matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}$$

, n: rows

m: columns

ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3}$$

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 10 \end{bmatrix}_{2 \times 2}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$$

$$D = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$$

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 6 & 7 & 8 \\ 1 & 1 & 1 \\ 0 & 3 & 5 \end{bmatrix}_{4 \times 3}$$

types of matrix:

1 Square Matrix: (n=m)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2} \quad \text{(Square matrix)}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3} \quad \text{(Square matrix)}$$

$$C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & -3 \end{bmatrix}_{2\times3} \quad \text{(non-square matrix)}$$

---

2 Diagonal matrix: (n=m), $A=[a_{ij}]$, $a_{ij}=c$

$\forall\, i=j$ and $a_{ij}=0\ \forall\, i\neq j$, c is constant.

ex:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}_{2\times2} \quad \text{Diagonal matrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3} \quad \text{Diagonal matrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}_{2\times 3}$$ (non-Diagonal matrix)

$(n \neq m)$

---

③ Zero matrix:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ (Zero matrix)

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (Zero matrix)

$$C = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$ (non-Zero matrix)

---

④ Identity matrix:

$(n = m)$, $A = [a_{ij}]$, $a_{ij} = 1$ $\forall\ i = j$ and $a_{ij} = 0$ $\forall\ i \neq j$.

ex:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (Identity matrix)

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ (Identity matrix)

$$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (\text{non-Identity matrix})$$

---

5 Triangular matrix:

(i) Upper Triangular matrix: $(n=m)$, $A=[a_{ij}]$

, $a_{ij}=c \ \forall \ i \leq j$ and $a_{ij}=0 \ \forall \ i>j$, c is constant

(ii) Lower Triangular matrix: $(n=m)$, $A=[a_{ij}]$

, $a_{ij}=c \ \forall \ i \geq j$ and $a_{ij}=0 \ \forall \ i<j$, c is constant

ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3\times3} \quad (\text{upper Triangular matrix})$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}_{2\times2} \quad (\text{upper Triangular matrix})$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & -1 & 10 \end{bmatrix}_{3\times3} \quad (\text{lower Triangular matrix})$$

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 6 \end{bmatrix}_{2\times2} \quad (\text{lower Triangular matrix})$$

6 ones matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2} \quad \text{(ones matrix)}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3} \quad \text{(ones matrix)}$$

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2} \quad \text{(ones matrix)}$$

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2} \quad \text{(non-ones matrix)}$$

operation of matrix:

1 addition: (+)

ex:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}, \quad B = \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix}_{2\times2}$$

$$A+B = \begin{bmatrix} 1+3 & 2+4 \\ 3+6 & 4+7 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 9 & 11 \end{bmatrix}$$

ex:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -2 & -3 \end{bmatrix}_{2\times3}, \quad B = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 3 & 6 \end{bmatrix}_{2\times3}$$

$$A+B = \begin{bmatrix} 1+2 & 2+3 & 4+8 \\ -1+0 & -2+3 & -3+6 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 12 \\ -1 & 1 & 3 \end{bmatrix}$$

2 subtract: (−)

ex:

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}_{2\times2}, \quad B = \begin{bmatrix} 3 & 6 \\ 8 & 10 \end{bmatrix}_{2\times2}$$

$$A-B = \begin{bmatrix} 1-3 & 2-6 \\ 5-8 & 6-10 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -3 & -4 \end{bmatrix}$$

ex:

$$A=\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 6 \\ 8 & 3 & 1 \end{bmatrix}_{3\times 3}, \quad B=\begin{bmatrix} -1 & 2 & 4 \\ 6 & 10 & 15 \\ -1 & 0 & 0 \end{bmatrix}_{3\times 3}$$

$$A-B=\begin{bmatrix} 1-(-1) & 0-2 & 2-4 \\ 2-6 & 4-10 & 6-15 \\ 8-(-1) & 3-0 & 1-0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ -4 & -6 & -9 \\ 9 & 3 & 1 \end{bmatrix}$$

3 multiply: (X)

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}_{n\times m}, \quad B=\begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}_{n\times m}$$

$A \times B$ is exist when m of matrix A is equal to n of matrix B.

