidity of the bar.

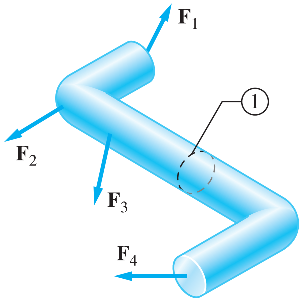
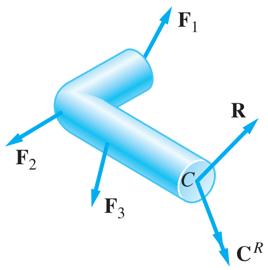
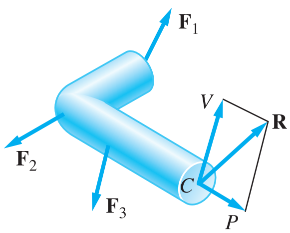


FIG. 1.2 External forces acting on a body.



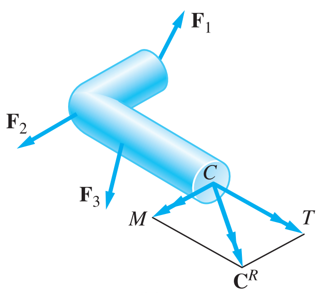
(a)

FIG. 1.3(a) Free-body diagram for determining the internal force system acting on section ①.



(b)

FIG. 1.3(b) Resolving the internal force **R** into the axial force P and the shear force V.



(c)

FIG. 1.3(c) Resolving the internal couple $\mathbf{C}^R$ into the torque T and the bending moment M.

The differences between rigid-body mechanics and mechanics of materials can be appreciated if we consider the bar shown in Fig. 1.1. The force P required to support the load W in the position shown can be found easily from equilibrium analysis. After we draw the free-body diagram of the bar, summing moments about the pin at O determines the value of P. In this solution, we assume that the bar is both rigid (the deformation of the bar is neglected) and strong enough to support the load W. In mechanics of materials, the statics solution is extended to include an analysis of the forces acting *inside* the bar to be certain that the bar will neither break nor deform excessively.

1.2 Analysis of Internal Forces; Stress

The equilibrium analysis of a rigid body is concerned primarily with the calculation of external reactions (forces that act external to a body) and internal reactions (forces that act at internal connections). In mechanics of materials, we must extend this analysis to determine *internal forces*—that is, forces that act on cross sections that are *internal* to the body itself. In addition, we must investigate the manner in which these internal forces are distributed within the body. Only after these computations have been made can the design engineer select the proper dimensions for a member and select the material from which the member should be fabricated.

If the external forces that hold a body in equilibrium are known, we can compute the internal forces by straightforward equilibrium analysis. For example, consider the bar in Fig. 1.2 that is loaded by the external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$. To determine the internal force system acting on the cross section labeled ①, we must first isolate the segments of the bar lying on either side of section ①. The free-body diagram of the segment to the left of section ① is shown in Fig. 1.3(a). In addition to the external forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, this free-body diagram shows the resultant force-couple system of the internal forces that are distributed over the cross section: the resultant force $\mathbf{R}$, acting at the centroid C of the cross section, and $\mathbf{C}^R$, the resultant couple[1] (we use double-headed arrows to represent couple-vectors). If the external forces are known, the equilibrium equations $\Sigma\mathbf{F} = \mathbf{0}$ and $\Sigma\mathbf{M}_C = \mathbf{0}$ can be used to compute $\mathbf{R}$ and $\mathbf{C}^R$.

It is conventional to represent both $\mathbf{R}$ and $\mathbf{C}^R$ in terms of two components: one perpendicular to the cross section and the other lying in the cross section, as shown in Figs. 1.3(b) and (c). These components are given the

[1] The resultant force **R** can be located at any point, provided that we introduce the correct resultant couple. The reason for locating **R** at the centroid of the cross section will be explained shortly.

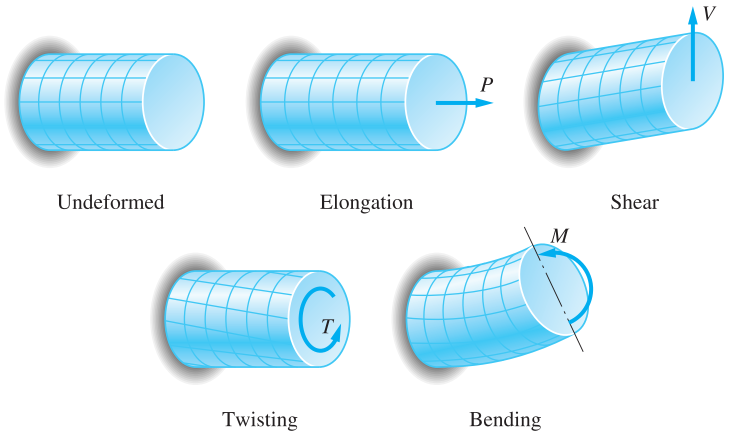


FIG. 1.4 Deformations produced by the components of internal forces and couples.

following physically meaningful names:

P: The component of the resultant force that is perpendicular to the cross section, tending to elongate or shorten the bar, is called the *normal force*.
V: The component of the resultant force lying in the plane of the cross section, tending to shear (slide) one segment of the bar relative to the other segment, is called the *shear force*.
T: The component of the resultant couple that tends to twist (rotate) the bar is called the *twisting moment* or *torque*.
M: The component of the resultant couple that tends to bend the bar is called the *bending moment.*

The deformations produced by these internal forces and internal couples are shown in Fig. 1.4.

Up to this point, we have been concerned only with the resultant of the internal force system. However, in design, the manner in which the internal forces are distributed is equally important. This consideration leads us to introduce the force intensity at a point, called *stress*, which plays a central role in the design of load-bearing members.

Figure 1.5(a) shows a small area element ΔA of the cross section located at the arbitrary point O. We assume that $\Delta\mathbf{R}$ is that part of the resultant force that is transmitted across ΔA, with its normal and shear components being ΔP and ΔV, respectively. The *stress vector* acting on the cross section at point O is defined as

$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{R}}{\Delta A} \tag{1.1}$$

Its normal component σ (lowercase Greek *sigma*) and shear component τ (lowercase Greek *tau*), shown in Fig. 1.5(b), are

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta P}{\Delta A} = \frac{dP}{dA} \qquad \tau = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A} = \frac{dV}{dA} \tag{1.2}$$

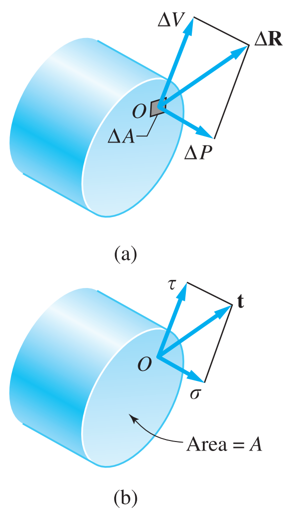


FIG. 1.5 Normal and shear stresses acting on the cross section at point O are defined in Eq. (1.2).

The dimension of stress is [F/L^2]—that is, force divided by area. In SI units, force is measured in newtons (N) and area in square meters, from which the unit of stress is *newtons per square meter* (N/m^2) or, equivalently, *pascals* (Pa): 1.0 Pa = 1.0 N/m^2. Because 1 pascal is a very small quantity in most engineering applications, stress is usually expressed with the SI prefix M (read as "mega"), which indicates multiples of 10^6: 1.0 MPa = 1.0×10^6 Pa. In U.S. Customary units, force is measured in pounds and area in square inches, so that the unit of stress is *pounds per square inch* (lb/in.^{2}), frequently abbreviated as psi. Another unit commonly used is *kips per square inch* (ksi) (1.0 ksi = 1000 psi), where "kip" is the abbreviation for kilopound.

The commonly used *sign convention* for axial forces is to define tensile forces as positive and compressive forces as negative. This convention is carried over to normal stresses: Tensile stresses are considered to be positive, compressive stresses negative. A simple sign convention for shear stresses does not exist; a convention that depends on a coordinate system will be introduced later in the text. If the stresses are *uniformly distributed*, Eq. (1.2) gives

$$\sigma = \frac{P}{A} \qquad \tau = \frac{V}{A} \tag{1.3}$$

where A is the area of the cross section. If the stress distribution is not uniform, then Eqs. (1.3) should be viewed as the *average stress* acting on the cross section.

1.3 Axially Loaded Bars

a. Centroidal (axial) loading

Figure 1.6(a) shows a bar of constant cross-sectional area A. The ends of the bar carry uniformly distributed normal loads of intensity p (units: Pa or psi). We know from statics that

when the loading is uniform, its resultant passes through the centroid of the loaded area.

Therefore, the resultant $P = pA$ of each end load acts along the centroidal axis (the line connecting the centroids of cross sections) of the bar, as shown in Fig. 1.6(b). The loads shown in Fig. 1.6 are called *axial* or *centroidal loads.*

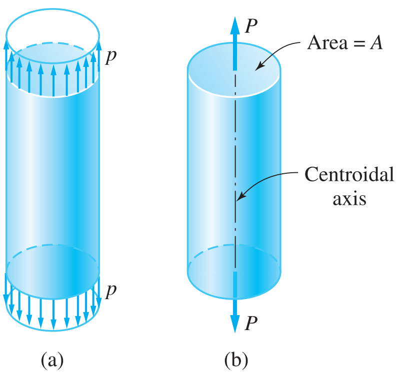


FIG. 1.6 A bar loaded axially by (a) uniformly distributed load of intensity p; and (b) a statically equivalent centroidal force $P = pA$.

Although the loads in Figs. 1.6(a) and (b) are statically equivalent, they do not result in the same stress distribution in the bar. In the case of the uniform loading in Fig. 1.6(a), the internal forces acting on all cross sections are also uniformly distributed. Therefore, the normal stress acting at any point on a cross section is

$$\boxed{\sigma = \frac{P}{A}} \tag{1.4}$$

The stress distribution caused by the concentrated loading in Fig. 1.6(b) is more complicated. Advanced methods of analysis show that on cross sections close to the ends, the maximum stress is considerably higher than the average stress P/A. As we move away from the ends, the stress

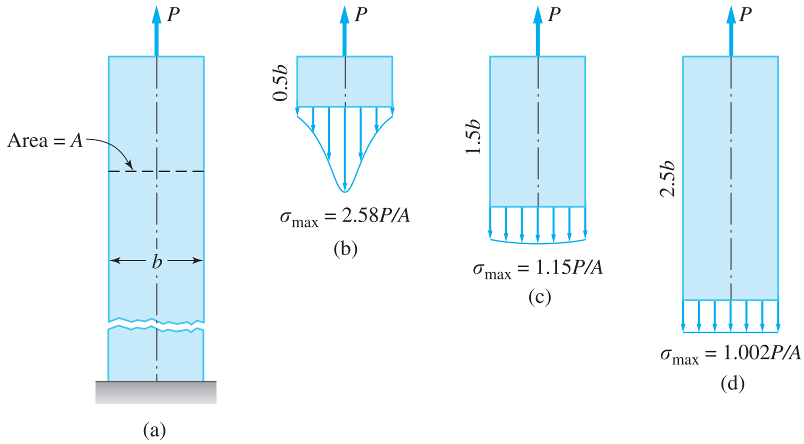


FIG. 1.7 Normal stress distribution in a strip caused by a concentrated load.

becomes more uniform, reaching the uniform value P/A in a relatively short distance from the ends. In other words, the stress distribution is approximately uniform in the bar, except in the regions close to the ends.

As an example of concentrated loading, consider the thin strip of width b shown in Fig. 1.7(a). The strip is loaded by the centroidal force P. Figures 1.7(b)–(d) show the stress distribution on three different cross sections. Note that at a distance $2.5b$ from the loaded end, the maximum stress differs by only 0.2% from the average stress P/A.

b. *Saint Venant's principle*

About 150 years ago, the French mathematician Saint Venant studied the effects of statically equivalent loads on the twisting of bars. His results led to the following observation, called *Saint Venant's principle*:

> *The difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from the load.*

The example in Fig. 1.7 is an illustration of Saint Venant's principle. The principle also applies to the effects caused by abrupt changes in the cross section. Consider, as an example, the grooved cylindrical bar of radius R shown in Fig. 1.8(a). The loading consists of the force P that is uniformly distributed over the end of the bar. If the groove were not present, the normal stress acting at all points on a cross section would be P/A. Introduction of the groove disturbs the uniformity of the stress, but this effect is confined to the vicinity of the groove, as seen in Figs. 1.8(b) and (c).

Most analysis in mechanics of materials is based on simplifications that can be justified with Saint Venant's principle. We often replace loads (including support reactions) by their resultants and ignore the effects of holes, grooves, and fillets on stresses and deformations. Many of the simplifications are not only justified but necessary. Without simplifying assumptions, analysis would be exceedingly difficult. However, we must always keep in mind the approximations that were made, and make allowances for them in the final design.

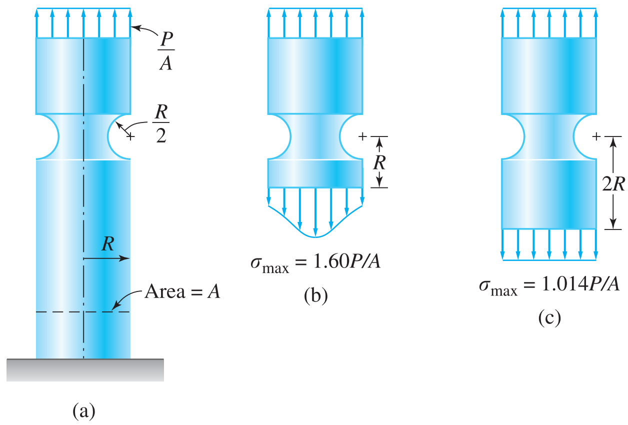


FIG. 1.8 Normal stress distribution in a grooved bar.

c. Stresses on inclined planes

When a bar of cross-sectional area A is subjected to an axial load P, the normal stress P/A acts on the cross section of the bar. Let us now consider the stresses that act on plane a-a that is inclined at the angle θ to the cross section, as shown in Fig. 1.9(a). Note that the area of the inclined plane is $A/\cos\theta$. To investigate the forces that act on this plane, we consider the free-body diagram of the segment of the bar shown in Fig. 1.9(b). Because the segment is a two-force body, the resultant internal force acting on the inclined plane must be the axial force P, which can be resolved into the normal component $P\cos\theta$ and the shear component $P\sin\theta$. Therefore, the corresponding stresses, shown in Fig. 1.9(c), are

$$\sigma = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta \tag{1.5a}$$

$$\tau = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\sin\theta\cos\theta = \frac{P}{2A}\sin 2\theta \tag{1.5b}$$

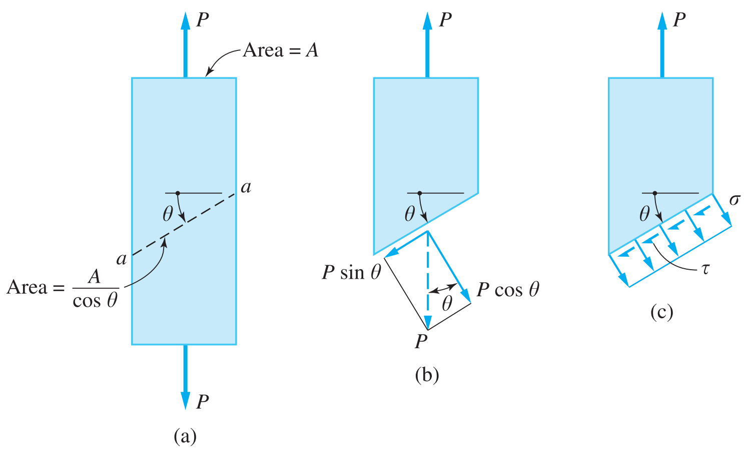


FIG. 1.9 Determining the stresses acting on an inclined section of a bar.

From these equations we see that the maximum normal stress is P/A, and it acts on the cross section of the bar (that is, on the plane $\theta = 0$). The shear stress is zero when $\theta = 0$, as would be expected. The maximum shear stress is $P/2A$, which acts on the planes inclined at $\theta = 45°$ to the cross section.

In summary, an axial load causes not only normal stress but also shear stress. The magnitudes of both stresses depend on the orientation of the plane on which they act.

By replacing θ with $\theta + 90°$ in Eqs. (1.5), we obtain the stresses acting on plane a'-a', which is perpendicular to a-a, as illustrated in Fig. 1.10(a):

$$\sigma' = \frac{P}{A}\sin^2\theta \qquad \tau' = -\frac{P}{2A}\sin 2\theta \tag{1.6}$$

where we used the identities $\cos(\theta + 90°) = -\sin\theta$ and $\sin 2(\theta + 90°) = -\sin 2\theta$. Because the stresses in Eqs. (1.5) and (1.6) act on mutually perpendicular, or "complementary" planes, they are called *complementary stresses*. The traditional way to visualize complementary stresses is to draw them on a small (infinitesimal) element of the material, the sides of which are parallel to the complementary planes, as in Fig. 1.10(b). When labeling the stresses, we made use of the following important result that follows from Eqs. (1.5) and (1.6):

$$\boxed{\tau' = -\tau} \tag{1.7}$$

In other words,

> *The shear stresses that act on complementary planes have the same magnitude but opposite sense.*

Although Eq. (1.7) was derived for axial loading, we will show later that it also applies to more complex loadings.

The design of axially loaded bars is usually based on the maximum normal stress in the bar. This stress is commonly called simply the *normal stress* and denoted by σ, a practice that we follow in this text. The design criterion thus is that $\sigma = P/A$ must not exceed the *working stress* of the material from which the bar is to be fabricated. The working stress, also called the *allowable stress*, is the largest value of stress that can be safely carried by the material. Working stress, denoted by σ_w, will be discussed more fully in Sec. 2.2.

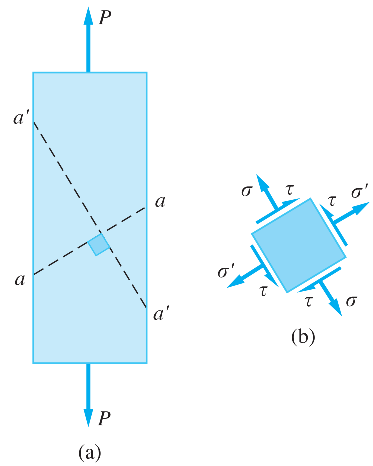


FIG. 1.10 Stresses acting on two mutually perpendicular inclined sections of a bar.

d. Procedure for stress analysis

In general, the stress analysis of an axially loaded member of a structure involves the following steps.

Equilibrium Analysis

- If necessary, find the external reactions using a free-body diagram (FBD) of the entire structure.
- Compute the axial force P in the member using the method of sections. This method introduces an imaginary cutting plane that isolates a segment of the structure. The cutting plane must include the cross section of the member of interest. The axial force acting in the member can

then be found from the FBD of the isolated segment because it now appears as an external force on the FBD.

Computation of Stress

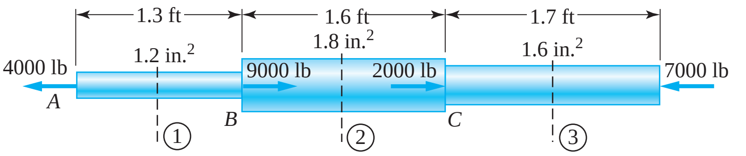
- After the axial force has been found by equilibrium analysis, the *average* normal stress in the member can be obtained from $\sigma = P/A$, where A is the cross-sectional area of the member at the cutting plane.
- In slender bars, $\sigma = P/A$ is the normal stress if the section is sufficiently far from applied loads and abrupt changes in the cross section (Saint Venant's principle).

Design Considerations For purposes of design, the computed stress must be compared with the allowable stress, also called the *working stress*. The working stress, which we denote by σ_w, is discussed in detail in the next chapter. To prevent failure of the member, the computed stress must be less than the working stress.

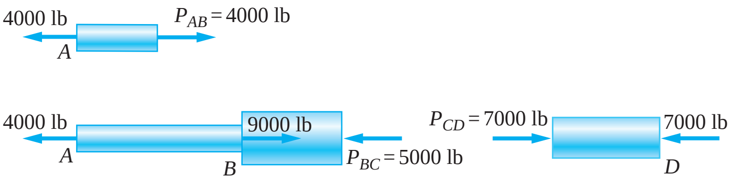
Note on the Analysis of Trusses The usual assumptions made in the analysis of trusses are: (1) weights of the members are negligible compared to the applied loads; (2) joints behave as smooth pins; and (3) all loads are applied at the joints. Under these assumptions, each member of the truss is an axially loaded bar. The internal forces in the bars can be obtained by the method of sections or the method of joints (utilizing the free-body diagrams of the joints).

Sample Problem **1.1**

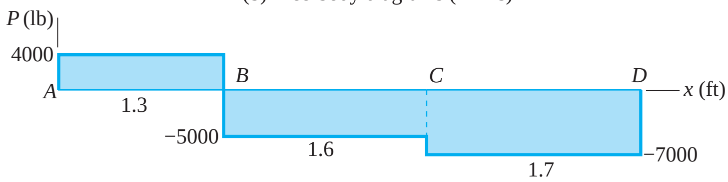
The bar *ABCD* in Fig. (a) consists of three cylindrical steel segments with different lengths and cross-sectional areas. Axial loads are applied as shown. Calculate the normal stress in each segment.



(a)



(b) Free-body diagrams (FBDs)



(c) Axial force diagram
(tension assumed positive)

Solution

We begin by using equilibrium analysis to compute the axial force in each segment of the bar (recall that equilibrium analysis is the first step in stress analysis). The required free body diagrams (FBDs), shown in Fig. (b), were drawn by isolating the portions of the beam lying to the left of sections ① and ②, and to the right of section ③. From these FBDs, we see that the internal forces in the three segments of the bar are $P_{AB} = 4000$ lb (T), $P_{BC} = 5000$ lb (C), and $P_{CD} = 7000$ lb (C), where (T) denotes tension and (C) denotes compression.

The axial force diagram in Fig. (c) shows how the how the internal forces vary with the distance x measured along the bar from end A. Note that the internal forces vary from segment to segment, but the force in each segment is constant. Because the internal forces are discontinuous at points A, B, C, and D, our stress calculations will be valid only for sections that are not too close to these points (Saint Venants principle).

The normal stresses in the three segments are

$$\sigma_{AB} = \frac{P_{AB}}{A_{AB}} = \frac{4000 \text{ lb}}{1.2 \text{ in.}^2} = 3330 \text{ psi (T)}$$ *Answer*

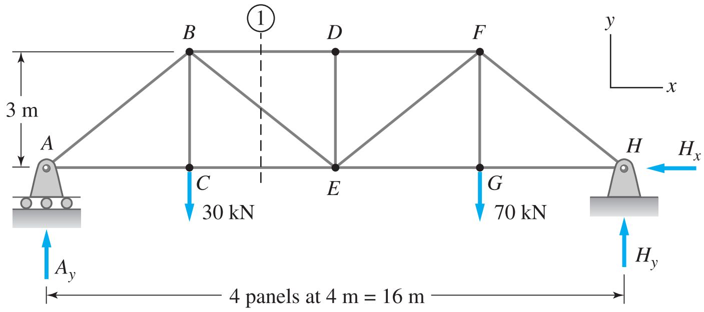
$$\sigma_{BC} = \frac{P_{BC}}{A_{BC}} = \frac{5000 \text{ lb}}{1.8 \text{ in.}^2} = 2780 \text{ psi (C)}$$ *Answer*

$$\sigma_{CD} = \frac{P_{CD}}{A_{CD}} = \frac{7000 \text{ lb}}{1.6 \text{ in.}^2} = 4380 \text{ psi (C)}$$ *Answer*

Observe that the lengths of the segments do not affect the calculations of the stresses. Also, the fact that the bar is made of steel is irrelevant; the stresses in the segments would be as calculated, regardless of the materials from which the segments of the bar are fabricated.

Sample Problem 1.2

For the truss shown in Fig. (a), calculate the normal stresses in (1) member AC; and (2) member BD. The cross-sectional area of each member is 900 mm^2.

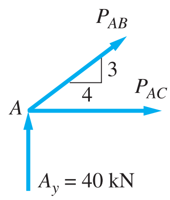


(a)

Solution

Equilibrium analysis using the FBD of the entire truss in Fig. (a) gives the following values for the external reactions: $A_y = 40$ kN, $H_y = 60$ kN, and $H_x = 0$.

Part 1



(b) FBD of pin A

Recall that according to the assumptions used in truss analysis, each member of the truss is an axially loaded bar. To find the force in member AC, we draw the FBD of pin A, as shown in Fig. (b). In this (FBD), P_{AB} and P_{AC} are the forces in members AB and AC, respectively. Note that we have assumed both of these forces to be tensile. Because the force system is concurrent and coplanar, there are two independent equilibrium equations. From the FBD in Fig. (b), we get

$$\sum F_y = 0 \quad +\uparrow \quad 40 + \frac{3}{5} P_{AB} = 0$$

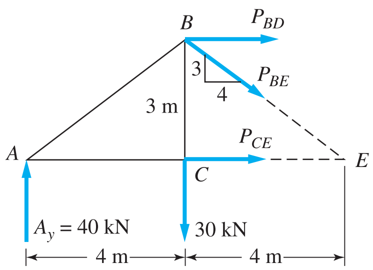
$$\sum F_x = 0 \quad \overset{+}{\rightarrow} \quad P_{AC} + \frac{4}{5} P_{AB} = 0$$

Solving the equations gives $P_{AC} = 53.33$ kN (tension). Thus, the normal stress in member AC is

$$\sigma_{AC} = \frac{P_{AC}}{A_{AC}} = \frac{53.33 \text{ kN}}{900 \text{ mm}^2} = \frac{53.33 \times 10^3 \text{ N}}{900 \times 10^{-6} \text{ m}^2}$$

$$= 59.3 \times 10^6 \text{ N/m}^2 = 59.3 \text{ MPa (T)}$$ *Answer*

Part 2



(c) FBD

To determine the force in member BD, we see that section ① in Fig. (a) cuts through members BD, BE, and CE. Because three equilibrium equations are available for a portion of the truss separated by this section, we can find the forces in all three members, if needed.

The FBD of the portion of the truss lying to the left of section ① is shown in Fig. (c) (the portion lying to the right could also be used). We have again assumed that the forces in the members are tensile. To calculate the force in member BD, we use the equilibrium equation

$$\sum M_E = 0 \quad +\circlearrowleft \quad -40(8) + 30(4) - P_{BD}(3) = 0$$

which yields

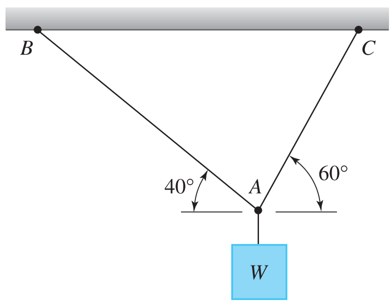
$$P_{BD} = -66.67\ \text{kN} = 66.67\ \text{kN (C)}$$

Therefore, the normal stress in member BD is

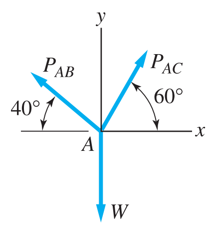
$$\sigma_{BD} = \frac{P_{BD}}{A_{BD}} = \frac{-66.67\ \text{kN}}{900\ \text{mm}^2} = \frac{-66.67 \times 10^3\ \text{N}}{900 \times 10^{-6}\ \text{m}^2}$$

$$= -74.1 \times 10^6\ \text{N/m}^2 = 74.1\ \text{MPa (C)}$$ *Answer*

Sample Problem 1.3



(a)



(b) FBD of pin A

Figure (a) shows a two-member truss supporting a block of weight W. The cross-sectional areas of the members are 800 mm^2 for AB and 400 mm^2 for AC. Determine the maximum safe value of W if the working stresses are 110 MPa for AB and 120 MPa for AC.

Solution

Being members of a truss, AB and AC can be considered to be axially loaded bars. The forces in the bars can be obtained by analyzing the FBD of pin A in Fig. (b). The equilibrium equations are

$$\sum F_x = 0 \quad \overset{+}{\rightarrow} \quad P_{AC} \cos 60^\circ - P_{AB} \cos 40^\circ = 0$$

$$\sum F_y = 0 \quad +\uparrow \quad P_{AC} \sin 60^\circ + P_{AB} \sin 40^\circ - W = 0$$

Solving simultaneously, we get

$$P_{AB} = 0.5077W \qquad P_{AC} = 0.7779W$$

Design for Normal Stress in Bar AB
The value of W that will cause the normal stress in bar AB to equal its working stress is given by

$$P_{AB} = (\sigma_w)_{AB} A_{AB}$$

$$0.5077W = (110 \times 10^6\ \text{N/m}^2)(800 \times 10^{-6}\ \text{m}^2)$$

$$W = 173.3 \times 10^3\ \text{N} = 173.3\ \text{kN}$$

Design for Normal Stress in Bar AC
The value of W that will cause the normal stress in bar AC to equal its working stress is found from

$$P_{AC} = (\sigma_w)_{AC} A_{AC}$$

$$0.7779W = (120 \times 10^6\ \text{N/m}^2)(400 \times 10^{-6}\ \text{m}^2)$$

$$W = 61.7 \times 10^3\ \text{N} = 61.7\ \text{kN}$$

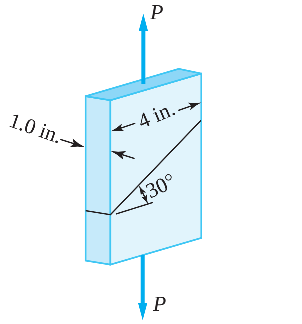
Choose the Correct Answer
The maximum safe value of W is the smaller of the preceding two values—namely,

$$W = 61.7\ \text{kN}$$ *Answer*

We see that the stress in bar AC determines the safe value of W. The other “solution,” $W = 173.3$ kN, must be discarded because it would cause the stress in AC to exceed its working stress of 120 MPa.

Sample Problem 1.4

The rectangular wood panel is formed by gluing together two boards along the 30-degree seam as shown in the figure. Determine the largest axial force P that can be carried safely by the panel if the working stress for the wood is 1120 psi, and the normal and shear stresses in the glue are limited to 700 psi and 450 psi, respectively.



Solution

The most convenient method for analyzing this design-type problem is to calculate the largest safe value of P that satisfies each of the three design criteria. The smallest of these three values is the largest safe value of P for the panel.

Design for Working Stress in Wood

The value of P for which the wood would reach its working stress is found as follows:

$$P = \sigma_w A = 1120(4 \times 1.0) = 4480 \text{ lb}$$

Design for Normal Stress in Glue

The axial force P that would cause the normal stress in the glue to equal its maximum allowable value is computed from Eq. (1.5a):

$$\sigma = \frac{P}{A}\cos^2\theta$$
$$700 = \frac{P}{(4 \times 1.0)}\cos^2 30^\circ$$
$$P = 3730\text{lb}$$

Design for Shear Stress in Glue

The value of P that would cause the shear stress in the glue to equal its maximum value is computed from Eq. (1.5b):

$$\sigma = \frac{P}{2A}\sin 2\theta$$
$$450 = \frac{P}{2(4 \times 1.0)}\sin 60^\circ$$
$$P = 4160\text{lb}$$

Choose the Correct Answer

Comparing the above three solutions, we see that the largest safe axial load that can be safely applied is governed by the normal stress in the glue, its value being

$$P = 3730 \text{ lb}$$ *Answer*

Problems

1.1 A hollow steel tube with an inside diameter of 80 mm must carry an axial tensile load of 330 kN. Determine the smallest allowable outside diameter of the tube if the working stress is 110 MN/m^2.

1.2 The cross-sectional area of bar *ABCD* is 600 mm^2. Determine the maximum normal stress in the bar.

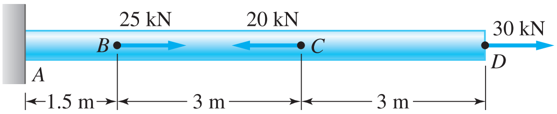


FIG. P1.2

1.3 Determine the largest weight W that can be supported by the two wires AB and AC. The working stresses are 100 MPa for AB and 150 MPa for AC. The cross-sectional areas of AB and AC are 400 mm^2 and 200 mm^2, respectively.

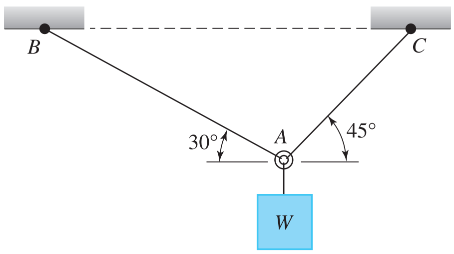


FIG. P1.3

1.4 Axial loads are applied to the compound rod that is composed of an aluminum segment rigidly connected between steel and bronze segments. What is the stress in each material given that $P = 10$ kN?

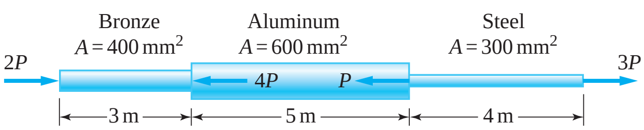


FIG. P1.4, P1.5

1.5 Axial loads are applied to the compound rod that is composed of an aluminum segment rigidly connected between steel and bronze segments. Find the largest safe value of P if the working stresses are 120 MPa for steel, 68 MPa for aluminum, and 110 MPa for bronze.

1.6 The wood pole is supported by two cables of 1/4-in. diameter. The turnbuckles in the cables are tightened until the stress in the cables reaches 60 000 psi. If the working compressive stress for wood is 200 psi, determine the smallest permissible diameter of the pole.

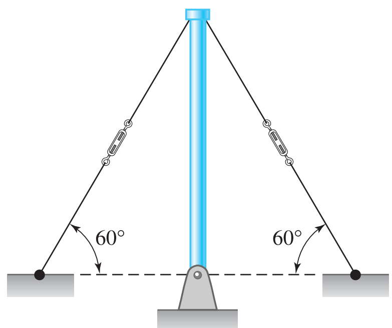


FIG. P1.6

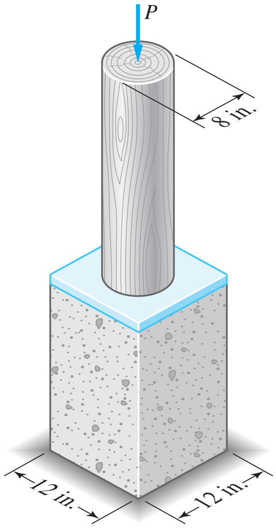


FIG. P1.7

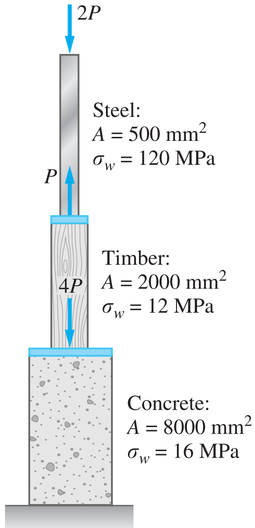


FIG. P1.8

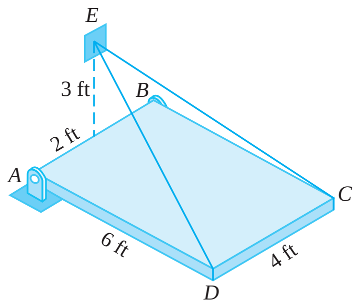


FIG. P1.9

1.7 The column consists of a wooden post and a concrete footing, separated by a steel bearing plate. Find the maximum safe value of the axial load P if the working stresses are 1000 psi for wood and 450 psi for concrete.

1.8 Find the maximum allowable value of P for the column. The cross-sectional areas and working stresses (σ_w) are shown in the figure.

1.9 The 1200-lb uniform plate $ABCD$ can rotate freely about the hinge AB. The plate is supported by the cables DE and CE. If the working stress in the cables is 18 000 psi, determine the smallest safe diameter of the cables.

1.10 The homogeneous bar AB weighing 1800 lb is supported at either end by a steel cable. Calculate the smallest safe area of each cable if the working stress is 18 000 psi for steel.

1.11 The homogeneous 6000-lb bar ABC is supported by a pin at C and a cable that runs from A to B around the frictionless pulley at D. Find the stress in the cable if its diameter is 0.6 in.

1.12 Determine the largest weight W that can be supported safely by the structure shown in the figure. The working stresses are 16 000 psi for the steel cable AB and 720 psi for the wood strut BC. Neglect the weight of the structure.

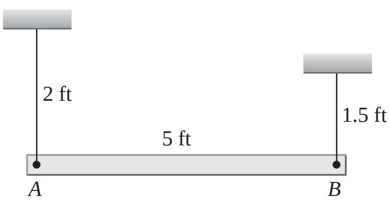


FIG. P1.10

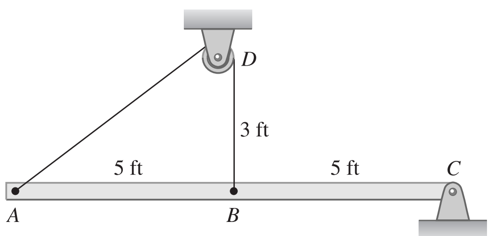


FIG. P1.11

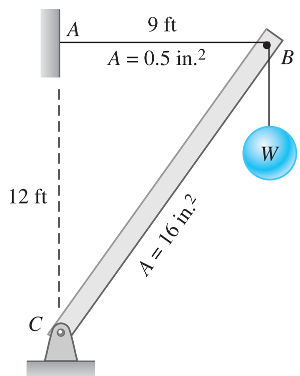


FIG. P1.12

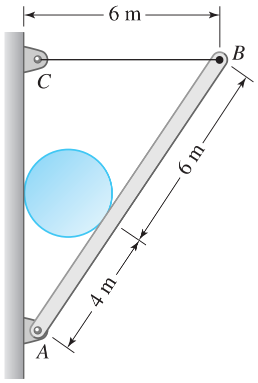


FIG. P1.13

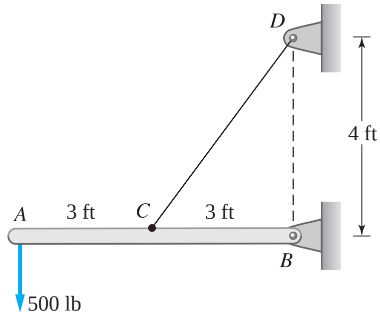


FIG. P1.14

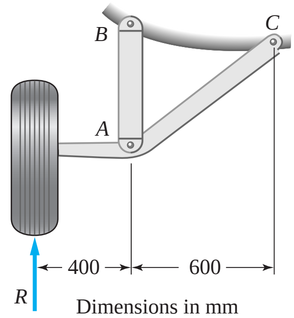


FIG. P1.15

1.13 Determine the mass of the heaviest uniform cylinder that can be supported in the position shown without exceeding a stress of 50 MPa in cable *BC*. Neglect friction and the weight of bar *AB*. The cross-sectional area of *BC* is 100 mm^2.

1.14 The uniform 300-lb bar *AB* carries a 500-lb vertical force at *A*. The bar is supported by a pin at *B* and the 0.5-in. diameter cable *CD*. Find the stress in the cable.

1.15 The figure shows the landing gear of a light airplane. Determine the compressive stress in strut *AB* caused by the landing reaction $R = 40$ kN. Neglect the weights of the members. The strut is a hollow tube, with 50-mm outer diameter and 40-mm inner diameter.

1.16 The 1000-kg uniform bar *AB* is suspended from two cables *AC* and *BD*, each with cross-sectional area 400 mm^2. Find the magnitude *P* and location *x* of the largest additional vertical force that can be applied to the bar. The stresses in *AC* and *BD* are limited to 100 MPa and 50 MPa, respectively.

1.17 The cross-sectional area of each member of the truss is 1.8 in.^{2}. Calculate the stresses in members *CE*, *DE*, and *DF*. Indicate tension or compression.

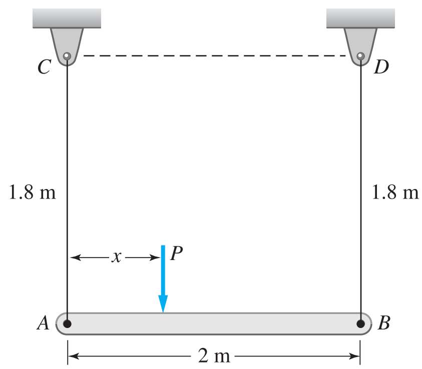


FIG. P1.16

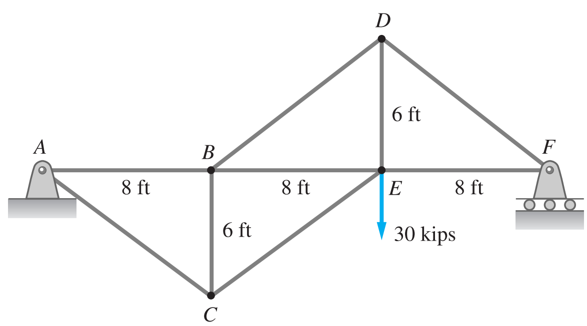


FIG. P1.17

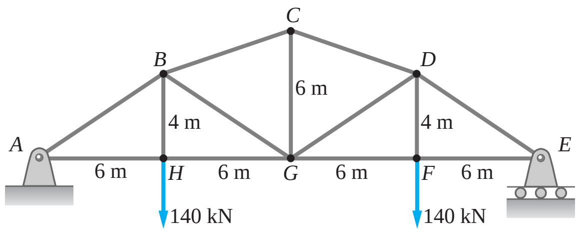


FIG. P1.18

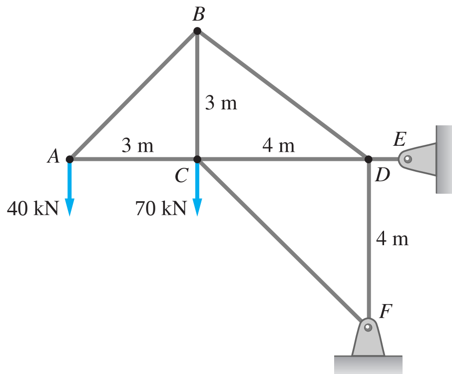


FIG. P1.19

1.18 Determine the smallest safe cross-sectional areas of members *CD, GD*, and *GF* for the truss shown. The working stresses are 140 MPa in tension and 100 MPa in compression. (The working stress in compression is smaller to reduce the danger of buckling.)

1.19 Find the stresses in members *BC*, *BD*, and *CF* for the truss shown. Indicate tension or compression. The cross-sectional area of each member is 1400 mm^2.

1.20 Determine the smallest allowable cross-sectional areas of members *CE, BE*, and *EF* for the truss shown. The working stresses are 20 ksi in tension and 14 ksi in compression. (The working stress in compression is smaller to reduce the danger of buckling.)

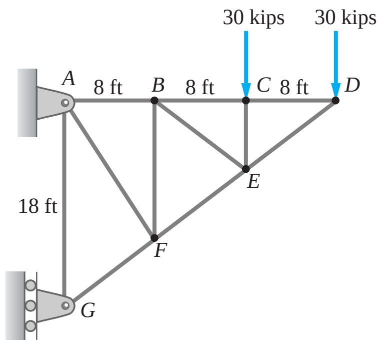


FIG. P1.20

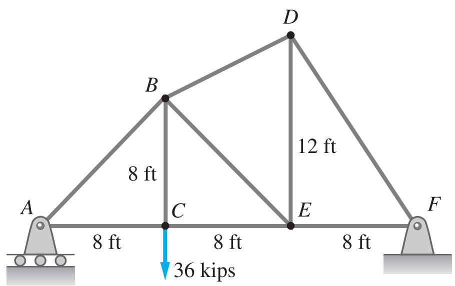


FIG. P1.21

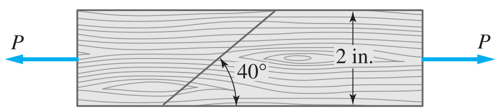


FIG. P1.22

1.21 Determine the smallest allowable cross-sectional areas of members *BD*, *BE*, and *CE* of the truss shown. The working stresses are 20 000 psi in tension and 12 000 psi in compression. (A reduced stress in compression is specified to reduce the danger of buckling.)

1.22 The two pieces of wood, 2 in. by 4 in., are glued together along the 40° joint. Determine the maximum safe axial load *P* that can be applied if the shear stress in the glue is limited to 250 psi.

1.23 The rectangular piece of wood, 50 mm by 100 mm, is used as a compression block. The grain of the wood makes a 20° angle with the horizontal, as shown in the figure. Determine the largest axial force *P* that can be applied safely if the allowable stresses on the plane of the grain are 18 MPa for compression and 4 MPa for shear.

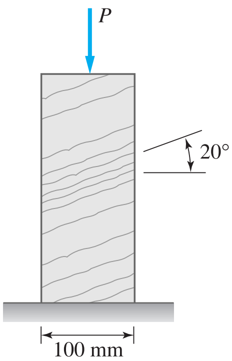


FIG. P1.23

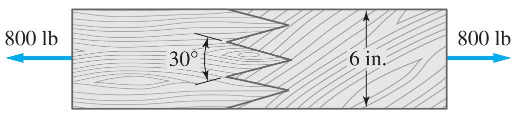


FIG. P1.24

1.24 The figure shows a glued joint, known as a finger joint, in a 6-in. by 3/4-in. piece of lumber. Find the normal and shear stresses acting on the surface of the joint.

1.25 The piece of wood, 100 mm by 100 mm in cross section, contains a glued joint inclined at the angle θ to the vertical. The working stresses are 20 MPa for wood in tension, 8 MPa for glue in tension, and 12 MPa for glue in shear. If $\theta = 50°$, determine the largest allowable axial force *P*.

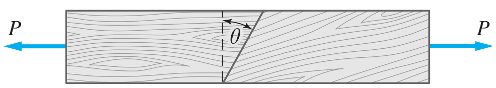


FIG. P1.25

1.4 Shear Stress

By definition, normal stress acting on an interior plane is directed perpendicular to that plane. Shear stress, on the other hand, is tangent to the plane on which it acts. Shear stress arises whenever the applied loads cause one section of a body to slide past its adjacent section. In Sec. 1.3, we examined how shear stress occurs in an axially loaded bar. Three other examples of shear stress are illustrated in Fig. 1.11. Figure 1.11(a) shows two plates that are joined by a rivet. As seen in the FBD, the rivet must carry the shear force $V = P$. Because only one cross section of the rivet resists the shear, the rivet is said to be in *single shear*. The bolt of the clevis in Fig. 1.11(b) carries the load P across two cross-sectional areas, the shear force being $V = P/2$ on each cross section. Therefore, the bolt is said to be in a state of *double shear*. In Fig. 1.11(c) a circular slug is being punched out of a metal sheet. Here the shear force is P and the shear area is similar to the milled edge of a coin. The loads shown in Fig. 1.11 are sometimes referred to as *direct shear* to distinguish them from the *induced shear* illustrated in Fig. 1.9.

