Differential Equations

Differential equation (DE): An equation involves one or more derivatives or differentials.

Ordinary Differential Equations (ODEs): Ordinary derivatives occur when the dependent variable "y" is a function of one independent variable "x"; *y= f(x).*

Partial Differential Equations (PDEs): Partial derivatives occur when the dependent variable "y" is a function of two or more independent variables; i.e. *y= f(x,t).*

Order of Differential Equation: The order of a differential equation is the order of the **highest derivative**.

Degree of Differential Equation: The degree of the differential equation is represented by the **power** of the **highest order derivative** in the differential equation.

Example: Classify the differential equations into **ordinary** or **partial** differential equation and then show the **order** and **degree** of each equation.

$x(\frac{d^2y}{dx^2})^2 = 2y(\frac{dy}{dx})$	ODE, 2nd order, 2nd degree
$\frac{d^2y}{dx^2} + 2\ \frac{dy}{dx} = 8x^2 + cosx$	ODE, 2nd order, 1st degree
$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} + \frac{\partial^2 y}{\partial x \partial z} = 0$	PDE, 2nd order, 1st degree
$(\frac{d^2u}{dx^2})^3 - 2\ (\frac{du}{dx})^4 + xu = 0$	ODE, 2nd order, 3rd degree

Homogeneous Linear Differential Equations of the Second-Order

A second-order linear differential equation has the form:

$$\boldsymbol{P_{(x)}\,\frac{d^2y}{dx^2} + Q_{(x)}\frac{dy}{dx} + R_{(x)}\,y = F_{(x)}} \quad \cdots\cdots \mathbf{1}$$

Or

$$\boldsymbol{P_{(x)}\,\overline{\overline{y}} + Q_{(x)}\overline{y} + R_{(x)}\,y = F_{(x)}}$$

Where: P, Q, R are continuous functions.

If $\boldsymbol{F_{(x)} = 0}$**, the** equation is called **homogeneous** linear equations. Thus, the form of a second-order linear **homogeneous** differential equation is:

$$\boldsymbol{P_{(x)}\,\frac{d^2y}{dx^2} + Q_{(x)}\frac{dy}{dx} + R_{(x)}\,y = 0} \quad \cdots\cdots \mathbf{2}$$

If $\boldsymbol{F_{(x)} \neq 0}$**, the** equation is called **Nonhomogeneous** linear equations.

∴ equation (1) is **Nonhomogeneous** linear equations
∴ equation (2) is **homogeneous** linear equations.

Homogeneous Linear ODEs of Second Order with constant coefficients

If the coefficient functions P, Q, and R are **constant functions**, that is, if the differential equation has the form:

$$\boldsymbol{a\,\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0} \quad \cdots\cdots \mathbf{(3)}$$

Where: a, b, and c are **constants** and a ≠ 0

To solve equation (3):

- Assume $\frac{d}{dx} = D, \frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y$
- Put $D = m$

$$a\,D^2y + bDy + c\,y = 0$$

$$a\,m^2y + bmy + c\,y = 0$$

$$y\,(a\,m^2 + \ bm + c) = 0$$

$$y \neq 0$$

$$\boldsymbol{a\,m^2 + \ bm + \ c = 0} \quad \textbf{(Auxiliary equation)}$$

To solve the auxiliary equation:

$$\boldsymbol{m_{1,2} = \frac{-b \ \mp \ \sqrt{b^2 - 4\,a\,c}}{2a}}$$

Case I: $\boldsymbol{b^2 > 4\,a\,c}$ → Two different real roots and the general solution is:

$$\boldsymbol{y_c = \ c_1\,e^{m1x} + \ c_2\,e^{m2x}}$$

Case II: $\boldsymbol{b^2 = 4\,a\,c}$ → Two equal real roots (double roots) and the general solution is:

$$m_1 = \ m_2 = m$$

$$y_c = \ c_1\,e^{mx} + \ c_2\,x\,e^{mx}$$

$$\boldsymbol{y_c = \ (c_1 \ + \ c_2\,x\,)e^{mx}}$$

Case III: $\boldsymbol{b^2 \ < 4\,a\,c}$ → Two complex (conjugate) roots and the general solution is:

$$m_{1,2} = \ \alpha \mp i\beta$$

$$y_c = \ c_1\,e^{\alpha x}cos\beta x + \ c_2 \ \ e^{\alpha x}sin\beta x$$

$$\boldsymbol{y_c = \ e^{\alpha x}(c_1\,cos\beta x + \ c_2\,sin\beta x)}$$

Summary: Solution of ($\boldsymbol{a\,m^2 + \ bm + c = 0}$)

Roots	General Solution
m_1, m_2 real and different roots	$\boldsymbol{y_c = \ c_1\,e^{m1x} + \ c_2\,e^{m2x}}$
$m_1 = \ m_2 = m$ real and equal	$\boldsymbol{y_c = \ (c_1 \ + \ c_2\,x\,)e^{mx}}$
m_1, m_2 complex (conjugate) roots: $\alpha \mp i\beta$	$y_c = \ c_1\,e^{\alpha x}cos\beta x + \ c_2 \ \ e^{\alpha x}sin\beta x$

Example: Solve the Ordinary differential equation (ODE):

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2\,y = 0$$

$$D^2y + Dy - 2\,y = 0$$

$$m^2y + my - 2y = 0$$

$$\mathrm{y}(m^2 + m - 2) = 0$$

$$\mathrm{y} \neq 0$$

$$m^2 + m - 2 = 0 \qquad \textbf{(Auxiliary equation)}$$

$$(m + 2)(m - 1) = 0$$

$$m_1 = -2$$

$$m_2 = 1$$

$m_1 \neq m_2$ → Two different real roots (Case I)

$$y_c = c_1\, e^{m1x} + c_2\, e^{m2x}$$

$$y_c = c_1\, e^{-2x} + c_2\, e^{x} \qquad \textbf{(General Solution)}$$

Example: Solve the Ordinary differential equation (ODE):

$$\bar{\bar{y}} + 8\bar{y} + 16\,y = 0$$

$$D^2y + 8Dy + 16\,y = 0$$

$$m^2y + 8my + 16y = 0$$

$$\mathrm{y}(m^2 + 8\,m + 16) = 0$$

$$\mathrm{y} \neq 0$$

$$m^2 + 8\,m + 16 = 0 \qquad \textbf{(Auxiliary equation)}$$

$$(m + 4)(m + 4) = 0$$

$$m_1 = -4$$

$$m_2 = -4$$

$m_1 = m_2 = m$ → Two equal real roots (Case II)

$$y_c = c_1\, e^{mx} + c_2\, x\, e^{mx}$$

$$y_c = c_1\, e^{-4x} + c_2\, x e^{-4x}$$

$$\boldsymbol{y_c = (c_1 + c_2\, x) e^{-4x}}$$ **(General Solution)**

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10\, y = 0}$$

$$D^2y - 2Dy + 10y = 0$$

$$m^2y - 2my + 10y = 0$$

$$\mathrm{y}(m^2 - 2\, m + 10) = 0$$

$$\mathrm{y} \neq 0$$

$$m^2 - 2\, m + 10 = 0$$ **(Auxiliary equation)**

$$m_{1,2} = \frac{-b\ \mp\ \sqrt{b^2 - 4\, a\, c}}{2a}$$

$$m_{1,2} = \frac{-\,(-2) \mp\ \sqrt{(-2)^2 - 4\ x\ 1\ x\ 10}}{2\ x\ 1}$$

$$m_{1,2} = \frac{2\ \mp\ \sqrt{4 - 40}}{2} = \frac{2\ \mp\ \sqrt{-36}}{2}$$

$\boldsymbol{m_{1,2} = 1\ \mp 3i}$ Two complex (conjugate) roots (Case III)

$$y_c = c_1\, e^x cos3x + c_2\, e^x sin3x$$

$$\boldsymbol{y_c = (c_1\, cos3x + c_2 sin3x) e^x}$$

Example: Find the general solution of the following ODEs:

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} = 0\ , \qquad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9\,y = 0, \qquad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13\,y = 0$$

$\boldsymbol{\frac{d^2y}{dx^2} + 7\frac{dy}{dx} = 0}$	$\boldsymbol{\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9\,y = 0}$	$\boldsymbol{\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13\,y = 0}$
$D^2y + 7Dy = 0$ $m^2y + 7\,my = 0$ $y\,(m^2 + 7m) = 0$ $y \neq 0$ $m^2 + 7m = 0$ $m(m+7) = 0$ $m_1 = 0$ $m_2 = -7$ $m_1 \neq m_2$ $y_c = c_1\,e^{m1x} + c_2\,e^{m2x}$ $y_c = c_1\,e^0 + c_2\,e^{-7x}$ $\boldsymbol{y_c = c_1 + c_2\,e^{-7x}}$ **(G. S.)**	$D^2y - 6Dy + 9\,y = 0$ $m^2y - 6my + 9y = 0$ $y\,(m^2 - 6\,m + 9) = 0$ $y \neq 0$ $m^2 - 6\,m + 9 = 0$ $(m-3)(m-3) = 0$ $m_1 = 3$ $m_2 = 3$ $m_1 = m_2 = m$ $y_c = c_1\,e^{mx} + c_2\,x\,e^{mx}$ $y_c = c_1\,e^{3x} + c_2\,xe^{3x}$ $\boldsymbol{y_c = (c_1 + c_2\,x)e^{3x}}$ **(G. S.)**	$D^2y - 6Dy + 13y = 0$ $m^2y - 6my + 13y = 0$ $y\,(m^2 - 6\,m + 13) = 0$ $y \neq 0$ $m^2 - 6\,m + 13 = 0$ $m_{1,2} = \frac{6 \mp \sqrt{36-52}}{2\,x\,1}$ $m_{1,2} = \frac{6 \mp \sqrt{-16}}{2} = 3 \mp 2i$ $m_{1,2} = 3 \mp 2i$ $y_c = c_1e^{3x}cos2x + c_2e^{3x}sin2x$ $\boldsymbol{y_c = (c_1\,cos2x + c_2\,sin2x)e^{3x}}$ **(G. S.)**

H.W.: Find the general solution to each of the following ODEs:

$\boldsymbol{\overline{\overline{y}} - 9y = 0}$	$\boldsymbol{\overline{\overline{y}} + 9y = 0}$
$\boldsymbol{\overline{\overline{y}} + 6\overline{y} - 9y = 0}$	$\boldsymbol{\overline{\overline{y}} + 6\overline{y} + 9y = 0}$
$\boldsymbol{\overline{\overline{y}} - 2\overline{y} - 15\,y = 0}$	$\boldsymbol{9\overline{\overline{y}} + \overline{y} = 0}$

Initial Value Problem

In most cases the unique solution of a given problem, hence a particular solution is obtained from a general solution by an initial condition $y(o) = y_o$,
Which given values x_o and y_o, that is used to determine a value of the arbitrary constant C.

