الرياضيات

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قسم تقنيات الاحصاء والمعلوماتية

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الفصل الاول

Equations in one Variable معادلات بمتغير واحد

1-1 Linear equation:

The equation is a statement that expresses the equality of two algebraic expression. Contains one variable or more. For example:

2-3(x-2)= $\frac{x}{5}$ -8 ... (1) $\frac{6x}{3} + 2(2x+1) = 8$ ... (2)

Equation (1) is first degree (or linear) in one variable x, and so is equation(2).

A solution is a number that when substituted for the variable makes the equation true.

The value of the variable that makes an equation a true statement is called a root or a solution of the given equation .we can add ,subtract, multiply or divide any constant or any algebraic expression (non zero) to both sides of an equation.

Definition:

An equation is a first degree or a linear equation in one variable (x) if it can be transformed into equation where the left side is of the form $(ax + b\ , a \neq 0)$ and the right side is equal to zero, where a and b are real constants.

For example : 4x+8=0 is a linear equation in one variable and subtracting 8 from both sides gives: 4x=-8 , then, dividing both sides by 4 we get x=-2 this is the only solution for the equation. Therefore, a first degree or a linear equation has only one solution (unique solution).

Steps for solving linear equations:

Step 1 : expand any parentheses which may be found in the equation.

Step 2 : remove any fractions which may be found in the equation by multiplying both sides by the common denominator of the fractions involved.

Step 3 : move all the terms containing the constants to the right side and all terms containing the variable to the left side, then simplify.

Example 1: solve the following equations:-

1) $\frac{7x}{5} - \frac{x-2}{3} = \frac{6}{3} - \frac{1}{3}\,(x - \frac{3x-2}{5})$

$(\frac{7x}{5} - \frac{x-2}{3} = \frac{6}{3} - \frac{x}{3} + \frac{3x-2}{15})\,(15) = 3(7x) - 5(x-2) = (5)(6) - 5x + (3x-2)$

21x-5x+10=30-5x+3x-2 → 18x=18 → x=1

2) 5x-3(7-x)=13-9x

5x-21+3x=13-9x → 5x+3x+9x=13+21 → 17x=34 → x=2

1-2 Quadratic equations:

The general form of a quadratic equation in one variable is: $ax^2 + bx + c = 0$,

a and b are real constant.

There are four methods for solving the quadratic that will be presented in this follows:

) Solution by square root.

2) Solution by factoring.

3) Solution by quadratic formula.

4) Solution by completing the square.

1-2-1 Solution by square root:

Solve the following quadratic equation using the square root method:

a) $x^2 - 13 = 0 \rightarrow x^2 = 13 \rightarrow x = \pm\sqrt{13}$

b) $2x^2 - 6 = 0 \rightarrow 2x^2 = 6 \rightarrow x^2 = 3 \rightarrow x = \pm\sqrt{3}$

c) $3x^2 + 12 = 0 \rightarrow 3x^2 = -12 \rightarrow x^2 = -4$

There is no real solution.

1-2-2 Solution by factoring:

If the left of a quadratic equation when written in standard form can be factored, then the equation can be solved very quickly. The method of solution by factoring depend on the following property of real numbers:

If A and B are real numbers. Then, AB=0 if and only if A=0 or B=0, or both are zero.

Solve the following quadratic equation by factoring:

a) $x^2+3x+2=0 \longrightarrow (x+2)(x+1)=0$

$x+2=0$ or $x+1=0$

$x=-2$ 0r $x=-1$

b) $3x^2-6x-24=0 \longrightarrow x^2-2x-8=0 \longrightarrow (x+2)(x-4)=0$

$x+2=0$ or $x-4=0$

$x=-2$ or $x=4$

c) $x^2-25=0 \longrightarrow (x-5)(x+5)=0$

$x-5=0$ or $x+5=0$

$x=5$ or $x=-5$

d) $6y^2=4y \longrightarrow 6y^2-4y=0 \longrightarrow y(6y-4)=0$

$6y-4=0$ or $y=0$

$6y=4 \longrightarrow y=\frac{4}{6} = \frac{2}{3}$

e) $6x^2+7x+1=0 \longrightarrow (6x+1)(x+1)=0$

$6x+1=0$ or $x+1=0$

$6x=-1$ or $x=-1$

$X=\frac{-1}{6}$

1-2-3 Solution by quadratic formula**:**

The solving method of a quadratic formula is $x = \frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$.