The distribution of direct shear stress is usually complex and not easily determined. It is common practice to assume that the shear force V is uniformly distributed over the shear area A, so that the shear stress can be computed from

$$\tau = \frac{V}{A} \tag{1.8}$$

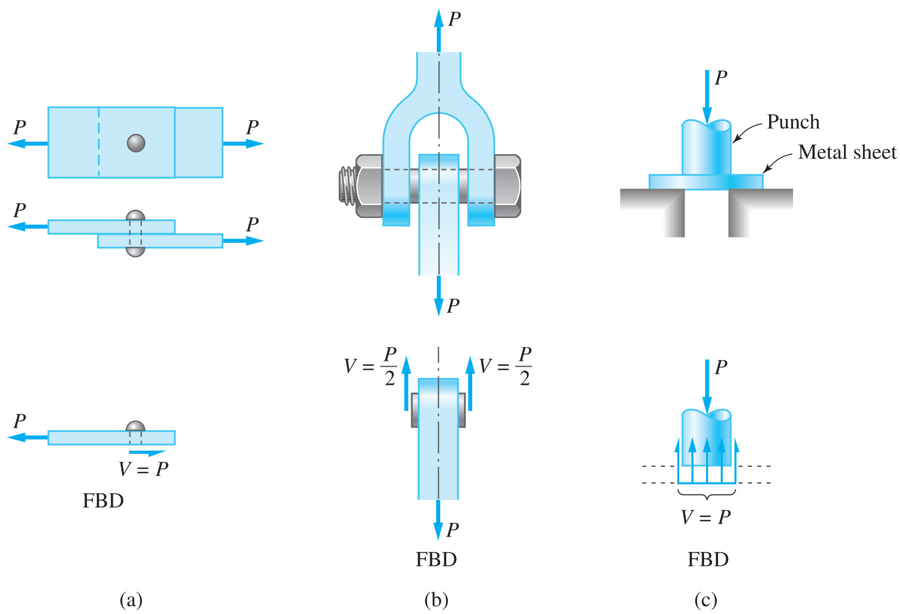


FIG. 1.11 Examples of direct shear: (a) single shear in a rivet; (b) double shear in a bolt; and (c) shear in a metal sheet produced by a punch.

Strictly speaking, Eq. (1.8) must be interpreted as the *average* shear stress. It is often used in design to evaluate the strength of connectors, such as rivets, bolts, and welds.

1.5 Bearing Stress

If two bodies are pressed against each other, compressive forces are developed on the area of contact. The pressure caused by these surface loads is called *bearing stress*. Examples of bearing stress are the soil pressure beneath a pier and the contact pressure between a rivet and the side of its hole. If the bearing stress is large enough, it can locally crush the material, which in turn can lead to more serious problems. To reduce bearing stresses, engineers sometimes employ bearing plates, the purpose of which is to distribute the contact forces over a larger area.

As an illustration of bearing stress, consider the lap joint formed by the two plates that are riveted together as shown in Fig. 1.12(a). The bearing stress caused by the rivet is not constant; it actually varies from zero at the sides of the hole to a maximum behind the rivet as illustrated in Fig. 1.12(b). The difficulty inherent in such a complicated stress distribution is avoided by the common practice of assuming that the bearing stress σ_b is uniformly distributed over a reduced area. The reduced area A_b is taken to be the *projected area* of the rivet:

$$A_b = td$$

where t is the thickness of the plate and d represents the diameter of the rivet, as shown in the FBD of the upper plate in Fig. 1.12(c). From this FBD we see that the bearing force P_b equals the applied load P (the bearing load will be reduced if there is friction between the plates), so that the bearing stress becomes

$$\sigma_b = \frac{P_b}{A_b} = \frac{P}{td} \tag{1.9}$$

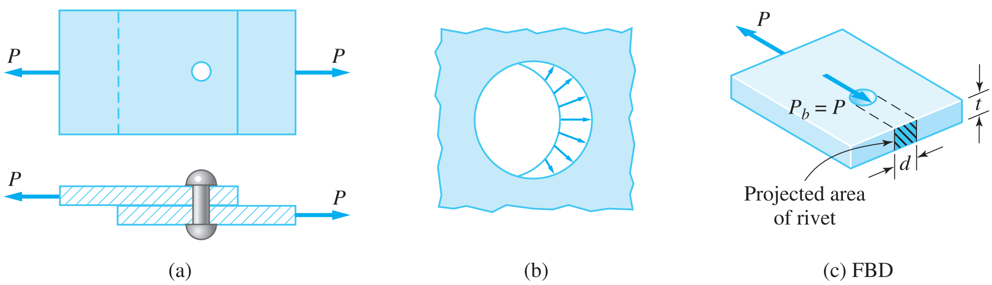
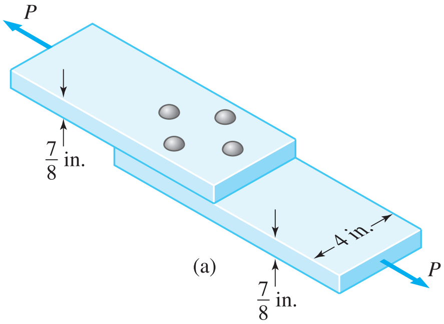


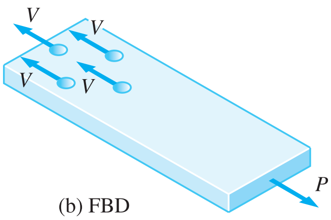
FIG. 1.12 Example of bearing stress: (a) a rivet in a lap joint; (b) bearing stress is not constant; (c) bearing stress caused by the bearing force P_b is assumed to be uniform on projected area td.

Sample Problem 1.5

The lap joint shown in Fig. (a) is fastened by four rivets of 3/4-in. diameter. Find the maximum load P that can be applied if the working stresses are 14 ksi for shear in the rivet and 18 ksi for bearing in the plate. Assume that the applied load is distributed evenly among the four rivets, and neglect friction between the plates.



(a)



(b) FBD

Solution

We will calculate P using each of the two design criteria. The largest safe load will be the smaller of the two values. Figure (b) shows the FBD of the lower plate. In this FBD, the lower halves of the rivets are in the plate, having been isolated from their top halves by a cutting plane. This cut exposes the shear forces V that act on the cross sections of the rivets. We see that the equilibrium condition is $V = P/4$.

Design for Shear Stress in Rivets

The value of P that would cause the shear stress in the rivets to reach its working value is found as follows:

$$V = \tau A$$

$$\frac{P}{4} = (14 \times 10^3)\left[\frac{\pi(3/4)^2}{4}\right]$$

$$P = 24\,700 \text{ lb}$$

Design for Bearing Stress in Plate

The shear force $V = P/4$ that acts on the cross section of one rivet is equal to the bearing force P_b due to the contact between the rivet and the plate. The value of P that would cause the bearing stress to equal its working value is computed from Eq. (1.9):

$$P_b = \sigma_b t d$$

$$\frac{P}{4} = (18 \times 10^3)(7/8)(3/4)$$

$$P = 47\,300 \text{ lb}$$

Choose the Correct Answer

Comparing the above solutions, we conclude that the maximum safe load P that can be applied to the lap joint is

$$P = 24\,700 \text{ lb} \qquad \textit{Answer}$$

with the shear stress in the rivets being the governing design criterion.

Problems

1.26 What force is required to punch a 20-mm-diameter hole in a plate that is 25 mm thick? The shear strength of the plate is 350 MN/m^2.

1.27 A circular hole is to be punched in a plate that has a shear strength of 40 ksi—see Fig. 1.11(c). The working compressive stress for the punch is 50 ksi. (a) Compute the maximum thickness of a plate in which a hole 2.5 in. in diameter can be punched. (b) If the plate is 0.25 in. thick, determine the diameter of the smallest hole that can be punched.

1.28 Find the smallest diameter bolt that can be used in the clevis in Fig. 1.11(b) if $P = 400$ kN. The working shear stress for the bolt is 300 MPa.

1.29 Referring to Fig. 1.11(a), assume that the diameter of the rivet that joins the plates is $d = 20$ mm. The working stresses are 120 MPa for bearing in the plate and 60 MPa for shear in the rivet. Determine the minimum safe thickness of each plate.

1.30 The lap joint is connected by three 20-mm-diameter rivets. Assuming that the axial load $P = 50$ kN is distributed equally among the three rivets, find (a) the shear stress in a rivet; (b) the bearing stress between a plate and a rivet; and (c) the maximum average tensile stress in each plate.

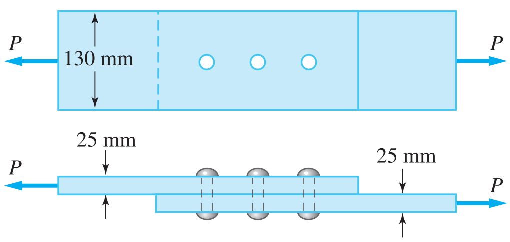


FIG. P1.30, P1.31

1.31 Assume that the axial load P applied to the lap joint is distributed equally among the three 20-mm-diameter rivets. What is the maximum load P that can be applied if the allowable stresses are 40 MPa for shear in rivets, 90 MPa for bearing between a plate and a rivet, and 120 MPa for tension in the plates?

1.32 A key prevents relative rotation between the shaft and the pulley. If the torque $T = 2200$ N·m is applied to the shaft, determine the smallest safe dimension b if the working shear stress for the key is 60 MPa.

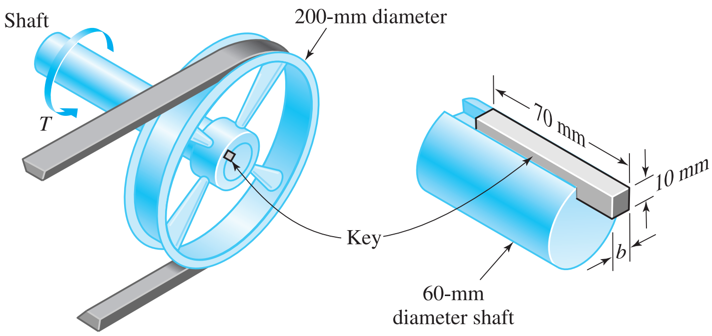


FIG. P1.32

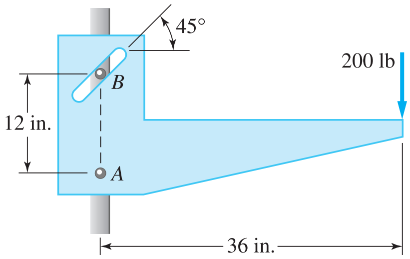


FIG. P1.33

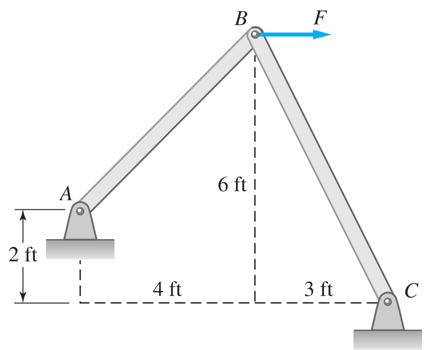


FIG. P1.34

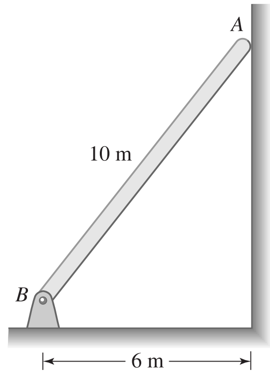


FIG. P1.35

1.33 The bracket is supported by 1/2-in.-diameter pins at *A* and *B* (the pin at *B* fits in the 45° slot in the bracket). Neglecting friction, determine the shear stresses in the pins, assuming single shear.

1.34 The 7/8-in.-diameter pins at *A* and *C* that support the structure are in single shear. Find the largest force *F* that can be applied to the structure if the working shear stress for these pins is 5000 psi. Neglect the weights of the members.

1.35 The uniform 2-Mg bar is supported by a smooth wall at *A* and by a pin at *B* that is in double shear. Determine the diameter of the smallest pin that can be used if its working shear stress is 60 MPa.

1.36 The bell crank, which is in equilibrium under the forces shown in the figure, is supported by a 20-mm-diameter pin at *D* that is in double shear. Determine (a) the required diameter of the connecting rod *AB*, given that its tensile working stress is 100 MPa; and (b) the shear stress in the pin.

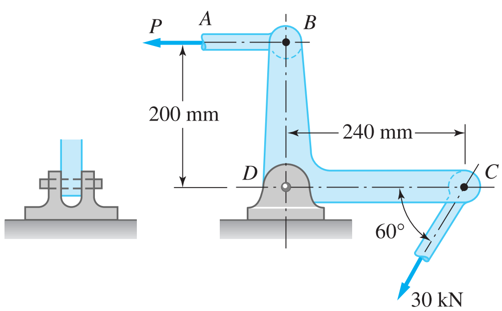


FIG. P1.36

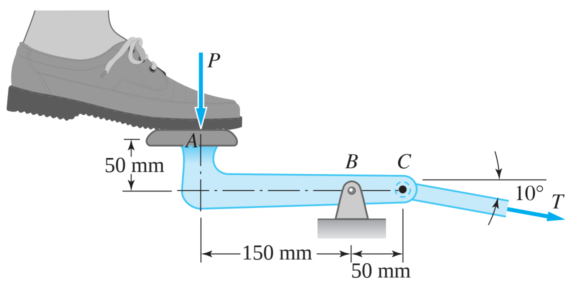


FIG. P1.37

1.37 Compute the maximum force *P* that can be applied to the foot pedal. The 6-mm.-diameter pin at *B* is in single shear, and its working shear stress is 28 MPa. The cable attached at *C* has a diameter of 3 mm. and a working normal stress of 140 MPa.

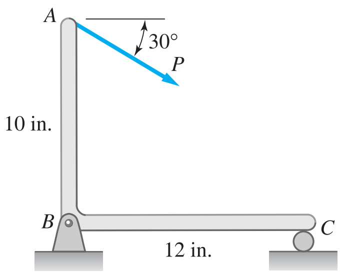


FIG. P1.38

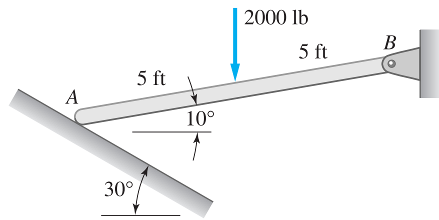


FIG. P1.39

1.38 The right-angle bar is supported by a pin at *B* and a roller at *C*. What is the maximum safe value of the load *P* that can be applied if the shear stress in the pin is limited to 20 000 psi? The 3/4-in.-diameter pin is in double shear.

1.39 The bar *AB* is supported by a frictionless inclined surface at *A* and a 7/8-in.-diameter pin at *B* that is in double shear. Determine the shear stress in the pin when the vertical 2000-lb force is applied. Neglect the weight of the bar.

1.40 A joint is made by gluing two plywood gussets of thickness *t* to wood boards. The tensile working stresses are 1200 psi for the plywood and 700 psi for the boards. The working shear stress for the glue is 50 psi. Determine the dimensions *b* and *t* so that the joint is as strong as the boards.

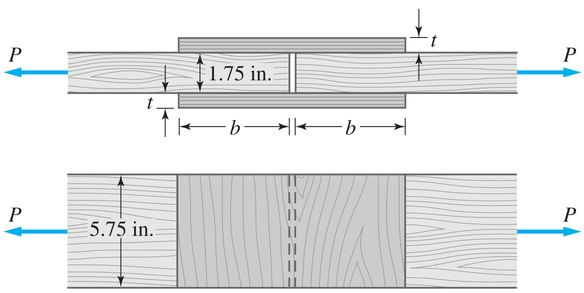


FIG. P1.40

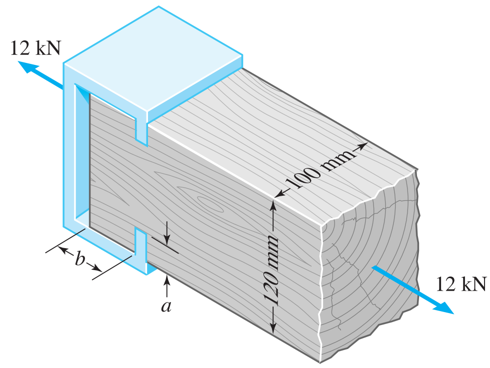


FIG. P1.41

1.41 The steel end-cap is fitted into grooves cut in the timber post. The working stresses for the post are 1.8 MPa in shear parallel to the grain and 5.5 MPa in bearing perpendicular to the grain. Determine the smallest safe dimensions *a* and *b*.

1.42 The halves of the coupling are held together by four 5/8-in.-diameter bolts. The working stresses are 12 ksi for shear in the bolts and 15 ksi for bearing in the coupling. Find the largest torque *T* that can be safely transmitted by the coupling. Assume that the forces in the bolts have equal magnitudes.

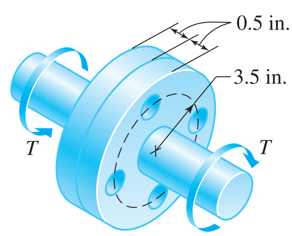


FIG. P1.42

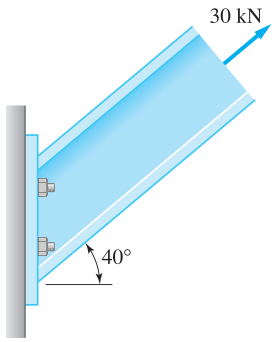


FIG. P1.43

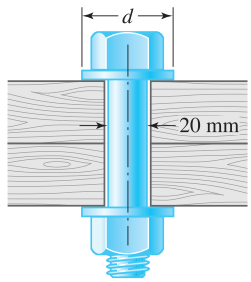


FIG. P1.44

1.43 The plate welded to the end of the I-beam is fastened to the support with four 10-mm-diameter bolts (two on each side). Assuming that the load is equally divided among the bolts, determine the normal and shear stresses in a bolt.

1.44 The 20-mm-diameter bolt fastens two wooden planks together. The nut is tightened until the tensile stress in the bolt is 150 MPa. Find the smallest safe diameter d of the washers if the working bearing stress for wood is 13 MPa.

1.45 The figure shows a roof truss and the detail of the connection at joint B. Members BC and BE are angle sections with the thicknesses shown in the figure. The working stresses are 70 MPa for shear in the rivets and 140 MPa for bearing stress due to the rivets. How many 19-mm-diameter rivets are required to fasten the following members to the gusset plate: (a) BC; and (b) BE?

1.46 Repeat Prob. 1.45 if the rivet diameter is 22 mm, with all other data remaining unchanged.

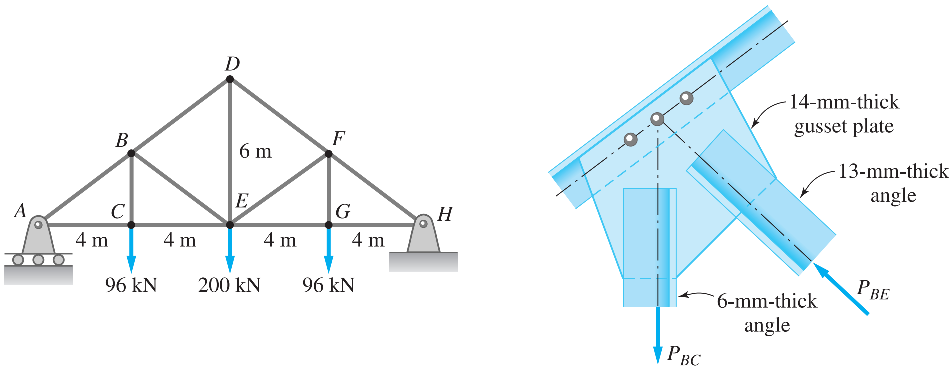


FIG. P1.45, P1.46

Review Problems

1.47 The cross-sectional area of each member of the truss is 1200 mm^2. Calculate the stresses in members *DF*, *CE*, and *BD*.

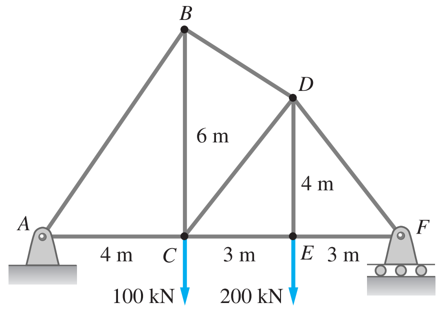


FIG. P1.47

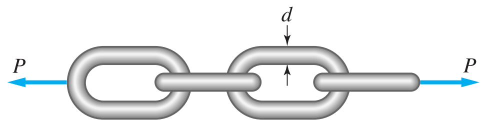


FIG. P1.48

1.48 The links of the chain are made of steel that has a working stress of 300 MPa in tension. If the chain is to support the force $P = 45$ kN, determine the smallest safe diameter d of the links.

1.49 Segment *AB* of the bar is a tube with an outer diameter of 1.5 in. and a wall thickness of 0.125 in. Segment *BC* is a solid rod of diameter 0.75 in. Determine the normal stress in each segment.

1.50 The cylindrical steel column has an outer diameter of 4 in. and inner diameter of 3.5 in. The column is separated from the concrete foundation by a square bearing plate. The working compressive stress is 26 000 psi for the column, and the working bearing stress is 1200 psi for concrete. Find the largest force P that can be applied to the column.

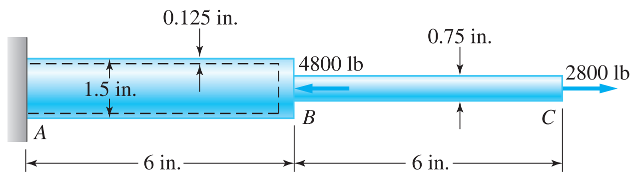


FIG. P1.49

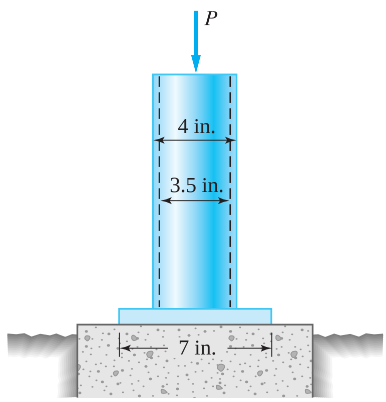


FIG. P1.50

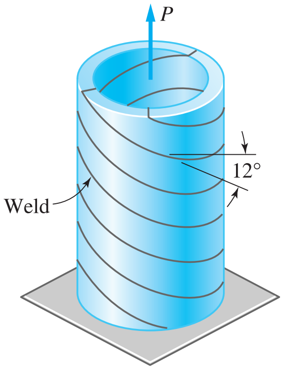


FIG. P1.51

1.51 The tubular tension member is fabricated by welding a steel strip into a 12° helix. The cross-sectional area of the resulting tube is 2.75 in.^{2}. If the normal stress acting on the plane of the weld is 12 ksi, determine (a) the axial force P; and (b) the shear stress acting on the plane of the weld.

1.52 An aluminum cable of 6 mm diameter is suspended from a high-altitude balloon. The density of aluminum is 2700 kg/m^3, and its breaking stress is 390 MPa. Determine the largest length of cable that can be suspended without breaking.

1.53 The 0.8-in-diameter steel bolt is placed in the aluminum sleeve. The nut is tightened until the normal stress in the bolt is 12 000 psi. Determine the normal stress in the sleeve.

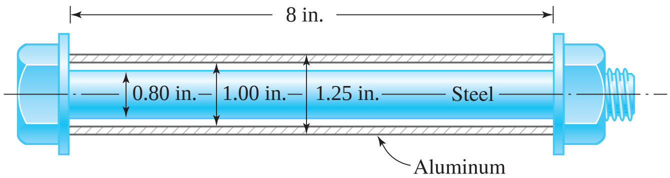


FIG. P1.53

1.54 For the joint shown in the figure, calculate (a) the largest bearing stress between the pin and the members; (b) the average shear stress in the pin; and (c) the largest average normal stress in the members.

1.55 The lap joint is fastened with four 3/4-in.-diameter rivets. The working stresses are 14 ksi for the rivets in shear and 18 ksi for the plates in bearing. Find the maximum safe axial load P that can be applied to the joint. Assume that the load is equally distributed among the rivets.

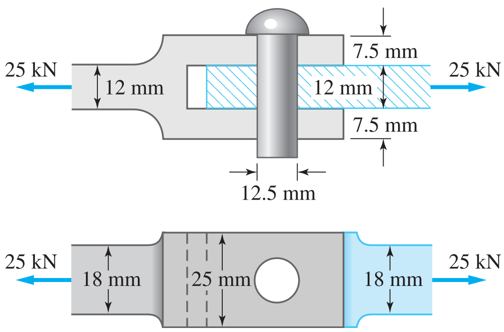


FIG. P1.54

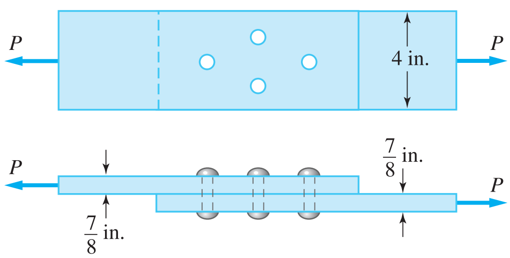


FIG. P1.55

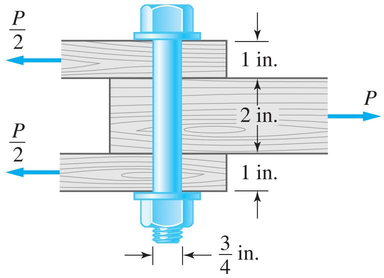


FIG. P1.56

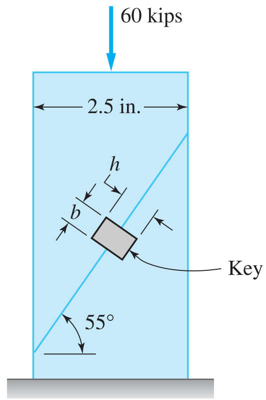


FIG. P1.57

1.56 Three wood boards, each 4 in. wide, are joined by the 3/4-in.-diameter bolt. If the working stresses for wood are 800 psi in tension and 1500 psi in bearing, find the largest allowable value of the force P.

1.57 The cast iron block with cross-sectional dimensions of 2.5 in. by 2.5 in. consists of two pieces. The pieces are prevented from sliding along the 55° inclined joint by the steel key, which is 2.5 in. long. Determine the smallest safe dimensions b and h of the key if the working stresses are 40 ksi for cast iron in bearing and 50 ksi for the key in shear.

1.58 Find the stresses in members BC and BE for the truss shown. The cross-sectional area of each member is 4.2 in.^{2}. Indicate whether the stresses are tensile (T) or compressive (C).

1.59 The boom AC is a 4-in. square steel tube with a wall thickness of 0.25 in. The boom is supported by the 0.5-in.-diameter pin at A, and the 0.375-in.-diameter cable BC. The working stresses are 25 ksi for the cable, 18 ksi for the boom, and 13.6 ksi for shear in the pin. Neglecting the weight of the boom, determine the largest safe load P that can be applied as shown.

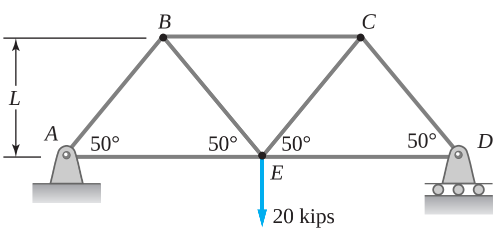


FIG. P1.58

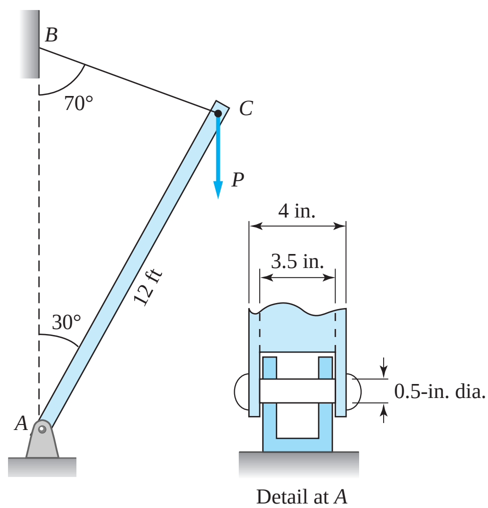


FIG. P1.59

Computer Problems

C1.1 The symmetric truss ABC of height h and span $2b$ carries the upward vertical force P at its apex C. The working stresses for the members are σ_t in tension and σ_c in compression. Given b, P, σ_t, and σ_c, write an algorithm to plot the required volume of material in the truss against h from $h = 0.5b$ to $4b$. Also find the value of h that results in the smallest volume of the material in the truss. Assume that the truss is fully stressed (each member is stressed to its working stress). Use the following data: $b = 6$ ft, $P = 120$ kips, $\sigma_t = 18$ ksi, and $\sigma_c = 12$ ksi.

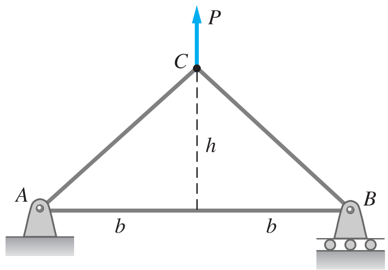


FIG. C1.1, C1.2

C1.2 Solve Prob. C1.1 assuming that P acts vertically downward.

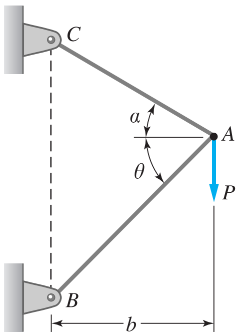


FIG. C1.3

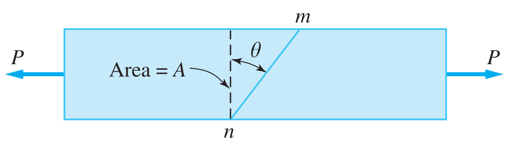


FIG. C1.4

C1.3 The truss ABC has an overhang b, and its two members are inclined at angles α and θ to the horizontal, both angles being positive. A downward vertical force P acts at A. The working stresses for the members are σ_t in tension and σ_c in compression. Given b, P, α, σ_t, and σ_c, construct an algorithm to plot the required volume of material in the truss against θ from $\theta = 0°$ to $75°$. Assume that each member of the truss is stressed to its working stress. What is the value of θ that results in the smallest material volume? Use the following data: $b = 1.8$ m, $P = 530$ kN, $\alpha = 30°$, $\sigma_t = 125$ MPa, and $\sigma_c = 85$ MPa.

C1.4 A high-strength adhesive is used to join two halves of a metal bar of cross-sectional area A along the plane m-n, which is inclined at the angle θ to the cross section. The working stresses for the adhesive are σ_w in tension and τ_w in shear. Given A, σ_w, and τ_w, write an algorithm that plots the maximum allowable axial force P that can be applied to the bar as a function of θ in the range $0° \le \theta \le 60°$. Assume that the metal is much stronger than the adhesive, so that P is determined by the stresses in the adhesive. Use the following data: $A = 4$ in.^{2}, $\sigma_w = 3500$ psi, and $\tau_w = 1800$ psi.

C1.5 The concrete cooling tower with a constant wall thickness of 1.5 ft is loaded by its own weight. The outer diameter of the tower varies as

$$d = 20 \text{ ft} - 0.1x + (0.35 \times 10^{-3} \text{ ft}^{-1})x^2$$

where x and d are in feet. Write an algorithm to plot the axial stress in the tower as a function of x. What is the maximum stress and where does it occur? Use 150 lb/ft^3 for the weight density of concrete.

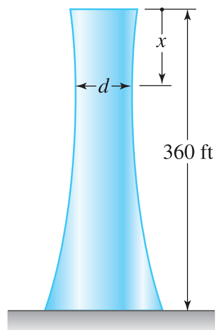


FIG. C1.5

2 Strain

Andrew Brookes, National Physical Laboratory/Photo Researchers, Inc

An assortment of tensile test specimens. The tensile test is a standard procedure for determining the mechanical properties of materials. An important material property is the stress-strain diagram, which is discussed in this chapter. Courtesy of Andrew Brookes, National Physical Laboratory/Photo Researchers, Inc.

2.1 Introduction

So far, we have dealt mainly with the strength, or load-carrying capacity, of structural members. Here we begin our study of an equally important topic of mechanics of materials—deformations, or strains. In general terms, *strain* is a geometric quantity that measures the deformation of a body. There are two types of strain: *normal strain*, which characterizes dimensional changes, and *shear strain*, which describes distortion (changes in angles). Stress and strain are two fundamental concepts of mechanics of materials. Their relationship to each other defines the mechanical properties of a material, the knowledge of which is of the utmost importance in design.

Although our emphasis in this chapter will be on axially loaded bars, the principles and methods developed here apply equally well to more complex cases of loading discussed later. Among other topics, we will learn how to use force-deformation relationships in conjunction with equilibrium analysis to solve statically indeterminate problems.

2.2 Axial Deformation; Stress-Strain Diagram

The strength of a material is not the only criterion that must be considered when designing machine parts or structures. The stiffness of a material is often equally important, as are mechanical properties such as hardness, toughness, and ductility. These properties are determined by laboratory tests. Many materials, particularly metals, have established standards that describe the test procedures in detail. We will confine our attention to only one of the tests—the tensile test of steel—and use its results to illustrate several important concepts of material behavior.

a. Normal (axial) strain

Before describing the tensile test, we must formalize the definition of normal (axial) strain. We begin by considering the elongation of the prismatic bar of length L in Fig. 2.1. The elongation δ may be caused by an applied axial force, or an expansion due to an increase in temperature, or even a force and a temperature increase acting simultaneously. Strain describes the geometry of deformation, independent of what actually causes the deformation. The normal strain ϵ (lowercase Greek *epsilon*) is defined as the *elongation per unit length*. Therefore, the *normal strain* in the bar in the axial direction, also known as the *axial strain*, is

$$\epsilon = \frac{\delta}{L} \tag{2.1}$$

If the bar deforms uniformly, then Eq. (2.1) represents the axial strain everywhere in the bar. Otherwise, this expression should be viewed as the *average axial strain*. Note that normal strain, being elongation per unit length, is

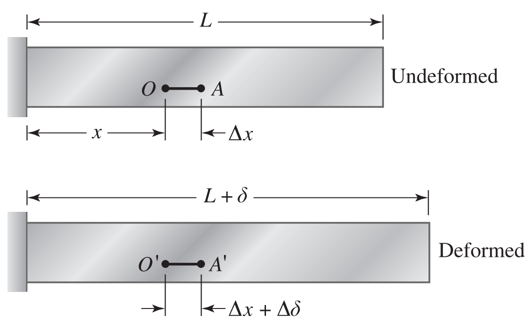


FIG. 2.1 Deformation of a prismatic bar.

a dimensionless quantity. However, "units" such as in./in. or mm/mm are frequently used for normal strain.

If the deformation is not uniform, we must define strain *at a point*. In Fig. 2.1, we let O be a point in the bar located at the distance x from the fixed end. To determine the axial strain at point O, we consider the deformation of an imaginary line element (fiber) OA of length Δx that is embedded in the bar at O. Denoting the elongation of OA by $\Delta\delta$, we define the *axial strain* at point O as

$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \tag{2.2}$$

Observe that normal strain, like normal stress, is defined *at a point in a given direction*.

We note that if the distribution of the axial strain is known, the elongation of the bar can be computed from

$$\delta = \int_0^L d\delta = \int_0^L \epsilon\, dx \tag{2.3}$$

For uniform strain distribution (the axial strain is the same at all points), Eq. (2.3) yields $\delta = \epsilon L$, which agrees with Eq. (2.1).

Although the preceding discussion assumed elongation, the results are also applicable to compression. By convention, compression (shortening) carries a negative sign. For example $\epsilon = -0.001$ means a compressive strain of magnitude 0.001.

b. *Tension test*

In the standard tension test, the specimen shown in Fig. 2.2 is placed in the grips of a testing machine. The grips are designed so that the load P applied by the machine is axial. Two gage marks are scribed on the specimen to define the *gage length* L. These marks are located away from the ends to avoid the local effects caused by the grips and to ensure that the stress and strain are uniform in the material between the marks.

The testing machine elongates the specimen at a slow, constant rate until the specimen ruptures. During the test, continuous readings are taken of the applied load and the elongation of the gage length. These data are then converted to stress and strain. The stress is obtained from $\sigma = P/A$, where P is the load and A represents the original cross-sectional area of the specimen. The strain is computed from $\epsilon = \delta/L$, where δ is the elongation

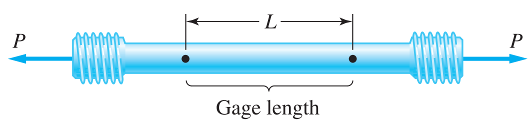


FIG. 2.2 Specimen used in the standard tension test.

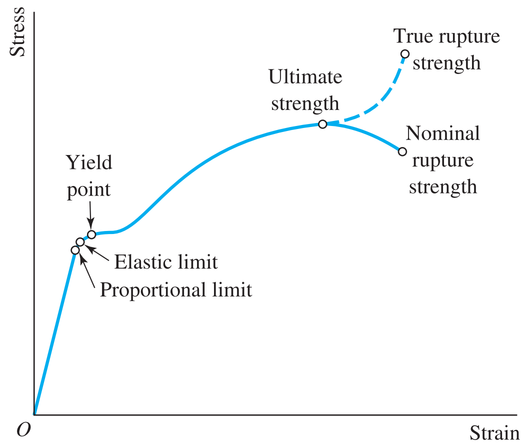


FIG. 2.3 Stress-strain diagram obtained from the standard tension test on a structural steel specimen.

between the gage marks and L is the original gage length. These results, which are based on the *original area* and the *original gage length*, are referred to as *nominal stress* and *nominal strain*.

As the bar is being stretched, its cross-sectional area becomes smaller and the length between the gage marks increases. Dividing the load by the actual (current) area of the specimen, we get the *true stress*. Similarly, the *true strain* is obtained by dividing the elongation δ by the current gage length. The nominal and true measures are essentially the same in the working range of metals. They differ only for very large strains, such as occur in rubber-like materials or in ductile metals just before rupture. With only a few exceptions, engineering applications use nominal stress and strain.

Plotting axial stress versus axial strain results in a *stress-strain diagram*. If the test is carried out properly, the stress-strain diagram for a given material is independent of the dimensions of the test specimen. That is, the characteristics of the diagram are determined solely by the mechanical properties of the material. A stress-strain diagram for structural steel is shown in Fig. 2.3. The following mechanical properties can be determined from the diagram.

Proportional Limit and Hooke's Law As seen in Fig. 2.3, the stress-strain diagram is a straight line from the origin O to a point called the *proportional limit*. This plot is a manifestation of *Hooke's law*:[1] Stress is proportional to strain; that is,

$$\sigma = E\epsilon \tag{2.4}$$

where E is a material property known as the *modulus of elasticity* or *Young's modulus*. The units of E are the same as the units of stress—that is, Pa or psi. For steel, $E = 29 \times 10^6$ psi, or 200 GPa, approximately. Note that Hooke's

[1] This law was first postulated by Robert Hooke in 1678.

law does not apply to the entire diagram; its validity ends at the proportional limit. Beyond this point, stress is no longer proportional to strain.[2]

Elastic Limit A material is said to be *elastic* if, after being loaded, the material returns to its original shape when the load is removed. The *elastic limit* is, as its name implies, the stress beyond which the material is no longer elastic. The permanent deformation that remains after the removal of the load is called the *permanent set*. The elastic limit is slightly larger than the proportional limit. However, because of the difficulty in determining the elastic limit accurately, it is usually assumed to coincide with the proportional limit.

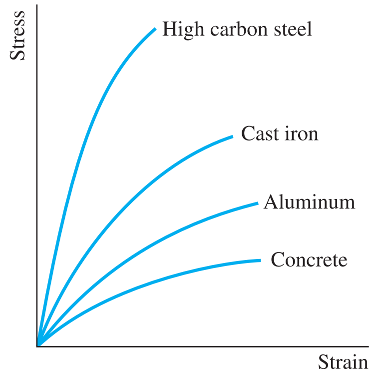


FIG. 2.4 Stress-strain diagrams for various materials that fail without significant yielding.

Yield Point The point where the stress-strain diagram becomes almost horizontal is called the *yield point*, and the corresponding stress is known as the *yield stress* or *yield strength*. Beyond the yield point there is an appreciable elongation, or yielding, of the material without a corresponding increase in load. Indeed, the load may actually decrease while the yielding occurs. However, the phenomenon of yielding is unique to structural steel. Other grades of steel, steel alloys, and other materials do not yield, as indicated by the stress-strain curves of the materials shown in Fig. 2.4. Incidentally, these curves are typical for a first loading of materials that contain appreciable residual stresses produced by manufacturing or aging processes. After repeated loading, these residual stresses are removed and the stress-strain curves become practically straight lines.

For materials that do not have a well-defined yield point, yield stress is determined by the *offset method*. This method consists of drawing a line parallel to the initial tangent of the stress-strain curve; this line starts at a prescribed offset strain, usually 0.2% ($\epsilon = 0.002$). The intersection of this line with the stress-strain curve, shown in Fig. 2.5, is called the *yield point at 0.2% offset*.

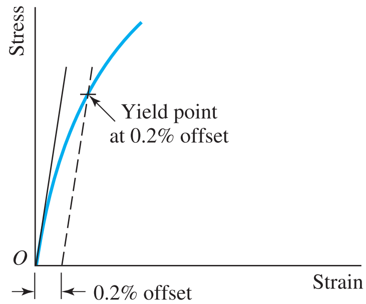


FIG. 2.5 Determining the yield point by the 0.2% offset method.

Ultimate Stress The *ultimate stress* or *ultimate strength*, as it is often called, is the highest stress on the stress-strain curve.

Rupture Stress The *rupture stress* or *rupture strength* is the stress at which failure occurs. For structural steel, the nominal rupture strength is considerably lower than the ultimate strength because the nominal rupture strength is computed by dividing the load at rupture by the original cross-sectional area. The true rupture strength is calculated using the reduced area of the cross section where the fracture occurred. The difference in the two values results from a phenomenon known as *necking*. As failure approaches, the material stretches very rapidly, causing the cross section to narrow, as shown in Fig. 2.6. Because the area where rupture occurs is smaller than the original area, the true rupture strength is larger than the ultimate strength. However, the ultimate strength is commonly used as the maximum stress that the material can carry.

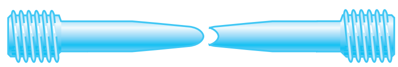


FIG. 2.6 Failed tensile test specimen showing necking, or narrowing, of the cross section.

[2] The stress-strain diagram of many materials is actually a curve on which there is no definite proportional limit. In such cases, the stress-strain proportionality is assumed to exist up to a stress at which the strain increases at a rate 50% greater than shown by the initial tangent to the stress-strain diagram.

c. Working stress and factor of safety

The *working stress* σ_w, also called the *allowable stress*, is the maximum safe axial stress used in design. In most designs, the working stress should be limited to values not exceeding the proportional limit so that the stresses remain in the elastic range (the straight-line portion of the stress-strain diagram). However, because the proportional limit is difficult to determine accurately, it is customary to base the working stress on either the yield stress σ_{yp} or the ultimate stress σ_{ult}, divided by a suitable number N, called the *factor of safety*. Thus,

$$\sigma_w = \frac{\sigma_{yp}}{N} \quad \text{or} \quad \sigma_w = \frac{\sigma_{ult}}{N} \tag{2.5}$$

The yield point is selected as the basis for determining σ_w in structural steel because it is the stress at which a prohibitively large permanent set may occur. For other materials, the working stress is usually based on the ultimate strength.

Many factors must be considered when selecting the working stress. This selection should not be made by the novice; usually the working stress is set by a group of experienced engineers and is embodied in building codes and specifications. A discussion of the factors governing the selection of a working stress starts with the observation that in many materials the proportional limit is about one-half the ultimate strength. To avoid accidental overloading, a working stress of one-half the proportional limit is usually specified for dead loads that are gradually applied. (The term *dead load* refers to the weight of the structure and other loads that, once applied, are not removed.) A working stress set in this way corresponds to a factor of safety of 4 with respect to σ_{ult} and is recommended for materials that are known to be uniform and homogeneous. For other materials, such as wood, in which unpredictable nonuniformities (such as knotholes) may occur, larger factors of safety are used. The dynamic effect of suddenly applied loads also requires higher factors of safety.

2.3 Axially Loaded Bars

Figure 2.7 shows a bar of length L and constant cross-sectional area A that is loaded by an axial tensile force P. We assume that the stress caused by P is below the proportional limit, so that Hooke's law $\sigma = E\epsilon$ is applicable. Because the bar deforms uniformly, the axial strain is $\epsilon = \delta/L$, which upon

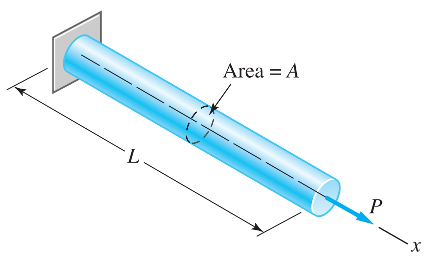


FIG. 2.7 Axially loaded bar.

substitution into Hooke's law yields $\sigma = E(\delta/L)$. Therefore, the elongation of the bar is

$$\delta = \frac{\sigma L}{E} = \frac{PL}{EA} \tag{2.6}$$

where in the last step we substituted $\sigma = P/A$. If the strain (or stress) in the bar is not uniform, then Eq. (2.6) is invalid. In the case where the axial strain varies with the x-coordinate, the elongation of the bar can be obtained by integration, as stated in Eq. (2.3): $\delta = \int_0^L \epsilon\, dx$. Using $\epsilon = \sigma/E = P/(EA)$, where P is the *internal* axial force, we get

$$\delta = \int_0^L \frac{\sigma}{E}\, dx = \int_0^L \frac{P}{EA}\, dx \tag{2.7}$$

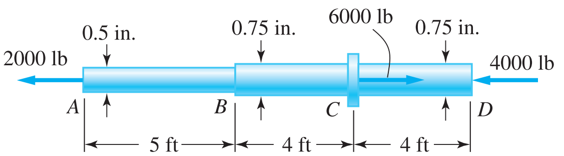
We see that Eq. (2.7) reduces to Eq. (2.6) only if P, E, and A are constants.

Notes on the Computation of Deformation

- The magnitude of the internal force P in Eqs. (2.6) and (2.7) must be found from equilibrium analysis. Note that a positive (tensile) P results in positive δ (elongation); conversely, a negative P (compression) gives rise to negative δ (shortening).
- Care must be taken to use consistent units in Eqs. (2.6) and (2.7). It is common practice to let the units of E determine the units to be used for P, L, and A. In the U.S. Customary system, E is expressed in psi (lb/in.^{2}), so that the units of the other variables should be P [lb], L [in.], and A [in.^{2}]. In the SI system, where E is in Pa (N/m^2), the consistent units are P [N], L [m], and A [m^2].
- As long as the axial stress is in the elastic range, the elongation (or shortening) of a bar is very small compared to its length. This property can be utilized to simplify the computation of displacements in structures containing axially loaded bars, such as trusses.