Geometrically this condition means that the solution curve pass through the point $(x_{o,} y_o)$ in the x-y plane.

An ODE together with an initial condition, is called an **initial value problem**. Thus, if the ODE is explicit, $\bar{y} = f(x, y)$, the initial value problem is of the form:

$$\bar{y} = f(x,y) \, , y(x_o) = y_o$$

For second order homogeneous linear ODEs (2) an initial value problem consist of (2) and two initial value.

$$\boldsymbol{y}(x_o) = k_o, \bar{\boldsymbol{y}}(x_o) = k_1$$

Example: Solve the initial value problem.

$$\frac{dy}{dx} - 2xy = x, \;\; y(o) = 2$$

$$\frac{dy}{dx} = x + 2xy$$

$$\frac{dy}{dx} = \; x(1 + 2y)$$

$$\int \frac{dy}{1 + 2y} = \int x \;\, dx$$

$$\frac{1}{2} \, ln \, (1 + 2y) = \frac{x^2}{2} + c$$

$$ln \, (1 + 2y) = x^2 + 2c$$

$$(1 + 2y) = \; e^{x^2 + 2c}$$

$$1 + 2y = e^{x^2} . e^{2c}$$

$$1 + 2y = c\, e^{x^2}$$

$$2y = c\, e^{x^2} - 1$$

$$y = \frac{c\, e^{x^2} - 1}{2} \qquad \text{G. S}$$

$$2 = \frac{c\, e^0 - 1}{2} \rightarrow \boldsymbol{c = 5}$$

$$\therefore \boldsymbol{y = \frac{5\, e^{x^2} - 1}{2}} \quad \textbf{P. S.}$$

Example: Solve the following initial value problem:

$\boldsymbol{\frac{dy}{dx} = 10 - x, \quad y(o) = -1}$	$\boldsymbol{\frac{dy}{dx} = 9\, x^2 - 4x + 5, \quad y(-1) = 0}$
$dy = (10 - x)dx$	$dy = (9\, x^2 - 4x + 5)\, dx$
$\boldsymbol{y = 10\, x - \frac{x^2}{2} + c \quad G.S.}$	$\boldsymbol{y = 3\, x^3 - 2x^2 + 5x + c \quad G.S.}$
$x = 0\,, \qquad y = -1$	$x = -1\,, \qquad y = 0$
$-1 = 10(0) - \frac{0^2}{2} + c \rightarrow c = -1$	$0 = 3\,(-1)^3 - 2(-1)^2 + 5(-1) + c$
	$c = 10$
$\therefore \boldsymbol{y = 10\, x - \frac{x^2}{2} - 1 \quad P.S.}$	$\therefore \boldsymbol{y = 3\, x^3 - 2x^2 + 5x + 10 \quad P.S.}$

H. W.: Solve the following initial value problem:

$$\bar{y} = y + 1 , \qquad y(0) = 5$$

Example: Find the solution of the ODE, give the initial value problem:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3\,y = 0$$

$$y(0) = 2, \ \frac{dy}{dx}(0) = -1$$

$$D^2y + 4\,Dy + 3\,y = 0$$
$$m^2y + 4\,my + 3y = 0$$
$$\mathrm{y}(m^2 + 4\,m + 3) = 0$$
$$\mathrm{y} \neq 0$$

$$m^2 + 4\,m + 3 = 0 \qquad \textbf{(Auxiliary equation)}$$
$$(m+3)(m+1) = 0$$
$$m_1 = -3$$
$$m_2 = -1$$

$m_1 \neq m_2 \rightarrow$ Two different real roots (Case I)

$$y_c = c_1\,e^{m1x} + c_2\,e^{m2x}$$

$$y_c = c_1\,e^{-3x} + c_2\,e^{-x} \qquad \textbf{(General Solution)}$$

$$y(0) = 2$$

$$2 = c_1\,e^{-3(0)} + c_2\,e^{-(0)}$$

$$2 = c_1 + c_2 \ \dots\dots\dots (1)$$

$$\bar{y}\,(0) = -1$$

$$\bar{y} = -3\,c_1\,e^{-3x} - c_2\,e^{-x}$$

$$-1 = -3\,c_1\,e^{-3(0)} - c_2\,e^{-(0)}$$

$$-1 = -3\,c_1 - c_2 \dots\dots\dots (2)$$

By solving equs. (1) and (2):

$$c_1 = -0.5, \quad c_2 = 2.5$$

$$\boldsymbol{y_c = -\frac{1}{2}\, e^{-3x} + 2.5\, e^{-x}}$$ **(Particular Solution)**

Example: Solve the initial value problem:

$$\boldsymbol{\bar{\bar{y}} + 8\bar{y} + 25\, y = 0}$$

$$\boldsymbol{y(0) = 4, \qquad \bar{y}(0) = 5}$$

$$D^2y + 8Dy + 25\, y = 0$$

$$m^2y + 8my + 25y = 0$$

$$\mathrm{y}(m^2 + 8\, m + 25) = 0$$

$$\mathrm{y} \neq 0$$

$$m^2 + 8\, m + 25 = 0$$ **(Auxiliary equation)**

$$m_{1,2} = \frac{-b \mp \sqrt{b^2 - 4\, a\, c}}{2a}$$

$$m_{1,2} = \frac{-8 \mp \sqrt{(8)^2 - 4\, x\, 1\, x\, 25}}{2\, x\, 1}$$

$$m_{1,2} = \frac{-8 \mp \sqrt{64 - 100}}{2} = \frac{-8 \mp \sqrt{-36}}{2}$$

$$m_1 = -4 + 3i$$

$$m_2 = -4 - 3i$$

Two complex (conjugate) roots (Case III)

$$\boldsymbol{y_c = c_1\, e^{-4x} cos3x + c_2\, e^{-4x} sin3x}$$ **(General Solution)**

$$y(0) = 4$$

$$4 = c_1\, e^0 cos3(0) + c_2\, e^0 sin3(0)$$

$$\boldsymbol{c_1 = 4}$$

$$\bar{y}(0) = 5$$

$$\bar{y} = c_1\, [e^{-4x} . (-3\, sin3x) + cos3x . (-4e^{-4x})] + c_2[\, e^{-4x} . 3\, cos3x + sin3x . (-4e^{-4x})]$$

$$5 = c_1 [0 - 4] + c_2[\ 3 + 0]$$

$$5 = -16 + 3c_2 \rightarrow c_2 = 7$$

$$y_p = 4\, e^{-4x} cos3x + 7e^{-4x} sin3x$$

$$\boldsymbol{y_p = e^{-4x}(4\, cos3x + 7sin3x)}$$ **(Particular Solution)**

H. W: Solve the initial value problem:

$$\boldsymbol{\overline{\overline{y}} + 9\, y = 0}$$

$$\boldsymbol{y(0) = 3, \qquad \overline{y}(0) = 7}$$

Homogeneous Linear ODEs of high - Order with constant coefficients

If the coefficient functions P, Q, and R are **constant functions**, that is, if the differential equation has the form:

$$\boldsymbol{a_o \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots\cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0}$$

- Assume $\frac{d}{dx} = D, \frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2 y, \frac{d^3y}{dx^3} = D^3 y, \ldots\ldots.$
- Put $D = m$

$$a_o D^n y + a_1 D^{n-1} y + \cdots\cdots + a_{n-1} Dy + a_n y = 0$$

$$a_o m^n y + a_1 m^{n-1} y + \cdots\cdots + a_{n-1} my + a_n y = 0$$

$$y\,(a_o m^n + a_1 m^{n-1} + \cdots\cdots + a_{n-1} m + a_n) = 0$$

$$y \neq 0$$

$$a_o m^n + a_1 m^{n-1} + \cdots\cdots + a_{n-1} m + a_n = 0$$

Solve to obtain the values of roots:

1. When the roots m_1, m_2, m_3, ……., m_n are different roots, then the general solution is:

$$\boldsymbol{y_c = c_1 e^{m1x} + c_2 e^{m2x} + c_3 e^{m3x} + \cdots\cdots\cdots}$$

2. If the roots are equal $(m_1 = m_2 = m_3 = \cdots\cdots\cdots)$, the general solution is:

$$y_c = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + c_4 x^3 e^{mx} + \cdots\cdots)$$

$$\boldsymbol{y_c = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \cdots\cdots)e^{mx}}$$

3. If the roots are complex (conjugate) number say $\alpha \mp i\beta$, then the general solution is:

$$y_c = c_1 e^{\alpha x} cos\beta x + c_2 e^{\alpha x} sin\beta x$$

$$\boldsymbol{y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)}$$

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^3y}{dx^3} - 2\,\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2\,y = 0}$$

$$D^3y - 2D^2y - Dy + 2\,y = 0$$

$$m^3y - 2m^2y - my + 2\,y = 0$$

$$\text{y}\,(m^3 - 2m^2 - m + 2) = 0$$

$$\text{y} \neq 0$$

$$m^3 - \ 2m^2 - m + 2 = \mathbf{0}$$

$$m^2(m-2) - (m-2) = 0$$

$$(m-2)(m^2-1) = 0$$

$$m_1 = 2$$

$$m_2 = 1$$

$$m_3 = -1$$

$$\boldsymbol{y_c = \ c_1\, e^{2x} + \ c_2\, e^{x} + \ c_3\, e^{-x}} \quad \textbf{(General Solution)}$$