ex:

$$A=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}_{2\times 3}, \quad B=\begin{bmatrix} 1 & 2 \\ 0 & 4 \\ 3 & 2 \end{bmatrix}_{3\times 2}$$

$$A \times B = \begin{bmatrix} 1\times1+2\times0+1\times3 & 1\times2+2\times4+1\times2 \\ 2\times1+1\times0+3\times3 & 2\times2+1\times4+3\times2 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 4 & 12 \\ 11 & 14 \end{bmatrix}$$

ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 4 \end{bmatrix}_{3\times 3} , \quad B = \begin{bmatrix} 6 & 2 & 15 \\ 1 & 8 & 3 \\ 1 & 3 & 1 \end{bmatrix}_{3\times 3}$$

$$A \times B = \begin{bmatrix} 1\times 6+2\times 1+3\times 1 & 1\times 2+2\times 8+3\times 3 & 1\times 15+2\times 3+3\times 1 \\ 1\times 6+0\times 1+2\times 1 & 1\times 2+0\times 8+2\times 3 & 1\times 15+0\times 3+2\times 1 \\ 0\times 6+3\times 1+4\times 1 & 0\times 2+3\times 8+4\times 3 & 0\times 15+3\times 3+4\times 1 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 11 & 27 & 24 \\ 8 & 8 & 17 \\ 7 & 36 & 13 \end{bmatrix}$$

4 Transpose of matrix:

$$A \to A^T , \quad B \to B^T$$

ex:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

ex:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 4 & 3 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 3 \\ 3 & 2 & 0 \end{bmatrix}$$

ex:

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 10 & 20 & 30 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 10 \\ -2 & 20 \\ -3 & 30 \end{bmatrix}$$

Determinants: (det) او | |

Determinant of (2x2 matrix):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = a \times d - b \times c$$

ex:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det(A) = 1 \times 4 - 2 \times 3 = 4 - 6 = -2$$

ex:

$$B = \begin{bmatrix} 10 & 3 \\ 5 & 2 \end{bmatrix}$$

$$\det(B) = 10 \times 2 - 3 \times 5 = 20 - 15 = 5$$

ex:

$$C = \begin{bmatrix} -2 & -1 \\ 0 & 6 \end{bmatrix}$$

$$\det(C) = -2 \times 6 - (-1 \times 0)$$

$$= -12 - 0$$

$$\det(C) = -12$$

ex:

$$A = \begin{bmatrix} -1 & 10 \\ -1 & 2 \end{bmatrix}$$

$$\det(A) = -1 \times 2 - (10 \times -1)$$

$$= -2 - (-10)$$

$$= -2 + 10$$

$$\det(A) = 8$$

Determinant of (3x3 matrix):

ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \\ -1 & 6 & -2 \end{bmatrix}$$

$$\det(A) \text{ او } |A| = \begin{array}{ccc|cc} + & + & + & - \;\; - & - \\ 1 & 2 & 3 & 1 & 2 \\ 1 & 0 & 4 & 1 & 0 \\ -1 & 6 & -2 & -1 & 6 \end{array}$$

$$|A| = +(1 \times 0 \times -2) + (2 \times 4 \times -1) + (3 \times 1 \times 6) - (2 \times 1 \times -2) - (1 \times 4 \times 6) - (3 \times 0 \times -1)$$

$$= +(0) + (-8) + (18) - (-4) - (24) - (0)$$

$$= 0 - 8 + 18 + 4 - 24 - 0$$

$$= +10 - 20$$

$$|A| = -10$$

ex:

$$B = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 3 & 1 \\ 1 & 10 & 2 \end{bmatrix}$$

$$|B| = \begin{array}{ccccc} + & + & + & - & - \; - \\ \left| \begin{matrix} 2 & 0 & 4 \\ 1 & 3 & 1 \\ 1 & 10 & 2 \end{matrix} \right| & & & \begin{matrix} 2 & 0 \\ 1 & 3 \\ 1 & 10 \end{matrix} & \end{array}$$

$$|B| = +(2 \times 3 \times 2) + (0 \times 1 \times 1) + (4 \times 1 \times 10) - (0 \times 1 \times 2)$$