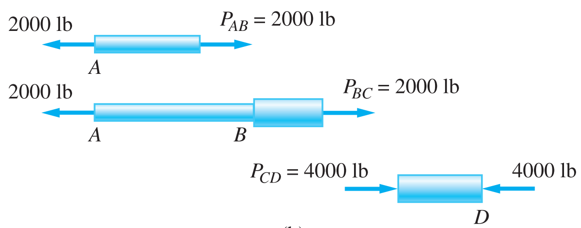
Sample Problem 2.1

The steel propeller shaft $ABCD$ carries the axial loads shown in Fig. (a). Determine the change in the length of the shaft caused by these loads. Use $E = 29 \times 10^6$ psi for steel.



(a)

Solution



(b)

From the free-body diagrams in Fig. (b) we see that the internal forces in the three segments of the shaft are

$$P_{AB} = P_{BC} = 2000 \text{ lb (T)} \qquad P_{CD} = 4000 \text{ lb (C)}$$

Because the axial force and the cross-sectional area are constant within each segment, the changes in the lengths of the segments can be computed from Eq. (2.6): $\delta = PL/(EA)$. The change in the length of the shaft is obtained by adding the contributions of the segments. Noting that tension causes elongation and compression results in shortening, we obtain for the elongation of the shaft

$$\delta = \sum \frac{PL}{EA} = \frac{1}{E}\left[\left(\frac{PL}{A}\right)_{AB} + \left(\frac{PL}{A}\right)_{BC} - \left(\frac{PL}{A}\right)_{CD}\right]$$

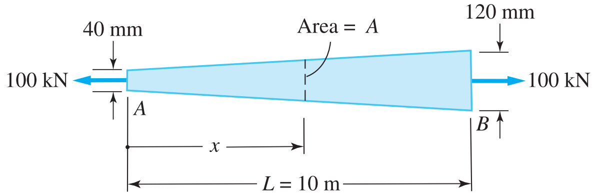
$$= \frac{1}{29 \times 10^6}\left[\frac{2000(5 \times 12)}{\pi(0.5)^2/4} + \frac{2000(4 \times 12)}{\pi(0.75)^2/4} - \frac{4000(4 \times 12)}{\pi(0.75)^2/4}\right]$$

$$= 0.013\,58 \text{ in.} \quad \text{(elongation)}$$

Answer

Sample Problem 2.2

The cross section of the 10-m-long flat steel bar AB has a constant thickness of 20 mm, but its width varies as shown in the figure. Calculate the elongation of the bar due to the 100-kN axial load. Use $E = 200$ GPa for steel.



Solution

Equilibrium requires that the internal axial force $P = 100$ kN is constant along the entire length of the bar. However, the cross-sectional area A of the bar varies with the x-coordinate, so that the elongation of the bar must be computed from Eq. (2.7).

We start by determining A as a function of x. The cross-sectional areas at A and B are $A_A = 20 \times 40 = 800\ \text{mm}^2$ and $A_B = 20 \times 120 = 2400\ \text{mm}^2$. Between A and B the cross-sectional area is a linear function of x:

$$A = A_A + (A_B - A_A)\frac{x}{L} = 800\ \text{mm}^2 + (1600\ \text{mm}^2)\frac{x}{L}$$

Converting the areas from mm^2 to m^2 and substituting $L = 10$ m, we get

$$A = (800 + 160x) \times 10^{-6}\ \text{m}^2 \qquad \text{(a)}$$

Substituting Eq. (a) together with $P = 100 \times 10^3$ N and $E = 200 \times 10^9$ Pa into Eq. (2.7), we obtain for the elongation of the rod

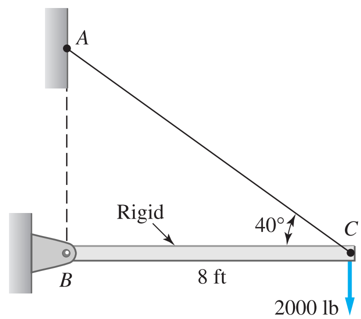
$$\delta = \int_0^L \frac{P}{EA}\,dx = \int_0^{10\ \text{m}} \frac{100 \times 10^3}{(200 \times 10^9)[(800 + 160x) \times 10^{-6}]}\,dx$$

$$= 0.5 \int_0^{10\ \text{m}} \frac{dx}{800 + 160x} = \frac{0.5}{160}\left[\ln(800 + 160x)\right]_0^{10}$$

$$= \frac{0.5}{160} \ln \frac{2400}{800} = 3.43 \times 10^{-3}\ \text{m} = 3.43\ \text{mm} \qquad \textit{Answer}$$

Sample Problem **2.3**

The rigid bar BC in Fig. (a) is supported by the steel rod AC of cross-sectional area 0.25 in.^{2}. Find the vertical displacement of point C caused by the 2000-lb load. Use $E = 29 \times 10^6$ psi for steel.



(a)

Solution

We begin by computing the axial force in rod AC. Noting that bar BC is a two-force body, the FBD of joint C in Fig. (b) yields

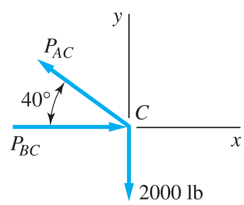
$$\Sigma F_y = 0 \quad +\uparrow \quad P_{AC} \sin 40^\circ - 2000 = 0 \qquad P_{AC} = 3111\ \text{lb}$$

The elongation of AC can now be obtained from Eq. (2.6). Noting that the length of the rod is

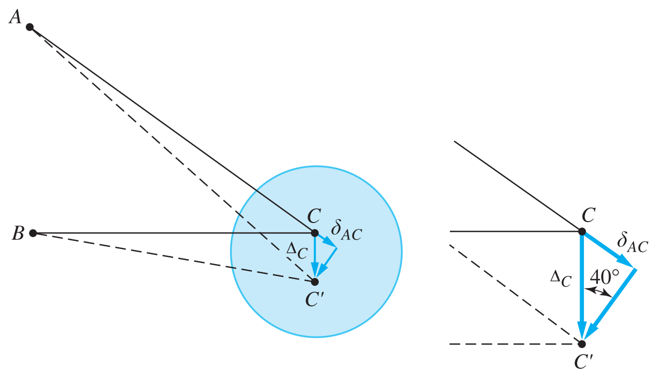
$$L_{AC} = \frac{L_{BC}}{\cos 40^\circ} = \frac{8 \times 12}{\cos 40^\circ} = 125.32\ \text{in.}$$

we get

$$\delta_{AC} = \left(\frac{PL}{EA}\right)_{AC} = \frac{3111(125.32)}{(29 \times 10^6)(0.25)} = 0.053\,78\ \text{in.} \quad \text{(elongation)}$$



(b)



(c)

The geometric relationship between δ_{AC} and the displacement Δ_C of C is illustrated in the displacement diagram in Fig. (c). Because bar BC is rigid, the movement of point C is confined to a circular arc centered at B. Observing that the displacements are very small relative to the lengths of the bars, this arc is practically the straight line CC', perpendicular to BC. Having established the direction of Δ_C, we now resolve Δ_C into components that are parallel and perpendicular to AC. The perpendicular component is due to the rotation of bar AC about A, whereas the parallel component is the elongation of AC. From geometry, the enlarged portion of the displacement diagram in Fig. (c) yields

$$\Delta_C = \frac{\delta_{AC}}{\sin 40^\circ} = \frac{0.053\,78}{\sin 40^\circ} = 0.0837 \text{ in.} \downarrow \qquad \textit{Answer}$$

Problems

2.1 The following data were recorded during a tensile test of a 14.0-mm-diameter mild steel rod. The gage length was 50.0 mm.

Load (N)	Elongation (mm)	Load (N)	Elongation (mm)
0	0	46 200	1.25
6 310	0.010	52 400	2.50
12 600	0.020	58 500	4.50
18 800	0.030	65 400	7.50
25 100	0.040	69 000	12.50
31 300	0.050	67 800	15.50
37 900	0.060	65 000	20.00
40 100	0.163	61 500	Fracture
41 600	0.433		

Plot the stress-strain diagram and determine the following mechanical properties: (a) proportional limit; (b) modulus of elasticity; (c) yield stress; (d) ultimate stress; and (e) nominal rupture stress.

2.2 The following data were obtained during a tension test of an aluminum alloy. The initial diameter of the test specimen was 0.505 in., and the gage length was 2.0 in.

Load (lb)	Elongation (in.)	Load (lb)	Elongation (in.)
0	0	14 000	0.020
2 310	0.0022	14 400	0.025
4 640	0.0044	14 500	0.060
6 950	0.0066	14 600	0.080
9 290	0.0088	14 800	0.100
11 600	0.0110	14 600	0.120
13 000	0.0150	13 600	Fracture

Plot the stress-strain diagram and determine the following mechanical properties: (a) proportional limit; (b) modulus of elasticity; (c) yield stress at 0.2% offset; (d) ultimate stress; and (e) nominal rupture stress.

2.3 The bar ABC in Fig. (a) consists of two cylindrical segments. The material of the bar has the stress-strain diagram shown in Fig. (b). Determine the approximate elongation of the bar caused by the 20-kN axial load.

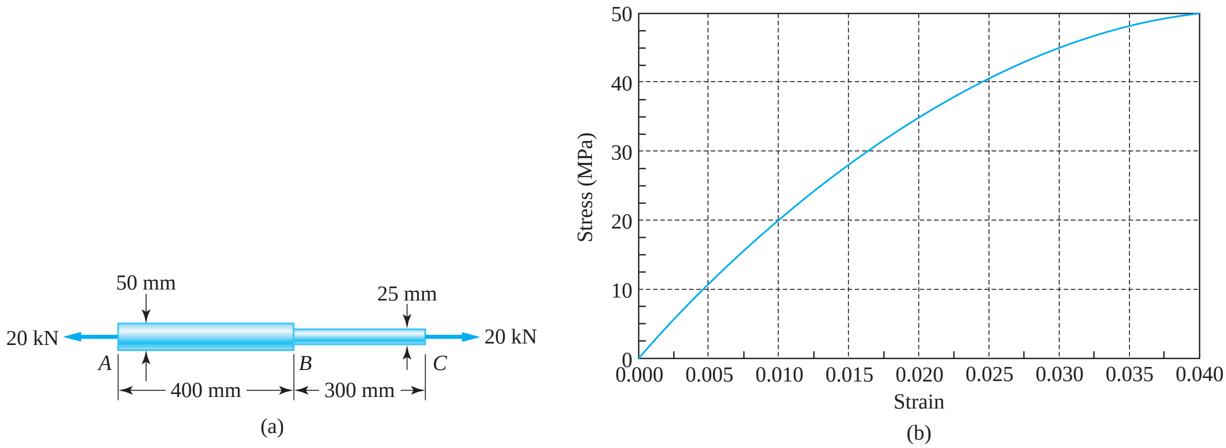


FIG. P2.3

2.4 A uniform bar of length L, cross-sectional area A, and mass density ρ is suspended vertically from one end. (a) Show that the elongation of the bar is $\delta = \rho g L^2/(2E)$, where g is the gravitational acceleration and E is the modulus of elasticity. (b) If the mass of the bar is M, show that $\delta = MgL/(2EA)$.

2.5 A steel rod having a cross-sectional area of 300 mm^2 and a length of 150 m is suspended vertically from one end. The rod supports a tensile load of 20 kN at its free end. Given that the mass density of steel is 7850 kg/m^3 and $E = 200$ GPa, find the total elongation of the rod. (*Hint:* Use the results of Prob. 2.4.)

2.6 Determine the elongation of the tapered cylindrical aluminum bar caused by the 30-kN axial load. Use $E = 72$ GPa.

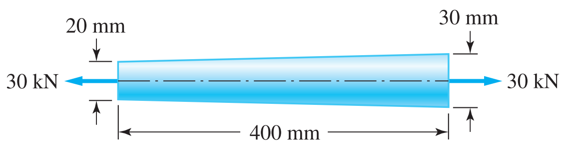


FIG. P2.6

2.7 The steel strip has a uniform thickness of 50 mm. Compute the elongation of the strip caused by the 500-kN axial force. The modulus of elasticity of steel is 200 GPa.

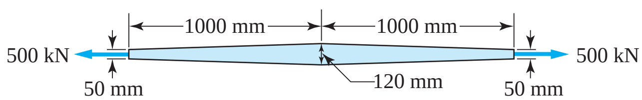


FIG. P2.7

2.8 A 4-mm-diameter steel wire, 3.2 m long, carries an axial tensile load P. Find the maximum safe value of P if the allowable normal stress is 280 MPa and the elongation of the wire is limited to 4 mm. Use $E = 200$ GPa.

2.9 The compound bar $ABCD$ has a uniform cross-sectional area of 0.25 in.^{2} When the axial force P is applied, the length of the bar is reduced by 0.018 in. Determine the magnitude of the force P. The moduli of elasticity are 29×10^6 psi for steel and 10×10^6 psi for aluminum.

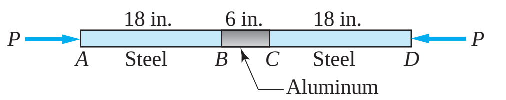


FIG. P2.9

2.10 The steel rod is placed inside the copper tube, the length of each being exactly 15 in. If the assembly is compressed by 0.0075 in., determine the stress in each component and the applied force P. The moduli of elasticity are 29×10^6 psi for steel and 17×10^6 psi for copper.

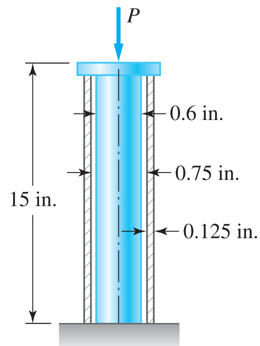


FIG. P2.10

2.11 A steel hoop, 10 mm thick and 80 mm wide, with inside diameter 1500.0 mm, is heated and shrunk onto a steel cylinder 1500.5 mm in diameter. What is the normal force in the hoop after it has cooled? Neglect the deformation of the cylinder, and use $E = 200$ GPa for steel.

2.12 The timber member has a cross-sectional area of 1750 mm^2 and its modulus of elasticity is 12 GPa. Compute the change in the total length of the member after the loads shown are applied.

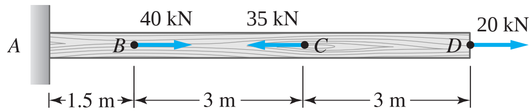


FIG. P2.12

2.13 The member consists of the steel rod AB that is screwed into the end of the bronze rod BC. Find the largest value of P that meets the following design criteria: (i) the overall length of the member is not to change by more than 3 mm; and (ii) the stresses are not to exceed 140 MPa in steel and 120 MPa in bronze. The moduli of elasticity are 200 GPa for steel and 80 GPa for bronze.

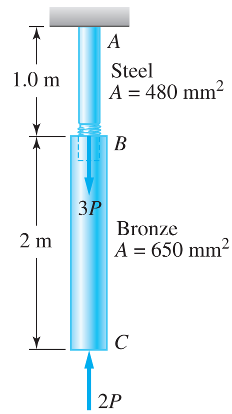


FIG. P2.13

2.14 The compound bar carries the axial forces P and $2P$. Find the maximum allowable value of P if the working stresses are 40 ksi for steel and 20 ksi for aluminum, and the total elongation of the bar is not to exceed 0.2 in.

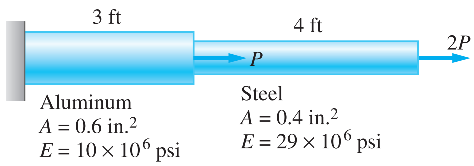


FIG. P2.14

2.15 The compound bar containing steel, bronze, and aluminum segments carries the axial loads shown in the figure. The properties of the segments and the working stresses are listed in the table.

	A (in.^{2})	E (psi)	σ_w (psi)
Steel	0.75	30×10^6	20 000
Bronze	1.00	12×10^6	18 000
Aluminum	0.50	10×10^6	12 000

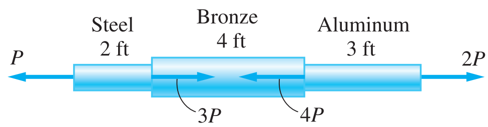


FIG. P2.15

Determine the maximum allowable value of P if the change in length of the entire bar is limited to 0.08 in. and the working stresses are not to be exceeded.

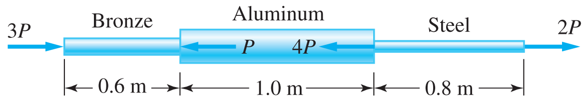


FIG. P2.16

2.16 A compound bar consisting of bronze, aluminum, and steel segments is loaded axially as shown in the figure. Determine the maximum allowable value of P if the change in length of the bar is limited to 2 mm and the working stresses prescribed in the table are not to be exceeded.

	A (mm^2)	E (GPa)	σ_w (MPa)
Bronze	450	83	120
Aluminum	600	70	80
Steel	300	200	140

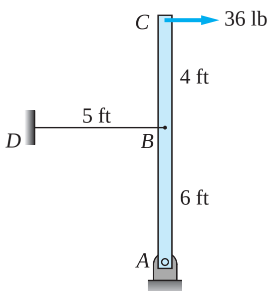


FIG. P2.17

2.17 The bar ABC is supported by a pin at A and a steel wire at B. Calculate the elongation of the wire when the 36-lb horizontal force is applied at C. The cross-sectional area of the wire is 0.0025 in.^{2} and the modulus of elasticity of steel is 29×10^6 psi.

2.18 The rigid bar AB is supported by two rods made of the same material. If the bar is horizontal before the load P is applied, find the distance x that locates the position where P must act if the bar is to remain horizontal. Neglect the weight of bar AB.

2.19 The rigid bar ABC is supported by a pin at A and a steel rod at B. Determine the largest vertical load P that can be applied at C if the stress in the steel rod is limited to 35 ksi and the vertical movement of end C must not exceed 0.12 in. Neglect the weights of the members.

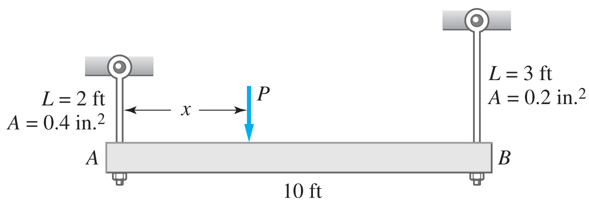


FIG. P2.18

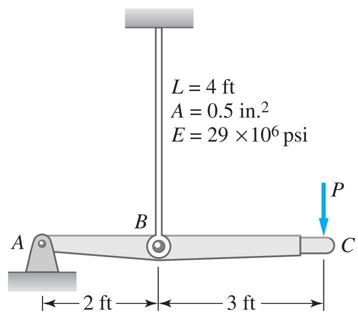


FIG. P2.19

2.20 The rigid bar AB, attached to aluminum and steel rods, is horizontal before the load P is applied. Find the vertical displacement of point C caused by the load $P = 50$ kN. Neglect all weights.

2.21 The rigid bars ABC and CD are supported by pins at A and D and by a steel rod at B. There is a roller connection between the bars at C. Compute the vertical displacement of point C caused by the 50-kN load.

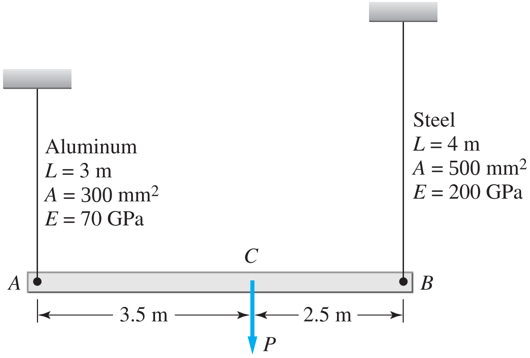


FIG. P2.20

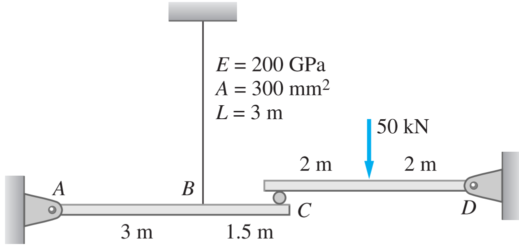


FIG. P2.21

2.22 The structure in the figure is composed of two rigid bars (AB and CD) and two vertical rods made of aluminum and steel. All connections are pin joints. Determine the maximum force P that can be applied to the structure if the vertical displacement of its point of application is limited to 6 mm. Neglect the weights of the members.

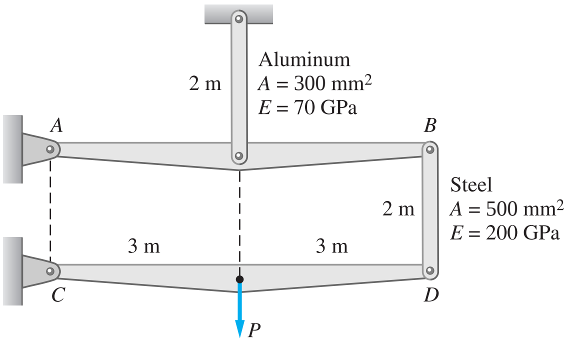


FIG. P2.22

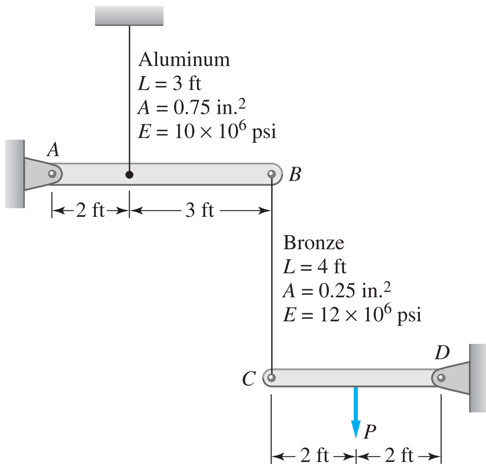


FIG. P2.23

2.23 The rigid bars AB and CD are supported by pins at A and D. The vertical rods are made of aluminum and bronze. Determine the vertical displacement of the point where the force $P = 10$ kips is applied. Neglect the weights of the members.

2.24 The uniform 2200-lb bar BC is supported by a pin at C and the aluminum wire AB. The cross-sectional area of the wire is 0.165 in.^{2}. Assuming bar BC to be rigid, find the vertical displacement of B due to the weight of the bar. Use $E = 10.6 \times 10^6$ psi for aluminum.

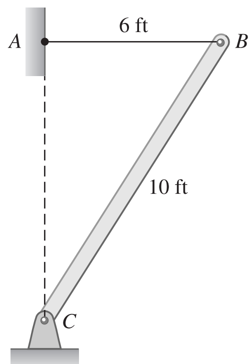


FIG. P2.24

2.25 The steel bars *AC* and *BC*, each of cross-sectional area 120 mm^2, are joined at *C* with a pin. Determine the displacement of point *C* caused by the 15-kN load. Use $E = 200$ GPa for steel.

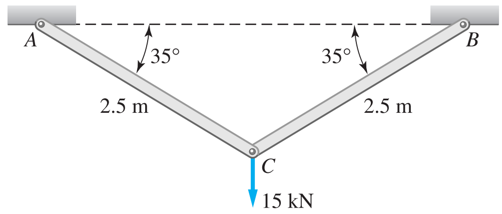


FIG. P2.25, P2.26

2.26 Solve Prob. 2.25 if the 15-kN load acts horizontally to the right.

2.27 The steel truss supports a 6000-lb load. The cross-sectional areas of the members are 0.5 $in.^2$ for *AB* and 0.75 $in.^2$ for *BC*. Compute the horizontal displacement of *B* using $E = 29 \times 10^6$ psi.

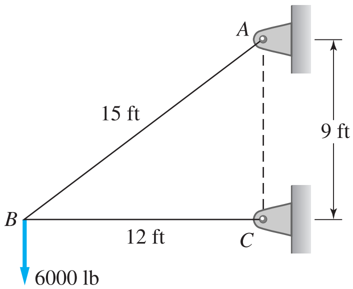


FIG. P2.27

2.4 Generalized Hooke's Law

a. Uniaxial loading; Poisson's ratio

Experiments show that when a bar is stretched by an axial force, there is a contraction in the transverse dimensions, as illustrated in Fig. 2.8. In 1811, Siméon D. Poisson showed that the ratio of the transverse strain to the axial strain is constant for stresses within the proportional limit. This constant, called *Poisson's ratio*, is denoted by ν (lowercase Greek *nu*). For uniaxial loading in the x-direction, as in Fig 2.8, Poisson's ratio is $\nu = -\epsilon_t/\epsilon_x$, where ϵ_t is the transverse strain. The minus sign indicates that a positive strain (elongation) in the axial direction causes a negative strain (contraction) in the transverse directions. The transverse strain is uniform throughout the cross section and is the *same in any direction in the plane of the cross section*. Therefore, we have for uniaxial loading

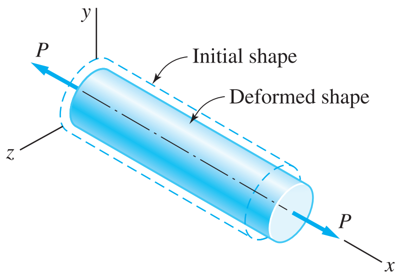


FIG. 2.8 Transverse dimensions contract as the bar is stretched by an axial force P.

$$\boxed{\epsilon_y = \epsilon_z = -\nu\epsilon_x} \tag{2.8}$$

Poisson's ratio is a dimensionless quantity that ranges between 0.25 and 0.33 for metals.

Using $\sigma_x = E\epsilon_x$ in Eq. (2.8) yields the generalized Hooke's law for uniaxial loading ($\sigma_y = \sigma_z = 0$):

$$\boxed{\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\nu\frac{\sigma_x}{E}} \tag{2.9}$$

b. Multiaxial Loading

Biaxial Loading Poisson's ratio permits us to extend Hooke's law for uniaxial loading to biaxial and triaxial loadings. Consider an element of the material that is subjected simultaneously to normal stresses in the x- and y-directions, as in Fig. 2.9(a). The strains caused by σ_x alone are given in Eqs. (2.9). Similarly, the strains due to σ_y are $\epsilon_y = \sigma_y/E$ and $\epsilon_x = \epsilon_z = -\nu\sigma_y/E$. Using superposition, we write the combined effect of the two normal stresses as

$$\boxed{\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)} \tag{2.10}$$

which is Hooke's law for biaxial loading in the xy-plane ($\sigma_z = 0$). The first two of Eqs. (2.10) can be inverted to express the stresses in terms of the strains:

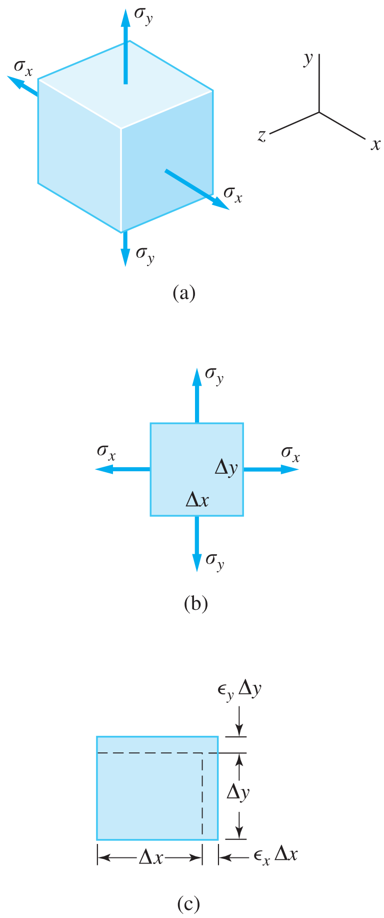


FIG. 2.9 (a) Stresses acting on a material element in biaxial loading; (b) two-dimensional view of stresses; (c) deformation of the element.

$$\boxed{\sigma_x = \frac{(\epsilon_x + \nu\epsilon_y)E}{1-\nu^2} \qquad \sigma_y = \frac{(\epsilon_y + \nu\epsilon_x)E}{1-\nu^2}} \tag{2.11}$$

Two-dimensional views of the stresses and the resulting deformation in the xy-plane are shown in Figs. 2.9(b) and (c). Note that Eqs. (2.10) show that for biaxial loading ϵ_z is not zero; that is, the strain is triaxial rather than biaxial.

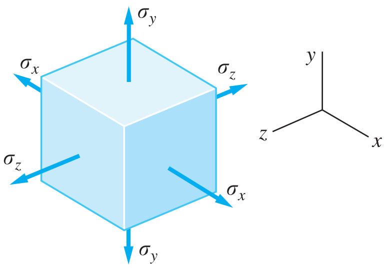


FIG. 2.10 Stresses acting on a material element in triaxial loading.

Triaxial Loading Hooke's law for the triaxial loading in Fig. 2.10 is obtained by adding the contribution of σ_z, $\epsilon_z = \sigma_z/E$ and $\epsilon_x = \epsilon_y = -\nu\sigma_z/E$, to the strains in Eqs. (2.10), which yields

$$\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \tag{2.12}$$

Equations (2.8)–(2.12) are valid for both tensile and compressive effects. It is only necessary to assign positive signs to elongations and tensile stresses and, conversely, negative signs to contractions and compressive stresses.

c. *Shear loading*

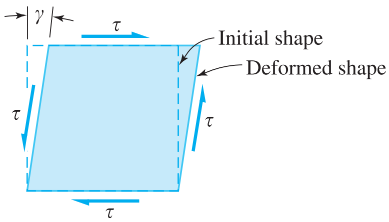


FIG. 2.11 Deformation of a material element caused by shear stress.

Shear stress causes the deformation shown in Fig. 2.11. The lengths of the sides of the element do not change, but the element undergoes a distortion from a rectangle to a parallelogram. The *shear strain*, which measures the amount of distortion, is the angle γ (lowercase Greek *gamma*), always expressed in radians. It can be shown that the relationship between shear stress τ and shear strain γ is linear within the elastic range; that is,

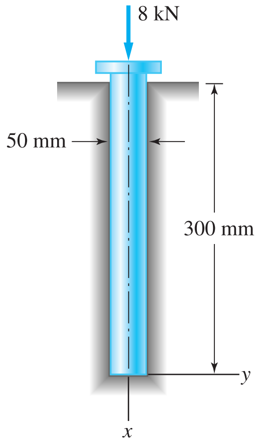
$$\tau = G\gamma \tag{2.13}$$

which is Hooke's law for shear. The material constant G is called the *shear modulus of elasticity* (or simply *shear modulus*), or the *modulus of rigidity*. The shear modulus has the same units as the modulus of elasticity (Pa or psi). We will prove later that G is related to the modulus of elasticity E and Poisson's ratio ν by

$$G = \frac{E}{2(1 + \nu)} \tag{2.14}$$

Sample Problem 2.4

The 50-mm-diameter rubber rod is placed in a hole with rigid, lubricated walls. There is no clearance between the rod and the sides of the hole. Determine the change in the length of the rod when the 8-kN load is applied. Use $E = 40$ MPa and $\nu = 0.45$ for rubber.



Solution

Lubrication allows the rod to contract freely in the axial direction, so that the axial stress throughout the bar is

$$\sigma_x = -\frac{P}{A} = -\frac{8000}{\frac{\pi}{4}(0.05)^2} = -4.074 \times 10^6 \text{ Pa}$$

(the negative sign implies compression). Because the walls of the hole prevent transverse strain in the rod, we have $\epsilon_y = \epsilon_z = 0$. The tendency of the rubber to expand laterally (Poisson's effect) is resisted by the uniform contact pressure p between the walls and the rod, so that $\sigma_y = \sigma_z = -p$. If we use the second of Eqs. (2.12) (the third equation would yield the same result), the condition $\epsilon_y = 0$ becomes

$$\frac{\sigma_y - \nu(\sigma_z + \sigma_x)}{E} = \frac{-p - \nu(-p + \sigma_x)}{E} = 0$$

which yields

$$p = -\frac{\nu\sigma_x}{1 - \nu} = -\frac{0.45(-4.074 \times 10^6)}{1 - 0.45} = 3.333 \times 10^6 \text{ Pa}$$

The axial strain is given by the first of Eqs. (2.12):

$$\epsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E} = \frac{\sigma_x - \nu(-2p)}{E}$$

$$= \frac{[-4.074 - 0.45(-2 \times 3.333)] \times 10^6}{40 \times 10^6} = -0.026\,86$$

The corresponding change in the length of the rod is

$$\delta = \epsilon_x L = -0.026\,86(300)$$

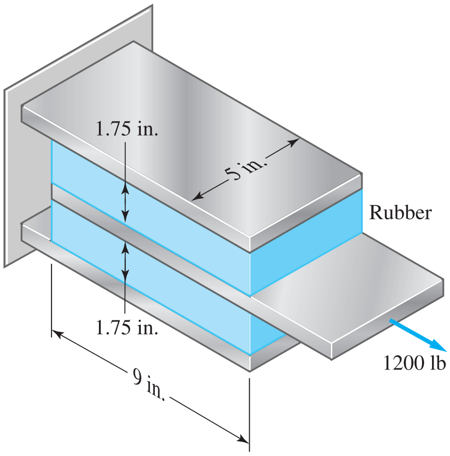
$$= -8.06 \text{ mm} = 8.06 \text{ mm} \quad \text{(contraction)}$$ *Answer*

For comparison, note that if the constraining effect of the hole were neglected, the deformation would be

$$\delta = -\frac{PL}{EA} = -\frac{8000(0.3)}{(40 \times 10^6)\left[\frac{\pi}{4}(0.05)^2\right]} = -0.0306 \text{ m} = -30.6 \text{ mm}$$

Sample Problem **2.5**

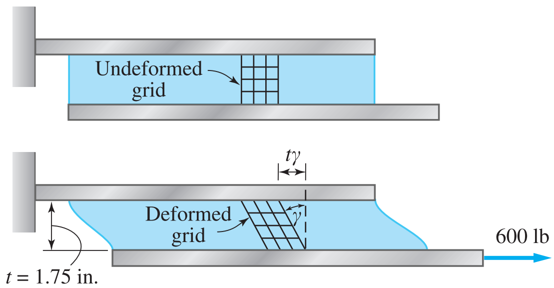
Two 1.75-in.-thick rubber pads are bonded to three steel plates to form the shear mount shown in Fig. (a). Find the displacement of the middle plate when the 1200-lb load is applied. Consider the deformation of rubber only. Use $E = 500$ psi and $\nu = 0.48$ for rubber.



(a)

Solution

To visualize the deformation of the rubber pads, we introduce a grid drawn on the edge of the upper pad—see Fig. (b). When the load is applied, the grid deforms as shown in the figure. Observe that the deformation represents uniform shear, except for small regions at the edges of the pad (Saint Venant's principle).



(b)

Each rubber pad has a shear area of $A = 5 \times 9 = 45$ in.^{2} that carries half the 1200-lb load. Hence, the average shear stress in the rubber is

$$\tau = \frac{V}{A} = \frac{600}{45} = 13.333 \text{ psi}$$

This stress is shown acting on the sides of a grid element in Fig. (c). The corresponding shear strain is $\gamma = \tau/G$, where from Eq. (2.14),

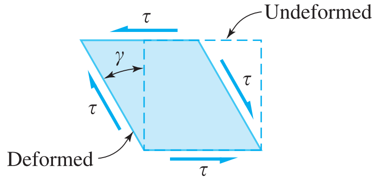
$$G = \frac{E}{2(1+\nu)} = \frac{500}{2(1+0.48)} = 168.92 \text{ psi}$$

Therefore,

$$\gamma = \frac{\tau}{G} = \frac{13.333}{168.92} = 0.07893$$

From Fig. (b) we see that the displacement of the middle plate (the lower plate in the figure) is

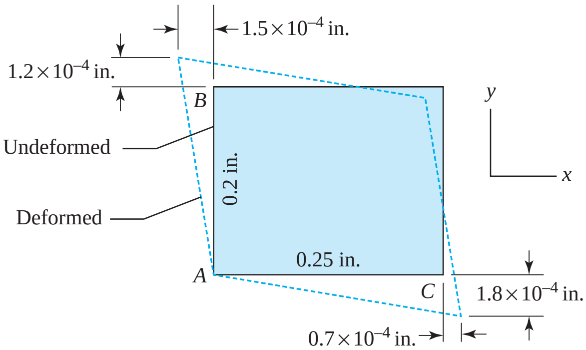
$$t\gamma = 1.75(0.078\,93) = 0.1381 \text{ in.} \qquad \textit{Answer}$$



(c)

Sample Problem **2.6**

An initially rectangular element of material is deformed as shown in the figure (note that the deformation is greatly exaggerated). Calculate the normal strains ε_x and ε_y, and the shear strain γ for the element.



Solution

The elongation of side AC is $\delta_{AC} = 0.7 \times 10^{-4}$ in. Therefore, the horizontal strain of the element is

$$\varepsilon_x = \frac{\delta_{AC}}{AC} = \frac{0.7 \times 10^{-4}}{0.25} = 280 \times 10^{-6} \qquad \textit{Answer}$$

The elongation of side AB is $\delta_{AB} = 1.2 \times 10^{-4}$ in., which yields for the vertical strain

$$\varepsilon_y = \frac{\delta_{AB}}{AB} = \frac{1.2 \times 10^{-4}}{0.2} = 600 \times 10^{-6} \qquad \textit{Answer}$$

The shear strain is the angle of distortion (change in the angle of a corner of the element), measured in radians. Referring to the corner at A, we have

$$\gamma = \text{rotation angle of } AC + \text{rotation angle of } AB$$

$$= \frac{1.8 \times 10^{-4}}{0.25} + \frac{1.5 \times 10^{-4}}{0.2} = 1470 \times 10^{-6} \qquad \textit{Answer}$$

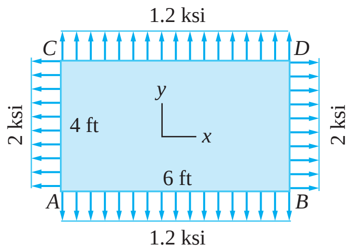


FIG. P2.29

Problems

2.28 A solid cylinder of diameter d carries an axial load P. Show that the change in diameter is $4P\nu/(\pi E d)$.

2.29 The polyethylene sheet is subjected to the biaxial loading shown. Determine the resulting elongations of sides AB and AC. The properties of polyethylene are $E = 300$ ksi and $\nu = 0.4$.

2.30 A sheet of copper is stretched biaxially in the xy-plane. If the strains in the sheet are $\epsilon_x = 0.40 \times 10^{-3}$ and $\epsilon_y = 0.30 \times 10^{-3}$, determine σ_x and σ_y. Use $E = 110$ GPa and $\nu = 0.35$.

2.31 The normal stresses at a point in a steel member are $\sigma_x = 8$ ksi, $\sigma_y = -4$ ksi, and $\sigma_z = 10$ ksi. Using $E = 29 \times 10^3$ ksi and $\nu = 0.3$, determine the normal strains at this point.

2.32 The rectangular block of material of length L and cross-sectional area A fits snugly between two rigid, lubricated walls. Derive the expression for the change in length of the block due to the axial load P.

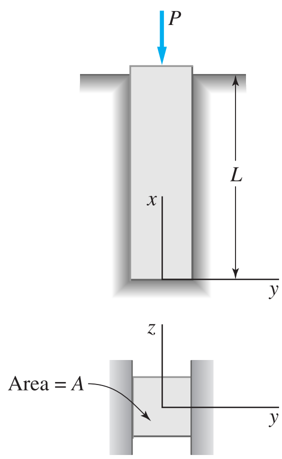


FIG. P2.32

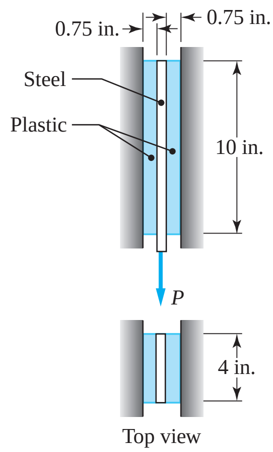


FIG. P2.33

2.33 The two sheets of soft plastic are bonded to the central steel strip. Determine the magnitude of the largest force P that can be safely applied to the steel strip and the corresponding displacement of the strip. For the plastic, use $\tau_w = 10$ ksi and $G = 800$ ksi. Neglect deformation of the steel strip.

2.34 A material specimen is subjected to a uniform, triaxial compressive stress (hydrostatic pressure) of magnitude p. Show that the volumetric strain of the material is $\Delta V/V = -3p(1-2\nu)/E$, where ΔV is the volume change and V is the initial volume.

2.35 A rubber sheet of thickness t and area A is compressed as shown in the figure. All contact surfaces are sufficiently rough to prevent slipping. Show that the change in the thickness of the rubber sheet caused by the load P is

$$\delta = \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \frac{Pt}{EA}$$

(*Hint*: The roughness of the surfaces prevents transverse expansion of the sheet.)

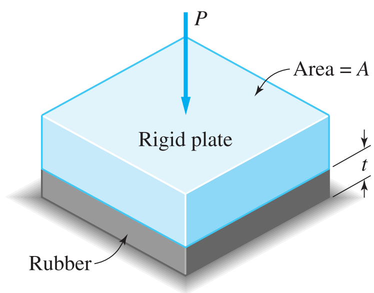


FIG. P2.35

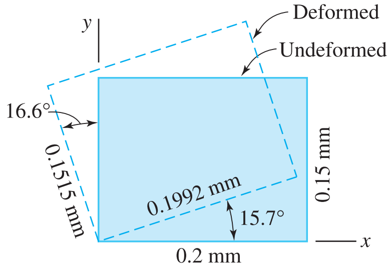


FIG. P2.37

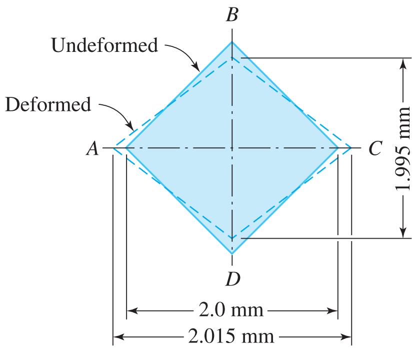


FIG. P2.38

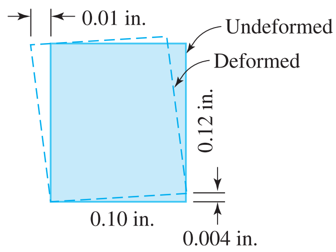


FIG. P2.39

2.36 A torsion test shows that the shear modulus of an aluminum specimen is 4.60×10^6 psi. When the same specimen is used in a tensile test, the modulus of elasticity is found to be 12.2×10^6 psi. Find Poisson's ratio for the specimen.

2.37 An initially rectangular element of a material is deformed into the shape shown in the figure. Find ϵ_x, ϵ_y, and γ for the element.

2.38 The initially square element of a material is deformed as shown. Determine the shear strain of the element and the normal strains of the diagonals AC and BD.

2.39 The rectangular element is deformed in shear as shown. Find the shear strain.

2.40 The square element of a material undergoes the shear strain γ. Assuming that $\gamma \ll 1$, determine the normal strains of the diagonals AC and BD.

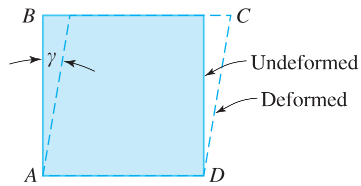


FIG. P2.40

2.41 The plastic sheet, 1/2 in. thick, is bonded to the pin-jointed steel frame. Determine the magnitude of the force P that would result in a 0.18-in. horizontal displacement of bar AB. Use $G = 70 \times 10^3$ psi for the plastic, and neglect the deformation of the steel frame.

2.42 The steel shaft of diameter D is cemented to the thin rubber sleeve of thickness t and length L. The outer surface of the sleeve is bonded to a rigid support. When the axial load P is applied, show that the axial displacement of the shaft is $\delta = Pt/(\pi GDL)$, where G is the shear modulus of rubber. Assume that $t \ll D$.

2.43 Show that if the rubber sleeve in Prob. 2.42 is thick, the displacement of the shaft is

$$\delta = \frac{P}{2\pi GL} \ln \frac{D + 2t}{D}$$

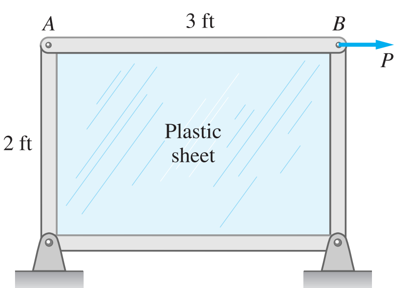


FIG. P2.41

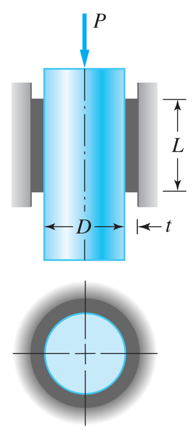


FIG. P2.42, P2.43

2.5 Statically Indeterminate Problems

If the equilibrium equations are sufficient to calculate all the forces (including support reactions) that act on a body, these forces are said to be *statically determinate*. In statically determinate problems, the number of unknown forces is always equal to the number of independent equilibrium equations. If the number of unknown forces exceeds the number of independent equilibrium equations, the problem is said to be *statically indeterminate*.

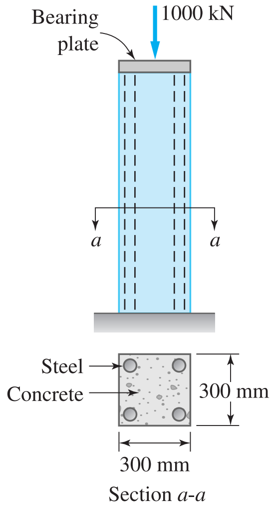
Static indeterminacy does not imply that the problem cannot be solved; it simply means that the solution cannot be obtained from the equilibrium equations alone. A statically indeterminate problem always has geometric restrictions imposed on its deformation. The mathematical expressions of these restrictions, known as the *compatibility equations*, provide us with the additional equations needed to solve the problem (the term *compatibility* refers to the geometric compatibility between deformation and the imposed constraints). Because the source of the compatibility equations is deformation, these equations contain as unknowns either strains or elongations. We can, however, use Hooke's law to express the deformation measures in terms of stresses or forces. The equations of equilibrium and compatibility can then be solved for the unknown forces.

Procedure for Solving Statically Indeterminate Problems In summary, the solution of a statically indeterminate problem involves the following steps:

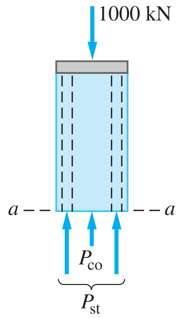
- Draw the required free-body diagrams and derive the equations of **equilibrium**.
- Derive the **compatibility** equations. To visualize the restrictions on deformation, it is often helpful to draw a sketch that exaggerates the magnitudes of the deformations.
- Use **Hooke's law** to express the deformations (strains) in the compatibility equations in terms of forces (or stresses).
- Solve the equilibrium and compatibility equations for the unknown forces.

Sample Problem 2.7

The concrete post in Fig. (a) is reinforced axially with four symmetrically placed steel bars, each of cross-sectional area 900 mm^2. Compute the stress in each material when the 1000-kN axial load is applied. The moduli of elasticity are 200 GPa for steel and 14 GPa for concrete.