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0}$$

$$D^3y - \ D^2y + Dy - y = 0$$

$$\text{y}\,(m^3 - m^2 + m - 1) = 0$$

$$\text{y} \neq 0$$

$$m^3 - m^2 + m - 1 = 0$$

$$m^2(m-1) + (m-1) = 0$$

$$(m-1)(m^2+1) = 0$$

$$m_1 = \ 1$$

$$m_2 = i$$

$$m_3 = -i$$

$$y_c = c_1 e^x + c_2 e^{0x} cosx + c_3 e^{0x} sinx$$

$$\boldsymbol{y_c = c_1 e^x + c_2 cosx + c_3 sinx}$$ (General Solution)

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^4y}{dx^4} - 4\frac{d^2y}{dx^2} + 4y = 0}$$

$$D^4y - 4D^2y + 4y = 0$$

$$\text{y}\,(m^4 - 4m^2 + 4) = 0$$

$$\text{y} \neq 0$$

$$m^4 - 4m^2 + 4 = 0$$

$$(m^2 - 2)(m^2 - 2) = 0$$

$$(m^2 - 2) = 0 \rightarrow m_1 = \sqrt{2}$$

$$m_2 = -\sqrt{2}$$

$$(m^2 - 2) = 0 \rightarrow m_3 = \sqrt{2}$$

$$m_4 = -\sqrt{2}$$

$$\boldsymbol{y_c = (c_1 + c_2 x)e^{\sqrt{2}x} + (c_3 + c_4 x)e^{-\sqrt{2}x}}$$ **(G. S.)**

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^4y}{dx^4} - 16y = 0}$$

$$D^4y - 16y = 0$$

$$\text{y}\,(m^4 - 16) = 0$$

$$\text{y} \neq 0$$

$$m^4 - 16 = 0$$

$$(m^2+4)(m^2-4)=0$$

$$(m^2+4)=0 \rightarrow m_1 = 2i$$

$$m_2 = -2i$$

$$(m^2-4)=0 \rightarrow m_3 = 2$$

$$m_4 = -2$$

$$\boldsymbol{y_c = c_1 e^{2x} + c_2\, e^{-2x} + c_3 cos2x + c_4\, sin2x} \quad \textbf{(G. S.)}$$

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^7y}{dx^7} + 18\frac{d^5y}{dx^5} + 81\frac{d^3y}{dx^3} = 0}$$

$$D^7y + 18D^5y + 81D^3y = 0$$

$$\text{y}\,(m^7 + 18m^5 + 81m^3) = 0$$

$$\text{y} \neq 0$$

$$m^3(m^4 + 18m^2 + 81) = 0$$

$$m^3(m^2+9)(m^2+9) = 0$$

$$m^3 = 0 \rightarrow m_1 = 0$$

$$m_2 = 0$$

$$m_3 = 0$$

$$m^2 = -9 \rightarrow m_4 = 3i$$

$$m_5 = -3i$$

$$m_6 = 3i$$

$$m_7 = -3i$$

$$\boldsymbol{y_c = c_1 + c_2 x + c_3 x^2 + c_4 cos3x + c_5\, sin3x + c_6 xcos3x + c_7 x\, sin3x}$$

(General Solution)

Non - Homogeneous Linear ODEs of Second Order with constant coefficients

A second-order linear differential equation has the form:

$$\boldsymbol{a\,\frac{d^2y}{dx^2} + \; b\frac{dy}{dx} + \; c\,y = F(x)}$$

Or

$$\boldsymbol{a\,\overline{\overline{y}} + \; b\overline{y} + \; c\,y = F_{(x)}}$$

Where: a, b, and c are **constants** and a ≠ 0

If $\boldsymbol{F_{(x)} = 0}$, **the** equation is called **homogeneous** linear equations. Thus, the form of a second-order linear **homogeneous** differential equation is:

If $\boldsymbol{F_{(x)} \neq 0}$, **the** equation is called **Nonhomogeneous** linear equations.

To solve the equation above:

$$\boldsymbol{y_x = \; y_c + \; y_p}$$

Where: y_c = Complementary equation (homogeneous).

$$\boldsymbol{a\,\overline{\overline{y}} + b\overline{y} + c\,y = 0}$$

y_p = Particular solution**.**

We need non homogeneous solution (particular solution), using **Undetermined Coefficients Method**.

Method of Undetermined Coefficients: This method for obtaining a particular solution (y_P) to a nonhomogeneous equation is called the method of undetermined coefficients.

f(x)	y_p
ke^{px}	Ce^{px}
kx^n ($n = 0,1,2,3,\ \ldots\ldots$)	$k_n x^n + k_{n-1} x^{n-1} + \ \ldots\ldots.\ + k_1 x^1 + k_o$
$k\cos\beta x$	$A\cos\beta x + B\sin\beta x$
$k\sin\beta x$	
$ke^{px}\cos\beta x$ $ke^{px}\sin\beta x$	$e^{px}(A\cos\beta x + B\sin\beta x)$
$k\cos\beta x + H\sin\beta x$	$A\cos\beta x + B\sin\beta x$

Examples:

f(x)	y_p
10	A
x	$Ax + B$
x^2	$Ax^2 + Bx + C$
x^4	$Ax^4 + Bx^3 + Cx^2 + Dx + E$
$x + 5$	$Ax + B$
$x^3 - 2$	$Ax^3 + Bx^2 + Cx + D$
$x^2 - 2x + 3$	$Ax^2 + Bx + C$
$5e^{-2x}$	Ae^{-2x}
$7e^{3x}$	Ae^{3x}
$5e^{-2x} + 7e^{3x}$	$Ae^{-2x} + Be^{3x}$
$5\sin 2x$	$A\cos 2x + B\sin 2x$
$7\cos 5x$	$A\cos 5x + B\sin 5x$
$5\sin 3x + 7\cos 3x$	$A\cos 3x + B\sin 3x$
$2\sin x + 5\cos 5x$	$A\cos x + B\sin x + C\cos 5x + D\sin 5x$
$3e^{2x} + 5x$	$Ae^{2x} + Bx + C$
$x^2 e^{-3x}$	$e^{-3x}(Ax^2 + Bx + C)$
$\sin 2x\ e^{5x}$	$e^{5x}(A\cos 2x + B\sin 2x)$
$x^3\cos 2x$	$(Ax^3 + Bx^2 + Cx + D)(E\cos 2x + F\sin 2x)$

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x}$$

$$y_x = y_c + y_p$$

$\boldsymbol{y_c}$:

$$\frac{d^2y}{d^2x} - 2\frac{dy}{dx} + y = 0$$

$$D^2y - 2Dy + y = 0$$

$$m^2y - 2\,my + y = 0$$

$$\text{y}\,(m^2 - 2m + 1) = 0$$

$$\text{y} \neq 0$$

$$\boldsymbol{m^2 - 2m + 1 = 0} \qquad \textbf{(Auxiliary equation)}$$

$$(m-1)(m-1) = 0$$

$$m_1 = 1$$

$$m_2 = 1$$

$$m_1 = m_2 = m \rightarrow \text{Two equal real roots (Case II)}$$

$$y_c = c_1\,e^x + c_2x\,e^x$$

$$\boldsymbol{y_c = e^x(c_1 + c_2x)} \quad \text{Complementary solution}$$

$\boldsymbol{y_p}$:

$$y_p = Ax + B$$

$$\bar{y}_p = A$$

$$\bar{\bar{y}}_p = 0$$

Substitute in the main equation:

$$0 - 2A + Ax + B = x$$

$$Ax - 2A + B = x$$

$$A = 1$$

$$-2A + B = 0 \rightarrow B = 2$$

$$\therefore y_p = x + 2 \qquad \textbf{Particular solution}$$

$$\therefore \boldsymbol{y = e^x(c_1 + c_2x) + x + 2} \quad \textbf{(G. S.)}$$

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\bar{\bar{y}} + 3\bar{y} + 2y = 12x^2}$$

$$y_x = y_c + y_p$$

$\boldsymbol{y_c}$:

$$\bar{\bar{y}} + 3\bar{y} + 2y = 0$$

$$D^2y + 3Dy + 2y = 0$$

$$m^2y + 3my + 2y = 0$$

$$\text{y}\,(m^2 + 3m + 2) = 0$$

$$\text{y} \neq 0$$

$$\boldsymbol{m^2 + 3m + 2 = 0} \qquad \textbf{(Auxiliary equation)}$$

$$(m + 2)(m + 1) = 0$$

$$m_1 = -2$$

$$m_2 = -1$$

$m_1 \neq m_2 \rightarrow$ Two different real roots (Case I)

$$y_c = c_1\, e^{-2x} + c_2\, e^{-x} \quad \text{Complementary solution}$$

$\boldsymbol{y_p}$:

$$y_p = Ax^2 + Bx + c$$

$$\bar{y}_p = 2Ax + B$$

$$\bar{\bar{y}}_p = 2A$$

Substitute in the main equation:

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + c) = 12x^2$$

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2c = 12x^2$$

$$2Ax^2 + (6A + 2B)x + (2A + 3B + 2c) = 12x^2$$

$$2A = 12 \ \rightarrow \boldsymbol{A = 6}$$

$$6A + 2B = 0 \rightarrow \boldsymbol{B = -18}$$

$$2A + 3B + 2c = 0 \rightarrow \boldsymbol{c = 21}$$

$$\therefore y_p = 6x^2 - 18x + 21$$ **Particular solution**

$$\therefore \boldsymbol{y = c_1\, e^{-2x} + c_2\, e^{-x} + 6x^2 - 18x + 21}$$ **(G. S.)**