$$- (2 \times 1 \times 10) - (4 \times 3 \times 1)$$

$$|B| = +(12) + (0) + (40) - (0) - (20) - (12)$$

$$= +12 + 40 - 20 - 12$$

$$|B| = +20$$

Cramer's Rule to solve system of equations:

$$X = \frac{|D_1|}{|D|} \quad , \quad y = \frac{|D_2|}{|D|}$$

ex:

$$X + y = 2 \quad \text{---} (1)$$

$$X + 2y = 4 \quad \text{---} (2)$$

Find the Solution of the above system by Cramer's Rule.

Solution:

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} , \ |D| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$|D| = 1 \times 2 - 1 \times 1 = 2 - 1 = 1$$

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} , \ b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} , \ |D_1| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$$

$$|D_1| = 2 \times 2 - 1 \times 4 = 4 - 4 = 0$$

$$D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}, \quad |D_2| = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$|D_2| = 1 \times 4 - 2 \times 1 = 4 - 2 = 2$$

$$X = \frac{|D_1|}{|D|} = \frac{0}{1} = 0$$

$$y = \frac{|D_2|}{|D|} = \frac{2}{1} = 2$$

ex:

$$x - y = 3 \quad \cdots (1)$$

$$2x + y = 4 \quad \cdots (2)$$

Find the solution of the above system by cramer's Rule.

Solution:

$$D = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad |D| = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$|D| = 1 \times 1 - (-1 \times 2) = 1 - (-2) = 1 + 2 = 3$$

$$D = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}, \quad |D_1| = \begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}$$

$$|D_1| = 3 \times 1 - (-1 \times 4)$$

$$|D_1| = 3 - (-4) = 3 + 4 = 7$$

$$D = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$|D_2| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$$

$$|D_2| = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$$

$$x = \frac{|D_1|}{|D|} = \frac{7}{3}$$

$$y = \frac{|D_2|}{|D|} = \frac{-2}{3}$$

Differentiation:

it denoted by y' or $\frac{dy}{dx}$.

they are many differentiation rules:

1.

$y = f(x)$ is function

$f(x) = a$, a is constant

$f'(x) = 0$

ex:

$f(x) = 10$

$f'(x) = 0$

ex:

$f(x) = \frac{-1}{2}$

$f'(x) = 0$

ex:

$f(x) = \sqrt{2}$

$f'(x) = 0$

(2) $f(x) = x^n$, n is constant

$f'(x) = n\,x^{n-1}$

ex: $f(x) = x$

$f'(x) = 1$

ex: $f(x) = x^2$

$f'(x) = 2x$

ex: $f(x) = x^3$

$f'(x) = 3x^2$

ex: $f(x) = 10x^2$

$f'(x) = 10(2x)$

$f'(x) = 20x$

ex: $f(x) = -6x^3$

$f'(x) = -6(3x^2)$

$f'(x) = -18x^2$

ex: $f(x) = \sqrt{x}$

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2} x^{\frac{1}{2}-1}$

$f'(x) = \frac{1}{2} x^{\frac{-1}{2}}$

ex: $f(x) = \sqrt[3]{x}$

$f(x) = x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1}$

$f'(x) = \frac{1}{3} x^{\frac{-2}{3}}$

ex: $f(x) = x^{\frac{-1}{5}}$

$f'(x) = \frac{-1}{5} x^{\frac{-1}{5}-1}$

$f'(x) = \frac{-1}{5} x^{\frac{-6}{5}}$

ex:

$$f(x) = x^{\frac{-1}{4}}$$

$$f'(x) = \frac{-1}{4} x^{\frac{-1}{4}-1}$$

$$f'(x) = \frac{-1}{4} x^{\frac{-5}{4}}$$

ex:

$$f(x) = x^{\frac{-2}{10}}$$

$$f'(x) = \frac{-2}{10} x^{\frac{-2}{10}-1}$$

$$f'(x) = \frac{-2}{10} x^{\frac{-12}{10}}$$

3) $f(x) = e^{ax}$, a is constant

$f'(x) = e^{ax} \cdot a$

Note: ex: $f(x) = 3x$
$f'(x) = 3$

ex: $f(x) = -2x$
$f'(x) = -2$

ex: $f(x) = 100x$
$f'(x) = 100$

ex: $f(x) = e^{x}$
$f'(x) = e^{x} \cdot 1$
$f'(x) = e^{x}$

ex: $f(x) = e^{2x}$
$f'(x) = e^{2x} \cdot 2$

ex: $f(x) = e^{10x}$
$f'(x) = e^{10x} \cdot 10$

ex: $f(x) = e^{-3x}$
$f'(x) = e^{-3x} \cdot (-3)$
$f'(x) = -3\, e^{-3x}$

ex: $f(x) = e^{2x^2}$
$f'(x) = e^{2x^2} \cdot (2 \cdot 2x)$
$f'(x) = e^{2x^2} \cdot (4x)$
$f'(x) = 4x\, e^{2x^2}$

ex: $f(x) = e^{x^2}$
$f'(x) = e^{x^2} \cdot (2x)$
$f'(x) = 2x\, e^{x^2}$

4

$f(x) = \ln(aX)$, a is constant

$$f'(x) = \frac{1}{aX} \cdot a$$

ex:

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{X} \cdot 1$$

$$f'(x) = \frac{1}{X}$$

ex:

$$f(x) = \ln(2X)$$

$$f'(x) = \frac{1}{2X} \cdot 2$$

ex:

$$f(x) = \ln(10X)$$

$$f'(x) = \frac{1}{10X} \cdot 10$$

ex:

$$f(x) = \ln(x^2)$$

$$f'(x) = \frac{1}{x^2} \cdot (2x)$$

ex:

$$f(x) = \ln(10x^2)$$

$$f'(x) = \frac{1}{10x^2} \cdot (10 \cdot 2x)$$

$$f'(x) = \frac{1}{10x^2} \cdot (20x)$$

5

$f(x) = \sin(ax)$, a is constant

$f'(x) = \cos(ax) \cdot a$

$f(x) = \cos(ax)$

$f'(x) = -\sin(ax) \cdot a$

$f(x) = \tan(ax)$

$f'(x) = \sec^2(ax) \cdot a$

$f(x) = \cot(ax)$

$f'(x) = -\csc^2(ax) \cdot a$

$f(x) = \sec(ax)$

$f'(x) = \sec(ax) \cdot \tan(ax) \cdot a$

$f(x) = \csc(ax)$

$f'(x) = -\csc(ax) \cdot \cot(ax) \cdot a$

---

ex:

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x) \cdot 1$$

$$f'(x) = \cos(x)$$

ex:

$$f(x) = \sin(2x)$$

$$f'(x) = \cos(2x) \cdot 2$$

ex:

$$f(x) = \sin(10x)$$

$$f'(x) = \cos(10x) \cdot 10$$

ex:

$$f(x) = \sin(-3x)$$

$$f'(x) = \cos(-3x) \cdot (-3)$$

ex:

$$f(x) = \sin(\tfrac{1}{2}x)$$

$$f'(x) = \cos(\tfrac{1}{2}x) \cdot (\tfrac{1}{2})$$

ex:

$f(x) = \cos(x)$

$f'(x) = -\sin(x) \cdot 1$

$f'(x) = -\sin(x)$

ex:

$f(x) = \cos(2x)$

$f'(x) = -\sin(2x) \cdot 2$

ex:

$f(x) = \sin(x^2)$

$f'(x) = \cos(x^2) \cdot (2x)$

ex:

$f(x) = \cos(3x^3)$

$f'(x) = -\sin(3x^3) \cdot (9x^2)$

ex:

$f(x) = \tan(2x)$

$f'(x) = \sec^2(2x) \cdot (2)$

ex:

$$f(x) = \cot(x^2)$$

$$f'(x) = -\csc^2(x^2)\cdot(2x)$$

ex:

$$f(x) = \sec(10x)$$

$$f'(x) = \sec(10x)\cdot\tan(10x)\cdot(10)$$

ex:

$$f(x) = \csc(x^3)$$

$$f'(x) = -\csc(x^3)\cdot\cot(x^3)\cdot(3x^2)$$

ex:

$$f(x) = \sec(\sqrt{x})$$

$$f'(x) = \sec(\sqrt{x})\cdot\tan(\sqrt{x})\cdot(\tfrac{1}{2}x^{\frac{-1}{2}})$$

ex:

$$f(x) = \cot(-10x)$$

$$f'(x) = -\csc^2(-10x)\cdot(-10)$$

6

$f(x) = u \cdot v$, u, v : functions

$f'(x) = u \cdot v' + v \cdot u'$

ex:

$f(x) = x \cdot \sin(x)$

$f'(x) = x \cdot \cos(x) \cdot 1 + \sin(x) \cdot 1$

$f'(x) = x \cdot \cos(x) + \sin(x)$

ex:

$f(x) = x^2 \cdot e^x$

$f'(x) = x^2 \cdot e^x \cdot 1 + e^x \cdot (2x)$

$f'(x) = x^2 e^x + e^x \cdot (2x)$

ex:

$f(x) = \sqrt{x} \cdot \ln(x)$

$f'(x) = \sqrt{x} \cdot \frac{1}{x} \cdot 1 + \ln(x) \cdot \frac{1}{2} x^{\frac{-1}{2}}$

$f'(x) = \sqrt{x} \cdot \frac{1}{x} + \ln(x) \cdot \frac{1}{2} x^{\frac{-1}{2}}$

ex: $f(x) = \sec(x^2) \cdot \tan(10x^3)$

$f'(x) = \sec(x^2) \cdot \sec^2(10x^3) \cdot 30x^2 + \tan(10x^3) \cdot \sec(x^2) \cdot \tan(x^2) \cdot 2x$

ex: $f(x) = \cot(x^3) \cdot \cos(10x)$

$f'(x) = \cot(x^3) \cdot (-\sin(10x)) \cdot 10 + \cos(10x) \cdot (-\csc^2(x^3)) \cdot 3x^2$

7

$$f(x) = \frac{u}{v} \quad , \quad u, v : \text{Functions}$$

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

ex:

$$f(x) = \frac{x^2}{\cos(x)}$$

$$f'(x) = \frac{\cos(x) \cdot 2x - x^2 \cdot (-\sin(x)) \cdot 1}{(\cos(x))^2}$$

$$f'(x) = \frac{\cos(x) \cdot 2x - x^2 \cdot (-\sin(x))}{(\cos(x))^2}$$

ex:

$$f(x) = \frac{e^{x^2}}{\sqrt{x}}$$

$$f'(x) = \frac{\sqrt{x} \cdot e^{x^2} \cdot 2x - e^{x^2} \cdot \frac{1}{2} x^{\frac{-1}{2}}}{(\sqrt{x})^2}$$

ex:

$$f(x)=\frac{\tan(10x)}{x^2+x^3}$$

$$f'(x)=\frac{(x^2+x^3)\cdot\sec^2(10x)\cdot 10-\tan(10x)\cdot(2x+3x^2)}{(x^2+x^3)^2}$$

ex:

$$f(x)=\frac{2x^{\frac{1}{3}}}{\ln(x^2)}$$

$$f'(x)=\frac{\ln(x^2)\cdot 2\cdot\frac{1}{3}\cdot x^{\frac{-2}{3}}-2x^{\frac{1}{3}}\cdot\frac{1}{x^2}\cdot 2x}{(\ln(x^2))^2}$$

ex:

$$f(x)=\frac{e^{-2x}}{\csc(3x)}$$

$$f'(x)=\frac{\csc(3x)\cdot(-2)\cdot e^{-2x}-e^{-2x}\cdot(-\csc(3x)\cdot\cot(3x)\cdot 3)}{(\csc(3x))^2}$$

Integration

There are two types of integration :

1- Indefinite integration .

2- definite integration .