(a)



(b) FBD

Solution

Equilibrium The FBD in Fig. (b) was drawn by isolating the portion of the post above section *a-a*, where P_{co} is the force in concrete and P_{st} denotes the total force carried by the steel rods. For equilibrium, we must have

$$\Sigma F = 0 \quad +\uparrow \quad P_{st} + P_{co} - 1.0 \times 10^6 = 0$$

which, written in terms of stresses, becomes

$$\sigma_{st} A_{st} + \sigma_{co} A_{co} = 1.0 \times 10^6 \text{ N} \tag{a}$$

Equation (a) is the only independent equation of equilibrium that is available in this problem. Because there are two unknown stresses, we conclude that the problem is statically indeterminate.

Compatibility For the deformations to be compatible, the changes in lengths of the steel rods and the concrete must be equal; that is, $\delta_{st} = \delta_{co}$. Because the lengths of steel and concrete are identical, the compatibility equation, written in terms of strains, is

$$\epsilon_{st} = \epsilon_{co} \tag{b}$$

Hooke's Law From Hooke's law, Eq. (b) becomes

$$\frac{\sigma_{st}}{E_{st}} = \frac{\sigma_{co}}{E_{co}} \tag{c}$$

Equations (a) and (c) can now be solved for the stresses. From Eq. (c) we obtain

$$\sigma_{st} = \frac{E_{st}}{E_{co}} \sigma_{co} = \frac{200}{14} \sigma_{co} = 14.286 \sigma_{co} \tag{d}$$

Substituting the cross-sectional areas

$$A_{st} = 4(900 \times 10^{-6}) = 3.6 \times 10^{-3} \text{ m}^2$$

$$A_{co} = 0.3^2 - 3.6 \times 10^{-3} = 86.4 \times 10^{-3} \text{ m}^2$$

and Eq. (d) into Eq. (a) yields

$$(14.286\sigma_{co})(3.6 \times 10^{-3}) + \sigma_{co}(86.4 \times 10^{-3}) = 1.0 \times 10^6$$

Solving for the stress in concrete, we get

$$\sigma_{co} = 7.255 \times 10^6 \text{ Pa} = 7.255 \text{ MPa}$$ *Answer*

From Eq. (d), the stress in steel is

$$\sigma_{st} = 14.286(7.255) = 103.6 \text{ MPa}$$ *Answer*

Sample Problem 2.8

Let the allowable stresses in the post described in Sample Problem 2.7 be $\sigma_{st} = 120$ MPa and $\sigma_{co} = 6$ MPa. Compute the maximum safe axial load P that may be applied.

Solution

The unwary student may attempt to obtain the forces by substituting the allowable stresses into the equilibrium equation—see Eq. (a) in Sample Problem 2.7. This approach is incorrect because it ignores the compatibility condition—that is, the equal strains of the two materials. From Eq. (d) in Sample Problem 2.7, we see that equal strains require the following relationship between the stresses:

$$\sigma_{st} = 14.286\sigma_{co}$$

Therefore, if the concrete were stressed to its limit of 6 MPa, the corresponding stress in the steel would be

$$\sigma_{st} = 14.286(6) = 85.72 \text{ MPa}$$

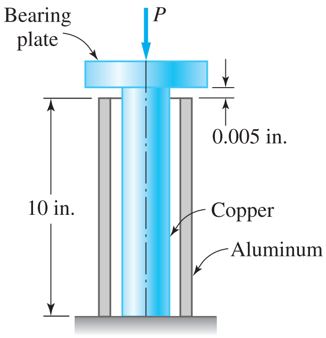
which is below the allowable stress of 120 MPa. The maximum safe axial load is thus found by substituting $\sigma_{co} = 6$ MPa and $\sigma_{st} = 85.72$ MPa into the equilibrium equation:

$$P = \sigma_{st}A_{st} + \sigma_{co}A_{co}$$

$$= (85.72 \times 10^6)(3.6 \times 10^{-3}) + (6 \times 10^6)(86.4 \times 10^{-3})$$

$$= 827 \times 10^3 \text{ N} = 827 \text{ kN}$$ *Answer*

Sample Problem 2.9



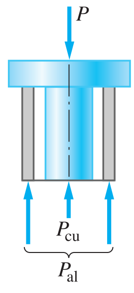
(a)

Figure (a) shows a copper rod that is placed in an aluminum tube. The rod is 0.005 in. longer than the tube. Find the maximum safe load P that can be applied to the bearing plate, using the following data:

	Copper	Aluminum
Area (in.^{2})	2	3
E (psi)	17×10^6	10×10^6
Allowable stress (ksi)	20	10

Solution

Equilibrium We assume that the rod deforms enough so that the bearing plate makes contact with the tube, as indicated in the FBD in Fig. (b). From this FBD we get



(b) FBD

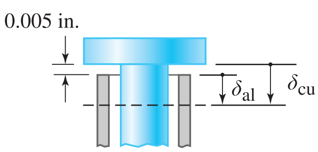
$$\Sigma F = 0 \quad +\uparrow \quad P_{cu} + P_{al} - P = 0 \tag{a}$$

Because no other equations of equilibrium are available, the forces P_{cu} and P_{al} are statically indeterminate.

Compatibility Figure (c) shows the changes in the lengths of the two materials (the deformations have been greatly exaggerated). We see that the compatibility equation is

$$\delta_{cu} = \delta_{al} + 0.005 \text{ in.} \tag{b}$$

Hooke's Law Substituting $\delta = \sigma L/E$ into Eq. (b), we get



(c)

$$\left(\frac{\sigma L}{E}\right)_{cu} = \left(\frac{\sigma L}{E}\right)_{al} + 0.005 \text{ in.}$$

or

$$\frac{\sigma_{cu}(10.005)}{17 \times 10^6} = \frac{\sigma_{al}(10)}{10 \times 10^6} + 0.005$$

which reduces to

$$\sigma_{cu} = 1.6992\sigma_{al} + 8496 \qquad \text{(c)}$$

From Eq. (c) we find that if $\sigma_{al} = 10\,000$ psi, the copper will be overstressed to 25 500 psi. Therefore, the allowable stress in the copper (20 000 psi) is the limiting condition. The corresponding stress in the aluminum is found from Eq. (c):

$$20\,000 = 1.6992\sigma_{al} + 8496$$

which gives

$$\sigma_{al} = 6770 \text{ psi}$$

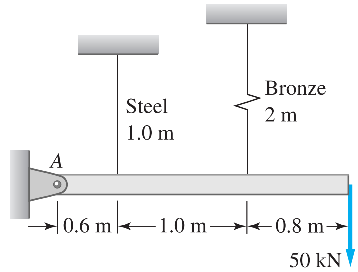
From Eq. (a), the safe load is

$$P = P_{cu} + P_{al} = \sigma_{cu}A_{cu} + \sigma_{al}A_{al}$$
$$= 20\,000(2) + 6770(3) = 60\,300 \text{ lb} = 60.3 \text{ kips} \qquad \textit{Answer}$$

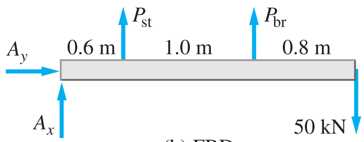
Sample Problem **2.10**

Figure (a) shows a rigid bar that is supported by a pin at A and two rods, one made of steel and the other of bronze. Neglecting the weight of the bar, compute the stress in each rod caused by the 50-kN load, using the following data:

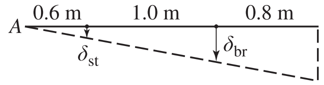
	Steel	**Bronze**
Area (mm^2)	600	300
E (GPa)	200	83



(a)



(b) FBD



(c)

Solution

Equilibrium The free-body diagram of the bar, shown in Fig. (b), contains four unknown forces. Since there are only three independent equilibrium equations, these forces are statically indeterminate. The equilibrium equation that does not involve the pin reactions at A is

$$\Sigma M_A = 0 \quad +\circlearrowleft \quad 0.6P_{st} + 1.6P_{br} - 2.4(50 \times 10^3) = 0 \qquad \text{(a)}$$

Compatibility The displacement of the bar, consisting of a rigid-body rotation about A, is shown greatly exaggerated in Fig. (c). From similar triangles, we see that the elongations of the supporting rods must satisfy the compatibility condition

$$\frac{\delta_{st}}{0.6} = \frac{\delta_{br}}{1.6} \qquad \text{(b)}$$

Hooke's Law When we substitute $\delta = PL/(EA)$ into Eq. (b), the compatibility equation becomes

$$\frac{1}{0.6}\left(\frac{PL}{EA}\right)_{st} = \frac{1}{1.6}\left(\frac{PL}{EA}\right)_{br}$$

Using the given data, we obtain

$$\frac{1}{0.6}\,\frac{P_{st}(1.0)}{(200)(600)} = \frac{1}{1.6}\,\frac{P_{br}(2)}{(83)(300)}$$

which simplifies to

$$P_{st} = 3.614P_{br} \qquad \text{(c)}$$

Note that we did not convert the areas from mm^2 to m^2, and we omitted the factor 10^9 from the moduli of elasticity. Since these conversion factors appear on both sides of the equation, they would cancel out.

Solving Eqs. (a) and (c), we obtain

$$P_{\mathrm{st}} = 115.08 \times 10^3 \text{ N} \qquad P_{\mathrm{br}} = 31.84 \times 10^3 \text{ N}$$

The stresses are

$$\sigma_{\mathrm{st}} = \frac{P_{\mathrm{st}}}{A_{\mathrm{st}}} = \frac{115.08 \times 10^3}{600 \times 10^{-6}} = 191.8 \times 10^6 \text{ Pa} = 191.8 \text{ MPa}$$ *Answer*

$$\sigma_{\mathrm{br}} = \frac{P_{\mathrm{br}}}{A_{\mathrm{br}}} = \frac{31.84 \times 10^3}{300 \times 10^{-6}} = 106.1 \times 10^6 \text{ Pa} = 106.1 \text{ MPa}$$ *Answer*

Problems

2.44 The figure shows the cross section of a circular steel tube that is filled with concrete and topped with a rigid cap. Calculate the stresses in the steel and in the concrete caused by the 200-kip axial load. Use $E_{st} = 29 \times 10^6$ psi and $E_{co} = 3.5 \times 10^6$ psi.

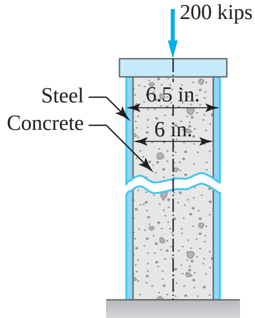


FIG. P2.44

2.45 A reinforced concrete column 200 mm in diameter is designed to carry an axial compressive load of 320 kN. Determine the required cross-sectional area of the reinforcing steel if the allowable stresses are 6 MPa for concrete and 120 MPa for steel. Use $E_{co} = 14$ GPa and $E_{st} = 200$ GPa.

2.46 A timber column, 8 in. by 8 in. in cross section, is reinforced on all four sides by steel plates, each plate being 8 in. wide and t in. thick. Determine the smallest value of t for which the column can support an axial load of 300 kips if the working stresses are 1200 psi for timber and 20 ksi for steel. The moduli of elasticity are 1.5×10^6 psi for timber and 29×10^6 psi for steel.

2.47 The rigid block of mass M is supported by the three symmetrically placed rods. The ends of the rods were level before the block was attached. Determine the largest allowable value of M if the properties of the rods are as listed (σ_w is the working stress):

	E (GPa)	A (mm^2)	σ_w (MPa)
Copper	120	900	70
Steel	200	1200	140

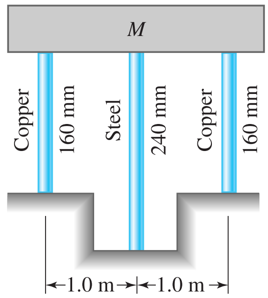


FIG. P2.47

2.48 The concrete column is reinforced by four steel bars of total cross-sectional area 1250 mm^2. If the working stresses for steel and concrete are 180 MPa and 15 MPa, respectively, determine the largest axial force P that can be safely applied to the column. Use $E_{st} = 200$ GPa and $E_{co} = 24$ GPa.

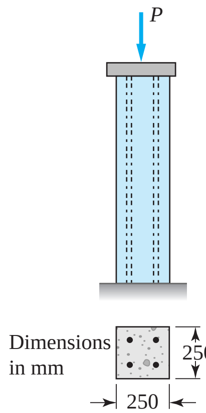


FIG. P2.48

2.49 The rigid slab of weight W, with center of gravity at G, is suspended from three identical steel wires. Determine the force in each wire.

2.50 Before the 400-kN load is applied, the rigid platform rests on two steel bars, each of cross-sectional area 1400 mm^2, as shown in the figure. The cross-sectional area of the aluminum bar is 2800 mm^2. Compute the stress in the aluminum bar after the 400-kN load is applied. Use $E = 200$ GPa for steel and $E = 70$ GPa for aluminum. Neglect the weight of the platform.

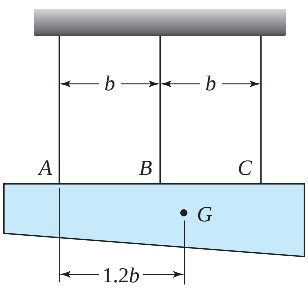


FIG. P2.49

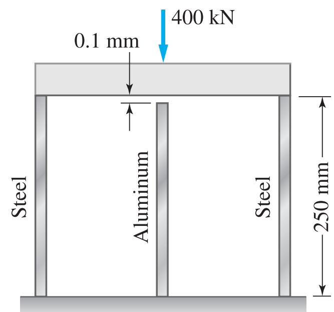


FIG. P2.50

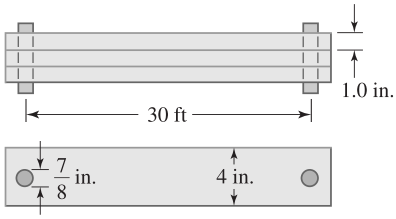


FIG. P2.51

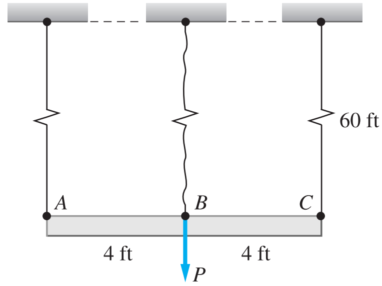


FIG. P2.52

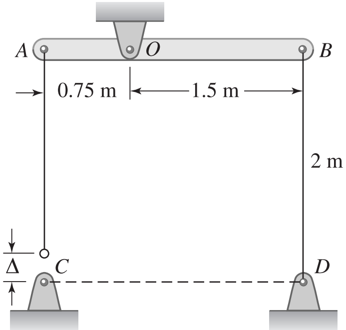


FIG. P2.53, P2.54

2.51 The three steel ($E = 29 \times 10^6$ psi) eye-bars, each 4 in. by 1.0 in. in cross section, are assembled by driving 7/8-in.-diameter drift pins through holes drilled in the ends of the bars. The distance between the holes is 30 ft in the two outer bars, but 0.045 in. less in the middle bar. Find the shear stress developed in the drift pins. Neglect local deformation at the holes.

2.52 The rigid bar ABC of negligible weight is suspended from three aluminum wires, each of cross-sectional area 0.3 in.^{2}. Before the load P is applied, the middle wire is slack, being 0.2 in. longer than the other two wires. Determine the largest safe value of P if the working stress for the wires is 12 ksi. Use $E = 10 \times 10^6$ psi for aluminum.

2.53 The rigid bar AB of negligible weight is supported by a pin at O. When the two steel rods are attached to the ends of the bar, there is a gap $\Delta = 4$ mm between the lower end of the left rod and its pin support at C. Compute the stress in the left rod after its lower end is attached to the support. The cross-sectional areas are 300 mm^2 for rod AC and 250 mm^2 for rod BD. Use $E = 200$ GPa for steel.

2.54 The rigid bar AB of negligible weight is supported by a pin at O. When the two steel rods are attached to the ends of the bar, there is a gap Δ between the lower end of the left rod and its pin support at C. After attachment, the strain in the left rod is 1.5×10^{-3}. What is the length of the gap Δ? The cross-sectional areas are 300 mm^2 for rod AC and 250 mm^2 for rod BD. Use $E = 200$ GPa for steel.

2.55 The homogeneous rod of constant cross section is attached to unyielding supports. The rod carries an axial load P, applied as shown in the figure. Show that the reactions are given by $R_1 = Pb/L$ and $R_2 = Pa/L$.

2.56 The homogeneous bar with a cross-sectional area of 600 mm^2 is attached to rigid supports. The bar carries the axial loads $P_1 = 20$ kN and $P_2 = 60$ kN, as shown. Determine the stress in segment BC. (*Hint*: Use the results of Prob. 2.55 to compute the reactions caused by P_1 and P_2 acting separately. Then use superposition to compute the reactions when both loads are applied.)

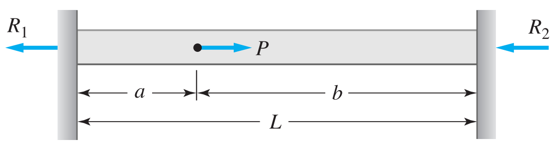


FIG. P2.55

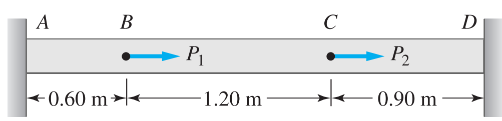


FIG. 2.56

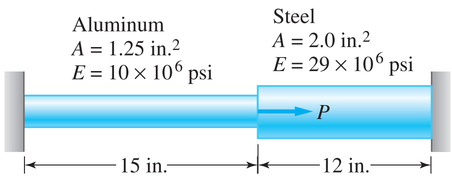


FIG. P2.57, P2.58

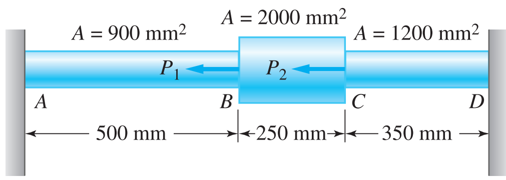


FIG. P2.59

2.57 The composite bar is firmly attached to unyielding supports. Compute the stress in each material caused by the application of the axial load $P = 40$ kips.

2.58 The composite bar, firmly attached to unyielding supports, is initially stress-free. What maximum axial load P can be applied if the allowable stresses are 10 ksi for aluminum and 18 ksi for steel?

2.59 The steel rod is stress-free before the axial loads $P_1 = 150$ kN and $P_2 = 90$ kN are applied to the rod. Assuming that the walls are rigid, calculate the axial force in each segment after the loads are applied. Use $E = 200$ GPa.

2.60 The bar BCD of length L has a constant thickness t, but its width varies as shown. The cross-sectional area A of the bar is given by $A = bt(1 + x/L)$. The ends of the bar are attached to the rigid walls, and the bar is initially stress-free. Compute the reactions at B and D after the force P is applied at the midpoint C of the bar.

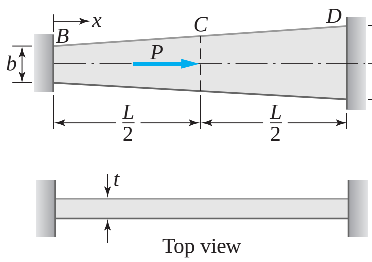


FIG. P2.60

2.61 The steel column of circular cross section is attached to rigid supports at A and C. Find the maximum stress in the column caused by the 25-kN load.

2.62 The assembly consists of a bronze tube and a threaded steel bolt. The pitch of the thread is 1/32 in. (one turn of the nut advances it 1/32 in.). The cross-sectional areas are 1.5 in.^{2} for the tube and 0.75 in.^{2} for the bolt. The nut is turned until there is a compressive stress of 4000 psi in the tube. Find the stresses in the bolt and the tube if the nut is given one additional turn. Use $E = 12 \times 10^6$ psi for bronze and $E = 29 \times 10^6$ psi for steel.

2.63 The two vertical rods attached to the rigid bar are identical except for length. Before the 6600-lb weight was attached, the bar was horizontal. Determine the axial force in each bar caused by the application of the weight. Neglect the weight of the bar.

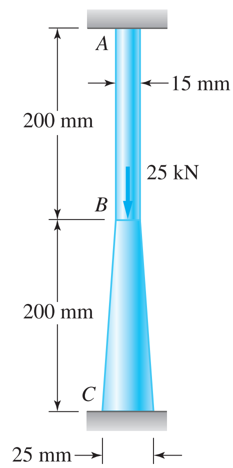


FIG. P2.61

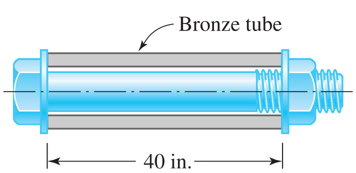


FIG. P2.62

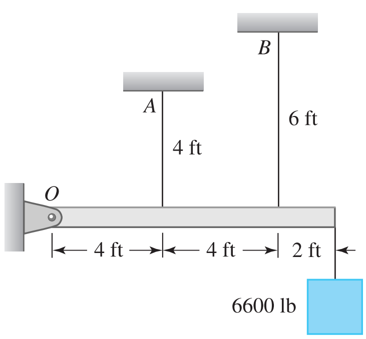


FIG. P2.63

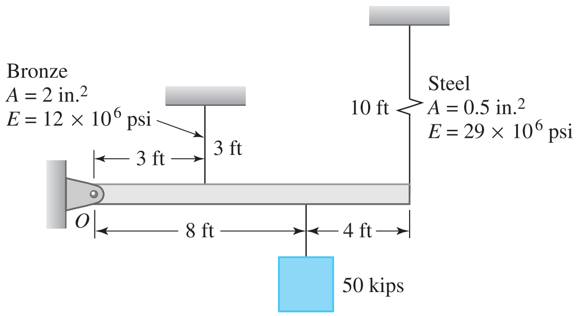


FIG. P2.64

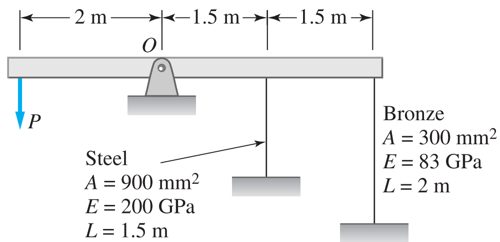


FIG. P2.65

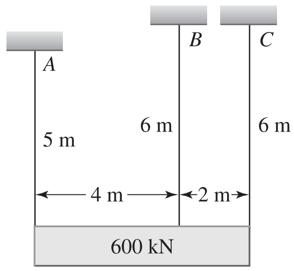


FIG. P2.66

2.64 The rigid beam of negligible weight is supported by a pin at O and two vertical rods. Find the vertical displacement of the 50-kip weight.

2.65 The rigid bar of negligible weight is pinned at O and attached to two vertical rods. Assuming that the rods were initially stress-free, what is the largest load P that can be applied without exceeding stresses of 150 MPa in the steel rod and 70 MPa in the bronze rod?

2.66 The rigid, homogeneous slab weighing 600 kN is supported by three rods of identical material and cross section. Before the slab was attached, the lower ends of the rods were at the same level. Compute the axial force in each rod.

2.67 The rigid bar BCD of negligible weight is supported by two steel cables of identical cross section. Determine the force in each cable caused by the applied weight W.

2.68 The three steel rods, each of cross-sectional area 250 mm^2, jointly support the 7.5-kN load. Assuming that there was no slack or stress in the rods before the load was applied, find the force in each rod. Use $E = 200$ GPa for steel.

2.69 The bars AB, AC, and AD are pinned together as shown in the figure. Horizontal movement of the pin at A is prevented by the rigid horizontal strut AE. Calculate the axial force in the strut caused by the 10-kip load. For each steel bar, $A = 0.3$ in.^{2} and $E = 29 \times 10^6$ psi. For the aluminum bar, $A = 0.6$ in.^{2} and $E = 10 \times 10^6$ psi.

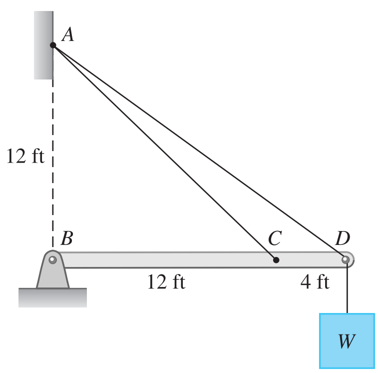


FIG. P2.67

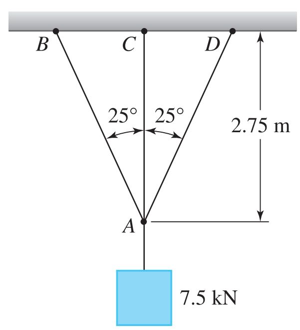


FIG. P2.68

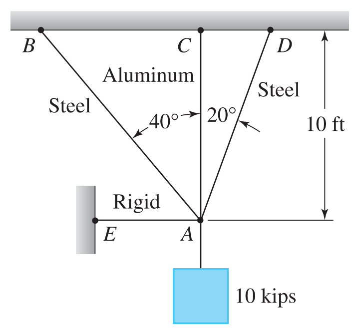


FIG. P2.69

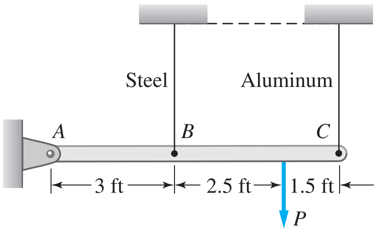


FIG. P2.70

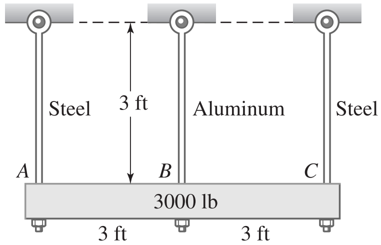


FIG. P2.71, P2.72

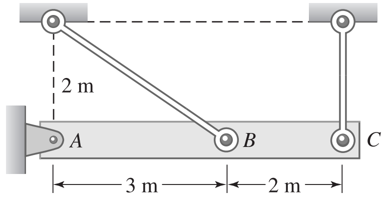


FIG. P2.73

2.70 The horizontal bar ABC is supported by a pin at A and two rods with identical cross-sectional areas. The rod at B is steel and the rod at C is aluminum. Neglecting the weight of the bar, determine the force in each rod when the force $P = 10$ kips is applied. Use $E_{st} = 29 \times 10^6$ psi and $E_{al} = 10 \times 10^6$ psi.

2.71 The lower ends of the three vertical rods were at the same level before the uniform, rigid bar ABC weighing 3000 lb was attached. Each rod has a cross-sectional area of 0.5 in.^{2}. The two outer rods are steel and the middle rod is aluminum. Find the force in the middle rod. Use $E_{st} = 29 \times 10^6$ psi and $E_{al} = 10 \times 10^6$ psi.

2.72 Solve Prob. 2.71 if the steel rod attached at C is replaced by an aluminum rod of the same size.

2.73 The uniform rigid bar ABC of weight W is supported by two rods that are identical except for their lengths. Assuming that the bar was held in the horizontal position when the rods were attached, determine the force in each rod after the attachment.

2.6 Thermal Stresses

It is well known that changes in temperature cause dimensional changes in a body: An increase in temperature results in expansion, whereas a temperature decrease produces contraction. This deformation is isotropic (the same in every direction) and proportional to the temperature change. It follows that the associated strain, called *thermal strain*, is

$$\epsilon_T = \alpha(\Delta T) \tag{2.15}$$

where the constant α is a material property known as the *coefficient of thermal expansion*, and ΔT is the temperature change. The coefficient of thermal expansion represents the normal strain caused by a one-degree change in temperature. By convention, ΔT is taken to be positive when the temperature increases, and negative when the temperature decreases. Thus, in Eq. (2.15), positive ΔT produces positive strain (elongation) and negative ΔT produces negative strain (contraction). The units of α are $1/°\text{C}$ (per degree Celsius) in the SI system, and $1/°\text{F}$ (per degree Fahrenheit) in the U.S. Customary system. Typical values of α are $23 \times 10^{-6}/°\text{C}$ ($13 \times 10^{-6}/°\text{F}$) for aluminum and $12 \times 10^{-6}/°\text{C}$ ($6.5 \times 10^{-6}/°\text{F}$) for steel.

If the temperature change is uniform throughout the body, the thermal strain is also uniform. Consequently, the change in any dimension L of the body is given by

$$\delta_T = \epsilon_T L = \alpha(\Delta T)L \tag{2.16}$$

If thermal deformation is permitted to occur freely (by using expansion joints or roller supports, for example), no internal forces will be induced in the body—there will be strain, but no stress. In cases where the deformation of a body is restricted, either totally or partially, internal forces will develop that oppose the thermal expansion or contraction. The stresses caused by these internal forces are known as *thermal stresses*.

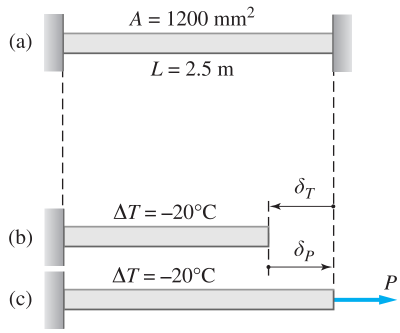
The forces that result from temperature changes cannot be determined by equilibrium analysis alone; that is, these forces are statically indeterminate. Consequently, the analysis of thermal stresses follows the same principles that we used in Sec. 2.5: equilibrium, compatibility, and Hooke's law. The only difference here is that we must now include thermal expansion in the analysis of deformation.

Procedure for Deriving Compatibility Equations We recommend the following procedure for deriving the equations of compatibility:

- Remove the constraints that prevent the thermal deformation to occur freely (this procedure is sometimes referred to as "relaxing the supports"). Show the thermal deformation on a sketch using an exaggerated scale.
- Apply the forces that are necessary to restore the specified conditions of constraint. Add the deformations caused by these forces to the sketch that was drawn in the previous step. (Draw the magnitudes of the deformations so that they are compatible with the geometric constraints.)
- By inspection of the sketch, write the relationships between the thermal deformations and the deformations due to the constraint forces.

Sample Problem **2.11**

The horizontal steel rod, 2.5 m long and 1200 mm^2 in cross-sectional area, is secured between two walls as shown in Fig. (a). If the rod is stress-free at 20°C, compute the stress when the temperature has dropped to −20°C. Assume that (1) the walls do not move and (2) the walls move together a distance $\Delta = 0.5$ mm. Use $\alpha = 11.7 \times 10^{-6}/°C$ and $E = 200$ GPa.



Solution

Part 1

Compatibility We begin by assuming that the rod has been disconnected from the right wall, as shown in Fig. (b), so that the contraction δ_T caused by the temperature drop ΔT can occur freely. To reattach the rod to the wall, we must stretch the rod to its original length by applying the tensile force P. Compatibility of deformations requires that the resulting elongation δ_P, shown in Fig. (c), must be equal to δ_T; that is,

$$\delta_T = \delta_P$$

Hooke's Law If we substitute $\delta_T = \alpha(\Delta T)L$ and $\delta_P = PL/(EA) = \sigma L/E$, the compatibility equation becomes

$$\frac{\sigma L}{E} = \alpha(\Delta T)L$$

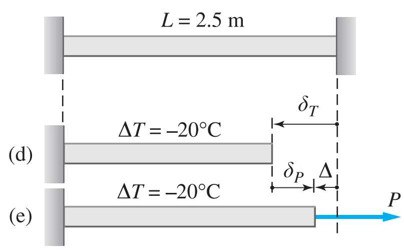
Therefore, the stress in the rod is

$$\sigma = \alpha(\Delta T)E = (11.7 \times 10^{-6})(40)(200 \times 10^{9})$$

$$= 93.6 \times 10^{6} \text{ Pa} = 93.6 \text{ MPa} \qquad \textit{Answer}$$

Note that L canceled out in the preceding equation, which indicates that the stress is independent of the length of the rod.

Part 2



Compatibility When the walls move together a distance Δ, we see from Figs. (d) and (e) that the free thermal contraction δ_T is related to Δ and the elongation δ_P caused by the axial force P by

$$\delta_T = \delta_P + \Delta$$

Hooke's Law Substituting for δ_T and δ_P as in Part 1, we obtain

$$\alpha(\Delta T)L = \frac{\sigma L}{E} + \Delta$$

The solution for the stress σ is

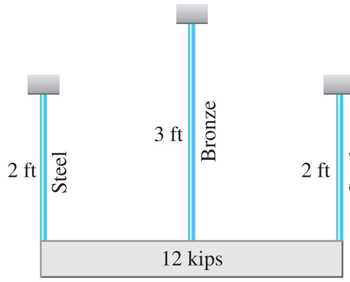
$$\sigma = E\left[\alpha(\Delta T) - \frac{\Delta}{L}\right]$$

$$= (200 \times 10^{9})\left[(11.7 \times 10^{-6})(40) - \frac{0.5 \times 10^{-3}}{2.5}\right]$$

$$= 53.6 \times 10^{6} \text{ Pa} = 53.6 \text{ MPa} \qquad \textit{Answer}$$

We see that the movement of the walls reduces the stress considerably. Also observe that the length of the rod does not cancel out as in Part 1.

Sample Problem **2.12**



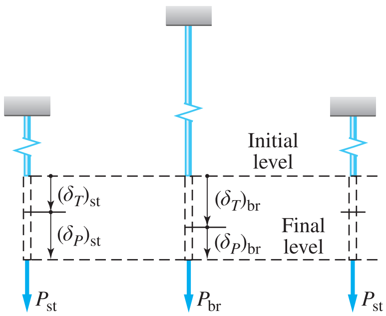
(a)

Figure (a) shows a homogeneous, rigid block weighing 12 kips that is supported by three symmetrically placed rods. The lower ends of the rods were at the same level before the block was attached. Determine the stress in each rod after the block is attached and the temperature of all bars increases by 100°F. Use the following data:

	A (in.^{2})	E (psi)	α (/°F)
Each steel rod	0.75	29×10^6	6.5×10^{-6}
Bronze rod	1.50	12×10^6	10.0×10^{-6}

Solution

Compatibility Note that the block remains horizontal because of the symmetry of the structure. Let us assume that the block is detached from the rods, as shown in Fig. (b). With the rods unconstrained, a temperature rise will cause the elongations $(\delta_T)_{st}$ in the steel rods and $(\delta_T)_{br}$ in the bronze rod. To reattach the block to the rods, the rods must undergo the additional deformations $(\delta_P)_{st}$ and $(\delta_P)_{br}$, both assumed to be elongations. From the deformation diagram in Fig. (b), we obtain the following compatibility equation (recall that the block remains horizontal):



(b)

$$(\delta_T)_{st} + (\delta_P)_{st} = (\delta_T)_{br} + (\delta_P)_{br}$$

Hooke's Law Using Hooke's law, we can write the compatibility equation as

$$[\alpha(\Delta T)L]_{st} + \left[\frac{PL}{EA}\right]_{st} = [\alpha(\Delta T)L]_{br} + \left[\frac{PL}{EA}\right]_{br}$$

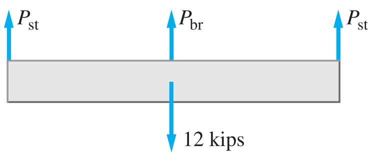
Substituting the given data, we have

$$(6.5 \times 10^{-6})(100)(2 \times 12) + \frac{P_{st}(2 \times 12)}{(29 \times 10^6)(0.75)}$$

$$= (10.0 \times 10^{-6})(100)(3 \times 12) + \frac{P_{br}(3 \times 12)}{(12 \times 10^6)(1.50)}$$

If we rearrange terms and simplify, the compatibility equation becomes

$$0.091\,95P_{st} - 0.1667P_{br} = 1700 \qquad \text{(a)}$$



(c) FBD

Equilibrium From the free-body diagram in Fig. (c) we obtain

$$\Sigma F = 0 \quad +\uparrow \quad 2P_{st} + P_{br} - 12\,000 = 0 \qquad \text{(b)}$$

Solving Eqs. (a) and (b) simultaneously yields

$$P_{st} = 8700 \text{ lb} \quad \text{and} \quad P_{br} = -5400 \text{ lb}$$

The negative sign for P_{br} means that the force in the bronze rod is compressive (it acts in the direction opposite to that shown in the figures). The stresses in the rods are:

$$\sigma_{st} = \frac{P_{st}}{A_{st}} = \frac{8700}{0.75} = 11\,600 \text{ psi (T)} \qquad \textit{Answer}$$

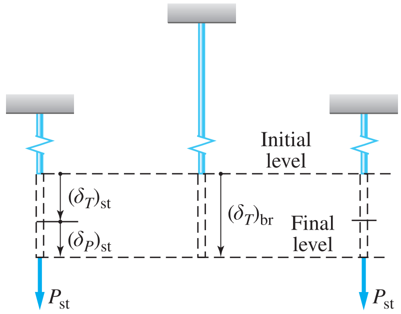
$$\sigma_{br} = \frac{P_{br}}{A_{br}} = \frac{-5400}{1.50} = -3600 \text{ psi} = 3600 \text{ psi (C)} \qquad \textit{Answer}$$

Sample Problem **2.13**

Using the data in Sample Problem 2.12, determine the temperature increase that would cause the entire weight of the block to be carried by the steel rods.

Solution

Equilibrium The problem statement implies that the bronze rod is stress-free. Thus, each steel rod carries half the weight of the rigid block, so that $P_{st} = 6000$ lb.



Compatibility The temperature increase causes the elongations $(\delta_T)_{st}$ and $(\delta_T)_{br}$ in the steel and bronze rods, respectively, as shown in the figure. Because the bronze rod is to carry no load, the ends of the steel rods must be at the same level as the end of the unstressed bronze rod before the rigid block can be reattached. Therefore, the steel rods must elongate by $(\delta_P)_{st}$ due to the tensile forces $P_{st} = 6000$ lb, which gives

$$(\delta_T)_{br} = (\delta_T)_{st} + (\delta_P)_{st}$$

Hooke's Law Using Hooke's law, the compatibility equation becomes

$$[\alpha(\Delta T)L]_{br} = [\alpha(\Delta T)L]_{st} + \left[\frac{PL}{EA}\right]_{st}$$

$$(10 \times 10^{-6})(\Delta T)(3 \times 12) = (6.5 \times 10^{-6})(\Delta T)(2 \times 12) + \frac{6000(2 \times 12)}{(29 \times 10^6)(0.75)}$$

which yields

$$\Delta T = 32.5°\text{F} \qquad \textit{Answer}$$

as the temperature increase at which the bronze rod would be unstressed.

Problems

2.74 A steel rod with a cross-sectional area of 0.25 in.^{2} is stretched between two fixed points. The tensile force in the rod at 70°F is 1200 lb. (a) What will be the stress at 0°F? (b) At what temperature will the stress be zero? Use $\alpha = 6.5 \times 10^{-6}/°\text{F}$ and $E = 29 \times 10^6$ psi.

2.75 A steel rod is stretched between two walls. At 20°C, the tensile force in the rod is 5000 N. If the stress is not to exceed 130 MPa at −20°C, find the minimum allowable diameter of the rod. Use $\alpha = 11.7 \times 10^{-6}/°\text{C}$ and $E = 200$ GPa.

2.76 Steel railroad rails 10 m long are laid with end-to-end clearance of 3 mm at a temperature of 15°C. (a) At what temperature will the rails just come in contact? (b) What stress would be induced in the rails at that temperature if there were no initial clearance? Use $\alpha = 11.7 \times 10^{-6}/°\text{C}$ and $E = 200$ GPa.

2.77 A steel rod 3 ft long with a cross-sectional area of 0.3 in.^{2} is stretched between two fixed points. The tensile force in the rod is 1200 lb at 40°F. Using $\alpha = 6.5 \times 10^{-6}/°\text{F}$ and $E = 29 \times 10^6$ psi, calculate the temperature at which the stress in the rod will be (a) 10 ksi; and (b) zero.

2.78 The bronze bar 3 m long with a cross-sectional area of 350 mm^2 is placed between two rigid walls. At a temperature of −20°C, there is a gap $\Delta = 2.2$ mm, as shown in the figure. Find the temperature at which the compressive stress in the bar will be 30 MPa. Use $\alpha = 18.0 \times 10^{-6}/°\text{C}$ and $E = 80$ GPa.

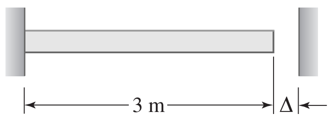


FIG. P2.78

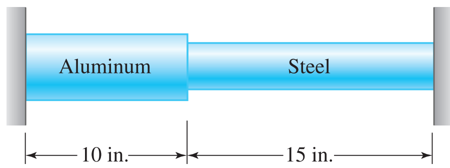


FIG. P2.79

2.79 Calculate the increase in stress in each segment of the compound bar if the temperature is increased by 80°F. Assume that the supports are unyielding and use the following data:

	A (in.^{2})	E (psi)	α (/°F)
Aluminum	1.5	10×10^6	12.8×10^{-6}
Steel	2.0	29×10^6	6.5×10^{-6}

2.80 A prismatic bar of length L fits snugly between two rigid walls. If the bar is given a temperature increase that varies linearly from ΔT_A at one end to ΔT_B at the other end, show that the resulting stress in the bar is $\sigma = \alpha E(\Delta T_A + \Delta T_B)/2$.

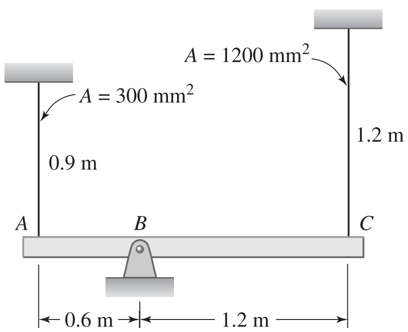


FIG. P2.81

2.81 The rigid bar ABC is supported by a pin at B and two vertical steel rods. Initially the bar is horizontal and the rods are stress-free. Determine the stress in each rod if the temperature of the rod at A is decreased by 40°C. Neglect the weight of bar ABC. Use $\alpha = 11.7 \times 10^{-6}/°\text{C}$ and $E = 200$ GPa for steel.

2.82 The rigid, horizontal slab is attached to two identical copper rods. There is a gap $\Delta = 0.18$ mm between the middle bar, which is made of aluminum, and the slab. Neglecting the mass of the slab, calculate the stress in each rod when the temperature in the assembly is increased by 85°C. Use the following data:

	A (mm^2)	α (/°C)	E (GPa)
Each copper rod	500	16.8×10^{-6}	120
Aluminum rod	400	23.1×10^{-6}	70

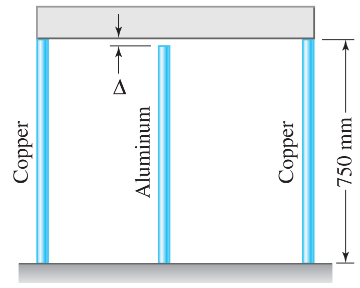


FIG. P2.82

2.83 A bronze sleeve is slipped over a steel bolt and held in place by a nut that is tightened to produce an initial stress of 2000 psi in the bronze. Find the stress in each material after the temperature of the assembly is increased by 100°F. The properties of the components are listed in the table.

	A (in.^{2})	α (/°F)	E (psi)
Bronze sleeve	1.50	10.5×10^{-6}	12×10^6
Steel bolt	0.75	6.5×10^{-6}	29×10^6

2.84 The rigid bar of negligible weight is supported as shown in the figure. If $W = 80$ kN, compute the temperature change of the assembly that will cause a tensile stress of 50 MPa in the steel rod. Use the following data:

	A (mm^2)	α (/°C)	E (GPa)
Steel rod	300	11.7×10^{-6}	200
Bronze rod	1400	18.9×10^{-6}	83

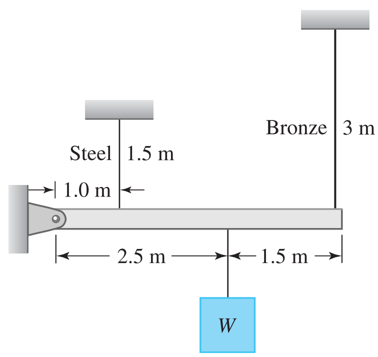


FIG. P2.84, P2.85

2.85 The rigid bar of negligible weight is supported as shown. The assembly is initially stress-free. Find the stress in each rod if the temperature rises 20°C after a load $W = 120$ kN is applied. Use the properties of the bars given in Prob. 2.84.

2.86 The composite bar is firmly attached to unyielding supports. The bar is stress-free at 60°F. Compute the stress in each material after the 50-kip force is applied and the temperature is increased to 120°F. Use $\alpha = 6.5 \times 10^{-6}$/°F for steel and $\alpha = 12.8 \times 10^{-6}$/°F for aluminum.

2.87 At what temperature will the aluminum and steel segments in Prob. 2.86 have stresses of equal magnitude after the 50-kip force is applied?

2.88 All members of the steel truss have the same cross-sectional area. If the truss is stress-free at 10°C, determine the stresses in the members at 90°C. For steel, $\alpha = 11.7 \times 10^{-6}$/°C and $E = 200$ GPa.

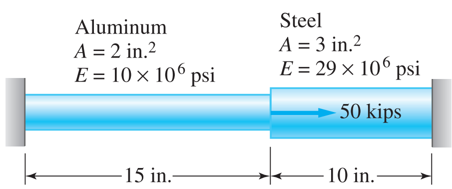


FIG. P2.86, P2.87

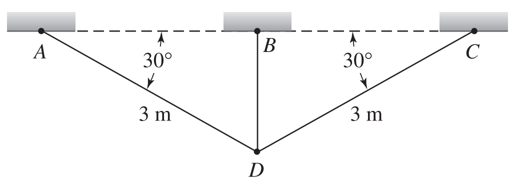


FIG. P2.88

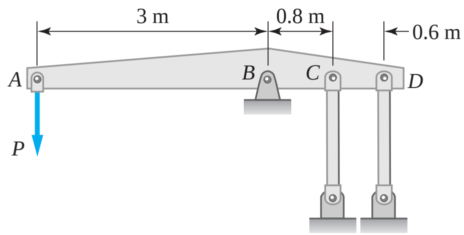


FIG. P2.89

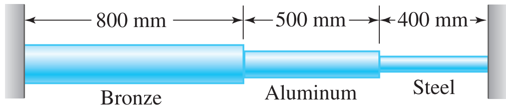


FIG. P2.90

2.89 The rigid bar *ABCD* is supported by a pin at *B* and restrained by identical steel bars at *C* and *D*, each of area 250 mm^2. If the temperature is increased by 80°C, determine the force *P* that will cause the bar at *C* to be stress-free. Use $E = 200$ GPa and $\alpha = 12 \times 10^{-6}/°C$.

2.90 The compound bar, composed of the three segments shown, is initially stress-free. Compute the stress in each material if the temperature drops 25°C. Assume that the walls do not yield and use the following data:

	A (mm^2)	*α* (/°C)	*E* (GPa)
Bronze segment	2000	19.0×10^{-6}	83
Aluminum segment	1400	23.0×10^{-6}	70
Steel segment	800	11.7×10^{-6}	200

2.91 The rigid bar *AOB* is pinned at *O* and connected to aluminum and steel rods. If the bar is horizontal at a given temperature, determine the ratio of the areas of the two rods so that the bar will be horizontal at any temperature. Neglect the mass of the bar.