H. W.: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\bar{\bar{y}} - 2\bar{y} - 3y = 9x^2 + 1}$$

Example: Solve the Ordinary differential equation (ODE):

$$\bar{\bar{\boldsymbol{y}}} - \bar{\boldsymbol{y}} = \boldsymbol{x}$$

$$y_x = y_c + y_p$$

$\boldsymbol{y_c}$:

$$\bar{\bar{y}} - \bar{y} = 0$$

$$D^2 y - Dy = 0$$

$$m^2 y - my = 0$$

$$\mathrm{y}\,(m^2 - m) = 0$$

$$\mathrm{y} \neq 0$$

$$m^2 - m = \mathbf{0} \qquad \textbf{(Auxiliary equation)}$$

$$m(m-1) = 0$$

$$m_1 = 0$$

$$m_2 = 1$$

$m_1 \neq m_2 \rightarrow$ Two different real roots (Case I)

$$y_c = c_1\, e^{0x} + c_2\, e^{x}$$

$$\boldsymbol{y_c = c_1 + c_2\, e^{x}} \quad \textbf{Complementary solution}$$

$\boldsymbol{y_p}$:

$$y_p = Ax^2 + Bx$$

$$\bar{y}_p = 2Ax + B$$

$$\bar{\bar{y}}_p = 2A$$

> $y_p = Ax + B$
>
> لالغاء التشابة مع $\boldsymbol{y_c}$ نضرب المعادلة بـ $\boldsymbol{x}$
>
> $\boldsymbol{y_p = Ax^2 + Bx}$

Substitute in the main equation:

$$2A - (2Ax + B) = x$$

$$2A - 2Ax - B = x$$

$$-2Ax + (2A - B) = x$$

$$-2A = 1 \rightarrow \boldsymbol{A} = -\frac{1}{2}$$

$$2A - B = 0 \ \rightarrow \boldsymbol{B} = \boldsymbol{-1}$$

$$\therefore y_p = -\tfrac{1}{2}x^2 - x$$ **Particular solution**

$$\therefore \boldsymbol{y = c_1 + c_2\, e^{x} - \tfrac{1}{2}x^2 - x}$$ **(G. S.)**

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\bar{\bar{y}} + 2\bar{y} + y = e^{-x}}$$

$$y_x = y_c + y_p$$

$\boldsymbol{y_c}$:

$$\bar{\bar{y}} + 2\bar{y} + y = 0$$

$$D^2 y + 2Dy + y = 0$$

$$m^2 y + 2my + 1 = 0$$

$$\mathrm{y}(m^2 + 2m + 1) = 0$$

$$\mathrm{y} \neq 0$$

$$m^2 + 2m + 1 = \boldsymbol{0}$$ **(Auxiliary equation)**

$$(m+1)(m+1) = 0$$

$$m_1 = -1$$

$$m_2 = -1$$

$m_1 = m_2 \rightarrow$ Two equal real roots (Case II)

$$y_c = c_1\, e^{-x} + c_2\, x e^{-x}$$

$$\boldsymbol{y_c = e^{-x}(c_1 + c_2\, x)}$$ **Complementary solution**

$\boldsymbol{y_p}$:

$$y_p = Ax^2 e^{-x}$$

$$\bar{y}_p = 2Axe^{-x} - Ax^2 e^{-x}$$

$$\bar{\bar{y}}_p = Ax^2 e^{-x} - 4Axe^{-x} + 2Ae^{-x}$$

$y_p = Ae^{-x}$
لالغاء التشابة مع $\boldsymbol{y_c}$ نضرب المعادلة بـ x^2
$\boldsymbol{y_p = Ax^2 e^{-x}}$

Substitute in the main equation:

$$\bar{\bar{y}} + 2\bar{y} + y = e^{-x}$$

$$Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x} + 2(2Axe^{-x} - Ax^2e^{-x}) + Ax^2e^{-x} = e^{-x}$$

$$Ax^2e^{-x} - 4Axe^{-x} + 2Ae^{-x} + 4Axe^{-x} - 2Ax^2e^{-x} + Ax^2e^{-x} = e^{-x}$$

$$2Ae^{-x} = e^{-x}$$

$$2A = 1 \rightarrow \boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{2}}$$

$$\therefore y_p = \tfrac{1}{2}x^2e^{-x}$$ **Particular solution**

$$\therefore y = e^{-x}(c_1 + c_2\, x) + \frac{1}{2}x^2e^{-x}$$

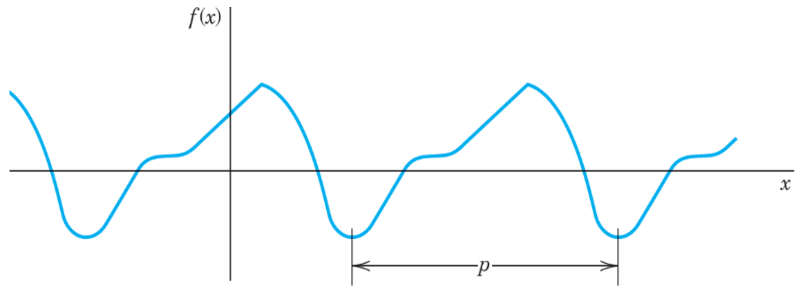
$$\therefore \boldsymbol{y} = \boldsymbol{e^{-x}}(\boldsymbol{c_1} + \boldsymbol{c_2}\, \boldsymbol{x} + \tfrac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x^2})$$ **(G. S.)**

H.W.: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\bar{\bar{y}} + 4\bar{y} + 5y = 2e^{x}}$$

Fourier Series

A function $f(x)$ is said to have a period **P** or to be periodic with period P if for all P, $f(x + \mathrm{P}) = f(\mathrm{P})$, where P is a positive constant. The least value of $\mathrm{P} > 0$ is called the period of $f(\mathrm{P})$.

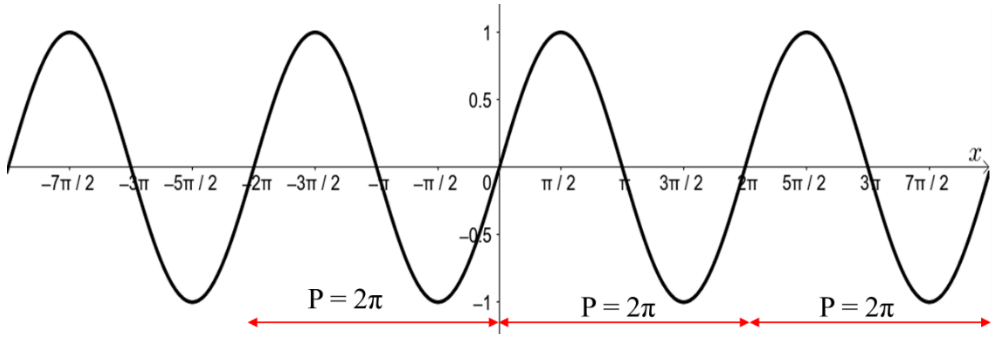


Periodic function of period p

$$f(x + \mathrm{P}) = f(\mathrm{P})$$

The function $\sin(x)$ has period 2π, since:

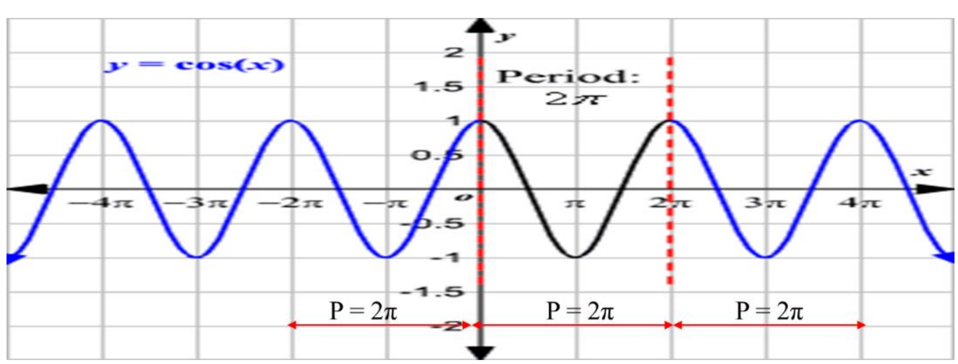
$$\sin(x + 2\pi) = \sin(x)$$



$$f(x + 2\pi) = f(x)$$

The function $\cos(x)$ has period 2π, since:

$$\cos(x + 2\pi) = \cos(x)$$



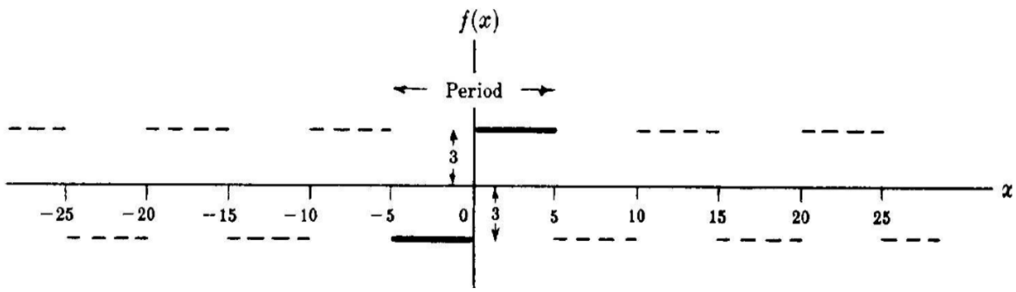
$$f(x + 2\pi) = f(x)$$

Example: Represent (Sketch) the following functions $f(x)$ by Fourier series:

$$f(x) = \begin{cases} -3, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$$

Period = 5 – (-5) = 10

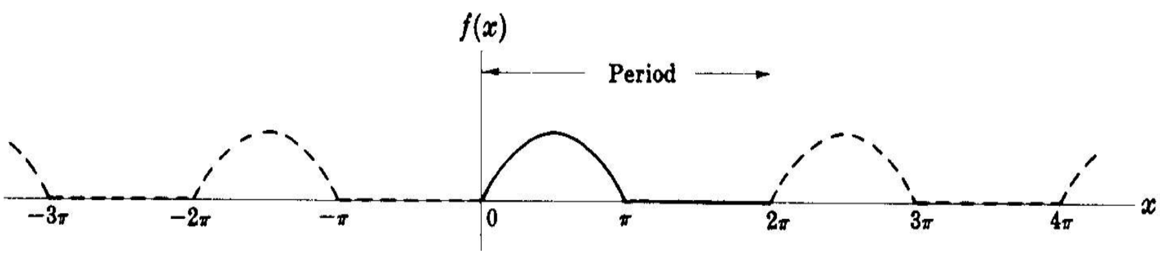
$$f(x+10) = f(x)$$



$$f(x) = \begin{cases} \sin(x), & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