We will study the indefinite integration , and the rules of indefinite integration are :

a) $\int a\, dx = ax + c \qquad , \; a\,, c : constants$

Example 1:

$$\int dx = x + c$$

Example 2:

$$\int 2\, dx = 2x + c$$

Example 3:

$$\int -4\, dx = -4x + c$$

b) $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \qquad , \; n \neq -1 \;,\; n\,, c : constants$

Example 1:

$$\int x\, dx = \frac{x^2}{2} + c$$

Example 2:

$$\int x^{10}\, dx = \frac{x^{11}}{11} + c$$

Example 3:

$$\int x^{-2}\, dx = \frac{x^{-1}}{-1} + c$$

Example 4:

$$\int x^{-20}\, dx = \frac{x^{-19}}{-19} + c$$

c) $\int (f(x) \pm g(x))dx = \int f(x)\, dx \pm \int g(x)\, dx$

Example 1:

$$\int (x+10)\, dx = \int x\, dx + \int 10\, dx$$

Example 2:

$$\int (x^3 - x)\, dx = \int x^3\, dx - \int x\, dx$$

d) $\int k.f(x)\, dx = k \int f(x)\, dx$, $k{:}\,constant$

Example:

$$\int 100\, x^8\, dx = 100 \int x^8\, dx$$

e) $\int e^{ax}\, a\, dx = e^{ax} + c$, $a, c{:}\,constants$

Example 1:

$$\int e^x\, dx = e^x + c$$

Example 2:

$$\int e^{10x}\ 10\, dx = e^{10x} + c$$

Example 3:

$$\int e^{4x}\, dx = \int e^{4x}\, dx\, \frac{4}{4} = \frac{1}{4} \int e^{4x}\, 4\, dx = \frac{1}{4}\, e^{4x} + c$$

Example 4:

$$\int e^{-6x}\, dx = \int e^{-6x}\, dx\, \frac{-6}{-6} = \frac{1}{-6} \int e^{-6x}\, (-6)\, dx = \frac{1}{-6}\, e^{-6x} + c$$

f) $\int \sin(ax)\,.\,a\,dx = -\cos(ax) + c \qquad , \qquad a, c: constants$

$$\int \cos(ax)\,.\,a\,dx = \sin(ax) + c$$

$$\int sec^2(ax)\,.\,a\,dx = \tan(ax) + c$$

$$\int csc^2(ax)\,.\,a\,dx = -\cot(ax) + c$$

$$\int \sec(ax)\,.\tan(ax)\,.\,a\,dx = \sec(ax) + c$$

$$\int \csc(ax)\,.\cot(ax)\,.\,a\,dx = -\csc(ax) + c$$

Example 1:

$$\int \sin(x)\;dx = -\cos(x) + c$$

Example 2:

$$\int \sin(2x)\,.2\,dx = -\cos(2x) + c$$

Example 3:

$$\int \cos(10x)\,.10\,dx = \sin(10x) + c$$

Example 4:

$$\int sec^2(-8x)\;dx = \int sec^2(-8x)\;dx\;\;\frac{-8}{-8} = \frac{1}{-8}\int sec^2(-8x)\,.\,(-8)\;dx$$

$$= \frac{1}{-8}\tan(-8x) + c$$

Example 5:

$$\int csc^2(20x)\ \ dx = \int csc^2(20x)\ dx\ \ \frac{20}{20} = \frac{1}{20}\int csc^2(20x)\ .20\ dx$$

$$= -\frac{1}{20}\cot(20x) + c$$

Example 6:

$$\int \sec(\sqrt{x}) . \tan(\sqrt{x}) . \frac{1}{2\sqrt{x}}\ dx = \sec(\sqrt{x}) + c$$

Example 7:

$$\int \csc(x^2) . \cot(x^2) . x\ dx = \int \csc(x^2) . \cot(x^2) . x\ dx\ \ \frac{2}{2}$$

$$= \frac{1}{2}\int \csc(x^2) . \cot(x^2)\ .2x\ dx = -\frac{1}{2}\csc(x^2) + c$$

Example 8:

$$\int \csc(\sqrt{x}) . \cot(\sqrt{x}) . \frac{1}{\sqrt{x}} dx$$

$$= \int \csc(\sqrt{x}) . \cot(\sqrt{x}) . \frac{1}{\sqrt{x}} dx\ \ \frac{2}{2}$$

$$= 2\int \csc(\sqrt{x}) . \cot(\sqrt{x}) . \frac{1}{2\sqrt{x}} dx = -2\csc(\sqrt{x}) + c$$

g) $\int (f(x))^n \,.\, f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + c \quad , \quad n \neq -1 \quad , \quad n, c: constants$