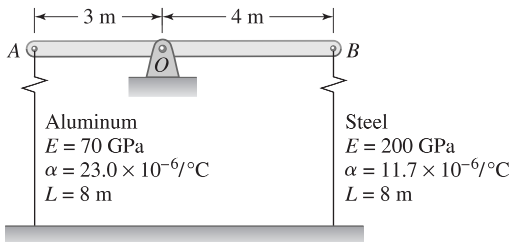


FIG. P2.91

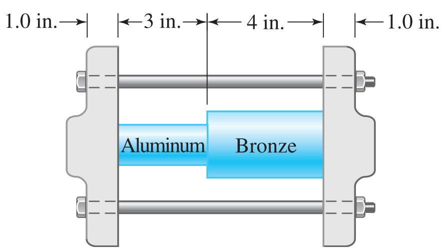


FIG. P2.92

2.92 The aluminum and bronze cylinders are centered and secured between two rigid end-plates by tightening the two steel bolts. There is no axial load in the assembly at a temperature of 50°F. Find the stress in the steel bolts when the temperature is increased to 200°F. Use the following data:

	A ($in.^2$)	*α* (/°F)	*E* (psi)
Aluminum cylinder	2.00	12.8×10^{-6}	10×10^6
Bronze cylinder	3.00	10.5×10^{-6}	12×10^6
Each steel bolt	0.75	6.5×10^{-6}	29×10^6

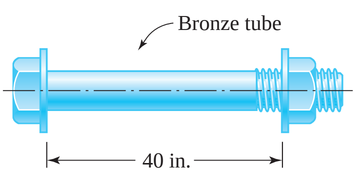


FIG. P2.93

2.93 The assembly consists of a bronze tube fitted over a threaded steel bolt. The nut on the bolt is turned until it is finger-tight. Determine the stresses in the sleeve and bolt when the temperature of the assembly is increased by 200°F. Use the following data:

	A ($in.^2$)	*α* (/°F)	*E* (psi)
Bronze	1.5	10×10^{-6}	12×10^6
Steel	0.75	6.5×10^{-6}	29×10^6

Review Problems

2.94 The elastic strip with a cutout is of length L, width b, and thickness t. Derive the expression for the elongation of the strip caused by the axial load P.

2.95 The aluminum bar of cross-sectional area 0.6 in.^{2} carries the axial loads shown in the figure. Compute the total change in length of the bar given that $E = 10 \times 10^6$ psi.

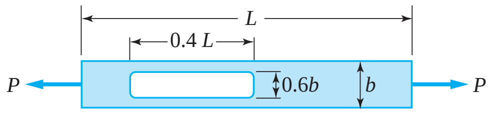


FIG. P2.94

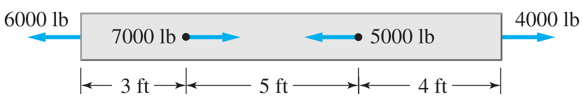


FIG. P2.95

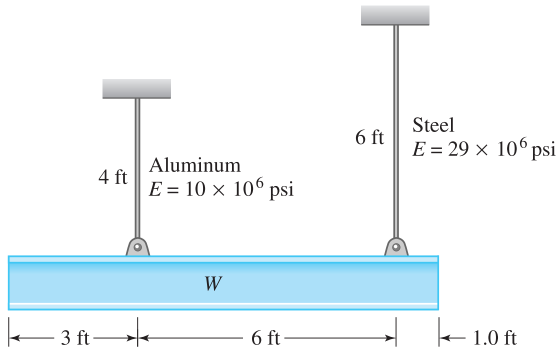


FIG. P2.96

2.96 The uniform beam of weight W is to be supported by the two rods, the lower ends of which were initially at the same level. Determine the ratio of the areas of the rods so that the beam will be horizontal after it is attached to the rods. Neglect the deformation of the beam.

2.97 A round bar of length L, modulus of elasticity E, and weight density γ tapers uniformly from a diameter $2D$ at one end to a diameter D at the other end. If the bar is suspended vertically from the larger end, find the elongation of the bar caused by its own weight.

2.98 The timber member BC, inclined at angle $\theta = 60°$ to the vertical, is supported by a pin at B and the 0.75-in.-diameter steel bar AC. (a) Determine the cross-sectional area of BC for which the displacement of C will be vertical when the 5000-lb force is applied. (b) Compute the corresponding displacement of C. The moduli of elasticity are 1.8×10^6 psi for timber and 29×10^6 psi for steel. Neglect the weight of BC.

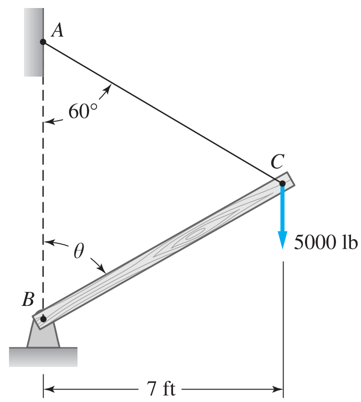


FIG. P2.98

2.99 The collar B is welded to the midpoint of the cylindrical steel bar AC of length $2L$. The left half of the bar is then inserted in a brass tube and the assembly is placed between rigid walls. Determine the forces in the steel bar and the brass tube when the force P is applied to the collar. Neglect the deformation of the collar and assume $(EA)_{st} = 3(EA)_{br}$.

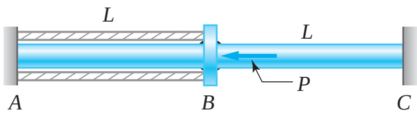


FIG. P2.99

2.100 A solid aluminum shaft of diameter 80 mm fits concentrically inside a hollow tube. Compute the minimum internal diameter of the tube so that no contact pressure exists when the aluminum shaft carries an axial compressive force of 400 kN. Use $\nu = 1/3$ and $E = 70$ GPa for aluminum.

2.101 The normal stresses in an aluminum block are $\sigma_x = -4000$ psi and $\sigma_y = \sigma_z = -p$. Determine (a) the value of p for which $\epsilon_x = 0$; and (b) the corresponding value of ϵ_y. Use $E = 10 \times 10^6$ psi and $\nu = 0.33$.

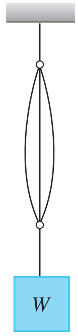


FIG. P2.102

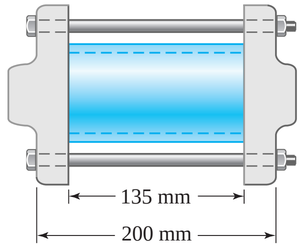


FIG. P2.103

2.102 The three steel wires, each of cross-sectional area 0.05 in.^{2}, support the weight W. Their unstressed lengths are 74.98 ft, 74.99 ft, and 75.00 ft. (a) Find the stress in the longest wire if $W = 1500$ lb. (b) Determine the stress in the shortest wire if $W = 500$ lb. Use $E = 29 \times 10^6$ psi.

2.103 The figure shows an aluminum tube that is placed between rigid bulkheads. After the two steel bolts connecting the bulkheads are turned finger-tight, the temperature of the assembly is raised by 90°C. Compute the resulting forces in the tube and bolts. Use the following data:

	E (GPa)	α (/°C)	Diameter (mm)
Aluminum tube	70	23×10^{-6}	outer: 68; inner: 60
Steel bolts	200	12×10^{-6}	each bolt: 12

2.104 The rigid bar $ABCD$ is supported by a pin at B and restrained by identical steel bars at C and D. Determine the forces in the bars caused by the vertical load P that is applied at A.

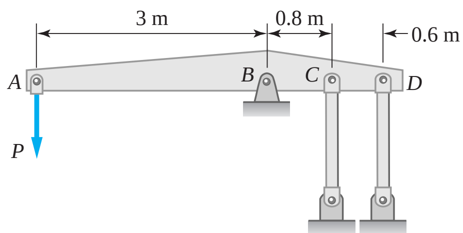


FIG. P2.104

2.105 The rigid bar ACE is supported by a pin at A and two horizontal aluminum rods, each of cross-sectional area 50 mm^2. When the 200-kN load is applied at point E, determine (a) the axial force in rod DE and (b) the vertical displacement of point E. Use $E = 70$ GPa for aluminum.

2.106 The two vertical steel rods that support the rigid bar $ABCD$ are initially stress-free. Determine the stress in each rod after the 20-kip load is applied. Neglect the weight of the bar and use $E = 29 \times 10^6$ psi for steel.

2.107 The rigid bar $ABCD$ of negligible weight is initially horizontal, and the steel rods attached at A and C are stress-free. The 20-kip load is then applied and the temperature of the steel rods is changed by ΔT. Find ΔT for which the stresses in the two steel rods will be equal. Use $\alpha = 6.5 \times 10^{-6}$/°F and $E = 29 \times 10^6$ psi for steel.

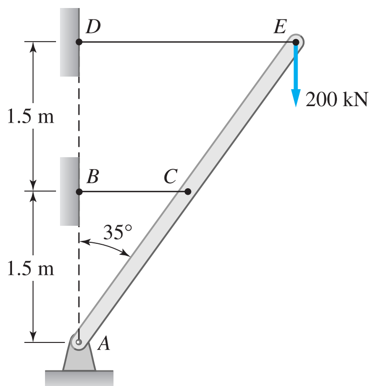


FIG. P2.105

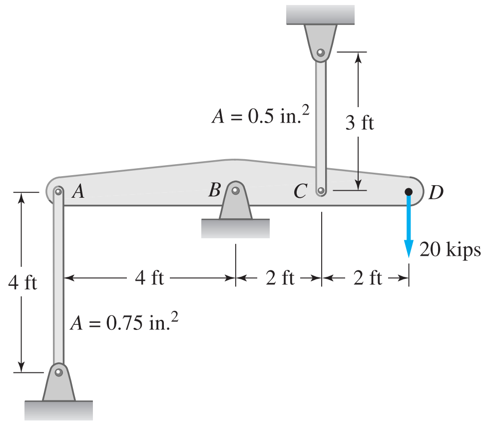


FIG. P2.106, P2.107

2.108 The rigid horizontal bar ABC of negligible mass is connected to two rods as shown in the figure. If the system is initially stress-free, calculate the temperature change that will cause a tensile stress of 90 MPa in the brass rod. Assume that both rods are subjected to the same change in temperature.

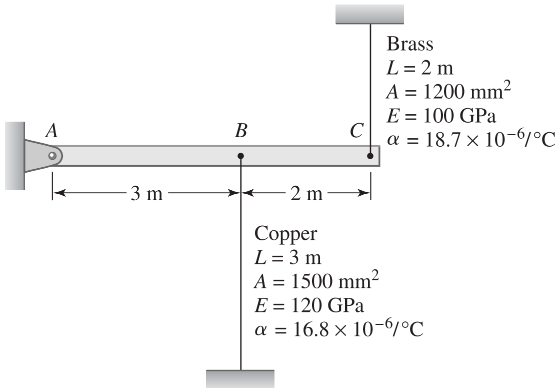


FIG. P2.108

Computer Problems

C2.1 The figure shows an aluminum bar of circular cross section with variable diameter. Use numerical integration to compute the elongation of the bar caused by the 6-kN axial force. Use $E = 70 \times 10^9$ Pa for aluminum.

C2.2 The flat aluminum bar shown in profile has a constant thickness of 10 mm. Determine the elongation of the bar caused by the 6-kN axial load using numerical integration. For aluminum $E = 70 \times 10^9$ Pa.

C2.3 The shaft of length L has diameter d that varies with the axial coordinate x. Given L, $d(x)$, and the modulus of elasticity E, write an algorithm to compute the axial stiffness $k = P/\delta$ of the bar. Use (a) $L = 500$ mm and

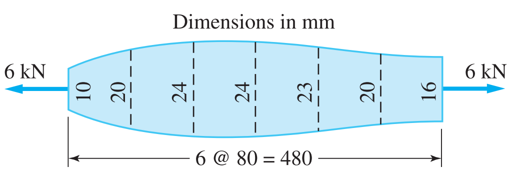


FIG. C2.1, C2.2

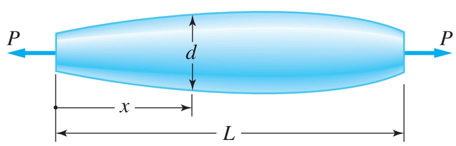


FIG. C2.3

$$d = (25\text{ mm})\left(1 + 3.8\frac{x}{L} - 3.6\frac{x^2}{L^2}\right)$$

and (b) $L = 200$ mm and

$$d = \begin{cases} (24 - 0.05x)\text{ mm} & \text{if } x \leq 120\text{ mm} \\ 18\text{ mm} & \text{if } x \geq 120\text{ mm} \end{cases}$$

C2.4 The symmetric truss carries a force P inclined at the angle θ to the vertical. Given P and the angle α, write an algorithm to plot the axial force in each member as a function of θ from $\theta = -90°$ to $90°$. Assume the cross-sectional areas of the members are the same. Use $P = 10$ kN and (a) $\alpha = 30°$; and (b) $\alpha = 60°$. (*Hint*: Compute the effects of the horizontal and vertical components of P separately, and then superimpose the effects.)

C2.5 The rigid bar BC of length b and negligible weight is supported by the wire AC of cross-sectional area A and modulus of elasticity E. The vertical displacement of point C can be expressed in the form

$$\Delta_C = \frac{Pb}{EA} f(\theta)$$

where θ is the angle between the wire and the rigid bar. (a) Derive the function $f(\theta)$ and plot it from $\theta = 20°$ to $85°$. (b) What value of θ yields the smallest vertical displacement of C?

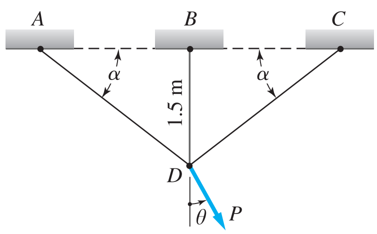


FIG. C2.4

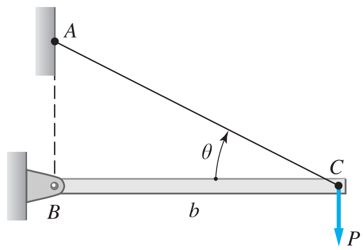


FIG. C2.5

C2.6 The steel bolt of cross-sectional area A_0 is placed inside the aluminum tube, also of cross-sectional area A_0. The assembly is completed by making the nut "finger-tight." The dimensions of the reduced segment of the bolt (length b and cross-sectional area A) are designed so that the segment will yield when the temperature of the assembly is increased by 200°F. Write an algorithm that determines the relationship between A/A_0 and b/L that satisfies this design requirement. Plot A/A_0 against b/L from $b/L = 0$ to 1.0. Use the properties of steel and aluminum shown in the figure.

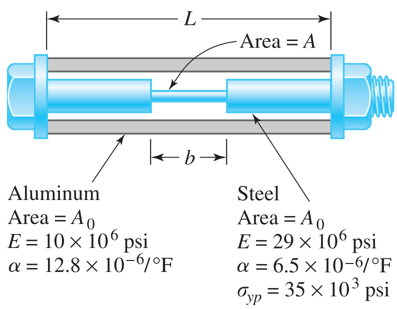


FIG. C2.6

3 Torsion

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The drive shaft of a twin-rotor helicopter. The power output of the turbine is transmitted to the rotors by the shaft. The relationship between transmitted power and shear stress in the shaft is one of the topics in this chapter. Courtesy of dutourdumonde/Shutterstock.

3.1 Introduction

In many engineering applications, members are required to carry torsional loads. In this chapter, we consider the torsion of circular shafts. Because a circular cross section is an efficient shape for resisting torsional loads, circular shafts are commonly used to transmit power in rotating machinery. We also discuss another important application—torsion of thin-walled tubes.

Torsion is our introduction to problems in which the stress is not uniform, or assumed to be uniform, over the cross section of the member. Another problem in this category, which we will treat later, is the bending of beams. Derivation of the equations used in the analysis of both torsion and bending follows these steps:

- Make simplifying assumptions about the deformation based on experimental evidence.
- Determine the strains that are geometrically compatible with the assumed deformations.
- Use Hooke's law to express the equations of compatibility in terms of stresses.
- Derive the equations of equilibrium. (These equations provide the relationships between the stresses and the applied loads.)

3.2 Torsion of Circular Shafts

a. Simplifying assumptions

Figure 3.1 shows the deformation of a circular shaft that is subjected to a twisting couple (torque) T. To visualize the deformation, we scribe the straight line AB on the surface of the shaft before the torque is applied. After loading, this line deforms into the helix AB' as the free end of the shaft rotates through the angle θ. During the deformation, the cross sections are not distorted in any manner—they remain plane, and the radius r does not change. In addition, the length L of the shaft remains constant. Based on these observations, we make the following assumptions:

- Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.
- Cross sections do not deform (there is no strain in the plane of the cross section).
- The distances between cross sections do not change (the axial normal strain is zero).

The deformation that results from the above assumptions is relatively simple: *Each cross section rotates as a rigid entity about the axis of the shaft.* Although this conclusion is based on the observed deformation of a cylindrical shaft carrying a constant internal torque, we assume that the result remains valid even if the diameter of the shaft or the internal torque varies along the length of the shaft.

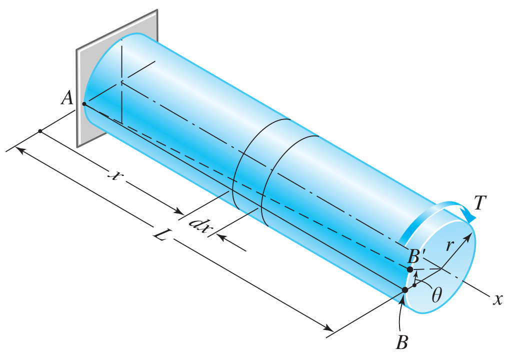


FIG. 3.1 Deformation of a circular shaft caused by the torque T. The initially straight line AB deforms into a helix.

b. Compatibility

To analyze the deformation in the interior of the shaft in Fig. 3.1, we consider the portion of the shaft shown in Fig. 3.2(a). We first isolate a segment of the shaft of infinitesimal length dx and then "peel" off its outer layer, leaving us with the cylindrical core of radius ρ. As the shaft deforms, the two cross sections of the segment rotate about the x-axis. Because the cross sections are separated by an infinitesimal distance, the difference in their rotations, denoted by the angle $d\theta$, is also infinitesimal. We now imagine that the straight line CD has been drawn on the cylindrical surface. As the cross sections undergo the relative rotation $d\theta$, CD deforms into the helix CD'. By observing the distortion of the shaded element, we recognize that the helix angle γ is the *shear strain* of the element.

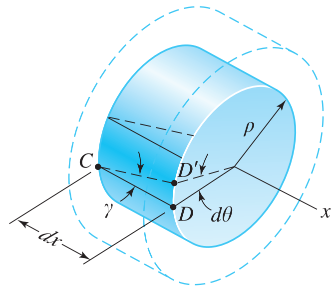
From the geometry of Fig. 3.2(a), we obtain $\overline{DD'} = \rho\, d\theta = \gamma\, dx$, from which the shear strain is

$$\gamma = \frac{d\theta}{dx}\rho \tag{3.1}$$

The quantity $d\theta/dx$ is the *angle of twist per unit length*, where θ is expressed in radians. The corresponding shear stress, illustrated in Fig. 3.2(b), is determined from Hooke's law:

$$\tau = G\gamma = G\frac{d\theta}{dx}\rho \tag{3.2}$$

Note that $G(d\theta/dx)$ in Eq. (3.2) is independent of the radial distance ρ. Therefore, *the shear stress varies linearly with the radial distance ρ from the axis of the shaft*. The variation of the shear stress acting on the cross section is illustrated in Fig. 3.3. The maximum shear stress, denoted by τ_{max}, occurs at the surface of the shaft.



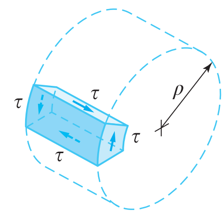


FIG. 3.2 (a) Shear strain of a material element caused by twisting of the shaft; (b) the corresponding shear stress.

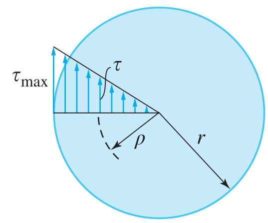


FIG. 3.3 Distribution of shear stress along the radius of a circular shaft.

c. Equilibrium

For the shaft to be in equilibrium, the resultant of the shear stress acting on a cross section must be equal to the internal torque T acting on that cross section. Figure 3.4 shows a cross section of the shaft containing a differential element of area dA located at the radial distance ρ from the axis of the shaft. The shear force acting on this area is $dP = \tau\, dA = G(d\theta/dx)\rho\, dA$, directed perpendicular to the radius. Hence, the moment (torque) of dP about the center O is $\rho\, dP = G(d\theta/dx)\rho^2\, dA$. Summing the contributions of all the differential elements across the cross-sectional area A and equating the result to the internal torque yields $\int_A \rho\, dP = T$, or

$$G\frac{d\theta}{dx}\int_A \rho^2\, dA = T$$

Recognizing that $\int_A \rho^2\, dA = J$ is (by definition) the *polar moment of inertia* of the cross-sectional area, we can write this equation as $G(d\theta/dx)J = T$, or

$$\frac{d\theta}{dx} = \frac{T}{GJ} \tag{3.3}$$

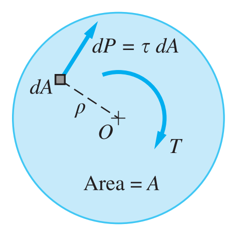


FIG. 3.4 Calculating the resultant of the shear stress acting on the cross section. Resultant is a couple equal to the internal torque T.

The rotation of the cross section at the free end of the shaft, called the *angle of twist*, is obtained by integration:

$$\theta = \int_0^L d\theta = \int_0^L \frac{T}{GJ}\, dx \tag{3.4a}$$

If the integrand is independent of x, as in the case of a prismatic bar carrying a constant torque, then Eq. (3.4a) reduces to the *torque-twist relationship*

$$\theta = \frac{TL}{GJ} \tag{3.4b}$$

Note the similarity between Eqs. (3.4) and the corresponding formulas for axial deformation: $\delta = \int_0^L (P/EA)\, dx$ and $\delta = PL/(EA)$.

Notes on the Computation of Angle of Twist

- It is common practice to let the units of G determine the units of the other terms in Eqs. (3.4). In the U.S. Customary system, the consistent units are G [psi], T [lb · in.], L [in.], and J [in.^{4}]; in the SI system, the consistent units are G [Pa], T [N · m], L [m], and J [m^4].
- The unit of θ in Eqs. (3.4) is radians, regardless of which system of units is used in the computation.
- In problems where it is convenient to use a sign convention for torques and angles of twist, we represent torques as vectors (we use double-headed arrows to represent couples and rotations) using the right-hand rule, as illustrated in Fig. 3.5. A torque vector is considered positive if it points away from the cross section, and negative if it points toward the cross section. The same sign convention applies to the angle of twist θ.

d. Torsion formulas

From Eq. (3.3) we see that $G(d\theta/dx) = T/J$, which, upon substitution into Eq. (3.2), gives the shear stress acting at the distance ρ from the center of the shaft:

$$\tau = \frac{T\rho}{J} \tag{3.5a}$$

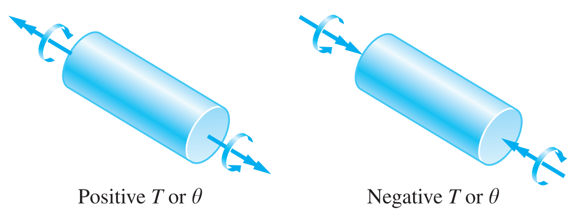


FIG. 3.5 Sign conventions for torque T and angle of twist θ.

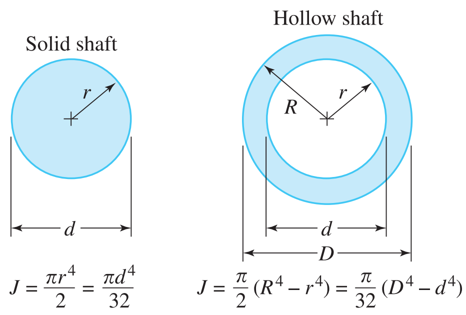


FIG. 3.6 Polar moments of inertia of circular areas.

The maximum shear stress is found by replacing ρ by the radius r of the shaft:

$$\tau_{max} = \frac{Tr}{J} \tag{3.5b}$$

Because Hooke's law was used in the derivation of Eqs. (3.2)–(3.5), these formulas are valid only if the shear stresses do not exceed the proportional limit of the material in shear.[1] Furthermore, these formulas are applicable only to circular shafts, either solid or hollow.

The expressions for the polar moments of circular areas are given in Fig. 3.6. Substituting these formulas into Eq. (3.5b), we obtain:

Solid shaft: $$\tau_{max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3} \tag{3.5c}$$

Hollow shaft: $$\tau_{max} = \frac{2TR}{\pi(R^4 - r^4)} = \frac{16TD}{\pi(D^4 - d^4)} \tag{3.5d}$$

Equations (3.5c) and (3.5d) are called the *torsion formulas*.

e. Power transmission

In many practical applications, shafts are used to transmit power. The power $\mathscr{P}$ transmitted by a torque T rotating at the angular speed ω is given by $\mathscr{P} = T\omega$, where ω is measured in radians per unit time. If the shaft is rotating with a frequency of f revolutions per unit time, then $\omega = 2\pi f$, which gives $\mathscr{P} = T(2\pi f)$. Therefore, the torque can be expressed as

[1] Equation (3.5b) is sometimes used to determine the "shear stress" corresponding to the torque at rupture, although the proportional limit is exceeded. The value so obtained is called the *torsional modulus of rupture*. It is used to compare the ultimate strengths of different materials and diameters.

$$T = \frac{\mathscr{P}}{2\pi f} \tag{3.6a}$$

In SI units, $\mathscr{P}$ in usually measured in watts (1.0 W = 1.0 N·m/s) and f in hertz (1.0 Hz = 1.0 rev/s); Eq. (3.6a) then determines the torque T in N·m. In U.S. Customary units with $\mathscr{P}$ in lb·in./s and f in hertz, Eq. (3.6a) calculates the torque T in lb·in. Because power in U.S. Customary units is often expressed in horsepower (1.0 hp = 550 lb·ft/s = 396×10^3 lb·in./min), a convenient form of Eq. (3.6a) is

$$T\ (\text{lb}\cdot\text{in.}) = \frac{\mathscr{P}\ (\text{hp})}{2\pi f\ (\text{rev/min})} \times \frac{396 \times 10^3\ (\text{lb}\cdot\text{in./min})}{1.0\ (\text{hp})}$$

which simplifies to

$$T\ (\text{lb}\cdot\text{in.}) = 63.0 \times 10^3 \frac{\mathscr{P}\ (\text{hp})}{f\ (\text{rev/min})} \tag{3.6b}$$

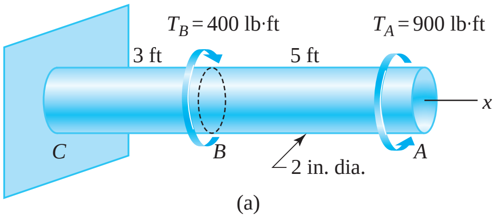
f. *Statically indeterminate problems*

The procedure for solving statically indeterminate torsion problems is similar to the steps presented in Sec. 2.5 for axially loaded bars:

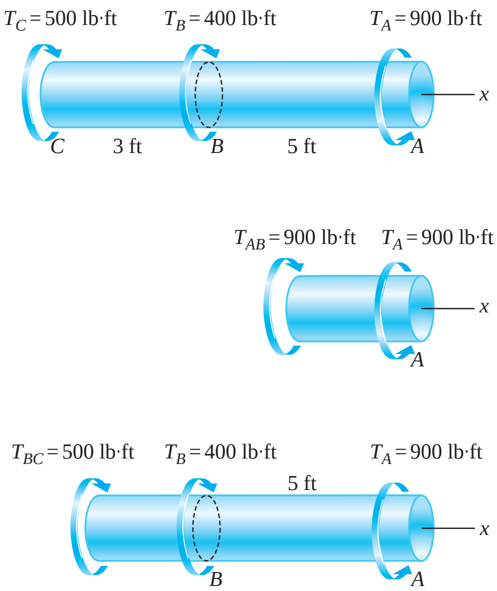
- Draw the required free-body diagrams and write the equations of **equilibrium**.
- Derive the **compatibility** equations from the restrictions imposed on the angles of twist.
- Use the **torque-twist relationships** in Eqs. (3.4) to express the angles of twist in the compatibility equations in terms of the torques.
- Solve the equations of equilibrium and compatibility for the torques.

Sample Problem **3.1**

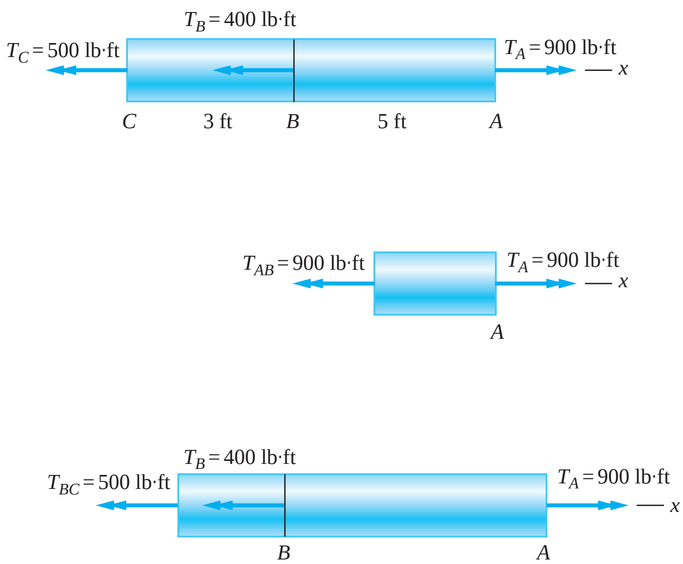
Figure (a) shows a 2-in.-diameter solid steel cylinder that is built into the support at C and subjected to the torques T_A and T_B. (1) Determine the maximum shear stresses in segments AB and BC of the cylinder; and (2) compute the angle of rotation of end A. Use $G = 12 \times 10^6$ psi for steel.



(a)



(b) FBDs



(c) FBDs (using the right-hand rule)

Solution

Preliminary calculations

Before we can find the required stresses and the rotation of end A, we must first use equilibrium analysis to determine the torque in each of the two segments of the cylinder.

Figure (b) displays three FBDs. The top FBD shows the torques acting upon the entire cylinder. The middle and bottom FBDs expose the internal torques acting on arbitrary sections of segments AB and BC, respectively. Applying the moment equilibrium equation, $\Sigma M_x = 0$, determines the reactive torque at C to be $T_C = 500 \text{ lb} \cdot \text{ft}$, with the torques in the segments being $T_{AB} = 900 \text{ lb} \cdot \text{ft}$ and $T_{BC} = 500 \text{ lb} \cdot \text{ft}$. Both internal torques are positive according to the sign convention in Fig. 3.5. Furthermore, note that the torque in each segment is constant.

You may find it convenient to use the equivalent FBDs shown in Fig. (c), where the torques are represented as double-headed vectors using the right-hand rule.

The polar moment of inertia for the cylinder is

$$J = \frac{\pi d^4}{32} = \frac{\pi (2)^4}{32} = 1.5708 \text{ in.}^4$$

Part 1

We calculate the maximum shear stress in each segment using Eq. (3.5b) as follows (converting the torques to pound-inches):

$$(\tau_{\max})_{AB} = \frac{T_{AB} r}{J} = \frac{(900 \times 12)(1.0)}{1.5708} = 6880 \text{ psi}$$ *Answer*

$$(\tau_{\max})_{BC} = \frac{T_{BC} r}{J} = \frac{(500 \times 12)(1.0)}{1.5708} = 3820 \text{ psi}$$ *Answer*

Part 2

The rotation of end A of the cylinder is obtained by summing the angles of twist of the two segments:

$$\theta_A = \theta_{A/B} + \theta_{B/C}$$

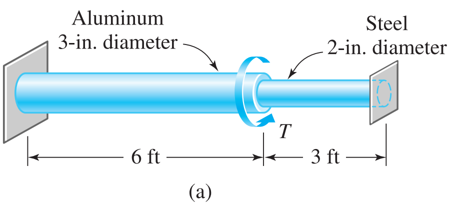
Using Eq. (3.4b), we obtain (converting the lengths to inches and the torques to pound-inches)

$$\theta_A = \frac{T_{AB}L_{AB} + T_{BC}L_{BC}}{GJ} = \frac{(900 \times 12)(5 \times 12) + (500 \times 12)(3 \times 12)}{(12 \times 10^6)(1.5708)}$$
$$= 0.045\,84 \text{rad} = 2.63^\circ$$ *Answer*

The positive result indicates that the rotation vector of A is in the positive x-direction; that is, θ_A is directed counterclockwise when viewed from A toward C.

Sample Problem **3.2**

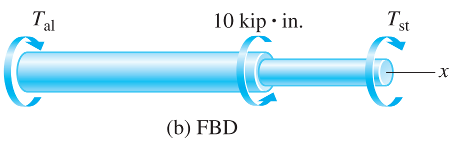
The shaft in Fig. (a) consists of a 3-in.-diameter aluminum segment that is rigidly joined to a 2-in.-diameter steel segment. The ends of the shaft are attached to rigid supports. Calculate the maximum shear stress developed in each segment when the torque $T = 10$ kip · in. is applied. Use $G = 4 \times 10^6$ psi for aluminum and $G = 12 \times 10^6$ psi for steel.



(a)

Solution

Equilibrium From the FBD of the entire shaft in Fig. (b), the equilibrium equation is



(b) FBD

$$\Sigma M_x = 0 \qquad (10 \times 10^3) - T_{st} - T_{al} = 0 \tag{a}$$

This problem is statically indeterminate because there are two unknown torques (T_{st} and T_{al}) but only one independent equilibrium equation.

Compatibility A second relationship between the torques is obtained by noting that the right end of the aluminum segment must rotate through the same angle as the left end of the steel segment. Therefore, the two segments must have the same angle of twist; that is, $\theta_{st} = \theta_{al}$. From Eq. (3.4b), this condition becomes

$$\left(\frac{TL}{GJ}\right)_{st} = \left(\frac{TL}{GJ}\right)_{al}$$

$$\frac{T_{st}(3 \times 12)}{(12 \times 10^6)\dfrac{\pi}{32}(2)^4} = \frac{T_{al}(6 \times 12)}{(4 \times 10^6)\dfrac{\pi}{32}(3)^4}$$

from which

$$T_{st} = 1.1852\ T_{al} \tag{b}$$

Solving Eqs. (a) and (b), we obtain

$$T_{al} = 4576\ \text{lb} \cdot \text{in.} \qquad T_{st} = 5424\ \text{lb} \cdot \text{in.}$$

From the torsion formula, Eq. (3.5c), the maximum shear stresses are

$$(\tau_{max})_{al} = \left(\frac{16T}{\pi d^3}\right)_{al} = \frac{16(4576)}{\pi(3)^3} = 863\ \text{psi}$$ *Answer*

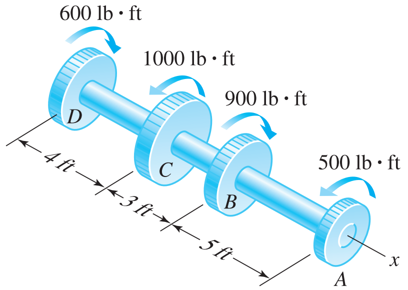
$$(\tau_{max})_{st} = \left(\frac{16T}{\pi d^3}\right)_{st} = \frac{16(5424)}{\pi(2)^3} = 3450\ \text{psi}$$ *Answer*

Sample Problem **3.3**

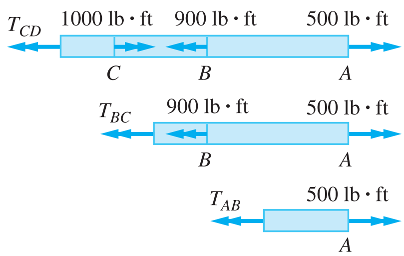
The four rigid gears, loaded as shown in Fig. (a), are attached to a 2-in.-diameter steel shaft. Compute the angle of rotation of gear A relative to gear D. Use $G = 12 \times 10^6$ psi for the shaft.

Solution

It is convenient to represent the torques as vectors (using the right-hand rule) on the FBDs in Fig. (b). We assume that the internal torques T_{AB}, T_{BC}, and T_{CD} are positive according to the sign convention introduced earlier (positive torque vectors point away from the cross section). Applying the equilibrium condition $\Sigma M_x = 0$ to each FBD, we obtain



(a)



(b) FBDs

$$500 - 900 + 1000 - T_{CD} = 0$$
$$500 - 900 - T_{BC} = 0$$
$$500 - T_{AB} = 0$$

which yield

$$T_{AB} = 500\ \text{lb}\cdot\text{ft} \qquad T_{BC} = -400\ \text{lb}\cdot\text{ft} \qquad T_{CD} = 600\ \text{lb}\cdot\text{ft}$$

The minus sign indicates that the sense of T_{BC} is opposite to that shown on the FBD.

The rotation of gear A relative to gear D can be viewed as the rotation of gear A if gear D were fixed. This rotation is obtained by summing the angles of twist of the three segments:

$$\theta_{A/D} = \theta_{A/B} + \theta_{B/C} + \theta_{C/D}$$

Using Eq. (3.4b), we obtain (converting the lengths to inches and torques to pound-inches)

$$\theta_{A/D} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}$$
$$= \frac{(500\times 12)(5\times 12) - (400\times 12)(3\times 12) + (600\times 12)(4\times 12)}{[\pi(2)^4/32](12\times 10^6)}$$
$$= 0.028\,27\ \text{rad} = 1.620^\circ$$ *Answer*

The positive result indicates that the rotation vector of A relative to D is in the positive x-direction; that is, θ_{AD} is directed counterclockwise when viewed from A toward D.

Sample Problem **3.4**

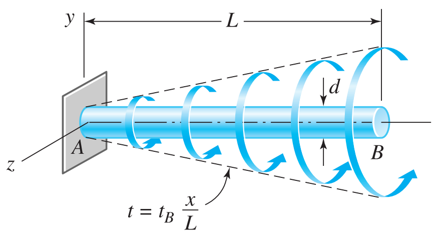
Figure (a) shows a steel shaft of length $L = 1.5$ m and diameter $d = 25$ mm that carries a distributed torque of intensity (torque per unit length) $t = t_B(x/L)$, where $t_B = 200\ \text{N}\cdot\text{m/m}$. Determine (1) the maximum shear stress in the shaft; and (2) the angle of twist of the shaft. Use $G = 80$ GPa for steel.

Solution

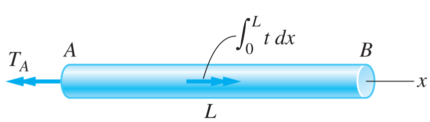
Part 1

Figure (b) shows the FBD of the shaft. The applied torque acting on a length dx of the shaft is $t\,dx$, so that the total torque applied to the shaft is $\int_0^L t\,dx$. The maximum torque in the shaft is T_A, which occurs at the fixed support. From the FBD we get

$$\Sigma M_x = 0 \qquad \int_0^L t\,dx - T_A = 0$$



(a)



(b) FBD

Therefore

$$T_A = \int_0^L t\,dx = \int_0^L t_B \frac{x}{L}\,dx = \frac{t_B L}{2}$$

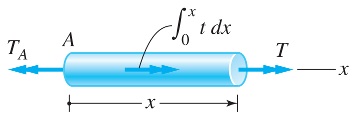
$$= \frac{1}{2}(200)(1.5) = 150\ \text{N} \cdot \text{m}$$

From Eq. (3.5c), the maximum shear stress in the shaft is

$$\tau_{\max} = \frac{16T_A}{\pi d^3} = \frac{16(150)}{\pi(0.025)^3} = 48.9 \times 10^6\ \text{Pa} = 48.9\ \text{MPa} \qquad \textit{Answer}$$

Part 2

The torque T acting on a cross section located at the distance x from the fixed end can be found from the FBD in Fig. (c):



(c) FBD

$$\Sigma M_x = 0 \qquad T + \int_0^x t\,dx - T_A = 0$$

which gives

$$T = T_A - \int_0^x t\,dx = \frac{t_B L}{2} - \int_0^x t_B \frac{x}{L}\,dx = \frac{t_B}{2L}(L^2 - x^2)$$

From Eq. (3.4a), the angle of twist of the shaft is

$$\theta = \int_0^L \frac{T}{GJ}\,dx = \frac{t_B}{2LGJ}\int_0^L (L^2 - x^2)\,dx = \frac{t_B L^2}{3GJ}$$

$$= \frac{200(1.5)^2}{3(80 \times 10^9)[(\pi/32)(0.025)^4]} = 0.0489\ \text{rad} = 2.80^\circ \qquad \textit{Answer}$$

Sample Problem **3.5**

A solid steel shaft in a rolling mill transmits 20 kW of power at 2 Hz. Determine the smallest safe diameter of the shaft if the shear stress is not to exceed 40 MPa and the angle of twist is limited to 6° in a length of 3 m. Use $G = 83$ GPa.

Solution

This problem illustrates a design that must possess sufficient strength as well as rigidity. We begin by applying Eq. (3.6a) to determine the torque:

$$T = \frac{\mathscr{P}}{2\pi f} = \frac{20 \times 10^3}{2\pi(2)} = 1591.5\ \text{N} \cdot \text{m}$$

To satisfy the strength condition, we apply the torsion formula, Eq. (3.5c):

$$\tau_{\max} = \frac{16T}{\pi d^3} \qquad 40 \times 10^6 = \frac{16(1591.5)}{\pi d^3}$$

which yields $d = 58.7 \times 10^{-3}$ m = 58.7 mm.

We next apply the torque-twist relationship, Eq. (3.4b), to determine the diameter necessary to satisfy the requirement of rigidity (remembering to convert θ from degrees to radians):

$$\theta = \frac{TL}{GJ} \qquad 6\left(\frac{\pi}{180}\right) = \frac{1591.5(3)}{(83 \times 10^9)(\pi d^4/32)}$$

from which we obtain $d = 48.6 \times 10^{-3}$ m = 48.6 mm.

To satisfy both strength and rigidity requirements, we must choose the larger diameter—namely,

$$d = 58.7\ \text{mm} \qquad \textit{Answer}$$

Problems

3.1 The steel shaft, 3 ft long and 4 in. in diameter, carries the end torque of 15 kip · ft. Determine (a) the maximum shear stress in the shaft; and (b) the angle of twist of the shaft. Use $G = 12 \times 10^6$ psi for steel.

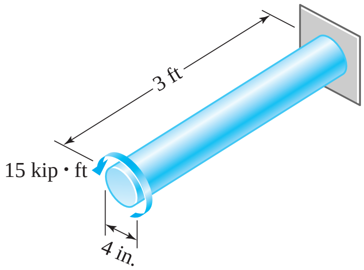


FIG. P3.1

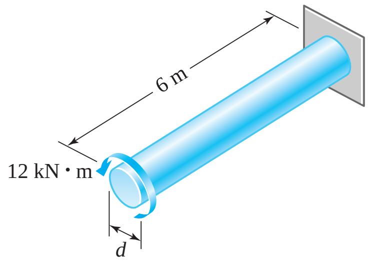


FIG. P3.2

3.2 The 12 kN · m torque is applied to the free end of the 6-m steel shaft. The angle of rotation of the shaft is to be limited to 3°. (a) Find the diameter d of the smallest shaft that can be used. (b) What will be the maximum shear stress in the shaft? Use $G = 83$ GPa for steel.

3.3 The torque of 100 kip · ft produces a maximum shear stress of 8000 psi in the 16-ft-long hollow steel shaft. Note that the inner diameter of the shaft is two-thirds of its outer diameter D. (a) Determine the outer diameter D. (b) Find the angle of twist of the shaft. Use $G = 12 \times 10^6$ psi for steel.

3.4 The inner diameter of the hollow shaft is one-half its outer diameter D. Show that the maximum torque that can be carried by this shaft is 15/16th of the maximum torque that could be carried by a solid shaft of diameter D that is made of the same material.

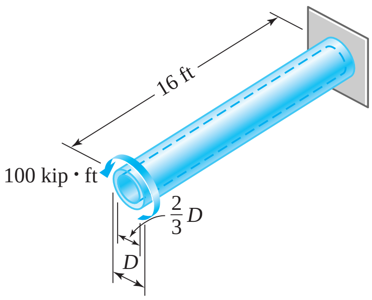


FIG. P3.3

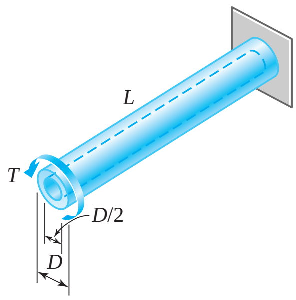


FIG. P3.4

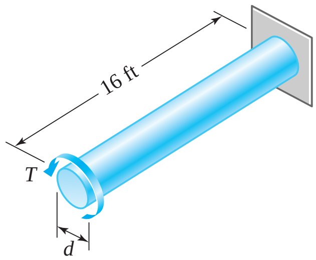


FIG. P3.5

3.5 The 16-ft solid steel shaft is twisted through 4°. If the maximum shear stress is 8000 psi, determine the diameter d of the shaft. Use $G = 12 \times 10^6$ psi for steel.

3.6 Two forces, each of magnitude P, are applied to the wrench. The diameter of the steel shaft AB is 15 mm. Determine the largest allowable value of P if the shear stress in the shaft is not to exceed 120 MPa and its angle of twist is limited to 5°. Use $G = 80$ GPa for steel.

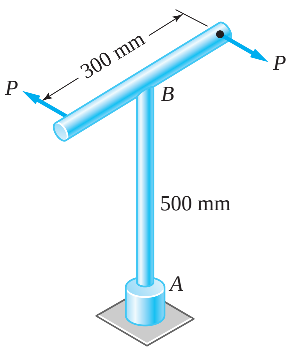


FIG. P3.6

3.7 The 1.25-in.-diameter steel shaft BC is built into the rigid wall at C and supported by a smooth bearing at B. The lever AB is welded to the end of the shaft. Determine the force P that will produce a 2-in. vertical displacement of end A of the lever. What is the corresponding maximum shear stress in the shaft? Use $G = 12 \times 10^6$ psi for steel, and neglect deformation of the lever.