Period = 2π – 0 = 2π

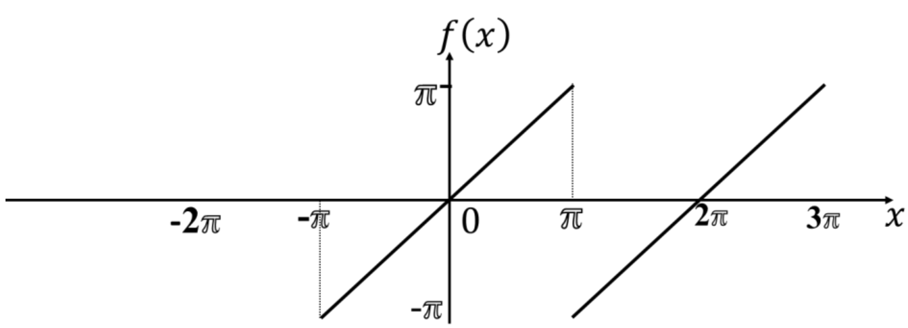
$$f(x+2\pi) = f(x)$$



$$f(x) = x, \quad -\pi < x < \pi$$

Period = π – (-π) = 2π

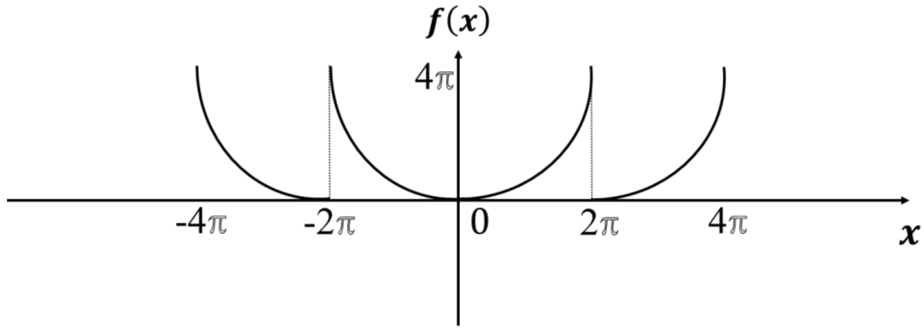
$$f(x+2\pi) = f(x)$$



$$f(x) = x^2, \quad 0 < x < 2\pi$$

Period = $2\pi - 0 = 2\pi$

$$f(x + 2\pi) = f(x)$$



HW.: Represent (Sketch) the following functions *f*(*x*) by Fourier series:

$$f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

All these functions have the period 2π. They form the so-called **trigonometric series,** that isa series of the form**:**

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin n\, x)$$

Where: a_o, a_1, a_2, …are constants, called the **coefficients** of the series.

Fourier Series: The Fourier series of *f*(*x*) is trigonometric series whose coefficients are determines from *f*(*x*) by certain formulas (Euler formula):

Euler function: If *f*(*x*) is periodic function of period **2π**, that can be represented by a trigonometric series:

Where:

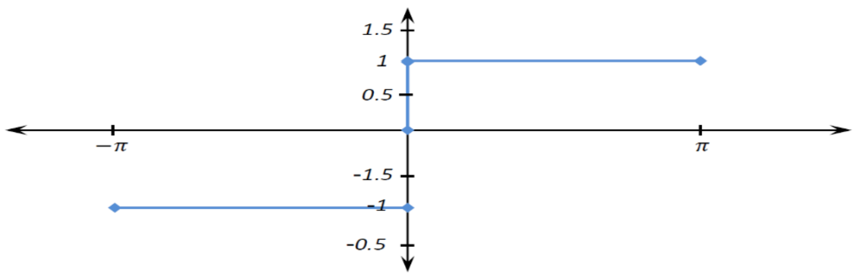
$$a_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, cos\; nx\; dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\ nx\,dx$$

Example: Find the Fourier series for the periodic function:

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$



$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi}(1)\,dx\right]$$

$$= \frac{1}{2\pi}\left[-x\Big|_{-\pi}^{0} + x\Big|_{0}^{\pi}\right] = \frac{1}{2\pi}(-\pi+\pi) = \mathbf{0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,cosnx\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-cosnx)\,dx + \int_{0}^{\pi}(cosnx)\,dx\right]$$

$$a_n = \frac{1}{n\pi}[-sinnx\Big|_{-\pi}^{0} + sinnx\Big|_{0}^{\pi} = \mathbf{0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}\sin nx\ dx + \int_{0}^{\pi}\sin nx\ dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\cos nx}{n}\Big|_{-\pi}^{0} - \frac{\cos nx}{n}\Big|_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right) - \left(\frac{\cos n\pi}{n} - \frac{1}{n}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2}{n} - \frac{2\cos n\pi}{n}\right\}$$

$$\boldsymbol{b_n = \frac{2}{n\pi}(1 - cos\pi), \quad n = 1, 2, 3, \dots..}$$

$$\therefore b_n = \begin{cases} 0, & n \ is \ even \\ \dfrac{4}{n\pi}, & n \ is \ odd \end{cases}$$

$$\boldsymbol{\therefore f(x) = \sum_{n=1,3,5,..}^{\infty} \frac{4}{n\pi} \, sin \, nx}$$

H.W.: Find the Fourier series for the periodic function:

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

Functions of any Period p = 2L

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n cos\frac{n\pi}{L}x \ + b_n sin\frac{n\pi}{L}x)$$

Where:

$$a_o = \frac{1}{2L}\int_{-L}^{L} f(x) \, dx$$

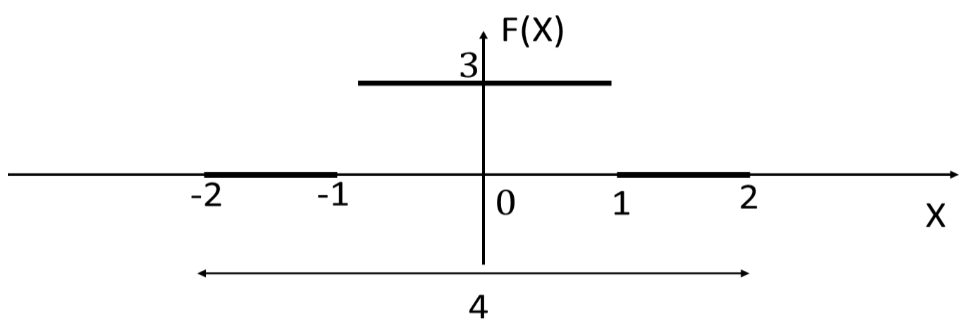
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \, cos\frac{n\pi}{L}x \, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \, sin\frac{n\pi}{L}x \, dx$$

$$2L = period$$

$$\boldsymbol{\therefore L = \frac{period}{2}}$$

Example: Find the Fourier series for the periodic function:



$$f(x) = \begin{cases} 0, & -2 \le x \le -1 \\ 3, & -1 \le x \le 1 \\ 0, & 1 \le x \le 2 \end{cases}$$

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n cos\frac{n\pi}{L} x \ + b_n sin\frac{n\pi}{L} x)$$

$$a_o = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{4}\,[\int_{-2}^{-1} 0\,dx + \int_{-1}^{1} 3\,dx + \int_{1}^{2} 0\,dx]$$

$$a_o = \frac{1}{4}\int_{-1}^{1} 3\,dx = \frac{1}{4}\left[3x \Big|_{-1}^{1}\right] = \frac{1}{4}\,[3+3] = \mathbf{\frac{3}{2}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,cos\frac{n\pi}{L} x\,dx = \frac{1}{2}\,[\int_{-2}^{-1} 0\,dx + \int_{-1}^{1} 3\,os\frac{n\pi}{2} x\,dx + \int_{1}^{2} 0\,dx]$$

$$a_n = \frac{1}{2}\,[\int_{-1}^{1} 3\,cos\frac{n\pi}{2} x\,dx]$$

$$a_n = \frac{3}{n\pi}\left(sin\frac{n\pi}{2} x \Big|_{-1}^{1}\right) = \frac{3}{n\pi}\left(sin\frac{n\pi}{2} + sin\frac{n\pi}{2}\right) = \boldsymbol{\frac{6}{n\pi}\left(sin\frac{n\pi}{2}\right)}$$

$$a_n = \begin{cases} \frac{6}{n\pi}, & n = 1, 5, 9, \ldots \\ 0, & n\ is\ even \\ -\frac{6}{n\pi}, & n = 3, 7, 11, \ldots \end{cases}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,sin\frac{n\pi}{L} x\,dx = \frac{1}{2}\,[\int_{-1}^{1} 3\,sin\frac{n\pi}{2} x\,dx]$$

$$b_n = \frac{3}{n\pi}\left(-cos\frac{n\pi}{2} x \Big|_{-1}^{1}\right) = \frac{3}{n\pi}\left(-cos\frac{n\pi}{2} + cos\frac{n\pi}{2}\right) = \mathbf{0}$$

$$\therefore \boldsymbol{f(x)} = \frac{3}{2} \sum_{\boldsymbol{n=1,5,9,..}}^{\infty} \frac{\boldsymbol{6}}{\boldsymbol{n\pi}} \cos\frac{n\pi}{2}x - \sum_{\boldsymbol{n=3,7,11,..}}^{\infty} \frac{\boldsymbol{6}}{\boldsymbol{n\pi}} \cos\frac{n\pi}{2}x$$

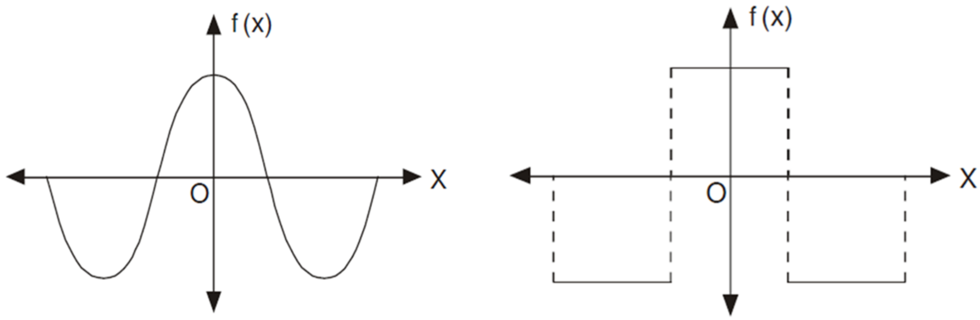
H. W.: Find the Fourier series for the periodic function:

$$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ 3, & 0 \le x \le 2 \end{cases}$$

Odd and Even Functions

A function $f(x)$ is said to be **even** (or symmetric) function if, $f(-x) = f(x)$, then there will not be any **sine** terms in the Fourier series for $f(x)$.