Example 1:

$$\int (x+1)^2\, dx = \frac{(x+1)^3}{3} + c$$

Example 2:

$$\int (x^3+2x)^4 \,.\, (3x^2+2)\, dx = \frac{(x^3+2x)^5}{5} + c$$

Example 3:

$$\int (\sqrt{x}\ +1)^{-3} \,.\left(\frac{1}{\sqrt{x}}\right) dx$$

$$= \int (\sqrt{x}\ +1)^{-3} \,.\left(\frac{1}{\sqrt{x}}\right) dx\ \ \frac{2}{2} = 2\int (\sqrt{x}\ +1)^{-3} \,.\left(\frac{1}{2\sqrt{x}}\right) dx$$

$$= 2.\frac{(\sqrt{x}\ +1)^{-2}}{-2} + c$$

Example 4:

$$\int (\sin(x))^{-2} \,.\, (\cos(x))\, dx = \frac{(\sin(x))^{-1}}{-1} + c$$

Example 5:

$$\int (\tan(x^3))^3 \,.\, (x^2.sec^2(x^3))\, dx = \int (\tan(x^3))^3 \,.\, (x^2.sec^2(x^3))\, dx\ \ \frac{3}{3}$$

$$= \frac{1}{3}\int (\tan(x^3))^3 \,.\, (3x^2.sec^2(x^3))\, dx = \frac{1}{3}.\frac{(\tan(x^3))^4}{4} + c$$

Example 6:

$$\int (\cot(6\text{x}))^{10} \,.\, (csc^2(6x))\, dx = \int (\cot(6\text{x}))^{10} \,.\, (csc^2(6x))\, dx\ \ \frac{-6}{-6}$$

$$= \frac{1}{-6}\int (\cot(6\text{x}))^{10} \,.\, ((-6).csc^2(6x))\, dx = \frac{1}{-6}.\frac{(\cot(6\text{x}))^{11}}{11} + c$$

h) $\int \frac{f'(x)}{f(x)}\,dx = \ln\big(f(x)\big) + c \qquad , \qquad c: constant$

Example 1:

$$\int \frac{1}{x}\,dx = \ln(x) + c$$

Example 2:

$$\int \frac{(3x^2+1)}{(x^3+x)}\,dx = \ln(x^3+\mathrm{x}) + c$$

Example 3:

$$\int \frac{(12x^2+2x)}{(4x^3+x^2)}\,dx = \ln(4x^3+x^2) + c$$

Example 4:

$$\int \frac{\cos(3x)}{\sin(3x)}\,dx = \int \frac{\cos(3x)}{\sin(3x)}\,dx\ \ \frac{3}{3} = \frac{1}{3}\int \frac{3.\cos(3x)}{\sin(3x)}\,dx = \frac{1}{3}\ln(\sin(3\mathrm{x})) + c$$

Example 5:

$$\int \frac{sec^2(10x)}{\tan(10\mathrm{x})}\,dx = \int \frac{sec^2(10x)}{\tan(10\mathrm{x})}\,dx\ \ \frac{10}{10}$$

$$= \frac{1}{10}\int \frac{10.\,sec^2(10x)}{\tan(10\mathrm{x})} = \frac{1}{10}\ln(\tan(10x)) + c$$

Example 6:

$$\int \frac{x.csc^2(x^2)}{\cot(x^2)}\,dx = \int \frac{x.csc^2(x^2)}{\cot(x^2)}\,dx\ \ \frac{-2}{-2} = \frac{1}{-2}\int \frac{(-2x).csc^2(x^2)}{\cot(x^2)}\,dx = \frac{1}{-2}\ln(\cot(x^2)) + c$$

Examples :