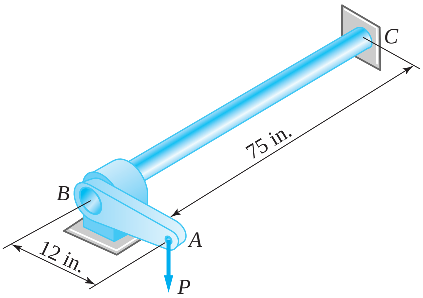


FIG. P3.7

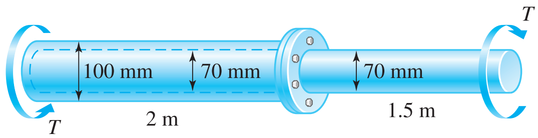


FIG. P3.8

3.8 The steel shaft is formed by attaching a hollow shaft to a solid shaft. Determine the maximum torque T that can be applied to the ends of the shaft without exceeding a shear stress of 70 MPa or an angle of twist of 2.5° in the 3.5-m length. Use $G = 83$ GPa for steel.

3.9 The compound shaft consists of bronze and steel segments, both having 120-mm diameters. If the torque T causes a maximum shear stress of 100 MPa in the bronze segment, determine the angle of rotation of the free end. Use $G = 83$ GPa for steel and $G = 35$ GPa for bronze.

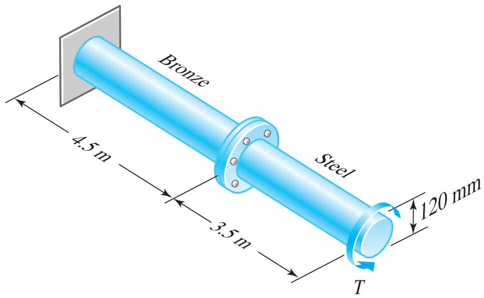


FIG. P3.9

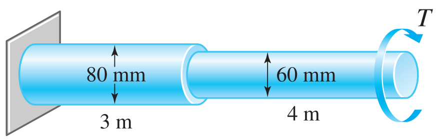


FIG. P3.10

3.10 The stepped steel shaft carries the torque T. Determine the maximum allowable magnitude of T if the working shear stress is 12 MPa and the rotation of the free end is limited to 4°. Use $G = 83$ GPa for steel.

3.11 The solid steel shaft carries the torques $T_1 = 750$ N · m and $T_2 = 1200$ N · m. Using $L_1 = L_2 = 2.5$ m and $G = 83$ GPa, determine the smallest allowable diameter of the shaft if the shear stress is limited to 60 MPa and the angle of rotation of the free end is not to exceed 4°.

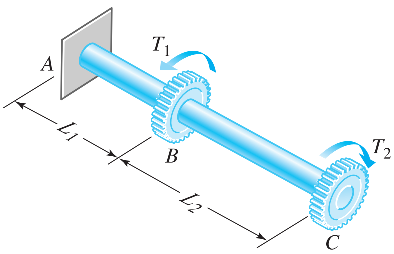


FIG. P3.11

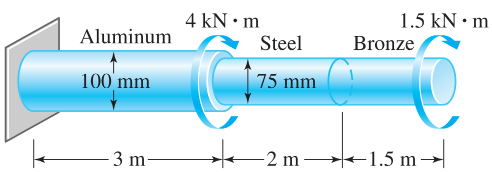


FIG. P3.12

3.12 The solid compound shaft, made of three different materials, carries the two torques shown. (a) Calculate the maximum shear stress in each material. (b) Find the angle of rotation of the free end of the shaft. The shear moduli are 28 GPa for aluminum, 83 GPa for steel, and 35 GPa for bronze.

3.13 The shaft consisting of steel and aluminum segments carries the torques T and $2T$. Find the largest allowable value of T if the working shear stresses are 14 000 psi for steel and 7500 psi for aluminum, and the angle of rotation at the free end must not exceed 8°. Use $G = 12 \times 10^6$ psi for steel and $G = 4 \times 10^6$ psi for aluminum.

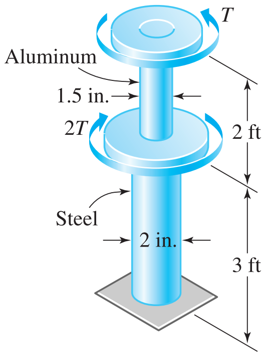


FIG. P3.13

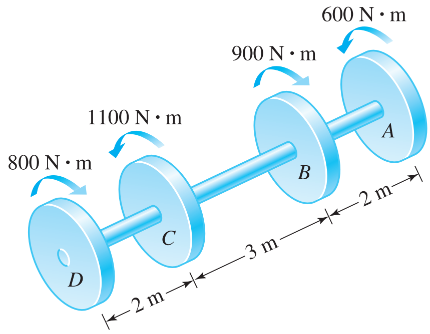


FIG. P3.14

3.14 Four pulleys are attached to the 50-mm-diameter aluminum shaft. If torques are applied to the pulleys as shown in the figure, determine the angle of rotation of pulley D relative to pulley A. Use $G = 28$ GPa for aluminum.

3.15 The tapered, wrought iron shaft carries the torque $T = 2000$ lb · in. at its free end. Determine the angle of twist of the shaft. Use $G = 10 \times 10^6$ psi for wrought iron.

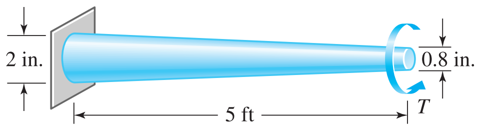


FIG. P3.15

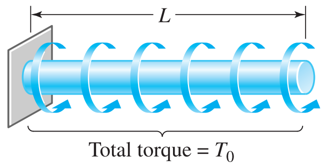


FIG. P3.16

3.16 The shaft carries a total torque T_0 that is uniformly distributed over its length L. Determine the angle of twist of the shaft in terms of T_0, L, G, and J.

3.17 The steel shaft of length $L = 1.5$ m and diameter $d = 25$ mm is attached to rigid walls at both ends. A distributed torque of intensity $t = t_A(L - x)/L$ is acting on the shaft, where $t_A = 200$ N · m/m. Determine the maximum shear stress in the shaft.

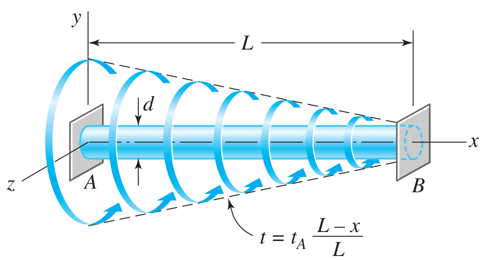


FIG. P3.17

3.18 The compound shaft is attached to a rigid wall at each end. For the bronze segment AB, the diameter is 75 mm and $G = 35$ GPa. For the steel segment BC, the diameter is 50 mm and $G = 83$ GPa. Given that $a = 2$ m and $b = 1.5$ m, compute the largest torque T that can be applied as shown in the figure if the maximum shear stress is limited to 60 MPa in the bronze and 80 MPa in the steel.

3.19 For the compound shaft described in Prob. 3.18, determine the torque T and the ratio b/a so that each material is stressed to its permissible limit.

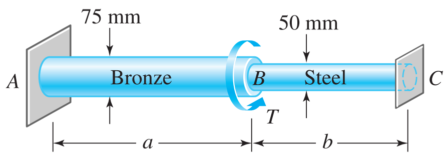


FIG. P3.18, P3.19

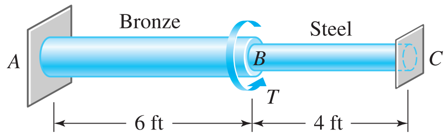


FIG. P3.20

3.20 The ends of the compound shaft are attached to rigid walls. The maximum shear stress is limited to 10 000 psi for the bronze segment AB and 14 000 psi for the steel segment BC. Determine the diameter of each segment so that each material is simultaneously stressed to its permissible limit when the torque $T = 16$ kip · ft is applied as shown. The shear moduli are 6×10^6 psi for bronze and 12×10^6 psi for steel.

3.21 Both ends of the steel shaft are attached to rigid supports. Find the distance a where the torque T must be applied so that the reactive torques at A and B are equal.

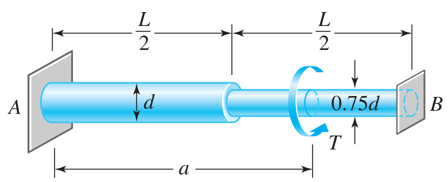


FIG. P3.21

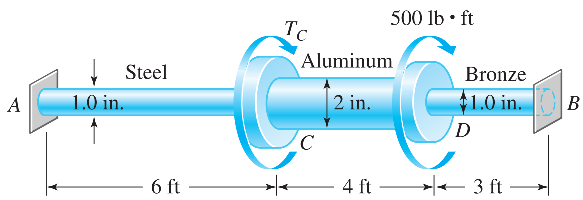


FIG. P3.22

3.22 The compound shaft, composed of steel, aluminum, and bronze segments, carries the two torques shown in the figure. If $T_C = 250$ lb · ft, determine the maximum shear stress developed in each material. The moduli of rigidity for steel, aluminum, and bronze are 12×10^6 psi, 4×10^6 psi, and 6×10^6 psi, respectively.

3.23 The stepped solid steel shaft ABC is attached to rigid supports at each end. Determine the diameter of segment BC for which the maximum shear stress in both segments will be equal when the torque T is applied at B. Note that the lengths of both segments are given and the diameter of segment AB is 60 mm.

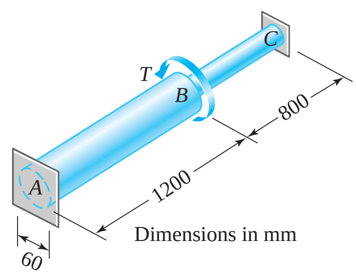


FIG. P3.23

3.24 The steel rod fits loosely inside the aluminum sleeve. Both components are attached to a rigid wall at A and joined together by a pin at B. Because of a slight misalignment of the pre-drilled holes, the torque $T_0 = 750$ N · m was applied to the steel rod before the pin could be inserted into the holes. Determine the torque in each component after T_0 was removed. Use $G = 80$ GPa for steel and $G = 28$ GPa for aluminum.

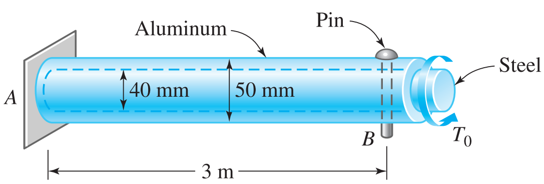


FIG. P3.24

3.25 A composite shaft is made by slipping a bronze tube of 3-in. outer diameter and 2-in. inner diameter over a solid steel shaft of the same length and 2-in. diameter. The two components are then fastened rigidly together at their ends. What is the largest torque that can be carried by the composite shaft if the working shear stresses are 10 ksi for bronze and 14 ksi for the steel? For bronze, $G = 6 \times 10^6$ psi, and for steel, $G = 12 \times 10^6$ psi.

3.26 If the composite shaft described in Prob. 3.25 carries a 2000-lb · ft torque, determine the maximum shear stress in each material.

3.27 The two identical shafts, 1 and 2, are built into supports at their left ends. Gears mounted on their right ends engage a third gear that is attached to shaft 3. Determine the torques in shafts 1 and 2 when the 500-N · m torque is applied to shaft 3.

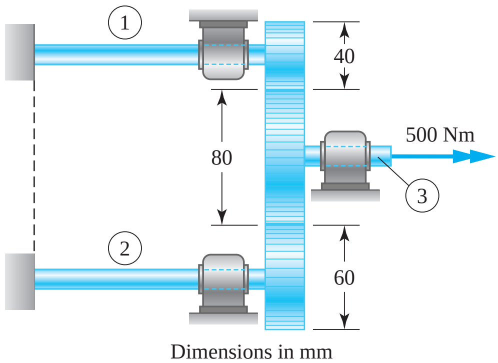


FIG. P3.27

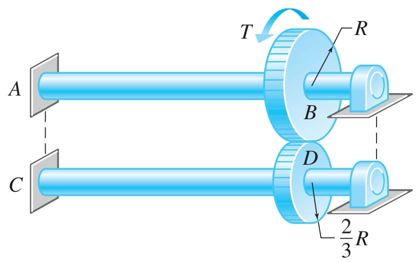


FIG. P3.28

3.28 Each of the two identical shafts is attached to a rigid wall at one end and supported by a bearing at the other end. The gears attached to the shafts are in mesh. Determine the reactive torques at A and C when the torque T is applied to gear B.

3.29 The two steel shafts, each with one end built into a rigid support, have flanges attached to their free ends. The flanges are to be bolted together. However, initially there is a 6° mismatch in the location of the bolt holes as shown in the figure. Determine the maximum shear stress in each shaft after the flanges have been bolted together. The shear modulus of elasticity for steel is 12×10^6 psi. Neglect deformations of the bolts and the flanges.

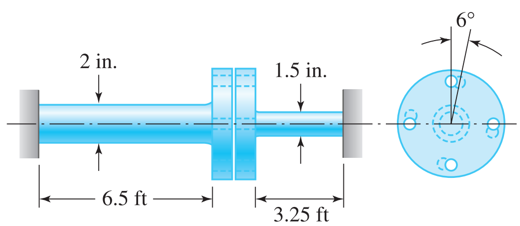


FIG. P3.29

3.30 A solid steel shaft transmits 20 hp while running at 120 rev/min. Find the smallest safe diameter of the shaft if the shear stress is limited to 5000 psi and the angle of twist of the shaft is not to exceed 9° in a length of 10 ft. Use $G = 12 \times 10^6$ psi for steel.

3.31 A hollow steel shaft, 6 ft long, has an outer diameter of 3 in. and an inner diameter of 1.5 in. The shaft is transmitting 200 hp at 120 rev/min. Determine (a) the maximum shear stress in the shaft; and (b) the angle of twist of the shaft in degrees. Use $G = 12 \times 10^6$ psi for steel.

3.32 A hollow steel propeller shaft, 18 ft long with 14-in. outer diameter and 10-in. inner diameter, transmits 5000 hp at 189 rev/min. Use $G = 12 \times 10^6$ psi for steel. Calculate (a) the maximum shear stress; and (b) the angle of twist of the shaft.

3.33 The figure shows an inboard engine, 8-ft long steel drive shaft, and propeller for a motor boat. The shaft is to be designed to safely transmit 200 hp at 3500 rev/min. Determine the diameter of the smallest shaft that can be used and its corresponding angle of twist. For the steel, use a working shear stress of 12 000 psi and $G = 12 \times 10^6$ psi.

FIG. P3.33

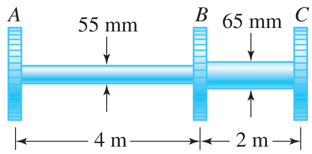


FIG. P3.34

3.34 The steel shaft with two different diameters rotates at 4 Hz. The power supplied to gear C is 55 kW, of which 35 kW is removed by gear A and 20 kW is removed by gear B. Find (a) the maximum shear stress in the shaft; and (b) the angle of rotation of gear A relative to gear C. Use $G = 83$ GPa for steel.

3.35 The motor A delivers 3000 hp to the shaft at 1500 rev/min, of which 1000 hp is removed by gear B and 2000 hp is removed by gear C. Determine (a) the maximum shear stress in the shaft; and (b) the angle of twist of end D relative to end A. Use $G = 12 \times 10^6$ psi for steel, and assume that friction at bearing D is negligible.

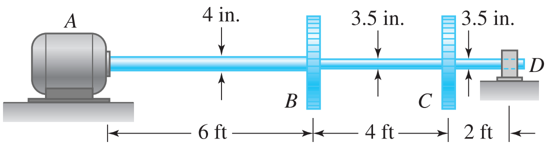


FIG. P3.35

3.3 Torsion of Thin-Walled Tubes

Although torsion of noncircular shafts requires advanced methods of analysis, fairly simple approximate formulas are available for thin-walled tubes. Such members are common in construction where light weight is of paramount importance, such as in automobiles and airplanes.

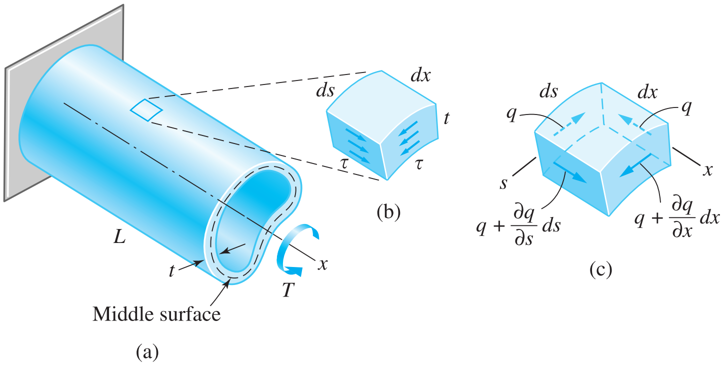


FIG. 3.7 (a) Thin-walled tube in torsion; (b) shear stress in the wall of the tube; (c) shear flows on wall element.

Consider the thin-walled tube subjected to the torque T shown in Fig. 3.7(a). We assume the tube to be prismatic (constant cross section), but the wall thickness t is allowed to vary within the cross section. The surface that lies midway between the inner and outer boundaries of the tube is called the *middle surface*. If t is small compared to the overall dimensions of the cross section, the shear stress τ induced by torsion can be shown to be almost constant through the wall thickness of the tube and directed tangent to the middle surface, as illustrated in Fig. 3.7(b). It is convenient to introduce the concept of *shear flow* q, defined as the shear force per unit edge length of the middle surface. Thus, the shear flow is

$$q = \tau t \tag{3.7}$$

If the shear stress is not constant through the wall thickness, then τ in Eq. (3.7) should be viewed as the average shear stress.

We now show that the *shear flow is constant throughout the tube*. This result can be obtained by considering equilibrium of the element shown in Fig. 3.7(c). In labeling the shear flows, we assume that q varies in the longitudinal (x) as well as the circumferential (s) directions. Thus, the terms $(\partial q/\partial x)\,dx$ and $(\partial q/\partial s)\,ds$ represent the changes in the shear flow over the distances dx and ds, respectively. The force acting on each side of the element is equal to the shear flow multiplied by the edge length, resulting in the equilibrium equations

$$\Sigma F_x = 0 \qquad \left(q + \frac{\partial q}{\partial s}\,ds\right) dx - q\,dx = 0$$

$$\Sigma F_s = 0 \qquad \left(q + \frac{\partial q}{\partial x}\,dx\right) ds - q\,ds = 0$$

which yield $\partial q/\partial x = \partial q/\partial s = 0$, thereby proving that the shear flow is constant throughout the tube.

To relate the shear flow to the applied torque T, consider the cross section of the tube in Fig. 3.8. The shear force acting over the infinitesimal edge length ds of the middle surface is $dP = q\,ds$. The moment of this force about an arbitrary point O in the cross section is $r\,dP = (q\,ds)r$, where r is the perpendicular distance of O from the line of action of dP. Equilibrium requires that the sum of these moments must be equal to the applied torque T; that is,

$$T = \oint_S qr\,ds \tag{a}$$

where the integral is taken over the closed curve formed by the intersection of the middle surface and the cross section, called the *median line*.

The integral in Eq. (a) need not be evaluated formally. Recalling that q is constant, we can take it outside the integral sign, so that Eq. (a) can be written as $T = q\oint_S r\,ds$. But from Fig. 3.8 we see that $r\,ds = 2\,dA_0$, where dA_0 is the area of the shaded triangle. Therefore, $\oint_S r\,ds = 2A_0$, where A_0 is the area of the cross section that is enclosed by the median line. Consequently, Eq. (a) becomes

$$T = 2A_0 q \tag{3.8a}$$

from which the shear flow is

$$q = \frac{T}{2A_0} \tag{3.8b}$$

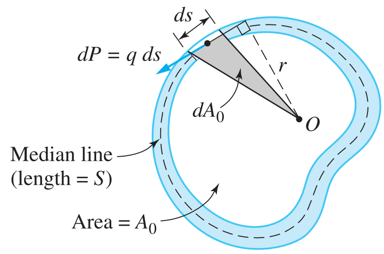


FIG. 3.8 Calculating the resultant of the shear flow acting on the cross section of the tube. Resultant is a couple equal to the internal torque T.

We can find the angle of twist of the tube by equating the work done by the shear stress in the tube to the work of the applied torque T. Let us start by determining the work done by the shear flow acting on the element in Fig. 3.7(c). The deformation of the element is shown in Fig. 3.9, where γ is the shear strain of the element. We see that work is done on the element by the shear force $dP = q\,ds$ as it moves through the distance $\gamma\,dx$. If we assume that γ is proportional to τ (Hooke's law), this work is

$$dU = \frac{1}{2}(\text{force} \times \text{distance}) = \frac{1}{2}(q\,ds)(\gamma\,dx)$$

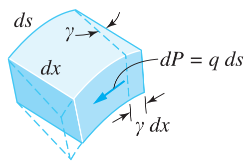


FIG. 3.9 Deformation of element caused by shear flow.

Substituting $\gamma = \tau/G = q/(Gt)$ yields

$$dU = \frac{q^2}{2Gt}\,ds\,dx \tag{b}$$

The work U of the shear flow for the entire tube is obtained by integrating Eq. (b) over the middle surface of the tube. Noting that q and G are constants and t is independent of x, we obtain

$$U = \frac{q^2}{2G}\int_0^L \left(\oint_S \frac{ds}{t}\right) dx = \frac{q^2 L}{2G}\oint_S \frac{ds}{t} \tag{c}$$

Conservation of energy requires U to be equal to the work of the applied torque; that is, $U = T\theta/2$. After substituting the expression for q from Eq. (3.8b) into Eq. (c), we obtain

$$\left(\frac{T}{2A_0}\right)^2 \frac{L}{2G}\oint_S \frac{ds}{t} = \frac{1}{2}T\theta$$

from which the angle of twist of the tube is

$$\theta = \frac{TL}{4GA_0^2} \oint_S \frac{ds}{t} \tag{3.9a}$$

If t is constant, we have $\oint_S (ds/t) = S/t$, where S is the length of the median line. Therefore, Eq. (3.9a) becomes

$$\theta = \frac{TLS}{4GA_0^2 t} \quad \text{(constant } t\text{)} \tag{3.9b}$$

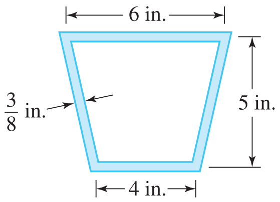
If the tube is not cylindrical, its cross sections do not remain plane but tend to warp. When the ends of the tube are attached to rigid plates or supports, the end sections cannot warp. As a result, the torsional stiffness of the tube is increased and the state of stress becomes more complicated—there are normal stresses in addition to the shear stress. However, if the tube is slender (length much greater than the cross-sectional dimensions), warping is confined to relatively small regions near the ends of the tube (Saint Venant's principle).

Tubes with very thin walls can fail by buckling (the walls "fold" like an accordion) while the stresses are still within their elastic ranges. For this reason, the use of very thin walls is not recommended. In general, the shear stress that results in buckling depends on the shape of the cross section and the material properties. For example, steel tubes of circular cross section require $r/t < 50$ to forestall buckling due to torsion.

Sharp re-entrant corners in the cross section of the tube should also be avoided because they cause stress concentration. It has been found that the shear stress at the inside boundary of a corner can be considerably higher than the average stress. The stress concentration effect diminishes as the radius a of the corner is increased, becoming negligible when $a/t > 2.5$, approximately.

Sample Problem **3.6**

A steel tube with the cross section shown carries a torque T. The tube is 6 ft long and has a constant wall thickness of 3/8 in. (1) Compute the torsional stiffness $k = T/\theta$ of the tube. (2) If the tube is twisted through 0.5°, determine the shear stress in the wall of the tube. Use $G = 12 \times 10^6$ psi, and neglect stress concentrations at the corners.



Solution

Part 1

Because the wall thickness is constant, the angle of twist is given by Eq. (3.9b):

$$\theta = \frac{TLS}{4GA_0^2 t}$$

Therefore, the torsional stiffness of the tube can be computed from

$$k = \frac{T}{\theta} = \frac{4GA_0^2 t}{LS}$$

The area enclosed by the median line is

$$A_0 = \text{average width} \times \text{height} = \left(\frac{6+4}{2}\right)(5) = 25 \text{ in.}^2$$

and the length of the median line is

$$S = 6 + 4 + 2\sqrt{1^2 + 5^2} = 20.20 \text{ in.}$$

Consequently, the torsional stiffness becomes

$$k = \frac{4(12 \times 10^6)(25)^2(3/8)}{(6 \times 12)(20.20)} = 7.735 \times 10^6 \text{ lb} \cdot \text{in./rad}$$

$$= 135.0 \times 10^3 \text{ lb} \cdot \text{in./deg}$$ *Answer*

Part 2

The torque required to produce an angle of twist of 0.5° is

$$T = k\theta = (135.0 \times 10^3)(0.5) = 67.5 \times 10^3 \text{ lb} \cdot \text{in.}$$

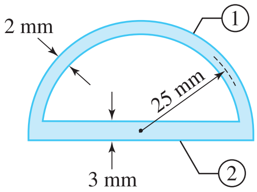
which results in the shear flow

$$q = \frac{T}{2A_0} = \frac{67.5 \times 10^3}{2(25)} = 1350 \text{ lb/in.}$$

The corresponding shear stress is

$$\tau = \frac{q}{t} = \frac{1350}{3/8} = 3600 \text{ psi}$$ *Answer*

Sample Problem 3.7



An aluminum tube, 1.2 m long, has the semicircular cross section shown in the figure. If stress concentrations at the corners are neglected, determine (1) the torque that causes a maximum shear stress of 40 MPa, and (2) the corresponding angle of twist of the tube. Use $G = 28$ GPa for aluminum.

Solution

Part 1

Because the shear flow is constant in a prismatic tube, the maximum shear stress occurs in the thinnest part of the wall, which is the semicircular portion with $t = 2$ mm. Therefore, the shear flow that causes a maximum shear stress of 40 MPa is

$$q = \tau t = (40 \times 10^6)(0.002) = 80 \times 10^3 \text{ N/m}$$

The cross-sectional area enclosed by the median line is

$$A_0 = \frac{\pi r^2}{2} = \frac{\pi (0.025)^2}{2} = 0.9817 \times 10^{-3} \text{ m}^2$$

which results in the torque—see Eq. (3.8a):

$$T = 2A_0 q = 2(0.9817 \times 10^{-3})(80 \times 10^3) = 157.07 \text{ N} \cdot \text{m}$$ *Answer*

Part 2

The cross section consists of two parts, labeled (1) and (2) in the figure, each having a constant thickness. Hence, we can write

$$\oint_S \frac{ds}{t} = \frac{1}{t_1}\int_{S_1} ds + \frac{1}{t_2}\int_{S_2} ds = \frac{S_1}{t_1} + \frac{S_2}{t_2}$$

where S_1 and S_2 are the lengths of the median lines of parts (1) and (2), respectively. Therefore,

$$\oint_S \frac{ds}{t} = \frac{\pi r}{t_1} + \frac{2r}{t_2} = \frac{\pi(25)}{2} + \frac{2(25)}{3} = 55.94$$

and Eq. (3.9a) yields for the angle of twist

$$\theta = \frac{TL}{4GA_0^2}\oint_S \frac{ds}{t} = \frac{157.07(1.2)}{4(28 \times 10^9)(0.9817 \times 10^{-3})^2}(55.94)$$

$$= 0.0977 \text{ rad} = 5.60^\circ$$ *Answer*

Problems

Neglect stress concentrations at the corners of the tubes in the following problems.

3.36 Consider a thin cylindrical tube of mean radius $\bar{r}$, constant thickness t, and length L. (a) Show that the polar moment of inertia of the cross-sectional area can be approximated by $J = 2\pi\bar{r}^3t$. (b) Use this approximation to show that Eqs. (3.8b) and (3.9b) are equivalent to $\tau = T\bar{r}/J$ and $\theta = TL/(GJ)$, respectively.

3.37 A cylindrical metal tube of mean radius $r = 5$ in., length $L = 14$ ft, and shear modulus $G = 11 \times 10^6$ psi carries the torque $T = 320$ kip · in. Determine the smallest allowable constant wall thickness t if the shear stress is limited to 12 ksi and the angle of twist is not to exceed 2°.

3.38 A cylindrical tube of constant wall thickness t and inside radius $r = 10t$ carries a torque T. Find the expression for the maximum shear stress in the tube using (a) the torsion formula for a hollow shaft in Eq. (3.5d); and (b) the thin-walled tube formula in Eq. (3.8b). What is the percentage error in the thin-walled tube approximation?

3.39 A torque of 800 N · m is applied to a tube with the rectangular cross section shown in the figure. Determine the smallest allowable constant wall thickness t if the shear stress is not to exceed 90 MPa.

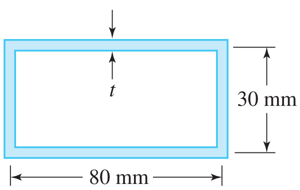


FIG. P3.39

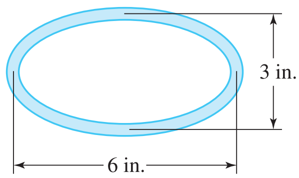


FIG. P3.40

3.40 The constant wall thickness of a tube with the elliptical cross section shown is 0.12 in. What torque will cause a shear stress of 6000 psi?

3.41 The constant wall thickness of a steel tube with the cross section shown is 2 mm. If a 600-N · m torque is applied to the tube, find (a) the shear stress in the wall of the tube; and (b) the angle of twist per meter of length. Use $G = 80$ GPa for steel.

3.42 Two identical metal sheets are formed into tubes with the circular and square cross sections shown. If the same torque is applied to each tube, determine the ratios (a) $\tau_{circle}/\tau_{square}$ of the shear stresses; and (b) $\theta_{circle}/\theta_{square}$ of the angles of twist.

3.43 A steel tube with the cross section shown carries a 50-kip · in. torque. Determine (a) the maximum shear stress in the tube; and (b) the angle of twist per foot of length. Use $G = 11 \times 10^6$ psi for steel.

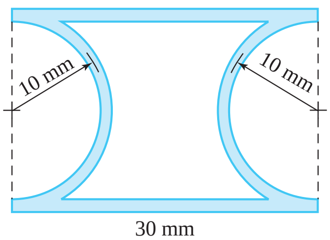


FIG. P3.41

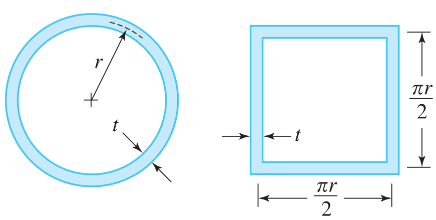


FIG. P3.42

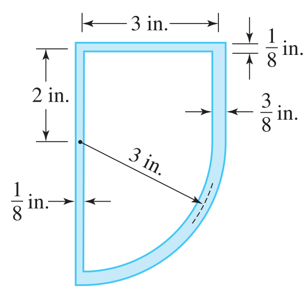


FIG. P3.43

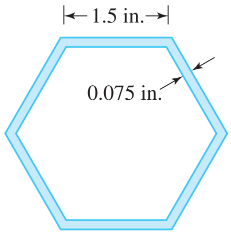


FIG. P3.44

3.44 An aluminum tube with the hexagonal cross section shown is 2.5 ft long and has a constant wall thickness of 0.080 in. Find (a) the largest torque that the tube can carry if the shear stress is limited to 7200 psi; and (b) the angle of twist caused by this torque. Use $G = 4 \times 10^6$ psi for aluminum.

3.45 A 4-ft-long tube with the cross section shown in the figure is made of aluminum. Find the torque that will cause a maximum shear stress of 10 000 psi. Use $G = 4 \times 10^6$ psi for aluminum.

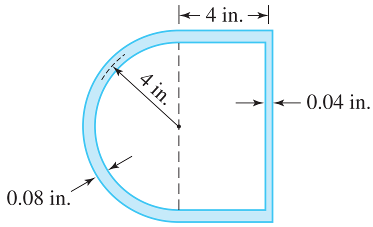


FIG. P3.45

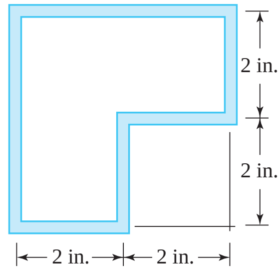


FIG. P3.46

3.46 A steel tube with the cross section shown is 6 ft long and has a wall thickness of 0.12 in. (a) If the allowable shear stress is 8000 psi, determine the largest torque that can be applied safely to the tube. (b) Compute the corresponding angle of twist. Use $G = 12 \times 10^6$ psi for steel.

3.47 The segment AB of the steel torsion bar is a cylindrical tube of constant 2-mm wall thickness. Segment BC is a square tube with a constant wall thickness of 3 mm. The outer dimensions of the cross sections are shown in the figure. The tubes are attached to a rigid bracket at B, which is loaded by a couple formed by the forces P. Determine the largest value of P if the shear stress in either tube is limited to 60 MPa.

***3.48** The tapered, circular, thin-walled tube of length L has a constant wall thickness t. Show that the angle of twist caused by the torque T is

$$\theta = \frac{20}{9\pi}\frac{TL}{Gtd_A^3}$$

(*Hint*: Apply Eq. (3.9b) to an infinitesimal length dx of the shaft.)

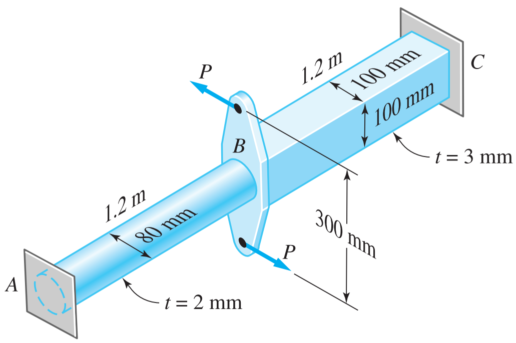


FIG. P3.47

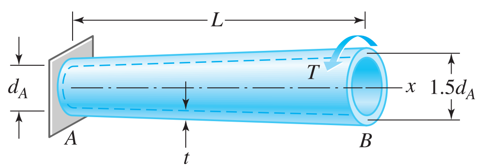


FIG. P3.48

*3.4 Torsion of Rectangular Bars

The analysis of circular shafts in Sec. 3.2 was based upon the assumption that plane cross sections remain plane and are undistorted. If the cross section of the shaft is not circular, experiments show that the cross sections distort and do not remain plane. Therefore, the formulas for shear stress distribution and torsional rigidity derived in Sec. 3.2 cannot be applied to noncircular members.

Figure 3.10 shows the distortion of a rectangular bar caused by the torque T. The two significant features of the deformation are:

- The cross sections become distorted.
- The shear strain (and thus the shear stress) is zero at the edges of the bar and largest at the middle of the sides.

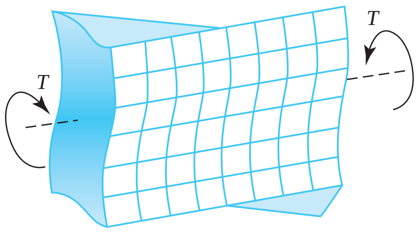


Fig. 3.10 Deformation of a rectangular bar due to torsion.

The reason for the shear strain vanishing at the edges of the bar is illustrated in Fig. 3.11. The small element labeled A in Fig. 3.11(a) is located at the edge of the bar. The shear stresses acting of the faces of this element, shown in Fig. 3.11(b), are denoted by τ_1, τ_2, and τ_3 (recall that shear stresses acting on complementary planes have the same magnitude but opposite sense). The two sides of the element that are shaded must be stress-free because they are free surfaces. Therefore, $\tau_1 = \tau_2 = \tau_3 = 0$, which proves that there are no shear stresses, and therefore no shear strains, at the corners of the bar.

The maximum shear stress τ_{max} occurs on element B in Fig. 3.11(a), which is located at the centerline of the *wider* face of the bar. This stress is shown in Fig. 3.11(c).

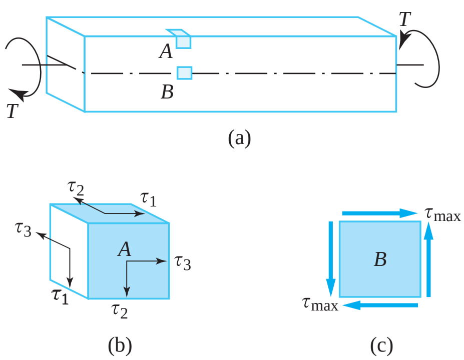


Fig. 3.11 Rectangular bar in torsion showing locations of zero and maximum stresses.

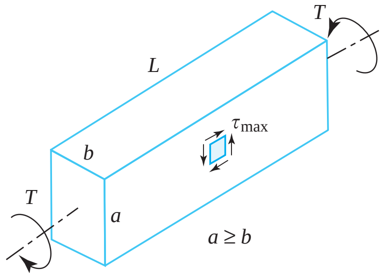


Fig. 3.12 Rectangular bar in torsion showing the dimensions used in Eqs. (3.10).

The analytical analysis of the torsion of noncircular bars lies in the realm of the theory of elasticity, a topic that is beyond the scope of this text. For the rectangular bar in Fig. 3.12 that carries the torque T, results obtained by numerical methods[2] determine that the maximum shear stress τ_{max} and the angle of twist θ are given by

$$\tau_{max} = \frac{T}{C_1 ab^2} \tag{3.10a}$$

and

$$\theta = \frac{TL}{C_2 ab^3 G} \tag{3.10b}$$

where G is the shear modulus. As shown in Fig. 3.12, a and b ($a \geq b$) are the cross-sectional dimensions of the bar and L is its length. The coefficients C_1 and C_2, which depend on the ratio a/b, are listed in Table 3.1.

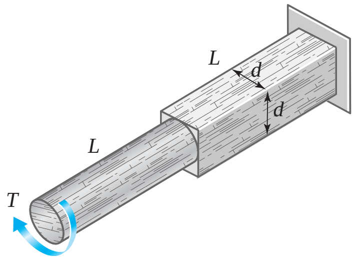
a/b	C_1	C_2
1.0	0.208	0.141
1.2	0.219	0.166
1.5	0.231	0.196
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Table 3-1

[2] S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3e, McGraw-Hill, New York, 1970.

Sample Problem **3.8**

The wooden bar consists of two segments, each of length L. One segment has a square cross section of width d; the cross section of the other segment is a circle of diameter d. The working stress for the wood is $\tau_w = 5$ MPa and the shear modulus is $G = 0.5$ GPa. Using $L = 0.6$ m and $d = 50$ mm, determine (1) the largest torque T that can be safely applied; and (2) the corresponding angle of twist for the bar.



Solution

Part 1

Assuming the circular segment governs, the largest safe torque from Eq. (3.5c) is

$$T = \frac{\tau_w \pi d^3}{16} = \frac{(5 \times 10^6)\pi(0.05)^3}{16} = 122.7 \text{ N} \cdot \text{m}$$

Assuming the square segment is critical, Eq. (3.10a) yields for the largest safe torque

$$T = C_1 d^3 \tau_w = 0.208(0.05)^3(5 \times 10^6) = 130.0 \text{ N} \cdot \text{m}$$

where $C_1 = 0.208$ was obtained from Table 3.1.

Comparing the above two values for T, we see that the stress in the circular segment governs. Therefore, the largest torque that can be applied safely is

$$T = 122.7 \text{ N} \cdot \text{m}$$ *Answer*

Part 2

The angle of twist of the bar is obtained by adding the contributions of the two segments using Eqs. (3.4b) and (3.10b):

$$\theta = \frac{TL}{GJ} + \frac{TL}{C_2 d^4 G} = \frac{TL}{G(\pi d^4/32)} + \frac{TL}{(0.141)d^4 G}$$

$$= \frac{TL}{Gd^4}\left(\frac{32}{\pi} + \frac{1}{0.141}\right) = 17.28\frac{TL}{Gd^4}$$

$$= 17.28\frac{122.7(0.6)}{(0.5 \times 10^9)(0.05)^4} = 0.4071 \text{ rad} = 23.3^\circ$$ *Answer*

Problems

3.49 (a) Determine the largest torque that can be safely applied to the rectangular steel bar if the maximum shear stress is limited to 120 MPa. (b) Compute the corresponding angle of twist using $G = 80$ GPa for steel.

3.50 Determine the torque required to produce a 5° twist in the piece of wood. Use $G = 1.0 \times 10^6$ psi for wood.

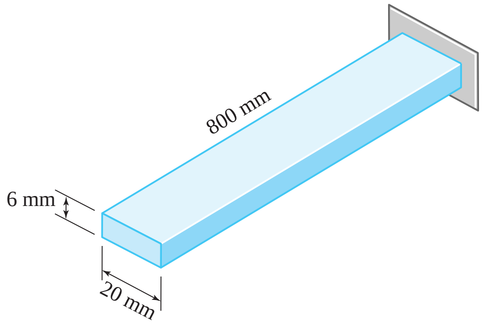


FIG. P3.49

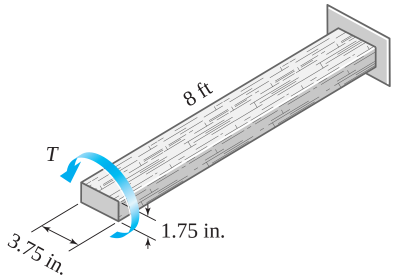


FIG. P3.50

3.51 The circular steel bar in Fig. (a) and the square steel bar in Fig. (b) are subjected to the same torque T. (Note that the volumes of the bars are equal.) Determine (a) the ratio $(\tau_{max})_a/(\tau_{max})_b$ of their maximum shear stresses; and (b) the ratio $(\theta_{max})_a/(\theta_{max})_b$ of their angles of twist.

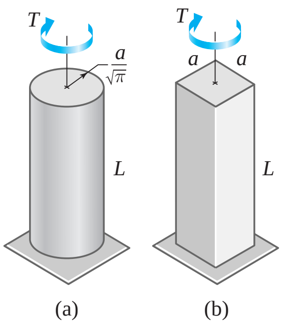


FIG. P3.51

3.52 Equal torques $T = 5$ kip · ft are applied to the two steel bars with the cross sections shown. (Note that the cross-sectional areas of the bars are equal.) The length of each bar is 8 ft. Calculate the maximum shear stress and angle of twist for each bar. Use $G = 12 \times 10^6$ psi for steel.

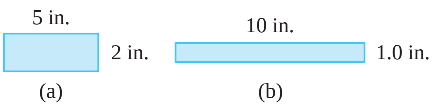


FIG. P3.52

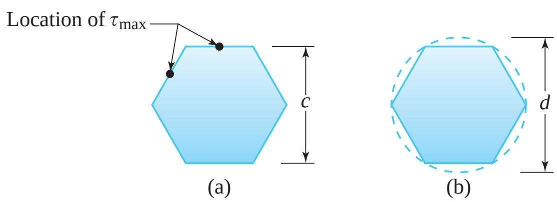


FIG. P3.53

3.53 When a bar with the hexagonal cross section shown in Fig. (a) is subjected to a torque T, numerical analysis shows that the maximum shear stress in the bar is $\tau_{max} = 5.7T/c^3$. Determine the percentage loss in strength that results when a circular bar of diameter d is machined into the hexagonal shape shown in Fig. (b).

3.54 A steel bar of length L with the cross section shown is twisted through 90°. Determine the smallest ratio L/b for which the maximum shear stress will not exceed 150 MPa. Use $G = 80$ GPa for steel.

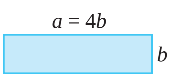


FIG. P3.54

Review Problems

3.55 The torque T is applied to the solid shaft of radius r_2. Determine the radius r_1 of the inner portion of the shaft that carries one-half of the torque.

3.56 The solid aluminium shaft $ABCD$ carries the three torques shown. (a) Determine the smallest safe diameter of the shaft if the allowable shear stress is 15 ksi. (b) Compute the angle of rotation of end A of the shaft using $G = 4 \times 10^6$ psi.

3.57 A circular tube of outer diameter D is slipped over a 40-mm-diameter solid cylinder. The tube and cylinder are then welded together. For what value of D will the torsional strengths of the tube and cylinder be equal?

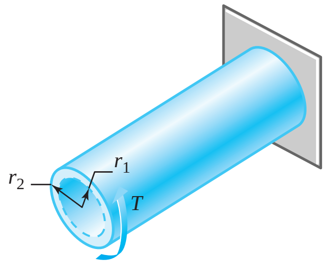


FIG. P3.55

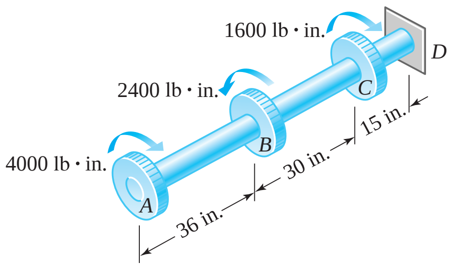


FIG. P3.56

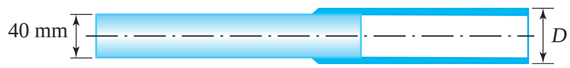


FIG. P3.57

3.58 A solid steel shaft 4 m long is stressed to 70 MPa when twisted through 3°. (a) Given that $G = 83$ GPa, find the diameter of the shaft. (b) What power does this shaft transmit when running at 18 Hz?

3.59 Determine the maximum torque that can be applied to a hollow circular steel shaft of 100-mm outer diameter and 80-mm inner diameter. The shear stress is limited to 70 MPa, and the angle of twist must not exceed 0.4° in a length of 1.0 m. Use $G = 83$ GPa for steel.

3.60 A 2-in.-diameter steel shaft rotates at 240 rev/min. If the shear stress is limited to 12 ksi, determine the maximum horsepower that can be transmitted at that speed.

3.61 The compound shaft, consisting of steel and aluminum segments, carries the two torques shown in the figure. Determine the maximum permissible value of T subject to the following design conditions: $\tau_{st} \leq 83$ MPa, $\tau_{al} \leq 55$ MPa, and $\theta \leq 6°$ (θ is the angle of rotation of the free end). Use $G = 83$ GPa for steel and $G = 28$ GPa for aluminum.

3.62 The four gears are attached to a steel shaft that is rotating at 2 Hz. Gear B supplies 70 kW of power to the shaft. Of that power, 20 kW are used by gear A, 20 kW by gear C, and 30 kW by gear D. (a) Find the uniform shaft diameter if the shear stress in the shaft is not to exceed 60 MPa. (b) If a uniform shaft diameter of 100 mm is specified, determine the angle by which one end of the shaft lags behind the other end. Use $G = 83$ GPa for steel.

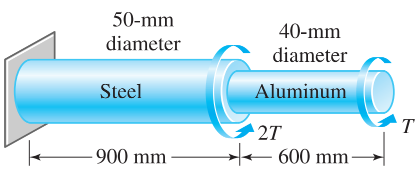


FIG. P3.61

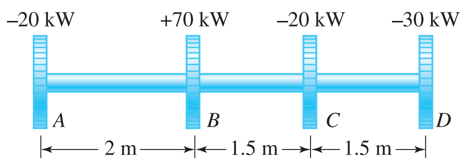


FIG. P3.62

3.63 The composite shaft consists of a copper rod that fits loosely inside an aluminum sleeve. The two components are attached to a rigid wall at one end and joined with an end-plate at the other end. Determine the maximum shear stress in each material when the 2-kN · m torque is applied to the end-plate. Use $G = 26$ GPa for aluminum and $G = 47$ GPa for copper.