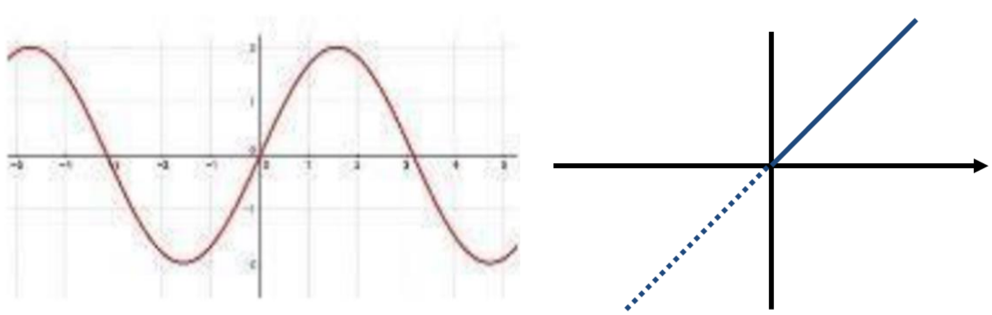
The graph of such a function is symmetric with respect to Y - axis. Here Y - axis is a mirror for the reflection of the curve.



Even Functions (Cosine Series)

$$\boldsymbol{f(x) = a_o + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x}$$

A function $f(x)$ is called **odd (Sine Series)** (or skew symmetric) function if $f(-x) = -f(x)$, then there will not be any **cosine** terms in the Fourier series for $f(x)$.



Odd Functions (Sine Series)

$$\boldsymbol{f_{(x)} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l} x}$$

Half-Range Expansions

Expansion of an even function:

$$\boldsymbol{b_o = 0}$$

$$a_o = \frac{1}{2L}\int_{-L}^{L} f_{(x)}\,dx \rightarrow\ a_o = \frac{1}{2L}\int_{0}^{L} f_{(x)}\,dx\ \ x\ 2$$

$$\boldsymbol{a_o = \frac{1}{L}\int_{0}^{L} f_{(x)}\,dx}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{(x)} \cos\frac{n\pi}{L} x\,dx \rightarrow\ a_n = \frac{1}{L}\int_{0}^{L} f_{(x)} \cos\frac{n\pi}{L} x\,dx\ \ x\ 2$$

$$\boldsymbol{a_n = \frac{2}{L}\int_{0}^{L} f_{(x)} \cos\frac{n\pi}{L} x\,dx}$$

$$\boldsymbol{f_{(x)} = a_o + \sum_{n=1}^{\infty} a_n\, \mathrm{scos}\frac{n\pi}{l} x}$$

Expansion of an odd function:

$$\boldsymbol{a_o = 0}$$

$$\boldsymbol{a_n = 0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{(x)} \sin\frac{n\pi}{L} x\,dx\ \ x\ 2$$

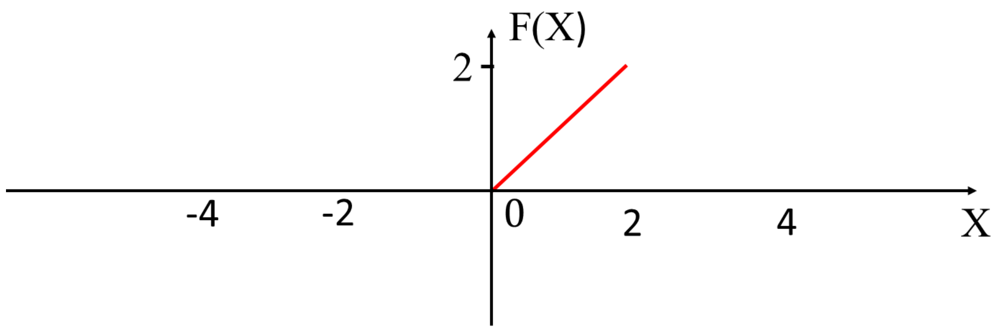
$$\boldsymbol{b_n = \frac{2}{L}\int_{0}^{L} f_{(x)} \sin\frac{n\pi}{L} x\,dx}$$

$$f_{(x)} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

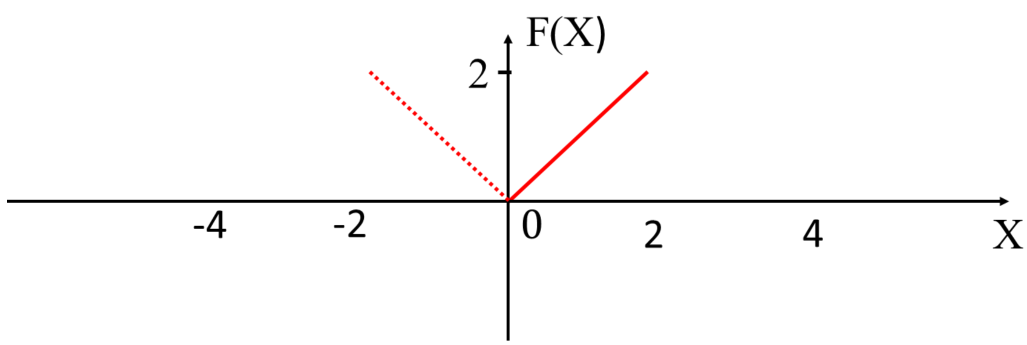
Example: Find half range series of the function:

$$f(x) = x, \qquad 0 \le x \le 2$$

1. Even expansion
2. Odd expansion



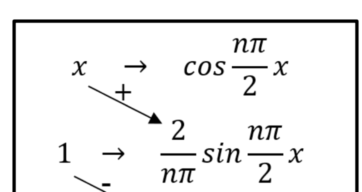
1. Even expansion: $f(-x) = f(x)$



$$b_n = 0$$

$$a_o = \frac{1}{L}\int_0^L f_{(x)}\, dx = \frac{1}{2}\int_0^2 x\, dx = \frac{1}{4}\, x^2 \Big|_0^2 = \mathbf{1}$$

$$a_n = \frac{2}{L}\int_0^L f_{(x)} \cos\frac{n\pi}{L} x\, dx = \frac{2}{L}\int_0^2 x \cos\frac{n\pi}{L} x\, dx = \frac{2}{n\pi}\, x \sin\frac{n\pi}{2} x \Big|_0^2 + \frac{4}{n^2\pi^2} \cos\frac{n\pi}{2} x \Big|_0^2$$



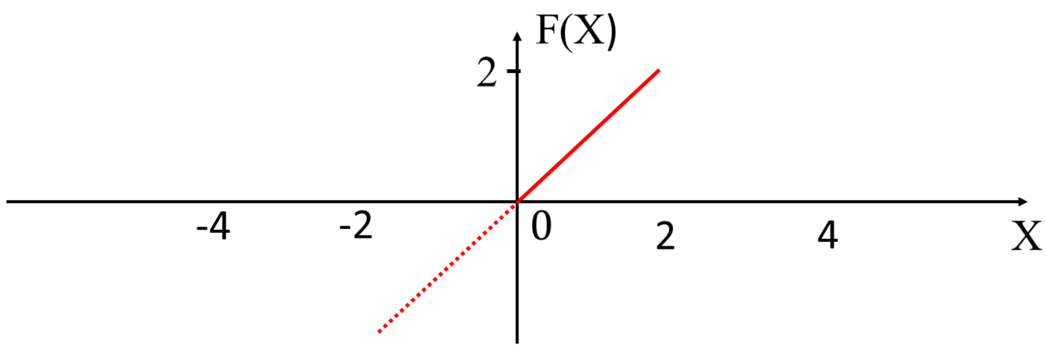
$$a_n = 0 + \frac{4}{n^2\pi^2}(cosn\pi - 1)$$

$$\boldsymbol{a_n = \frac{4}{n^2\pi^2}(cosn\pi - 1)}$$

$$a_n = \begin{cases} 0, & n \ is \ even \\ -\frac{8}{n^2\pi^2}, & n \ is \ odd \end{cases}$$

$$f_{(x)} = 1 + \sum_{n=1,3,5}^{\infty} -\frac{8}{n^2\pi^2} \cos\frac{n\pi}{2}x$$

2. Odd expansion: $f(-x) = -f(x)$



$$a_o = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L f_{(x)} \sin\frac{n\pi}{L}x \, dx = \frac{2}{2}\int_0^2 x \sin\frac{n\pi}{2}x \, dx$$

$$b_n = -\frac{2}{n\pi} \ x \ \cos\frac{n\pi}{2}x \Big|_0^2 + \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2}x \Big|_0^2$$

$$b_n = -\frac{4}{n\pi}cosn\pi$$

$$b_n = \begin{cases} \frac{4}{n\pi}, & n \ is \ odd \\ -\frac{4}{n\pi}, & n \ is \ even \end{cases}$$

$$f_{(x)} = \left[\sum_{n=1,3,5,...}^{\infty} \frac{4}{n\pi}\cos n\pi - \sum_{n=2,4,6,..}^{\infty} \frac{4}{n\pi}\cos n\pi\right] \sin\frac{n\pi}{2}x$$

Example 1

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Find the half-range sine series of the function

$$f(x) = \begin{cases} 4, & \text{if } 0 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < \pi. \end{cases}$$

Solution: $L = \pi$, so that or T/2=π

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \\ &= \frac{8}{\pi}\int_0^{\pi/2} \sin(nx)\,dx = \\ &= -\frac{8}{n\pi}[\cos nx]_0^{\pi/2} \\ &= \frac{8}{n\pi}[1 - \cos(n\pi/2)]. \end{aligned}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n\pi}[1 - \cos(n\pi/2)](\sin nx)$$

Example 2

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Find the half-range cosine series of the function

$$f(x) = \begin{cases} 4, & \text{if } 0 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < \pi. \end{cases}$$

NB same function as in the previous example!