Solve the following quadratic equation using the quadratic formula:

a) $x^2-2x-1=0$

$a=1$, $b=2$, $c=-1$, then:

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a} \quad \rightarrow \quad x=\frac{-(-2)\pm\sqrt{((-2)^2-4(1)(-1))}}{2(1)}$$

$$x=\frac{2\pm\sqrt{(4+4)}}{2}=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

b) $x^2-4x+4=0$

$a=1$, $b=-4$, $c=4$

$$x=\frac{-(-4)\pm\sqrt{((-4)^2-4(1)(4))}}{2(1)} \rightarrow x=\frac{4\pm\sqrt{(16-16)}}{2} \rightarrow =\frac{4\pm\sqrt{0}}{2}=\frac{4}{2}=2$$

c) $3x^2-5x+6=0$

$a=3$, $b=-5$, $c=6$

$$x=\frac{-(-5)\pm\sqrt{((-5)^2-4(3)(6))}}{2(3)}= \quad \rightarrow \quad x=\frac{5\pm\sqrt{-47}}{6} \quad .$$

1-2-4 Solution by completing the square:

This method is based on the process of arranging the equation of the standard form $ax^2+bx+c=0$ into the form $(x+A)^2=B$, where A and B are real constants. The equation can be solved by taking the square root of both sides of the equation, if it has a real solution.

The steps to solve the quadratic equation by completing the square method:

Step 1:- Add constant to both sides of the equation to remove the constant term from the left side.

Step 2:- Divide by the coefficient of x^2 if it is not equal to one.

Step 3:- To complete the square in equation, add the square of one-half the coefficient of x to both sides.

Step 4:- Now the left side is a complete square, take the square root to both sids and complete the solution.

Solve the following quadratic equation using the completing the square method :

a) $x^2-2x-1=0$

$x^2-2x=1 \rightarrow x^2-2x+1=1+1 \rightarrow x^2-2x+1=2 \rightarrow (x-1)^2=2 \rightarrow x-1=\pm\sqrt{2}$

$x-1=-\sqrt{2}$ or $x-1=\sqrt{2}$

$x=1-\sqrt{2}$ or

$x=1+\sqrt{2}$

b) $2x^2-8x+3=0$

$2x^2-8=-3 \rightarrow x^2-4x=\frac{-3}{2} \rightarrow (\frac{-4}{2})^2=(-2)^2=4 \rightarrow$

$x^2-4x+4=\frac{-3}{2}+4 \rightarrow x^2-4x+4=\frac{5}{2}$

$(x-2)^2=\frac{5}{2} \longrightarrow x-2= \pm\sqrt{\frac{5}{2}}$

$x-2 = -\sqrt{\frac{5}{2}}$ or $x-2 = \sqrt{\frac{5}{2}}$

$x= 2-\sqrt{\frac{5}{2}}$ or $x= 2+\sqrt{\frac{5}{2}}$

الفصل الثاني

Inequalities المتباينات

2-1 Intervals:

a) If a and b are real numbers, such that $a<b$. Then, the open interval from a to b , denoted by (a,b), is the set of all real numbers x that lie between a and b. Thus,

$(a,b) = \{x/x \text{ is real number and } a<x<b\}$

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a b

b) The closed interval from a to b, denoted by [a,b], is the set of all real numbers that lie between a and b together with a and b included. Thus,

$[a,b] = \{x/x \text{ is real number and } a \leq x \leq b\}$

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a b

c) Semi closed or semi open intervals are defined as follows:

$[a,b) = \{x/x \text{ is real number and } a\leq x<b\}$

$(a,b] = \{x/x \text{ is real number and } a<x\leq b\}$

——————[- - - - - - - - - -) ——————
a b

——————(- - - - - - - - - - -]——————
a b

Unbounded interval:

a) $(a,\infty) = \{x: x>a\}$

b) $[a,\infty) = \{x; x\geq a\}$

c) $(-\infty,b) = \{x: x<b\}$

d) $(-\infty,b] = \{x: x\leq b\}$

Write the following in the interval form:

a) $2\leq x\leq 8 \quad \rightarrow \quad [2,8] = \{x: 2\leq x\leq 8\}$

b) $x>-4 \quad \rightarrow \quad (-4,\infty) = \{x: x>-4\}$.

2-2 Linear inequalities in one variable:

The general form for the Linear inequalities in one variable are:

$ax+b<0$ or $ax+b>0$

$ax+b\leq0$ or $ax+b\geq0$

Where $a\neq0$, a and b are real numbers.