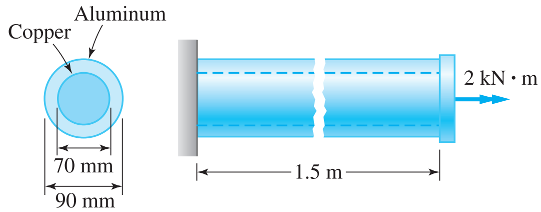


FIG. P3.63

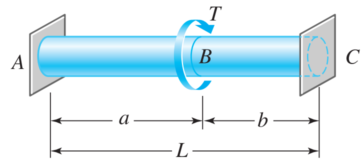


FIG. P3.64

3.64 The torque T is applied to the solid shaft with built-in ends. (a) Show that the reactive torques at the walls are $T_A = Tb/L$ and $T_C = Ta/L$. (b) How would the results of Part (a) change if the shaft were hollow?

3.65 A flexible shaft consists of a 0.20-in.-diameter steel rod encased in a stationary tube that fits closely enough to impose a torque of intensity 0.50 lb · in./in. on the rod. (a) Determine the maximum length of the shaft if the shear stress in the rod is not to exceed 20 ksi. (b) What will be the relative angular rotation between the ends of the rod? Use $G = 12 \times 10^6$ psi for steel.

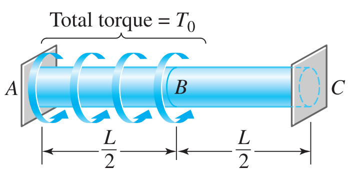


FIG. P3.66

3.66 The shaft ABC is attached to rigid walls at A and C. The torque T_0 is distributed uniformly over segment AB of the shaft. Determine the reactions at A and C.

3.67 A torque of 400 lb · ft is applied to the square tube with constant 0.10-in. wall thickness. Determine the smallest permissible dimension a if the shear stress is limited to 6500 psi.

3.68 The cross section of a brass tube is an equilateral triangle with a constant wall thickness, as shown in the figure. If the shear stress is limited to 8 ksi and the angle of twist is not to exceed 2° per foot length, determine the largest allowable torque that can be applied to the tube. Use $G = 5.7 \times 10^6$ psi for brass.

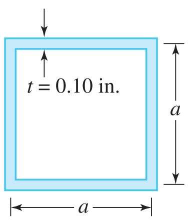


FIG. P3.67

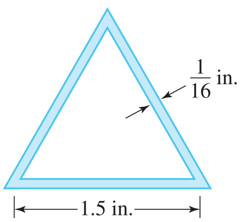


FIG. P3.68

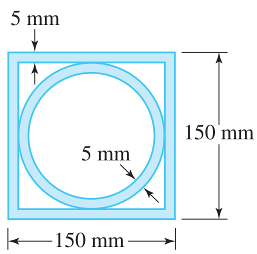


FIG. P3.69

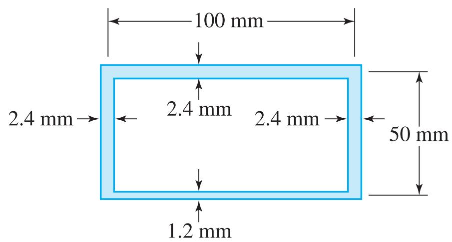


FIG. P3.70

3.69 A torsion member is made by placing a circular tube inside a square tube, as shown, and joining their ends by rigid end-plates. The tubes are made of the same material and have the same constant wall thickness $t = 5$ mm. If a torque T is applied to the member, what fraction of T is carried by each component?

3.70 A 3-m-long aluminum tube with the cross section shown carries a 200-N · m torque. Determine (a) the maximum shear stress in the tube; and (b) the relative angle of rotation of the ends of the tube. For aluminum, use $G = 28$ GPa.

Computer Problems

C3.1 An aluminum bar of circular cross section and the profile specified in Prob. C2.1 is subjected to a 15-N·m torque. Use numerical integration to compute the angle of twist of the bar. For aluminum, use $G = 30$ GPa.

C3.2 A steel bar of circular cross section has the profile shown in the figure. Use numerical integration to compute the torsional stiffness $k = T/\theta$ of the bar. For steel, use $G = 12 \times 10^6$ psi.

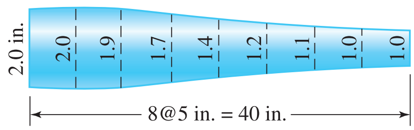


FIG. C3.2

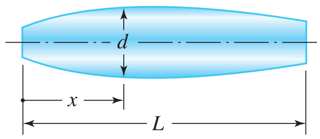


FIG. C3.3

C3.3 The diameter d of the solid shaft of length L varies with the axial coordinate x. Given L and $d(x)$, write an algorithm to calculate the constant diameter D of a shaft that would have the same torsional stiffness (assume that the two shafts have the same length and are made of the same material). Use (a) $L = 500$ mm and

$$d = (25 \text{ mm})\left(1 + 3.8\frac{x}{L} - 3.6\frac{x^2}{L^2}\right)$$

and (b) $L = 650$ mm and

$$d = \begin{cases} 20 \text{ mm} & \text{if } x \le 200 \text{ mm} \\ 20 \text{ mm} + \dfrac{x - 200 \text{ mm}}{10} & \text{if } 200 \text{ mm} \le x \le 350 \text{ mm} \\ 35 \text{ mm} & \text{if } x \ge 350 \text{ mm} \end{cases}$$

C3.4 The solid shaft ABC of length L and variable diameter d is attached to rigid supports at A and C. A torque T acts at the distance b from end A. Given L, b, and $d(x)$, write an algorithm to compute the fraction of T that is carried by segments AB and BC. Use (a) $L = 200$ mm, $b = 110$ mm, and

$$d = 30 \text{ mm} - (20 \text{ mm}) \sin\frac{\pi x}{L}$$

and (b) $L = 400$ mm, $b = 275$ mm, and

$$d = \begin{cases} 25 \text{ mm} & \text{if } x \le 200 \text{ mm} \\ 25 \text{ mm} + \dfrac{(x - 200 \text{ mm})^2}{250 \text{ mm}} & \text{if } 200 \text{ mm} \le x \le 250 \text{ mm} \\ 35 \text{ mm} & \text{if } x \ge 250 \text{ mm} \end{cases}$$

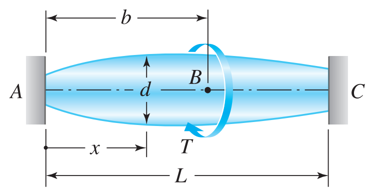


FIG. C3.4

C3.5 An extruded tube of length L has the cross section shown in the figure. The radius of the median line is $r = 75$ mm, and the wall thickness varies with the angle α as

$$t = t_1 + (t_2 - t_1)\sin\frac{\alpha}{2}$$

Given L, r, t_1, t_2, and G, write an algorithm to compute the angle of twist required to produce the maximum shear stress τ_{max}. Use $L = 1.8$ m, $r = 75$ mm, $t_1 = 2$ mm, $t_2 = 4$ mm, $G = 40$ GPa (brass), and $\tau_{max} = 110$ MPa.

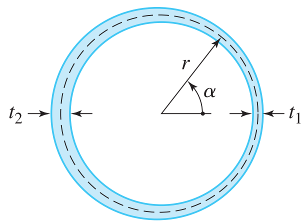


FIG. C3.5

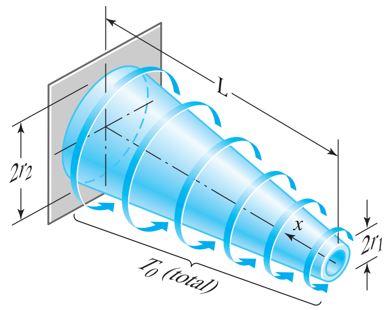


FIG. C3.6

C3.6 The thin-walled tube in the shape of a truncated cone carries a torque T_0 that is uniformly distributed over its length L. The radius of the median line varies linearly from r_1 to r_2 over the length of the tube. The wall thickness t is constant. Given L, r_1, r_2, t, T_0, and G, construct an algorithm that (a) plots the shear stress in the tube as a function of the axial distance x; and (b) computes the angle of rotation at the free end of the tube. Use $L = 10$ ft, $r_1 = 3$ in., $r_2 = 12$ in., $t = 0.2$ in., $T_0 = 60$ kip · ft, and $G = 12 \times 10^6$ psi (steel).

4
Shear and Moment in Beams

Power-generating turbines on a wind farm. The supporting columns can be modeled as beams subjected to wind loading. The determination of shear forces and bending moments in beams caused by various load conditions is the topic of this chapter. Courtesy of 2009fotofriends/Shutterstock.

4.1 Introduction

The term *beam* refers to a slender bar that carries transverse loading; that is, the applied forces are perpendicular to the bar. In a beam, the internal force system consists of a shear force and a bending moment acting on the cross section of the bar. As we have seen in previous chapters, axial and torsional loads often result in internal forces that are constant in the bar, or over portions of the bar. The study of beams, however, is complicated by the fact that the shear force and the bending moment usually vary continuously along the length of the beam.

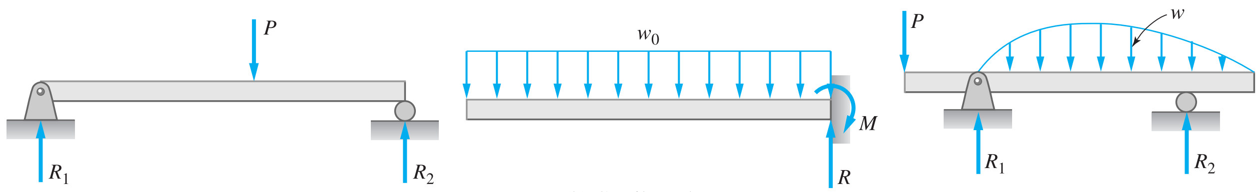
The internal forces give rise to two kinds of stresses on a transverse section of a beam: (1) normal stress that is caused by the bending moment and (2) shear stress due to the shear force. This chapter is concerned only with the variation of the shear force and the bending moment under various combinations of loads and types of supports. Knowing the distribution of the shear force and the bending moment in a beam is essential for the computation of stresses and deformations, which will be investigated in subsequent chapters.

4.2 Supports and Loads

Beams are classified according to their supports. A *simply supported beam*, shown in Fig. 4.1(a), has a pin support at one end and a roller support at the other end. The pin support prevents displacement of the end of the beam, but not its rotation. The term *roller support* refers to a pin connection that is free to move parallel to the axis of the beam; hence, this type of support suppresses only the transverse displacement. A *cantilever beam* is built into a rigid support at one end, with the other end being free, as shown in Fig. 4.1(b). The built-in support prevents displacements as well as rotations of the end of the beam. An *overhanging beam*, illustrated in Fig. 4.1(c), is supported by a pin and a roller support, with one or both ends of the beam extending beyond the supports. The three types of beams are statically determinate because the support reactions can be found from the equilibrium equations.

A *concentrated load*, such as P in Fig. 4.1(a), is an approximation of a force that acts over a very small area. In contrast, a *distributed load* is applied over a finite area. If the distributed load acts on a very narrow area, the load may be approximated by a *line load*. The intensity w of this loading is expressed as force per unit length (lb/ft, N/m, etc.). The load distribution may be uniform, as shown in Fig. 4.1(b), or it may vary with distance along the beam, as in Fig. 4.1(c). The weight of the beam is an example of distributed loading, but its magnitude is usually small compared to the loads applied to the beam.

Figure 4.2 shows other types of beams. These beams are over-supported in the sense that each beam has at least one more reaction than is necessary for support. Such beams are statically indeterminate; the presence of these *redundant supports* requires the use of additional equations obtained by considering the deformation of the beam. The analysis of statically indeterminate beams will be discussed in Chapter 7.



(a) Simply supported beam (b) Cantilever beam (c) Overhanging beam

FIG. 4.1 Statically determinate beams.

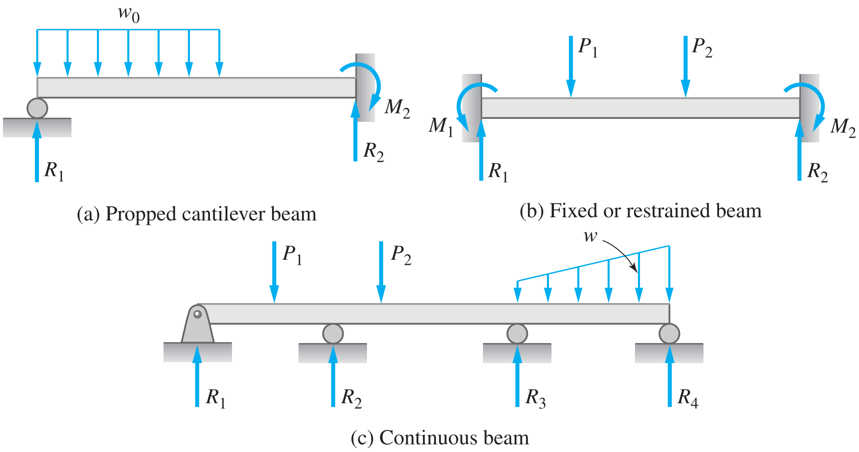


FIG. 4.2 Statically indeterminate beams.

4.3 Shear-Moment Equations and Shear-Moment Diagrams

The determination of the internal force system acting at a *given* section of a beam is straightforward: We draw a free-body diagram that exposes these forces and then compute the forces using equilibrium equations. However, the goal of beam analysis is more involved—we want to determine the shear force V and the bending moment M at *every* cross section of the beam. To accomplish this task, we must derive the expressions for V and M in terms of the distance x measured along the beam. By plotting these expressions to scale, we obtain the *shear force and bending moment diagrams* for the beam. The shear force and bending moment diagrams are convenient visual references to the internal forces in a beam; in particular, they identify the maximum values of V and M.

a. Sign conventions

For consistency, it is necessary to adopt sign conventions for applied loading, shear forces, and bending moments. We will use the conventions shown in Fig. 4.3, which assume the following to be *positive*:

- External forces that are directed downward; external couples that are directed clockwise.
- Shear forces that tend to rotate a beam element clockwise.
- Bending moments that tend to bend a beam element concave upward (the beam "smiles").

The main disadvantage of the above conventions is that they rely on such adjectives as "downward," "clockwise," and so on. To eliminate this obstacle, a convention based upon a Cartesian coordinate system is sometimes used.

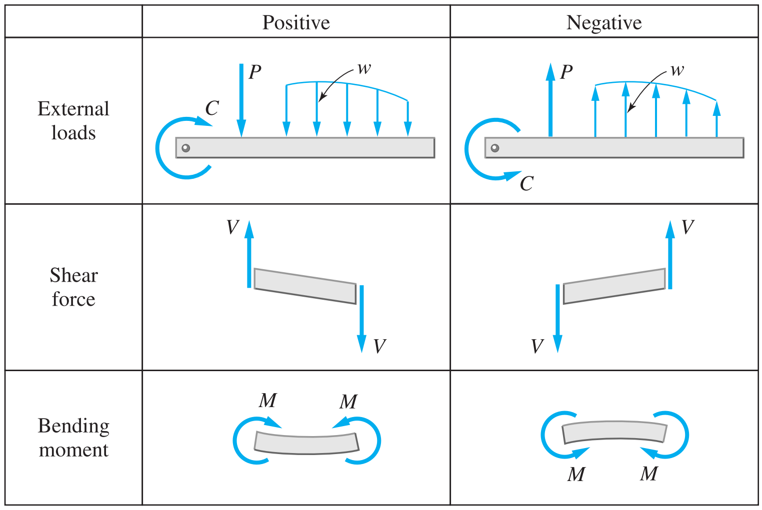


FIG. 4.3 Sign conventions for external loads, shear force, and bending moment.

b. *Procedure for determining shear force and bending moment diagrams*

The following is a general procedure for obtaining shear force and bending moment diagrams of a statically determinate beam:

- Compute the support reactions from the FBD of the entire beam.
- Divide the beam into segments so that the loading within each segment is continuous. Thus, the end-points of the segments are discontinuities of loading, including concentrated loads and couples.

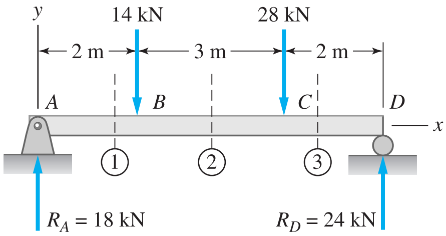
Perform the following steps for each segment of the beam:

- Introduce an imaginary cutting plane within the segment, located at a distance x from the left end of the beam, that cuts the beam into two parts.
- Draw a FBD for the part of the beam lying either to the left or to the right of the cutting plane, whichever is more convenient. At the cut section, show V and M acting in their positive directions.
- Determine the expressions for V and M from the equilibrium equations obtainable from the FBD. These expressions, which are usually functions of x, are the shear force and bending moment equations for the segment.
- Plot the expressions for V and M for the segment. It is visually desirable to draw the V-diagram below the FBD of the entire beam, and then draw the M-diagram below the V-diagram.

The bending moment and shear force diagrams of the beam are composites of the V- and M-diagrams of the segments. These diagrams are usually discontinuous and/or have discontinuous slopes at the end-points of the segments due to discontinuities in loading.

Sample Problem 4.1

The simply supported beam in Fig. (a) carries two concentrated loads. (1) Derive the expressions for the shear force and the bending moment for each segment of the beam. (2) Draw the shear force and bending moment diagrams. Neglect the weight of the beam. Note that the support reactions at A and D have been computed and are shown in Fig. (a).



(a)

Solution

Part 1

The determination of the expressions for V and M for each of the three beam segments (AB, BC, and CD) is explained below.

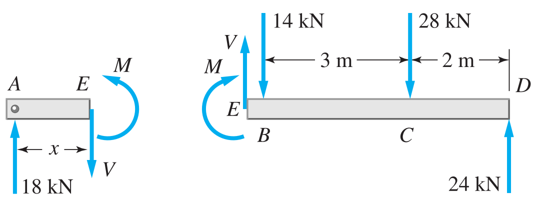
***Segment AB* (0 < x < 2 m)** Figure (b) shows the FBDs for the two parts of the beam that are separated by section ①, located within segment AB. Note that we show V and M acting in their positive directions according to the sign conventions in Fig. 4.3. Because V and M are equal in magnitude and oppositely directed on the two FBDs, they can be computed using either FBD. The analysis of the FBD of the part to the left of section ① yields

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - V = 0$$

$$V = +18 \text{ kN} \qquad \textit{Answer}$$

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad -18x + M = 0$$

$$M = +18x \text{ kN} \cdot \text{m} \qquad \textit{Answer}$$



(b) FBDs

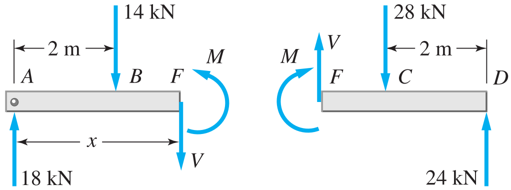
Segment BC (2 m < x < 5 m) Figure (c) shows the FBDs for the two parts of the beam that are separated by section ②, an arbitrary section within segment BC. Once again, V and M are assumed to be positive according to the sign conventions in Fig. 4.3. The analysis of the part to the left of section ② gives

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - V = 0$$

$$V = +18 - 14 = +4 \text{ kN} \qquad \textit{Answer}$$

$$\Sigma M_F = 0 \quad +\circlearrowleft \quad -18x + 14(x-2) + M = 0$$

$$M = +18x - 14(x-2) = 4x + 28 \text{ kN} \cdot \text{m} \qquad \textit{Answer}$$



(c) FBDs

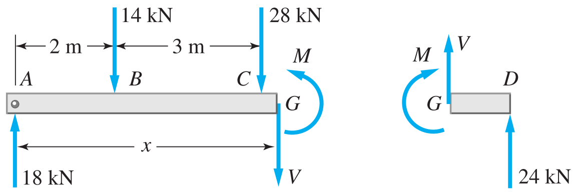
Segment CD (5 m < x < 7 m) Section ③ is used to find the shear force and bending moment in segment CD. The FBDs in Fig. (d) again show V and M acting in their positive directions. Analyzing the portion of the beam to the left of section ③, we obtain

$$\Sigma F_y = 0 \quad +\uparrow \quad 18 - 14 - 28 - V = 0$$

$$V = +18 - 14 - 28 = -24 \text{ kN} \qquad \textit{Answer}$$

$$\Sigma M_G = 0 \quad +\circlearrowleft \quad -18x + 14(x-2) + 28(x-5) + M = 0$$

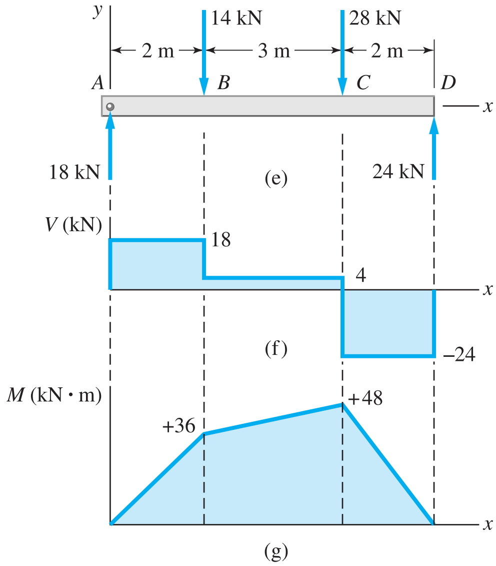
$$M = +18x - 14(x-2) - 28(x-5) = -24x + 168 \text{ kN} \cdot \text{m} \qquad \textit{Answer}$$



(d) FBDs

Part 2

The shear force and bending moment diagrams in Figs. (f) and (g) are the plots of the expressions for V and M derived in Part 1. By placing these plots directly below the sketch of the beam in Fig. (e), we establish a clear visual relationship between the diagrams and locations on the beam.

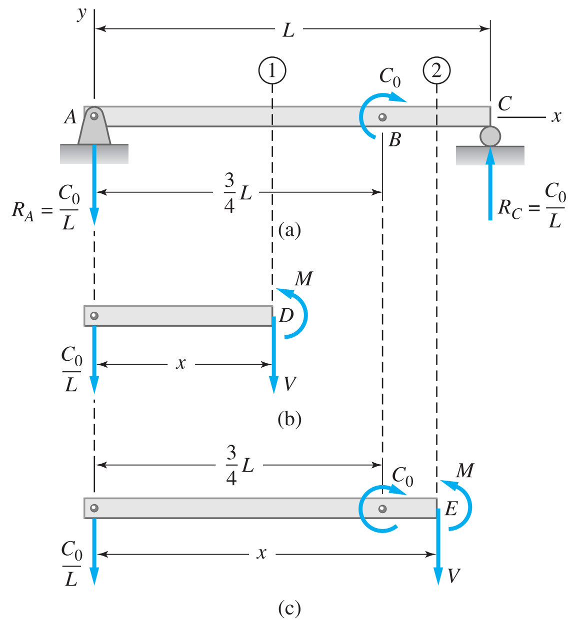


Shear force and bending moment diagrams

An inspection of the V-diagram reveals that the largest shear force in the beam is -24 kN and that it occurs at every cross section of the beam in segment CD. From the M-diagram we see that the maximum bending moment is $+48$ kN · m, which occurs under the 28-kN load at C. Note that at each concentrated force the V-diagram "jumps" by an amount equal to the force. Furthermore, there is a discontinuity in the slope of the M-diagram at each concentrated force.

Sample Problem 4.2

The simply supported beam in Fig. (a) is loaded by the clockwise couple C_0 at B. (1) Derive the shear force and bending moment equations, and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam. The support reactions A and C have been computed, and their values are shown in Fig. (a).



FBDs

Solution

Part 1

Due to the presence of the couple C_0, we must analyze segments AB and BC separately.

***Segment AB* $(0 < x < 3L/4)$** Figure (b) shows the FBD of the part of the beam to the left of section ① (we could also use the part to the right). Note that V and M are assumed to act in their positive directions according to the sign conventions in Fig. 4.3. The equilibrium equations for this portion of the beam yield

$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L}$$

Answer

$$\Sigma M_D = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x + M = 0 \qquad M = -\frac{C_0}{L}x$$

Answer

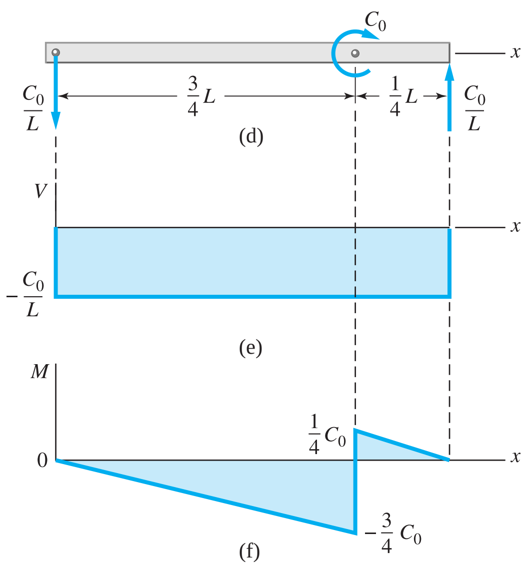
***Segment BC* $(3L/4 < x < L)$** Figure (c) shows the FBD of the portion of the beam to the left of section ② (the right portion could also be used). Once again, V and M are assumed to act in their positive directions. Applying the equilibrium equations to the beam segment, we obtain

$$\Sigma F_y = 0 \quad +\uparrow \quad -\frac{C_0}{L} - V = 0 \qquad V = -\frac{C_0}{L} \qquad \textit{Answer}$$

$$\Sigma M_E = 0 \quad +\circlearrowleft \quad \frac{C_0}{L}x - C_0 + M = 0 \qquad M = -\frac{C_0}{L}x + C_0 \qquad \textit{Answer}$$

Part 2

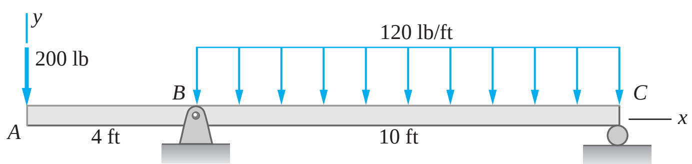
The sketch of the beam is repeated in Fig. (d). The shear force and bending moment diagrams shown in Figs. (e) and (f) are obtained by plotting the expressions for V and M found in Part 1. From the V-diagram, we see that the shear force is the same for all cross sections of the beam. The M-diagram shows a jump of magnitude C_0 at the point of application of the couple.



Shear force and bending moment diagrams

Sample Problem **4.3**

The overhanging beam ABC in Fig.(a) carries a concentrated load and a uniformly distributed load. (1) Derive the shear force and bending moment equations; and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam.



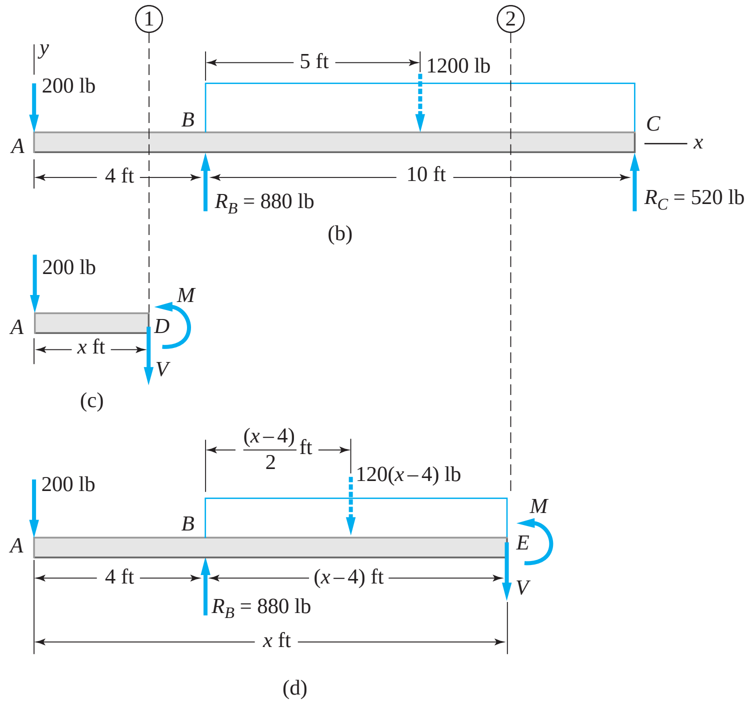
(a)

Solution

Inspection of the beam in Fig. (a) reveals that we must analyze segments AB and BC separately.

Part 1

The FBD of the beam is shown in Fig. (b). Note that the uniformly distributed load has been replaced by its resultant, which is the force $120(10) = 1200$ lb (area under the loading diagram) acting at the centroid of the loading diagram. The reactions shown at the supports at B and C were computed from the equilibrium equations.



***Segment AB* (0 < *x* < 4 ft)** Figure (c) shows the FBD of the portion of the beam that lies to the left of section ①. (The part of the beam lying to the right of the section could also be used.) The shearing force V and the bending moment M that act at the cut section were assumed to act in their positive directions following the sign convention in Fig. 4.3. The equilibrium equations for this part of the beam yield

$$\Sigma F_y = 0 \quad +\uparrow \quad -200 - V = 0 \qquad V = -200 \text{ lb} \qquad \textit{Answer}$$

$$\Sigma M_D = 0 \quad +\circlearrowleft \quad -200x + M = 0 \qquad M = -200x \text{ lb}\cdot\text{ft} \qquad \textit{Answer}$$

***Segment BC* (4 ft < *x* < 14 ft)** The FBD of the part of the beam that lies to the left of section ② is shown in Fig. (d). (The portion of the beam lying to the right of the section could also be used.) Once again, the shearing force V and the bending moment M are shown acting in their positive directions. Applying the equilibrium equations to the beam segment, we obtain

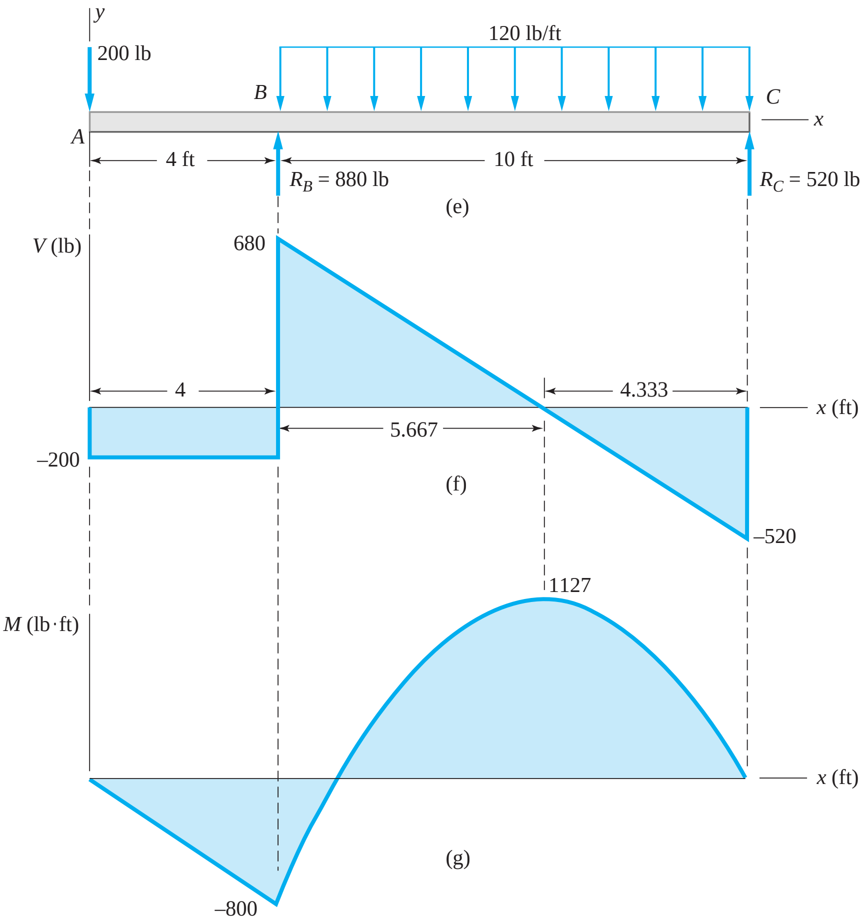
$$\Sigma F_y = 0 \; +\uparrow \quad -200 + 880 - 120(x-4) - V = 0 \qquad V = 1160 - 120x \text{ lb} \qquad \textit{Answer}$$

$$\Sigma M_E = 0 \; +\circlearrowleft \; 200x - 880(x-4) + 120(x-4)\frac{(x-4)}{2} + M = 0$$

$$M = -60x^2 + 1160x - 4480 \text{ lb}\cdot\text{ft} \qquad \textit{Answer}$$

Part 2

The FBD of the beam is repeated in Fig. (e). The plots of the shear force and bending moment diagrams are shown in Figs. (f) and (g), respectively. Note that the shear force diagram is composed of straight-line segments, and the bending moment diagram is a straight line between A and B, and a parabola between B and C.



The location of the section where the shear force is zero is determined as follows:

$$V = 1160 - 120x = 0$$

which gives

$$x = 9.667 \text{ ft}$$

The maximum bending moment occurs where the slope of the moment diagram is zero; that is, where $dM/dx = 0$, which yields

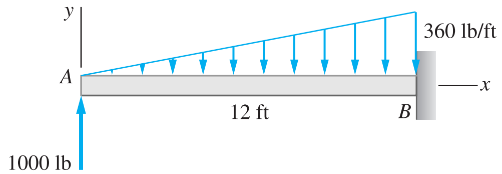
$$\frac{dM}{dx} = -120x + 1160 = 0$$

which again gives $x = 9.667$ ft. (The reason that the maximum bending moment occurs at the section where the shear force is zero will be explained in Sec. 4.4.) Substituting this value of x into the expression for the bending moment, we find that the maximum bending moment is

$$M_{\max} = -60(9.667)^2 + 1160(9.667) - 4480 = 1127 \text{ lb} \cdot \text{ft}$$

Sample Problem **4.4**

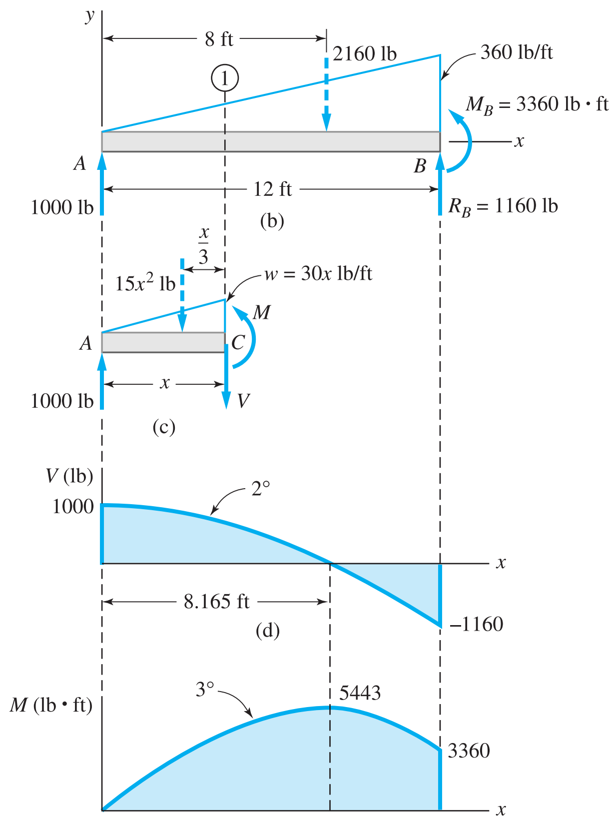
The cantilever beam in Fig. (a) carries a triangular load, the intensity of which varies from zero at the left end to 360 lb/ft at the right end. In addition, a 1000-lb upward vertical load acts at the free end of the beam. (1) Derive the shear force and bending moment equations, and (2) draw the shear force and bending moment diagrams. Neglect the weight of the beam.



(a)

Solution

The FBD of the beam is shown in Fig. (b). Note that the triangular load has been replaced by its resultant, which is the force $0.5(12)(360) = 2160$ lb (area under the loading diagram) acting at the centroid of the loading diagram. The support reactions at *B* can now be computed from the equilibrium equations; the results are shown in Fig. (b).



Because the loading is continuous, the beam does not have to be divided into segments. Therefore, only one expression for *V* and one expression for *M* apply to the entire beam.

Part 1

Figure (c) shows the FBD of the part of the beam that lies to the left of section ①. Letting w be the intensity of the loading at section ①, as shown in Fig. (b), we have from similar triangles, $w/x = 360/12$, or $w = 30x$ lb/ft. Now the triangular load in Fig. (c) can be replaced by its resultant force $15x^2$ lb acting at the centroid of the loading diagram, which is located at $x/3$ ft from section ①. The shear force V and bending moment M acting at section ① are shown acting in their positive directions according to the sign conventions in Fig. 4.3. Equilibrium analysis of the FBD in Fig. (c) yields

$$\Sigma F_y = 0 \quad +\uparrow \quad 1000 - 15x^2 - V = 0$$

$$V = 1000 - 15x^2 \text{ lb}$$ *Answer*

$$\Sigma M_C = 0 \quad +\circlearrowleft \quad -1000x + 15x^2\left(\frac{x}{3}\right) + M = 0$$

$$M = 1000x - 5x^3 \text{ lb} \cdot \text{ft}$$ *Answer*

Part 2

Plotting the expressions for V and M found in Part 1 gives the shear force and bending moment diagrams shown in Figs. (d) and (e). Observe that the shear force diagram is a parabola and the bending moment diagram is a third-degree polynomial in x.

The location of the section where the shear force is zero is found from

$$V = 1000 - 15x^2 = 0$$

which gives

$$x = 8.165 \text{ ft}$$

The maximum bending moment occurs where the slope of the M-diagram is zero—that is, where $dM/dx = 0$. Differentiating the expression for M, we obtain

$$\frac{dM}{dx} = 1000 - 15x^2 = 0$$

which again yields $x = 8.165$ ft. (In the next section, we will show that the slope of the bending moment is always zero where the shear force vanishes.) Substituting this value of x into the expression for M, we find that the maximum bending moment is

$$M_{\max} = 1000(8.165) - 5(8.165)^3 = 5443 \text{ lb} \cdot \text{ft}$$

Problems

4.1–4.18 For the beam shown, derive the expressions for V and M, and draw the shear force and bending moment diagrams. Neglect the weight of the beam.

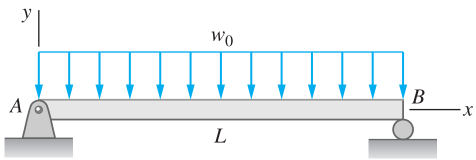


FIG. P4.1

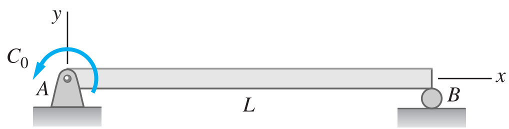


FIG. P4.2

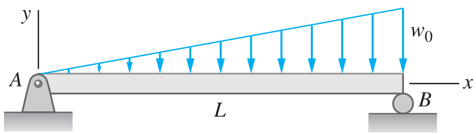


FIG. P4.3

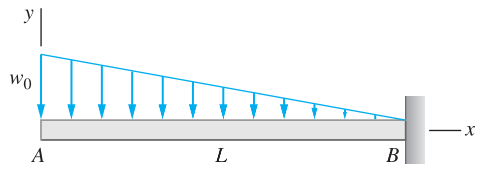


FIG. P4.4

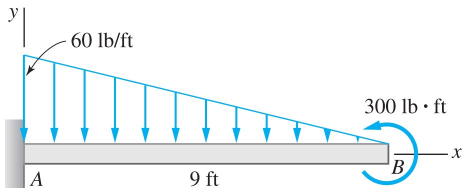


FIG. P4.5

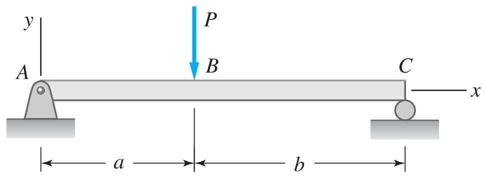


FIG. P4.6

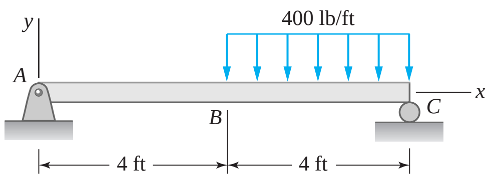


FIG. P4.7

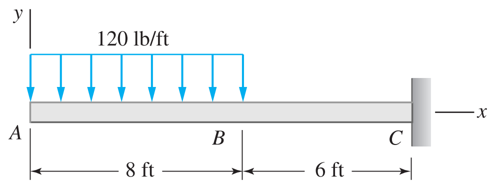


FIG. P4.8

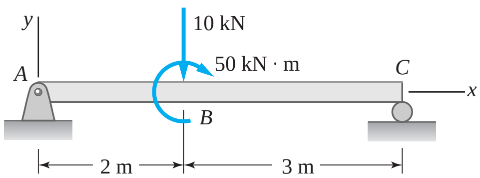


FIG. P4.9

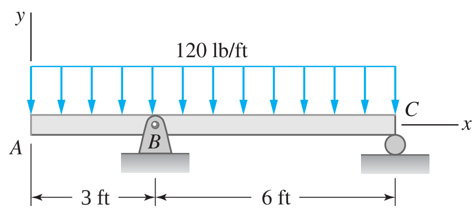


FIG. P4.10

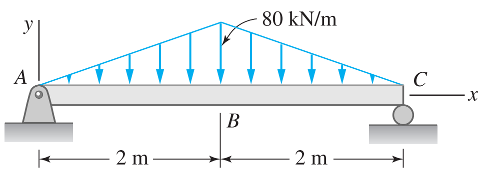


FIG. P4.11

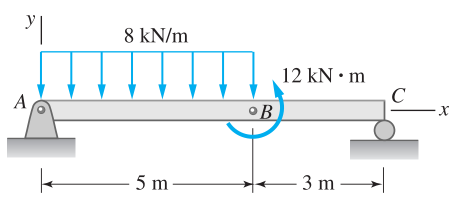


FIG. P4.12

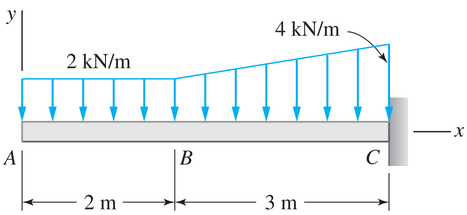


FIG. P4.13

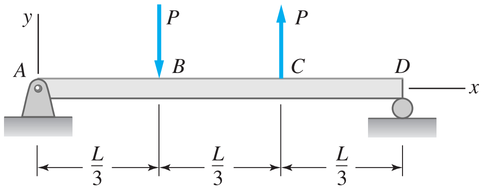


FIG. P4.14

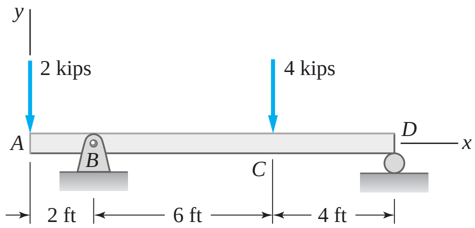


FIG. P4.15

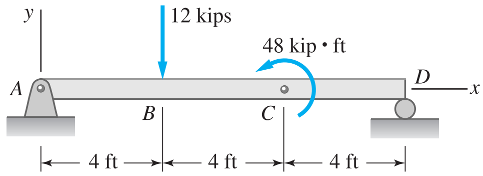


FIG. P4.16

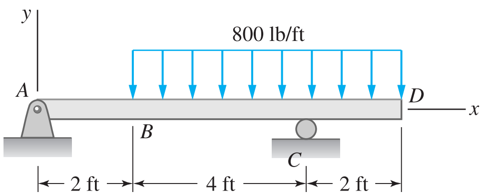


FIG. P4.17

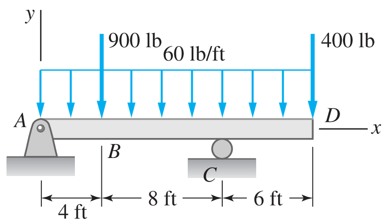


FIG. P4.18

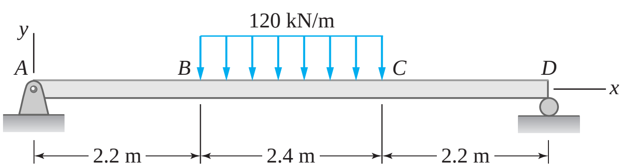


FIG. P4.19

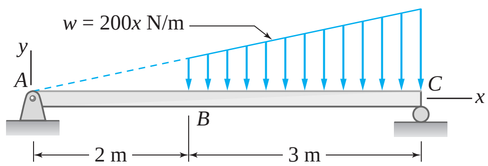


FIG. P4.20

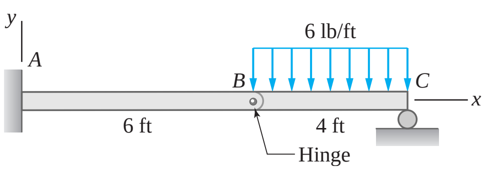


FIG. P4.21

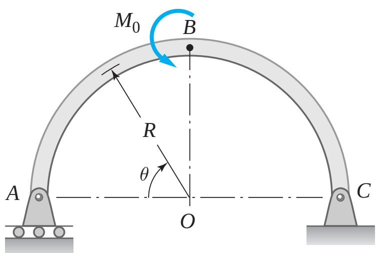


FIG. P4.22

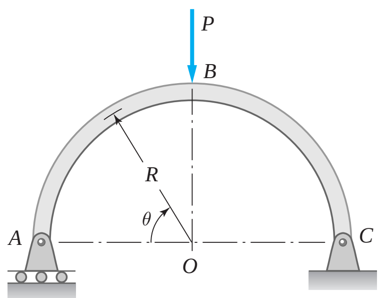


FIG. P4.23

4.22–4.23 Derive the shear force and the bending moment as functions of the angle θ for the arch shown. Neglect the weight of the arch.

4.4 Area Method for Drawing Shear-Moment Diagrams

Useful relationships between the loading, shear force, and bending moment can be derived from the equilibrium equations. These relationships enable us to plot the shear force diagram directly from the load diagram, and then construct the bending moment diagram from the shear force diagram. This technique, called the *area method*, allows us to draw the shear force and bending moment diagrams without having to derive the equations for V and M. We first consider beams subjected to distributed loading and then discuss concentrated forces and couples.

a. Distributed loading

Consider the beam in Fig. 4.4(a) that is subjected to a line load of intensity $w(x)$, where $w(x)$ is assumed to be a continuous function. The free-body diagram of an infinitesimal element of the beam, located at the distance x from the left end, is shown in Fig. 4.4(b). In addition to the distributed load $w(x)$, the segment carries a shear force and a bending moment at each end, which are denoted by V and M at the left end and by $V + dV$ and $M + dM$ at the right end. The infinitesimal differences dV and dM represent the changes that occur over the differential length dx of the element. Observe that all forces and bending moments are assumed to act in their positive directions, as defined in Fig. 4.3 (on p. 110).