Solution: again, $L = \pi$, so that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi/2} 4\,dx = 2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{8}{\pi}\int_0^{\pi/2}\cos(nx)\,dx$$

$$= \frac{8}{n\pi}[\sin nx]_0^{\pi/2}$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{8}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nx)\right) \quad \cos nx$$

8

H.W.: Draw the function, state whether it's **odd** or **even?** and determine the **Fourier series expansion** for the periodic function whose definition in one period is:

$$f(x) = \begin{cases} x+3, & -3 \le x \le 0 \\ x-3, & 0 \le x \le 3 \end{cases}$$

Homogeneous Linear ODEs of high - Order with constant coefficients

If the coefficient functions P, Q, and R are **constant functions**, that is, if the differential equation has the form:

$$\boldsymbol{a_o \frac{d^n y}{d^n x} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots\cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0}$$

- Assume $\frac{d}{dx} = D, \frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y, \frac{d^3y}{dx^3} = D^3y, \ldots\ldots.$
- Put $D = m$

$$a_o D^n y + a_1 D^{n-1} y + \cdots\cdots + a_{n-1} Dy + a_n y = 0$$

$$a_o m^n y + a_1 m^{n-1} y + \cdots\cdots + a_{n-1} my + a_n y = 0$$

$$y\,(a_o m^n + a_1 m^{n-1} + \cdots\cdots + a_{n-1} m + a_n) = 0$$

$$y \neq 0$$

$$a_o m^n + a_1 m^{n-1} + \cdots\cdots + a_{n-1} m + a_n = 0$$

Solve to obtain the values of roots:

1. When the roots m_1, m_2, m_3, ……., m_n are different roots, then the general solution is:

$$\boldsymbol{y_c = c_1 e^{m1x} + c_2 e^{m2x} + c_3 e^{m3x} + \cdots\cdots\cdots}$$

2. If the roots are equal $(m_1 = m_2 = m_3 = \cdots\cdots\cdots)$, the general solution is:

$$y_c = c_1 e^{mx} + c_2\, x\, e^{mx} + c_3 x^2 e^{mx} + c_4 x^3 e^{mx} + \cdots\cdots)$$

$$\boldsymbol{y_c = (c_1 + c_2\, x + c_3 x^2 + c_4 x^3 + \cdots\cdots\ )e^{mx}}$$

3. If the roots are complex (conjugate) number say $\alpha \mp i\beta$, then the general solution is:

$$y_c = c_1\, e^{\alpha x} cos\beta x + c_2\, e^{\alpha x} sin\beta x$$

$$\boldsymbol{y_c = e^{\alpha x}(c_1\, cos\beta x + c_2\, sin\beta x)}$$

Example: Solve the Ordinary differential equation (ODE):

$$\frac{d^3y}{d^3x} - 2\,\frac{d^2y}{d^2x} - \frac{dy}{dx} + 2\,y = 0$$

$$D^3y - 2D^2y - Dy + 2\,y = 0$$

$$m^3y - 2m^2y - my + 2\,y = 0$$

$$\text{y}\,(m^3 - 2m^2 - m + 2) = 0$$

$$\text{y} \neq 0$$

$$m^3 - 2m^2 - m + 2 = \mathbf{0}$$

$$m^2(m-2) - (m-2) = 0$$

$$(m-2)(m^2-1) = 0$$

$$m_1 = 2$$

$$m_2 = 1$$

$$m_3 = -1$$

$$y_c = c_1\,e^{2x} + c_2\,e^{x} + c_3\,e^{-x}$$ **(General Solution)**

Example: Solve the Ordinary differential equation (ODE):

$$\frac{d^3y}{d^3x} - \frac{d^2y}{d^2x} + \frac{dy}{dx} - y = 0$$

$$D^3y - D^2y + Dy - y = 0$$

$$\text{y}\,(m^3 - m^2 + m - 1) = 0$$

$$\text{y} \neq 0$$

$$m^3 - m^2 + m - 1 = \mathbf{0}$$

$$m^2(m-1) + (m-1) = 0$$

$$(m-1)(m^2+1) = 0$$

$$m_1 = 1$$

$$m_2 = i$$

$$m_3 = -i$$

$$y_c = c_1 e^x + c_2 e^{0x} cosx + c_3 e^{0x} sinx$$

$$\boldsymbol{y_c = c_1 e^x + c_2 cosx + c_3 sinx}$$ (General Solution)

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^4y}{d^4x} - 4\frac{d^2y}{d^2x} + 4y = 0}$$

$$D^4y - 4D^2y + 4y = 0$$

$$y\,(m^4 - 4m^2 + 4) = 0$$

$$y \neq 0$$

$$m^4 - 4m^2 + 4 = 0$$

$$(m^2 - 2)(m^2 - 2) = 0$$

$$(m^2 - 2) = 0 \rightarrow m_1 = \sqrt{2}$$

$$m_2 = -\sqrt{2}$$

$$(m^2 - 2) = 0 \rightarrow m_3 = \sqrt{2}$$

$$m_4 = -\sqrt{2}$$

$$\boldsymbol{y_c = (c_1 + c_2 x)e^{\sqrt{2}x} + (c_3 + c_4 x)e^{-\sqrt{2}x}}$$ **(G. S.)**

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^4y}{d^4x} - 16y = 0}$$

$$D^4y - 16y = 0$$

$$y\,(m^4 - 16) = 0$$

$$y \neq 0$$

$$m^4 - 16 = 0$$

$$(m^2 + 4)(m^2 - 4) = 0$$

$$(m^2+4) = 0 \rightarrow m_1 = 2i$$

$$m_2 = -2i$$

$$(m^2-4) = 0 \rightarrow m_3 = 2$$

$$m_4 = -2$$

$$\boldsymbol{y_c = c_1 e^{2x} + c_2\, e^{-2x} + c_3 cos2x + c_4\, sin2x} \quad \textbf{(G. S.)}$$

Example: Solve the Ordinary differential equation (ODE):

$$\boldsymbol{\frac{d^7y}{d^7x} + 18\frac{d^5y}{d^5x} + 81\frac{d^3y}{d^3x} = 0}$$

$$D^7y + 18D^5y + 81D^3y = 0$$

$$\text{y}\,(m^7 + 18m^5 + 81m^3) = 0$$

$$\text{y} \neq 0$$

$$m^3(m^4 + 18m^2 + 81) = 0$$

$$m^3(m^2+9)(m^2+9) = 0$$

$$m^3 = 0 \rightarrow m_1 = 0$$

$$m_2 = 0$$

$$m_3 = 0$$

$$m^2 = -9 \rightarrow m_4 = 3i$$

$$m_5 = -3i$$

$$m_6 = 3i$$

$$m_7 = -3i$$

$$\boldsymbol{y_c = c_1 + c_2x + c_3x^2 + c_4 cos3x + c_5\, sin3x + c_6 cos3x + c_7\, sin3x}$$

(General Solution)

Application of ODEs of integration method in beams

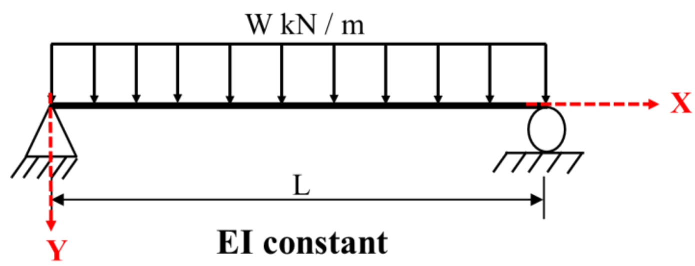
$$\frac{d^4y}{dx^4} = \frac{W_x}{EI} \qquad \textit{Load}$$

$$\frac{d^3y}{dx^3} = \frac{V_x}{EI} \qquad \textit{Shear}$$

$$\frac{d^2y}{dx^2} = \frac{M_x}{EI} \qquad \textit{Moment}$$

$$\frac{dy}{dx} \qquad \textit{Slope}$$

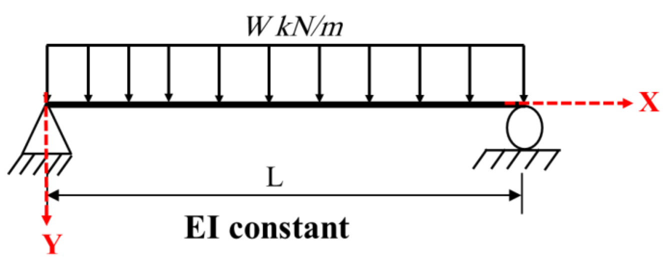
$$y \qquad \textit{Deflection}$$



Where:

E = Modulus of elasticity
I = Moment of inertia

<u>Example:</u> By using the ***Integration Method***, find the ***deflection equation*** for the loaded beam shown in figure below.