(1)

$$\int cos^2(2x)\sin(2x)\ dx =$$

$$\int(\cos(2x))^2\sin(2x)\ dx = \int(\cos(2x))^2\sin(2x)\ dx\ \left(\frac{-2}{-2}\right)$$

$$= \frac{1}{-2}\int(\cos(2x))^2\,(-2\sin(2x))\ dx = \frac{1}{-2}\ \frac{(\cos(2x))^3}{3} + c$$

$$= \frac{1}{-6}\ (\cos(2x))^3 + c$$

(2)

$$\int sec^3(x)\tan(x)\,dx =$$

$$\int sec^2(x)\sec(x)\tan(x)\ dx = \int(\sec(x))^2\sec(x)\tan(x)\ dx = \frac{(\sec(x))^3}{3} + c$$

(3) $\int\frac{1}{x\ln(x)}\ dx = \int\frac{\frac{1}{x}}{\ln(x)}\ dx = \ln(\ln(x)) + c$

(4) $\int\frac{e^{2x}+x}{e^{2x}+x^2}\ dx = \int\frac{e^{2x}+x}{e^{2x}+x^2}\ dx\ \frac{2}{2} = \frac{1}{2}\int\frac{(2e^{2x}+2x)}{e^{2x}+x^2}\ dx = \frac{1}{2}\ \ln(e^{2x}+x^2) + c$

(5) $\int\frac{e^{\sqrt{x}}-1}{\sqrt{x}}\ dx = \int\left(\frac{e^{\sqrt{x}}}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx = \int\frac{e^{\sqrt{x}}}{\sqrt{x}}dx - \int\frac{1}{\sqrt{x}}dx = \int e^{\sqrt{x}}\ \frac{1}{\sqrt{x}}dx - \int\frac{1}{x^{\frac{1}{2}}}\ dx$

$$= \int e^{\sqrt{x}}\ \frac{1}{\sqrt{x}}\ dx\ \frac{2}{2} - \int x^{-\frac{1}{2}}\ dx = 2\int e^{\sqrt{x}}\ \frac{1}{2\sqrt{x}}\ dx - \int x^{-\frac{1}{2}}\ dx$$

$$= 2\ e^{\sqrt{x}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\ e^{\sqrt{x}} - 2\ x^{\frac{1}{2}} + c$$

Integration Methods (udv Method):

udv method formula is :

$$\int udv = u\,.\,v - \int vdu$$

Examples :

(1) $\int x \ln(x)\, dx$

solution :

$u = \ln(x) \qquad ; \qquad dv = x$

$du = \frac{1}{x}\, dx \qquad ; \qquad v = \frac{x^2}{2}$

$$\int udv = u\,.\,v - \int vdu$$

$$\int x\, ln(x) = \ln(x).\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}\, dx$$

$$= ln(x).\frac{x^2}{2} - \frac{1}{2}\int x\, dx$$

$$= ln(x).\frac{x^2}{2} - \frac{1}{2}\,\frac{x^2}{2} + c$$

$$= ln(x).\frac{x^2}{2} - \frac{1}{4}\, x^2 + c$$

(2) $\int x\, e^x\, dx$

solution :

$u = x \qquad ; \qquad dv = e^x$

$du = dx \qquad ; \qquad v = e^x$

$$\int udv = u\,.\,v - \int vdu$$

$$\int x\, e^x\, dx = x\,.\,e^x - \int e^x\, dx$$

$$= x\,.\,e^x - e^x + c$$

(3) $\int x\, sin(x)\, dx$

solution :

$u = x \qquad ; \qquad dv = sin(x)$

$du = dx \qquad ; \qquad v = -cos(x)$

$$\int udv = u \,.\, v - \int vdu$$

$$\int x\, sin(x)\, dx = x\,.\,(-cos(x)) - \int (-cos(x))\, dx$$

$$= -x.\,cos(x) + \int cos(x)\, dx$$

$$= -x.\,cos(x) + sin(x) + c$$

Exercises :

(1) $\int x \cos(2x)\, dx$

(2) $\int x \sin(10x)\, dx$

(3) $\int x \sin(-20x)\, dx$

(4) $\int x\, sec^2(x)\, dx$

(5) $\int x\, csc^2(x)\, dx$

(6) $\int x \cos(-25x)\ \ dx$