The force equation of equilibrium for the element is

$$\Sigma F_y = 0 \quad +\uparrow \quad V - w\,dx - (V + dV) = 0$$

from which we get

$$w = -\frac{dV}{dx} \tag{4.1}$$

The moment equation of equilibrium yields

$$\Sigma M_O = 0 \quad +\circlearrowleft \quad -M - V\,dx + (M + dM) + w\,dx\,\frac{dx}{2} = 0$$

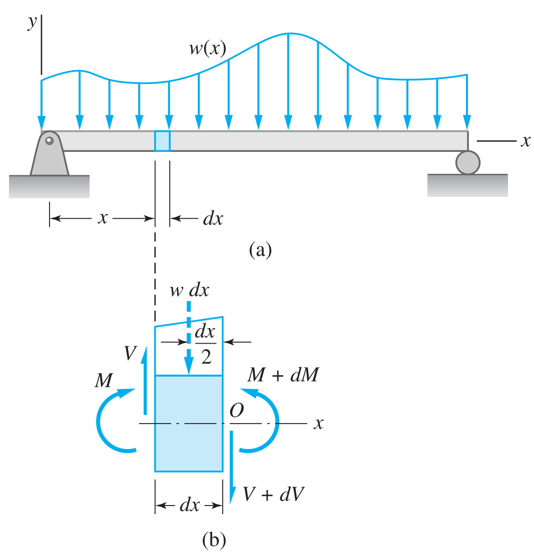


FIG. 4.4 (a) Simply supported beam carrying distributed loading; (b) free-body diagram of an infinitesimal beam segment.

After canceling M and dividing by dx, we get

$$-V + \frac{dM}{dx} + \frac{w\,dx}{2} = 0$$

Because dx is infinitesimal, the last term can be dropped (this is not an approximation), yielding

$$V = \frac{dM}{dx} \tag{4.2}$$

Equations (4.1) and (4.2) are called the *differential equations of equilibrium* for beams. The following five theorems relating the load, the shear force, and the bending moment diagrams follow from these equations.

1. The load intensity at any section of a beam is equal to the negative of the slope of the shear force diagram at the section.
   Proof—follows directly from Eq. (4.1).
2. The shear force at any section is equal to the slope of the bending moment diagram at that section.
   Proof—follows directly from Eq. (4.2).
3. The difference between the shear forces at two sections of a beam is equal to the negative of the area under the load diagram between those two sections.
   Proof—integrating Eq. (4.1) between sections A and B in Fig. 4.5, we obtain

$$\int_{x_A}^{x_B} \frac{dV}{dx}\,dx = V_B - V_A = -\int_{x_A}^{x_B} w\,dx$$

   Recognizing that the integral on the right-hand side of this equation represents the area under the load diagram between A and B, we get

$$V_B - V_A = -\text{area of } w\text{-diagram}\Big]_A^B \qquad \text{Q.E.D.}$$

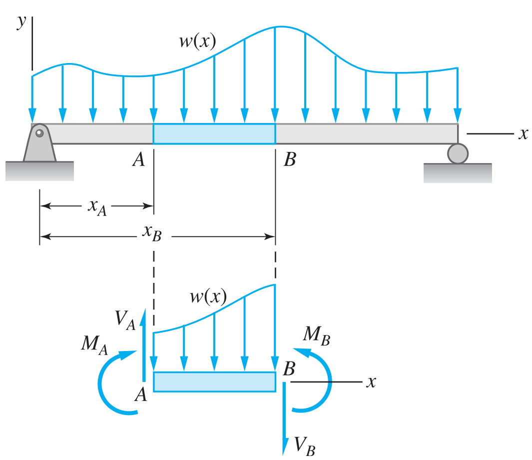


FIG. 4.5 (a) Simply supported beam carrying distributed loading; (b) free-body diagram of a finite beam segment.

For computational purposes, a more convenient form of this equation is

$$\boxed{V_B = V_A - \text{area of } w\text{-diagram}]_A^B} \tag{4.3}$$

Note that the signs in Eq. (4.3) are correct only if $x_B > x_A$.

4. The difference between the bending moments at two sections of a beam is equal to the area of the shear force diagram between these two sections.
Proof—integrating Eq. (4.2) between sections A and B (see Fig. 4.5), we have

$$\int_{x_A}^{x_B} \frac{dM}{dx}\, dx = M_B - M_A = \int_{x_A}^{x_B} V\, dx$$

Because the right-hand side of this equation is the area of the shear force diagram between A and B, we obtain

$$M_B - M_A = \text{area of } V\text{-diagram}]_A^B \qquad \text{Q.E.D.}$$

We find it convenient to use this equation in the form

$$\boxed{M_B = M_A + \text{area of } V\text{-diagram}]_A^B} \tag{4.4}$$

The signs in Eq. (4.4) are correct only if $x_B > x_A$.

5. If the load diagram is a polynomial of degree n, then the shear force diagram is a polynomial of degree $(n+1)$, and the bending moment diagram is a polynomial of degree $(n+2)$.
Proof—follows directly from the integration of Eqs. (4.1) and (4.2).

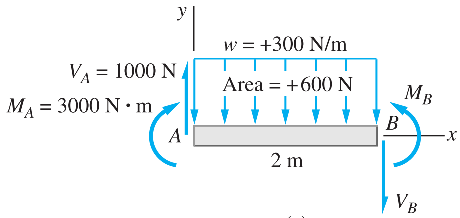
The area method for drawing shear force and bending moment diagrams is a direct application of the foregoing theorems. For example, consider the beam segment shown in Fig. 4.6(a), which is 2 m long and is subjected to a uniformly distributed load $w = 300$ N/m. Figure 4.6(b) shows the steps required in the construction of the shear force and bending moment diagrams for the segment, given that the shear force and the bending moment at the left end are $V_A = +1000$ N and $M_A = +3000$ N·m.

b. *Concentrated forces and couples*

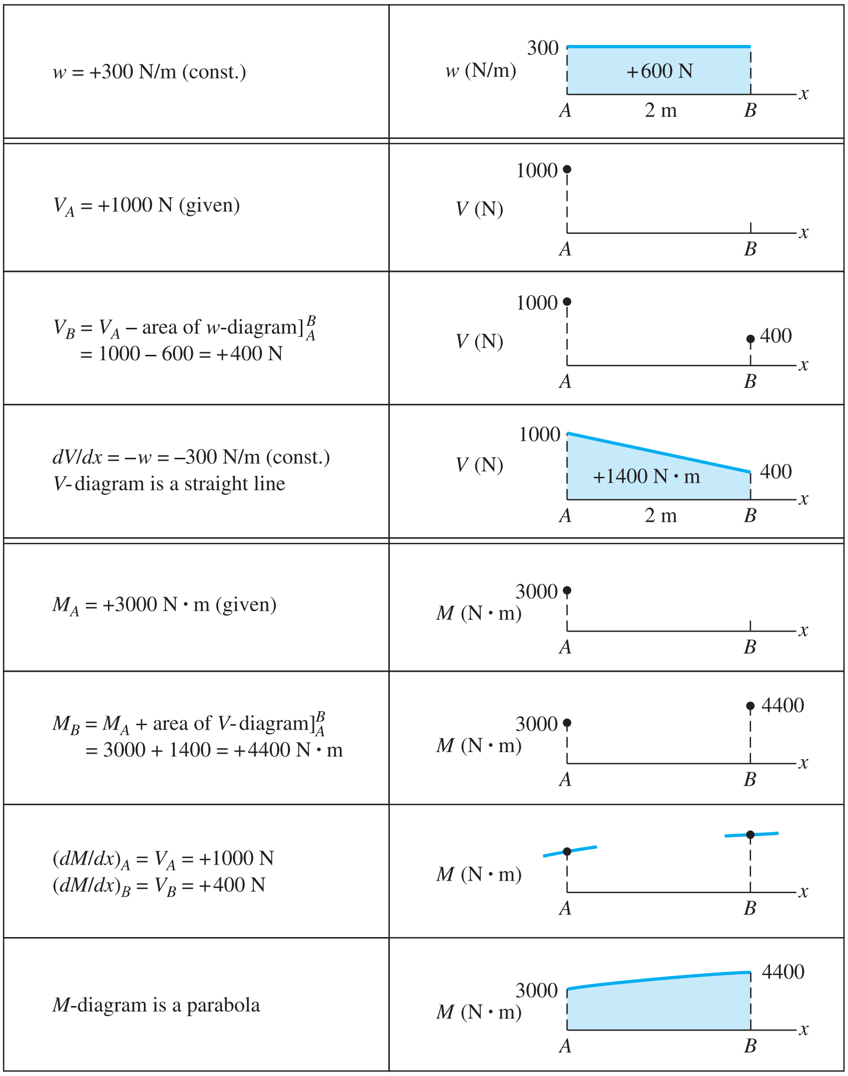
The area method for constructing shear force and bending moment diagrams described above for distributed loads can be extended to beams that are loaded by concentrated forces and/or couples. Figure 4.7 shows the free-body diagram of a beam element of infinitesimal length dx containing a point A where a concentrated force P_A and a concentrated couple C_A are applied. The shear force and the bending moment acting at the left side of the element are denoted by V_A^- and M_A^-, whereas the notation V_A^+ and M_A^+ is used for the right side of the element. Observe that all forces and moments in Fig. 4.7 are assumed to be positive according to the sign conventions in Fig. 4.3.

The force equilibrium equation gives

$$\Sigma F_y = 0 \quad +\uparrow \quad V_A^- - P_A - V_A^+ = 0$$



(a)



(b)

FIG. 4.6 (a) Free-body diagram of a beam segment carrying uniform loading; (b) constructing shear force and bending moment diagrams for the beam segment.

$$V_A^+ = V_A^- - P_A \tag{4.5}$$

Equation (4.5) indicates that a positive concentrated force causes a negative jump discontinuity in the shear force diagram at A (a concentrated couple does not affect the shear force diagram).

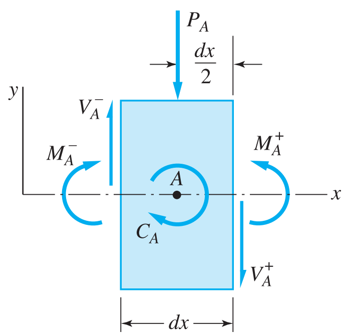


FIG. 4.7 Free-body diagram of an infinitesimal beam element carrying a concentrated force P_A and a concentrated couple C_A.

The moment equilibrium equation yields

$$\Sigma M_A = 0 \quad +\circlearrowleft \quad M_A^+ - M_A^- - C_A - V_A^+ \frac{dx}{2} - V_A^- \frac{dx}{2} = 0$$

Dropping the last two terms because they are infinitesimal (this is not an approximation), we obtain

$$\boxed{M_A^+ = M_A^- + C_A} \tag{4.6}$$

Thus, a positive concentrated couple causes a positive jump in the bending moment diagram.

c. Summary

Equations (4.1)–(4.6), which are repeated below, form the basis of the area method for constructing shear force and bending moment diagrams without deriving the expressions for V and M. The area method is useful only if the areas under the load and shear force diagrams can be computed easily.

$$w = -\frac{dV}{dx} \tag{4.1}$$

$$V = \frac{dM}{dx} \tag{4.2}$$

$$V_B = V_A - \text{area of } w\text{-diagram}\Big]_A^B \tag{4.3}$$

$$M_B = M_A + \text{area of } V\text{-diagram}\Big]_A^B \tag{4.4}$$

$$V_A^+ = V_A^- - P_A \tag{4.5}$$

$$M_A^+ = M_A^- + C_A \tag{4.6}$$

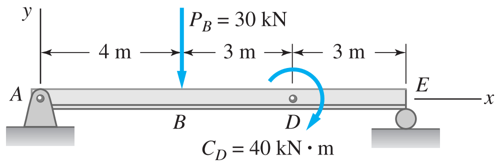
Procedure for the Area Method The following steps outline the procedure for constructing shear force and bending moment diagrams by the area method:

- Compute the support reactions from the FBD of the entire beam.
- Draw the load diagram of the beam (which is essentially a FBD) showing the values of the loads, including the support reactions. Use the sign conventions in Fig. 4.3 to determine the correct sign of each load.
- Working from left to right, construct the V- and M-diagrams for each segment of the beam using Eqs. (4.1)–(4.6).
- When you reach the right end of the beam, check to see whether the computed values of V and M are consistent with the end conditions. If they are not, you have made an error in the computations.

At first glance, using the area method may appear to be more cumbersome than plotting the shear force and bending moment equations. However, with practice you will find that the area method is not only much faster but also less susceptible to numerical errors because of the self-checking nature of the computations.

Sample Problem 4.5

The simply supported beam in Fig. (a) supports a 30-kN concentrated force at B and a 40-kN · m couple at D. Sketch the shear force and bending moment diagrams by the area method. Neglect the weight of the beam.

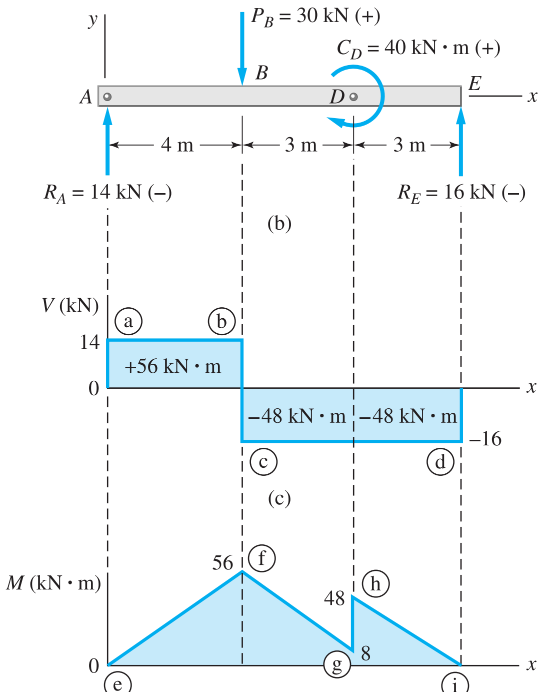


(a)

Solution

Load Diagram

The load diagram for the beam is shown in Fig. (b). The reactions at A and E were found from equilibrium analysis. The numerical value of each force (and the couple) is followed by a plus or minus sign in parentheses, indicating its sign as established by the sign conventions in Fig. 4.3.



Shear Force Diagram

We now explain the steps used to construct the shear force diagram in Fig. (c). From the load diagram, we see that there are concentrated forces at A, B, and E that will cause jumps in the shear force diagram at these points. Therefore, our discussion of shear force must distinguish between sections of the beam immediately to the left and to the right of each of these points.

We begin by noting that $V_A^- = 0$ because no loading is applied to the left of A. We then proceed across the beam from left to right, constructing the diagram as we go:

$$V_A^+ = V_A^- - R_A = 0 - (-14) = +14 \text{ kN}$$

Plot point ⓐ.

$$V_B^- = V_A^+ - \text{area of } w\text{-diagram}]_A^B = 14 - 0 = 14 \text{ kN}$$

Plot point ⓑ.

Because $w = -dV/dx = 0$ between A and B, the slope of the V-diagram is zero between these points.

Connect ⓐ and ⓑ with a horizontal straight line.

$$V_B^+ = V_B^- - P_B = 14 - (+30) = -16 \text{ kN}$$

Plot point ⓒ.

$$V_E^- = V_B^+ - \text{area of } w\text{-diagram}]_B^E = -16 - 0 = -16 \text{ kN}$$

Plot point ⓓ.

Noting that $w = -dV/dx = 0$ between B and E, we conclude that the slope of the V-diagram is zero in segment BE.

Connect ⓒ and ⓓ with a horizontal straight line.

Because there is no loading to the right of E, we should find that $V_E^+ = 0$.

$$V_E^+ = V_E^- - R_E = -16 - (-16) = 0$$

Checks!

Bending Moment Diagram

We now explain the steps required to construct the bending moment diagram shown in Fig. (d). Because the applied couple is known to cause a jump in the bending moment diagram at D, we must distinguish between the bending moments at sections just to the left and to the right of D. Before proceeding, we compute the areas under the shear force diagram for the different beam segments. The results of these computations are shown in Fig. (c). Observe that the areas are either positive or negative, depending on the sign of the shear force.

We begin our construction of the bending moment diagram by noting that $M_A = 0$ (there is no couple applied at A).

Plot point ⓔ.

Proceeding across the beam from left to right, we generate the moment diagram in Fig. (d) in the following manner:

$$M_B = M_A + \text{area of } V\text{-diagram}]_A^B = 0 + (+56) = 56 \text{ kN} \cdot \text{m}$$

Plot point ⓕ.

The V-diagram shows that the shear force between A and B is constant and positive. Therefore, the slope of the M-diagram between these two sections is also constant and positive (recall that $dM/dx = V$).

Connect ⓔ and ⓕ with a straight line.

$$M_D^- = M_B + \text{area of } V\text{-diagram}]_B^D = 56 + (-48) = 8 \text{ kN} \cdot \text{m}$$

Plot point ⓖ.

Because the slope of the V-diagram between B and D is negative and constant, the M-diagram has a constant, negative slope in this segment.

Connect ⓕ and ⓖ with a straight line.

$$M_D^+ = M_D^- + C_D = 8 + (+40) = 48 \text{ kN} \cdot \text{m}$$

Plot point (h).

Next, we note that $M_E = 0$ (there is no couple applied at E). Our computation based on the area of the V-diagram should verify this result.

$$M_E = M_D^+ + \text{area of } V\text{-diagram}]_D^E = 48 + (-48) = 0$$ *Checks!*

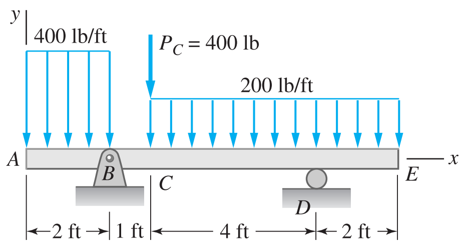
Plot point (i).

The shear force between D and E is negative and constant, which means that the slope of the M-diagram for this segment is also constant and negative.

Connect (h) and (i) with a straight line.

Sample Problem **4.6**

The overhanging beam in Fig. (a) carries two uniformly distributed loads and a concentrated load. Using the area method, draw the shear force and bending moment diagrams for the beam. Neglect the weight of the beam.

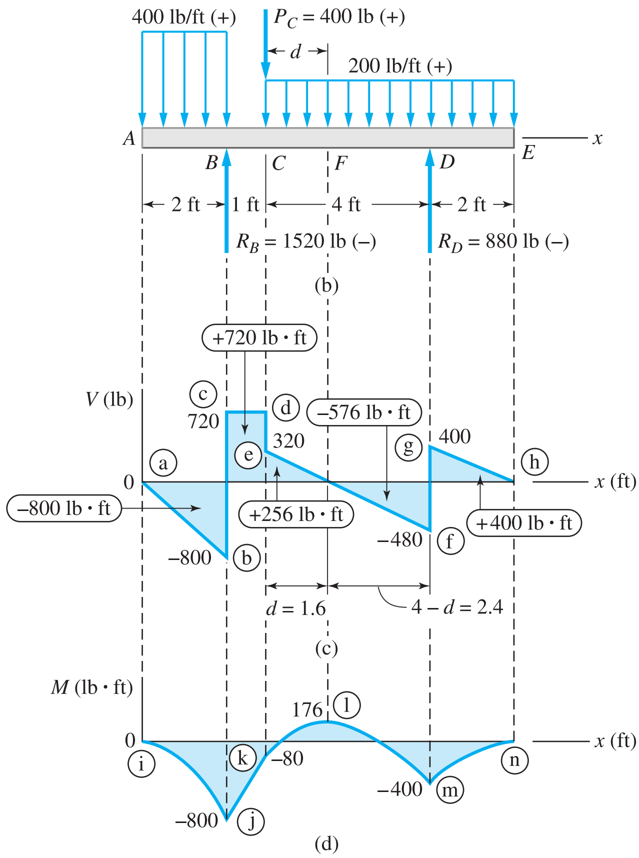


(a)

Solution

Load Diagram

The load diagram for the beam is given in Fig. (b); the reactions at B and D were determined by equilibrium analysis. Each of the numerical values is followed by a plus or minus sign in parentheses, determined by the sign conventions established in Fig. 4.3. The significance of the section labeled F will become apparent in the discussion that follows.



Shear Force Diagram

The steps required to construct the shear force diagram in Fig. (c) are now detailed. From the load diagram, we see that there are concentrated forces at B, C, and D, which means that there will be jumps in the shear diagram at these points. Therefore, we must differentiate between the shear force immediately to the left and to the right of each of these points.

We begin our construction of the V-diagram by observing that $V_A = 0$ because no force is applied at A.

Plot point ⓐ.

$$V_B^- = V_A - \text{area of } w\text{-diagram}\Big]_A^B = 0 - (+400)(2) = -800 \text{ lb}$$

Plot point ⓑ.

We observe from Fig. (b) that the applied loading between A and B is constant and positive, so the slope of the shear diagram between the two cross sections is constant and negative (recall that $dV/dx = -w$).

Connect ⓐ and ⓑ with a straight line.

$$V_B^+ = V_B^- - R_B = -800 - (-1520) = 720 \text{ lb}$$

Plot point ⓒ.

$$V_C^- = V_B^+ - \text{area of } w\text{-diagram}]_B^C = 720 - 0 = 720 \text{ lb}$$

Plot point ⓓ.

Because $w = -dV/dx = 0$ between B and C, the slope of the V-diagram is zero in this segment.

Connect ⓒ and ⓓ with a horizontal straight line.

$$V_C^+ = V_C^- - P_C = 720 - (+400) = 320 \text{ lb}$$

Plot point ⓔ.

$$V_D^- = V_C^+ - \text{area of } w\text{-diagram}]_C^D = 320 - (+200)4 = -480 \text{ lb}$$

Plot point ⓕ.

Because the loading between C and D is constant and positive, the slope of the V-diagram between these two sections is constant and negative.

Connect ⓔ and ⓕ with a straight line.

Our computations have identified an additional point of interest—the point where the shear force is zero, labeled F on the load diagram in Fig. (b). The location of F can be found from

$$V_F = V_C^+ - \text{area of } w\text{-diagram}]_C^F = 320 - (+200)d = 0$$

which gives $d = 1.60$ ft, as shown in Fig. (c).

Continuing across the beam, we have

$$V_D^+ = V_D^- - R_D = -480 - (-880) = 400 \text{ lb}$$

Plot point ⓖ.

Next, we note that $V_E = 0$ (there is no force acting at E). The computation based on the area of the load diagram should verify this result.

$$V_E = V_D^+ - \text{area of } w\text{-diagram}]_D^E = 400 - (+200)2 = 0$$ *Checks!*

Plot point ⓗ.

From Fig. (b), we see that the applied loading between D and E is constant and positive. Therefore, the slope of the V-diagram between these two cross sections is constant and negative.

Connect ⓖ and ⓗ with a straight line.

This completes the construction of the shear force diagram.

Bending Moment Diagram

We now explain the steps required to construct the bending moment diagram shown in Fig. (d). Because there are no applied couples, there will be no jumps in the M-diagram. The areas of the shear force diagram for the different segments of the beam are shown in Fig. (c).

We begin by noting that $M_A = 0$ because no couple is applied at A.

Plot point ⓘ.

Proceeding from left to right across the beam, we construct the bending moment diagram as follows:

$$M_B = M_A + \text{area of } V\text{-diagram}]_A^B = 0 + (-800) = -800 \text{ lb} \cdot \text{ft}$$

Plot point ⓙ.

We note from Fig. (c) that the V-diagram between A and B is a first-degree polynomial (inclined straight line). Therefore, the M-diagram between these two cross sections is a second-degree polynomial—that is, a parabola. From $dM/dx = V$, we see that the slope of the M-diagram is zero at A and -800 lb/ft at B.

Connect ⓘ and ⓙ with a parabola that has zero slope at ⓘ and negative slope at ⓙ. The parabola will be concave downward.

$$M_C = M_B + \text{area of } V\text{-diagram}\Big]_B^C = -800 + (+720) = -80 \text{ lb} \cdot \text{ft}$$

Plot point ⓚ.

Because the V-diagram is constant and positive between B and C, the slope of the M-diagram is constant and positive between those two cross sections.

Connect ⓙ and ⓚ with a straight line.

$$M_F = M_C + \text{area of } V\text{-diagram}\Big]_C^F = -80 + (+256) = +176 \text{ lb} \cdot \text{ft}$$

Plot point ⓛ.

Using $V = dM/dx$, we know that the slope of the M-diagram is $+320$ lb/ft at C and zero at F, and that the curve is a parabola between these two cross sections.

Connect ⓚ and ⓛ with a parabola that has positive slope at ⓚ and zero slope at ⓛ. The parabola will be concave downward.

$$M_D = M_F + \text{area of } V\text{-diagram}\Big]_F^D = 176 + (-576) = -400 \text{ lb} \cdot \text{ft}$$

Plot point ⓜ.

The M-diagram between F and D is again a parabola, with a slope of zero at F and -480 lb/ft at D.

Connect ⓛ and ⓜ with a parabola that has zero slope at ⓛ and negative slope at ⓜ. The parabola will be concave downward.

Next, we note that $M_E = 0$ because no couple is applied at E. Our computation based on the area of the V-diagram should verify this result.

$$M_E = M_D + \text{area of } V\text{-diagram}\Big]_D^E = -400 + (+400) = 0$$ *Checks!*

Plot point ⓝ.

From the familiar arguments, the M-diagram between D and E is a parabola with a slope equal to $+400$ lb/ft at D and zero at E.

Connect ⓜ and ⓝ with a parabola that has positive slope at ⓜ and zero slope at ⓝ. The parabola will be concave downward.

This completes the construction of the bending moment diagram. It is obvious in Fig. (d) that the slope of the M-diagram is discontinuous at ⓙ and ⓜ. Not so obvious is the slope discontinuity at ⓚ: From $dM/dx = V$, we see that the slope of the M-diagram to the left of ⓚ equals $+720$ lb/ft, whereas to the right of ⓚ the slope equals $+320$ lb/ft. Observe that the slope of the M-diagram is continuous at ⓛ because the shear force has the same value (zero) to the left and to the right of ⓛ.

Problems

4.24–4.47 Construct the shear force and bending moment diagrams for the beam shown by the area method. Neglect the weight of the beam.

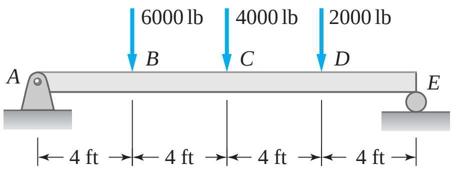


FIG. P4.24

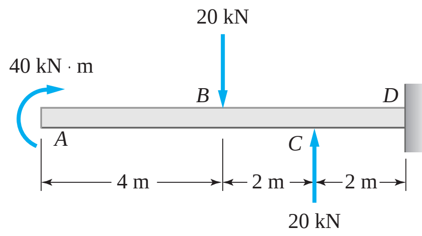


FIG. P4.25

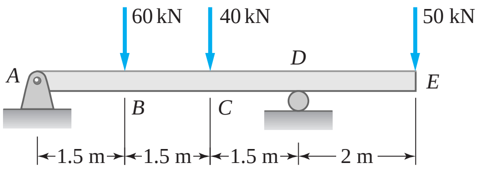


FIG. P4.26

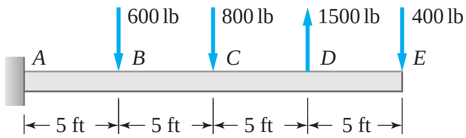


FIG. P4.27

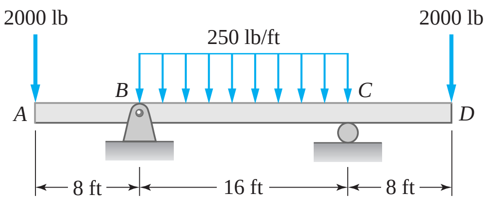


FIG. P4.28

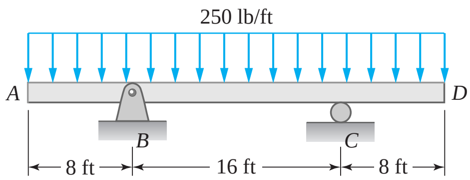


FIG. P4.29

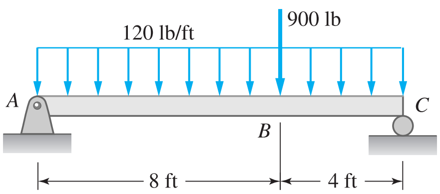


FIG. P4.30

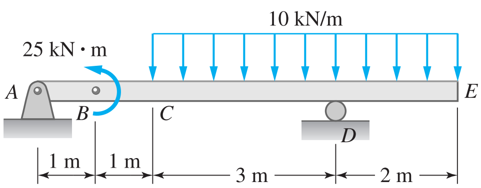


FIG. P4.31

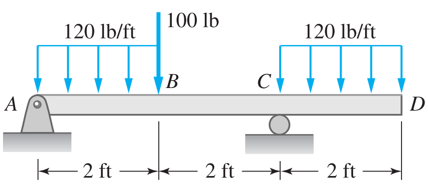


FIG. P4.32

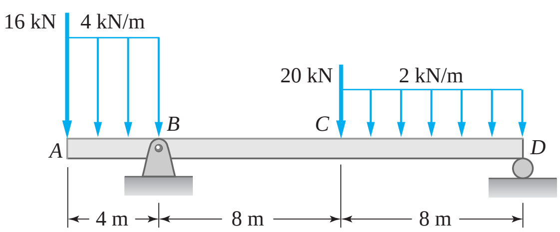


FIG. P4.33

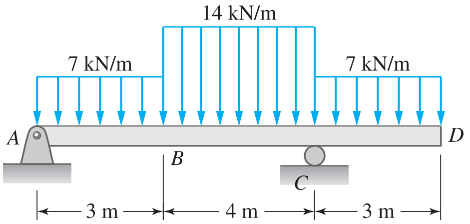


FIG. P4.34

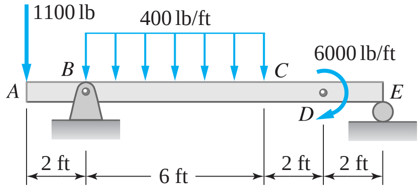


FIG. P4.35

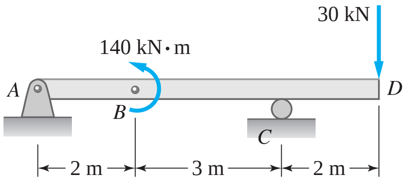


FIG. P4.36

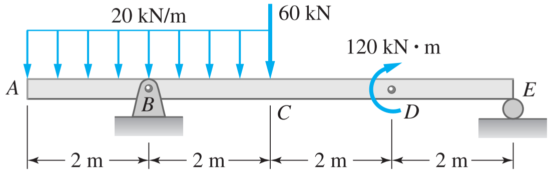


FIG. P4.37

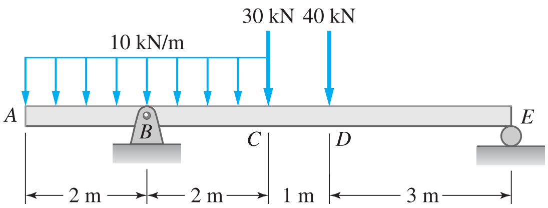


FIG. P4.38

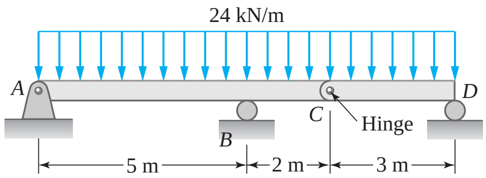


FIG. P4.39

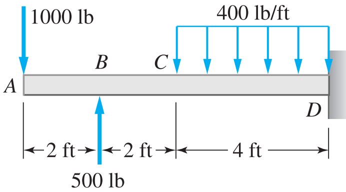


FIG. P4.40

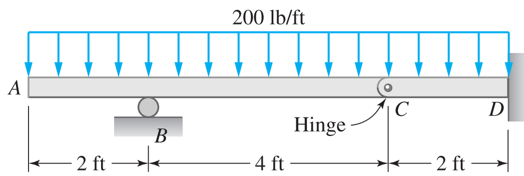


FIG. P4.41

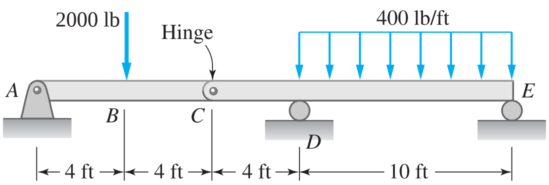


FIG. P4.42

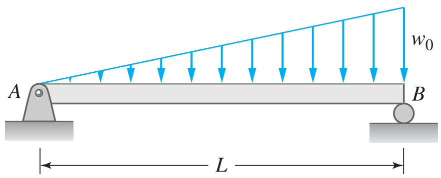


FIG. P4.43

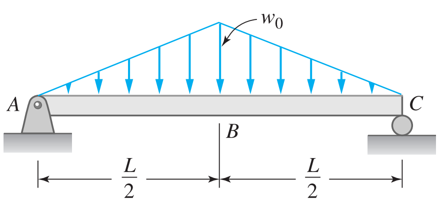


FIG. P4.44

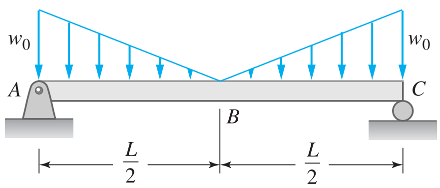


FIG. P4.45

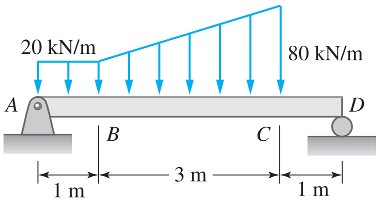


FIG. P4.46

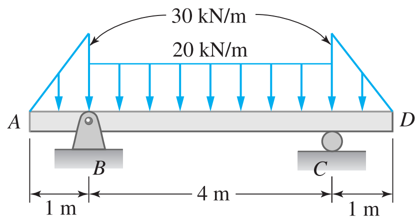


FIG. P4.47

4.48–4.52 Draw the load and the bending moment diagrams that correspond to the given shear force diagram. Assume no couples are applied to the beam.

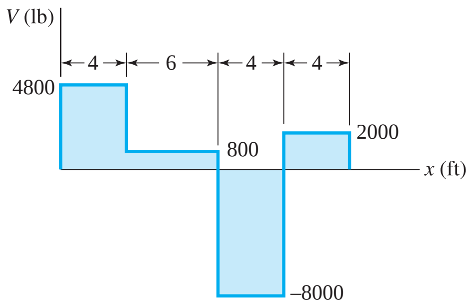


FIG. P4.48

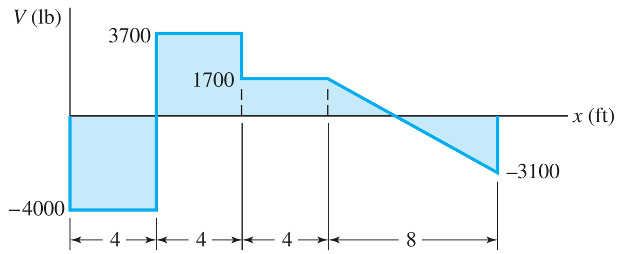


FIG. P4.49

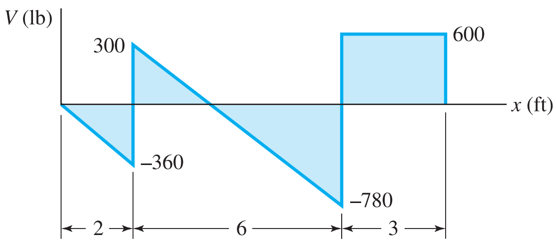


FIG. P4.50

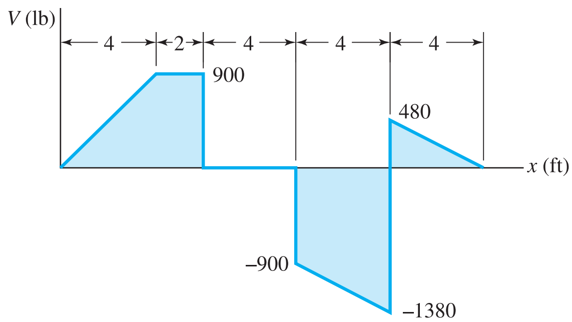


FIG. P4.51

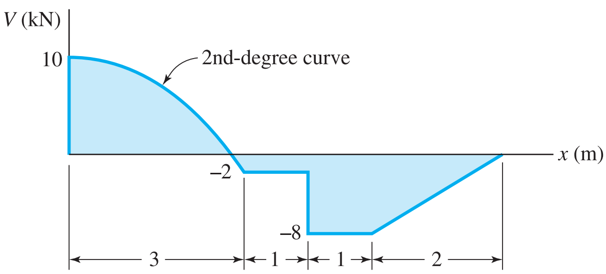


FIG. P4.52

Review Problems

4.53–4.67 Draw the shear force and bending moment diagrams for the beam shown. Neglect the weight of the beam.

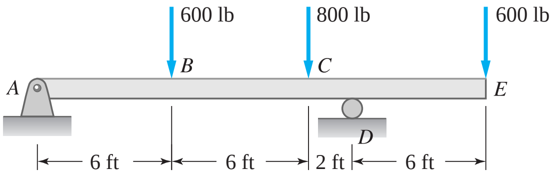


FIG. P4.53

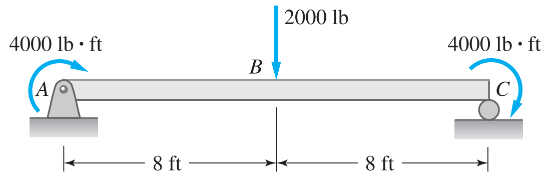


FIG. P4.54

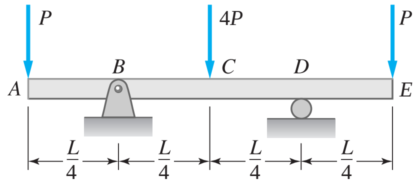


FIG. P4.55

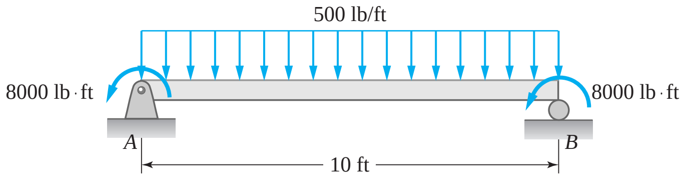


FIG. P4.56

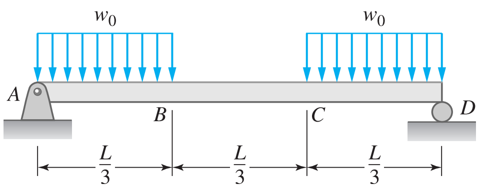


FIG. P4.57

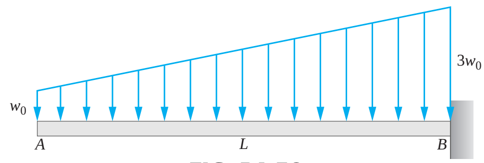


FIG. P4.58

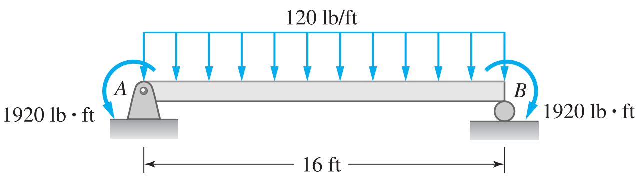


FIG. P4.59

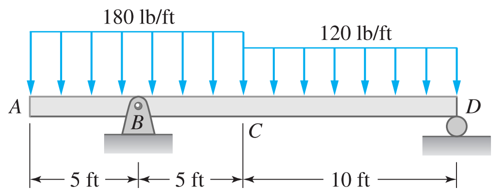


FIG. P4.60

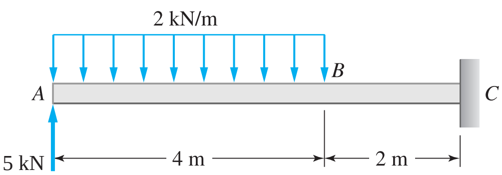


FIG. P4.61

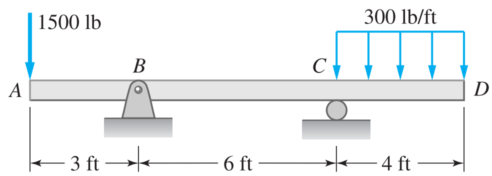


FIG. P4.62

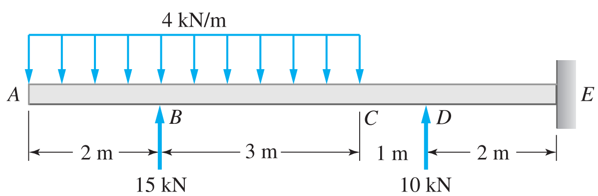


FIG. P4.63

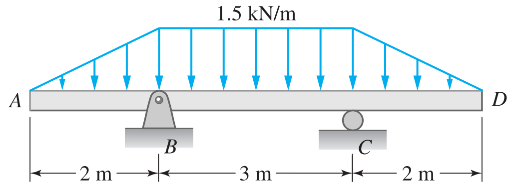


FIG. P4.64

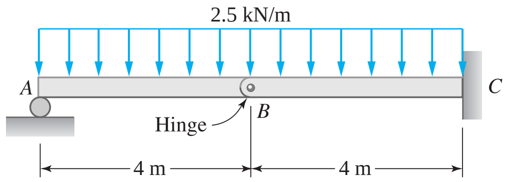


FIG. P4.65

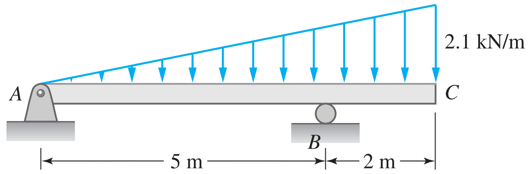


FIG. P4.66

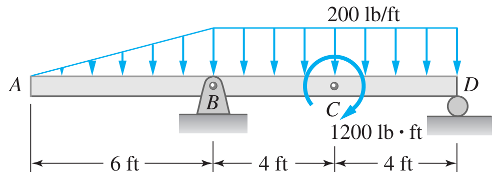


FIG. P4.67

4.68–4.69 Draw the load and the bending moment diagrams that correspond to the given shear force diagram. Assume that no couples are applied to the beam.

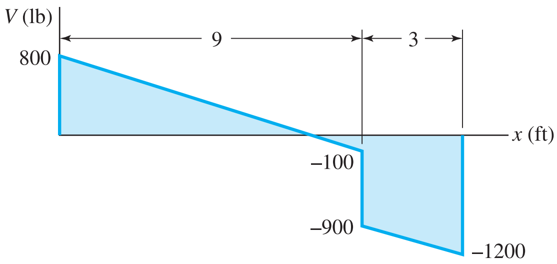


FIG. P4.68

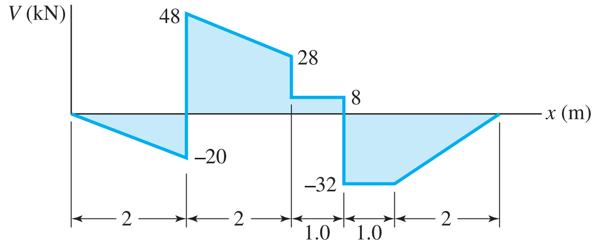
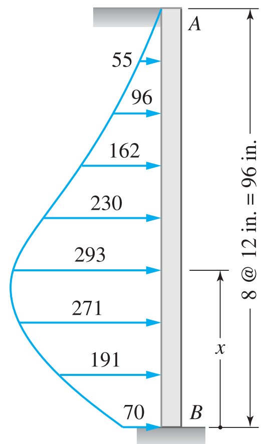


FIG. P4.69

Computer Problems

C4.1 The cantilever beam AB represents a pile that supports a retaining wall. Due to the pressure of soil, the pile carries the distributed loading shown in the figure. Use numerical integration to compute the shear force and the bending moment at B.

C4.2 The overhanging beam carries a distributed load of intensity w_0 over its length L and a concentrated load P at the free end. The distance between the supports is x. Given L, w_0, and P, plot the maximum bending moment in the beam as a function of x from $x = L/2$ to L. Use $L = 16$ ft, $w_0 = 200$ lb/ft, and (a) $P = 1200$ lb and (b) $P = 0$. What value of x minimizes the maximum bending moment in each case?



Load in lb/in.

FIG. C4.1

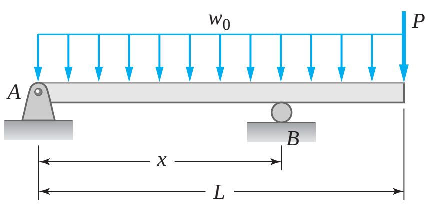


FIG. C4.2

C4.3 The concentrated loads P_1, P_2, and P_3, separated by the fixed distances a and b, travel across the simply supported beam AB of length L. The distance between A and P_1 is x. Given the magnitudes of the loads, a, b, and L, write an algorithm to plot the bending moment under each load as a function of x from $x = 0$ to $L - a - b$. Use (a) $P_1 = 4000$ lb, $P_2 = 8000$ lb, $P_3 = 6000$ lb, $a = 9$ ft, $b = 18$ ft, and $L = 44$ ft; and (b) $P_1 = 8000$ lb, $P_2 = 4000$ lb, $P_3 = 6000$ lb, $a = 5$ ft, $b = 28$ ft, and $L = 80$ ft.

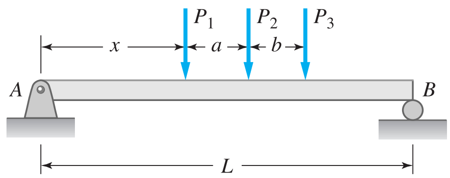


FIG. C4.3

C4.4 The cantilever beam AB of length L carries a distributed loading w that varies with the distance x. Given L and $w(x)$, construct an algorithm to plot the shear force and bending moment diagrams. Use (a) $L = 3$ m and $w = (50\text{ kN/m})\sin(\pi x/2L)$; and (b) $L = 5$ m and

$$w = \begin{cases} 20\text{ kN/m} & \text{if } x \le 1.0\text{ m} \\ (20\text{ kN/m})\dfrac{x}{1.0\text{ m}} & \text{if } 1.0\text{ m} \le x \le 4\text{ m} \\ 0 & \text{if } x > 4\text{ m} \end{cases}$$

C4.5 Solve Prob. C4.4 if the beam is simply supported at A and B.

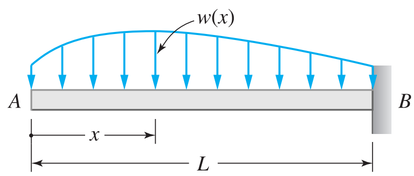


FIG. C4.4, C4.5