$$\frac{d^4y}{dx^4} = \frac{W_{(x)}}{EI}$$

$$W_x = w$$

$$\frac{d^4y}{dx^4} = \frac{w}{EI}$$

$$\frac{d^3y}{dx^3} = \frac{w}{EI}x + c_1$$

$$\frac{d^2y}{dx^2} = \frac{w}{2EI}x^2 + c_1x + c_2$$

$$\frac{dy}{dx} = \frac{w}{6EI}x^3 + \frac{c_1}{2}x^2 + c_2 x + c_3$$

$$\boldsymbol{y_{(x)} = \frac{w}{24EI}x^4 + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3 x + c_4}$$

$$B.C.$$

$$y(0) = 0, \quad \bar{\bar{y}}(0) = 0$$

$$y(L) = 0, \quad \bar{\bar{y}}(L) = 0$$

$$y(0) = 0 \rightarrow \boldsymbol{c_4 = 0}$$

$$\bar{\bar{y}}(0) = 0 \rightarrow \boldsymbol{c_2 = 0}$$

$$\bar{\bar{y}}(L) = 0$$

$$0 = \frac{w}{2EI}L^2 + c_1 L$$

$$\boldsymbol{c_1 = -\frac{wL}{2EI}}$$

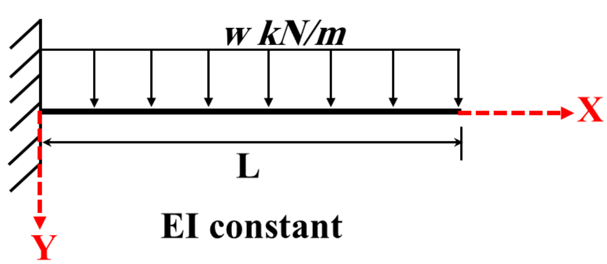
$$y(L) = 0$$

$$0 = \frac{wL^4}{24EI} - \frac{wL^4}{12EI} + c_3 L$$

$$\boldsymbol{c_3 = \frac{wL^3}{24EI}}$$

$$\therefore \boldsymbol{y_{(x)} = \frac{w}{24EI}x^4 - \frac{wL}{12EI}x^3 + \frac{wL^3}{24EI}x} \quad \textbf{(Deflection equation)}$$

Example: By using the ***method of Integration***, find the ***deflection equation*** for the loaded beam shown in figure below.



$$\frac{d^4y}{dx^4} = \frac{W_{(x)}}{EI}$$

$$W_x = w$$

$$\frac{d^4y}{dx^4} = \frac{w}{EI}$$

$$\frac{d^3y}{dx^3} = \frac{w}{EI}x + c_1$$

$$\frac{d^2y}{dx^2} = \frac{w}{2EI}x^2 + c_1x + c_2$$

$$\frac{dy}{dx} = \frac{w}{6EI}x^3 + \frac{c_1}{2}x^2 + c_2x + c_3$$

$$\boldsymbol{y}_{(x)} = \frac{w}{24EI}x^4 + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4$$

$$B.C.$$

$$y(0) = 0, \quad \bar{\bar{y}}(L) = 0$$

$$\bar{y}(0) = 0, \quad y'''(L) = 0$$

$$y(0) = 0 \rightarrow \boldsymbol{c_4} = 0$$

$$\bar{y}(0) = 0 \rightarrow \boldsymbol{c_3} = \boldsymbol{0}$$

$$y'''_{(L)} = 0$$

$$0 = \frac{w}{EI}L + c_1 \rightarrow \boldsymbol{c_1} = -\frac{\boldsymbol{wL}}{\boldsymbol{EI}}$$

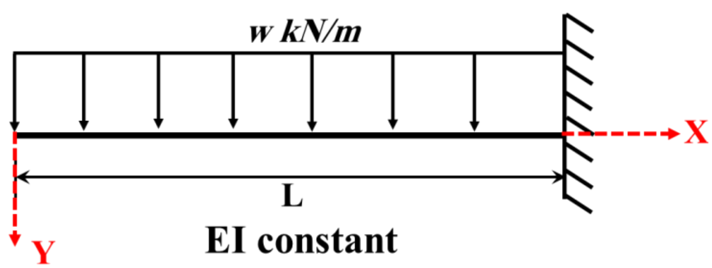
$$\bar{\bar{y}}(L) = 0$$

$$0 = \frac{wL^2}{2EI} - \frac{wL^2}{EI} + c_2$$

$$\boldsymbol{c_2} = \frac{\boldsymbol{wL^2}}{\boldsymbol{2EI}}$$

$$\boldsymbol{y}_{(x)} = \frac{\boldsymbol{w}}{\boldsymbol{24EI}}\boldsymbol{x^4} - \frac{\boldsymbol{wL}}{\boldsymbol{6EI}}\boldsymbol{x^3} + \frac{\boldsymbol{wL^2}}{\boldsymbol{4EI}}\boldsymbol{x^2} \quad \textbf{(D. E.)}$$

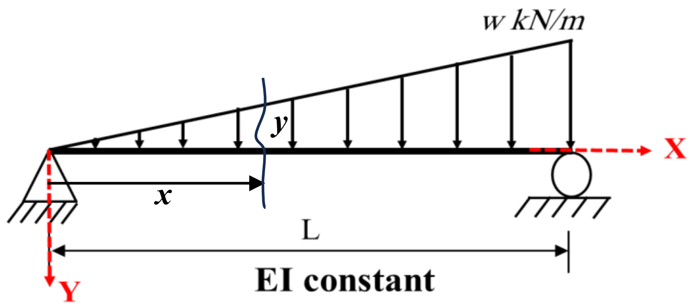
H.W.: By using the Integration ***Method***, find the **deflection equation, maximum shear force** and **maximum moment** for the loaded cantilever beam shown in figure.



Example: By using the ***Integration Method***, find the ***deflection equation*** for the loaded beam shown in figure below.

$$\frac{y}{x} = \frac{w}{L}$$

$$y = \frac{w}{L}x$$



$$\frac{d^4y}{dx^4} = \frac{W_{(x)}}{EI}$$

$$W_x = \frac{w}{L}x$$

$$\frac{d^4y}{dx^4} = \frac{w}{EIL}x$$

$$\frac{d^3y}{dx^3} = \frac{w}{2EIL}x^2 + c_1$$

$$\frac{d^2y}{dx^2} = \frac{w}{6EIL}x^3 + c_1x + c_2$$

$$\frac{dy}{dx} = \frac{w}{24EIL}x^4 + \frac{c_1}{2}x^2 + c_2x + c_3$$

$$\boldsymbol{y}_{(\boldsymbol{x})} = \frac{w}{120EIL}x^5 + \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4$$

$B.C.$

$$y(0) = 0, \quad y(L) = 0$$

$$\bar{\bar{y}}(0) = 0, \qquad \bar{\bar{y}}(L) = 0$$

$$y(0) = 0 \rightarrow \boldsymbol{c_4} = 0$$

$$\bar{\bar{y}}(0) = 0 \rightarrow \boldsymbol{c_2} = \boldsymbol{0}$$

$$\bar{\bar{y}}(L) = 0$$

$$0 = \frac{wL^2}{6EI} + c_1L$$

$$\boldsymbol{c_1} = -\frac{\boldsymbol{wL}}{\boldsymbol{6EI}}$$

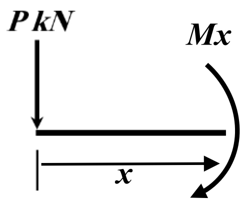
$$y(L) = 0$$

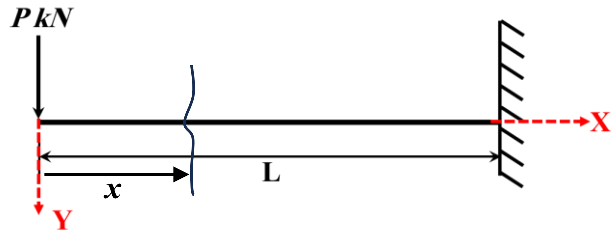
$$0 = \frac{wL^4}{120EI} - \frac{\boldsymbol{wL^4}}{\boldsymbol{6EI}} + c_3L$$

$$\boldsymbol{c_3} = \frac{\boldsymbol{7wL^3}}{\boldsymbol{360EI}}$$

$$\therefore \boldsymbol{y_x} = \frac{\boldsymbol{w}}{\boldsymbol{120LEI}}\boldsymbol{x^5} - \frac{\boldsymbol{wL}}{\boldsymbol{36EI}}\boldsymbol{x^3} + \frac{\boldsymbol{7wL^3}}{\boldsymbol{360EI}}\boldsymbol{x}$$

Example: find the ***deflection equation*** for the loaded beam shown in figure below using the ***Integration Method***. Then find the value of maximum deflection, If ($E = 21.72\times10^6$ kN/m^2, $I = 1330 \times 10^{-6}$ m^4, L = 2m, p = 20 kN).





Mx = p. x

$$\frac{d^2y}{dx^2} = \frac{M_{(x)}}{EI}$$

$$\frac{d^2y}{dx^2} = \frac{p}{EI}x$$

$$\frac{dy}{dx} = \frac{p}{2EI}x^2 + c_1$$

$$\boldsymbol{y_{(x)} = \frac{p}{6EI}x^3 + c_1x + c_2}$$

$$B.C.$$

$$\bar{\bar{y}}(0) = 0,\quad y(L) = 0$$

$$y_p'''(0) = P,\quad \bar{y}(L) = 0$$

$$\bar{y}(L) = 0$$

$$0 = \frac{p}{2EI}L^2 + c_1 \rightarrow \boldsymbol{c_1 = -\frac{pL^2}{2EI}}$$

$$y(L) = 0$$

$$0 = \frac{pL^3}{6EI} - \frac{pL^3}{2EI} + c_2$$

$$\boldsymbol{c_2 = \frac{pL^3}{3EI}}$$

$$\boldsymbol{y_{(x)} = \frac{p}{6EI}x^3 - \frac{pL^2}{2EI}x + \frac{pL^3}{3EI}}$$

Maximum deflection at x = 0

$$y_{(0)} = \frac{p}{6EI}(0)^3 - \frac{pL^2}{2EI}(0) + \frac{pL^3}{3EI}$$

$$\boldsymbol{y_{(0)} = \frac{pL^3}{3EI}}$$

$$y_{(0)} = \frac{20\ x\ (2)^3}{3\ x\ (21.71\ x\ 10^6)\ x\ (1330\ x\ 10^{-6})} = \boldsymbol{0.00185\ m = 1.85\ mm}$$