The general form for the solution of Linear inequalities in one variable are:

$x<-\frac{b}{a}$ or $x>-\frac{b}{a}$

$x\leq-\frac{b}{a}$ or $x\geq-\frac{b}{a}$

Find all real numbers that satisfy the inequality:

a) $2x\geq1 \rightarrow x\geq\frac{1}{2}$

b) $3x-5<10 \rightarrow 3x<15 \rightarrow x<5$

c) $3-x\leq2x+4 \rightarrow 3\leq3x+4 \rightarrow -1\leq3x$

$-\frac{1}{3}\leq x$ or $x\geq-\frac{1}{3}$

d) $5-2x<7 \rightarrow -2x<2 \rightarrow x>-1$

e) $5x-\frac{1}{2}<x+3 \rightarrow 10x-1<2x+6 \rightarrow 8x<7 \rightarrow x<\frac{7}{8}$

Solve the double inequality for x:

a) $5<2x+7<13 \rightarrow -2<2x<6 \rightarrow -1<x<3$

b) $2x+1<3-x<2x+5$

$2x+1<3-x$	$3-x<2x+5$
$3x<2$	$-3x<2$
$x<\frac{2}{3}$	$x>-\frac{2}{3}$

2-3 Quadratic inequalities in one variable

The general form for the quadratic inequalities in one variable are:

$ax^2+bx+c>0$

$ax^2+bx+c\geq0$

Where $a\neq0$. a, b, and c are real constants.

Solve the following inequalities:

a) $x^2-3x>0 \rightarrow x(x-3)>0$

$x>0$ or $(x-3)>0$

$x>0$ or $x>3$

b) $x^2+3x-4\leq0 \rightarrow (x+4)(x-1)\leq0$

$(x+4)\leq0$ or $(x-1)\leq0$

$x\leq-4$ or $x\leq1$

c) $(x+2)(x+3)>(x-2)^2 \rightarrow x^2+5x+6>x^2-4x+4$

$5x+4x > 4-6 \rightarrow 9x > -2 \rightarrow x > \frac{-2}{9}$

2-4 Absolute values

If x is a real number, then the absolute value of x, denoted by $|x|$, is defined by:

$$|x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$$

1) If $|a|$=b, where b≥0 then either a=b or a=-b.

2) If $|a| = |b|$, then either a=b or a=-b.

3) $|x|$=$|-x| = \sqrt{x^2}$

4) $|x|$<a if and only if –a<x<a.

5) $|x|$>a if and only if either x>a or x<-a

6) $|ab|$=$|a|.|b|$

7) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, b≠0

Solve for x :

a) $|2x - 4| = 6$

2x-4=6 or 2x-4=-6

X=5 or x=-1

b) $|2x + 5| = |3x - 1|$

2x+5=3x-1 or 2x+5=-3x+1

X=6 or x= $\frac{-4}{5}$

c) $|3x - 4| < 5$

-5<3x-4<5 $\rightarrow$ -1<3x<9 $\rightarrow$ -$\frac{1}{3}$<x<3

d) $|3x - 4| > 7$

3x-4>7 or 3x-4<-7

3x>11 or 3x<-3

x>$\frac{11}{3}$ or x<-1

e) $\left|\frac{2x-3}{7}\right| \geq 1$

2x-3≥7 or 2x-3≤-7

2x≥10 or 2x≤-4

x≥5 or x≤-2

الفصل الثالث
الخطوط المستقيمة وانظمة المعادلات الخطية
Straight lines and systems of linear equations

3-1 The distance formula

Distance between two points $(x_1,y_1)$and (x_2,y_2):

d = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Find the distance between the points (4,3) and (-3,-4):

d = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

=$\sqrt{(-3-4)^2+(-4-3)^2}=\sqrt{(-7)^2+(-7)^2}=\sqrt{49+49}=\sqrt{98}$

Find the distance between the points (6,6) and (-6,-6):

d = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

=$\sqrt{(-6-6)^2+(-6-6)^2}=\sqrt{(-12)^2+(-12)^2}=\sqrt{144+144}=\sqrt{288}$

3-2 The Slope

The slope of two points (x_1,y_1),(x_2,y_2) is the following formula:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{\Delta y}{\Delta x}$$

Find the slope of the line through each pair of points:

a) (-2,5) & (4,-7)

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-7-5}{4-(-2)}=\frac{-12}{4+2}=\frac{-12}{6}=-2$$

b) (-3,-1) & (-3,5)

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{5-(-1)}{-3-(-3)}=\frac{5+1}{-3+3}=\frac{6}{0}\ \ not\ defined$$

c) (6,4) & (2,2)

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{6-2}{4-2}=\frac{4}{2}=2$$

3-3 Point – Slope formula

$$m = \frac{y-y_1}{x-x_1} \quad \rightarrow \quad y - y_1 = m(x - x_1)$$

a) passing through (-4,4) with slope 1/4:

$(x_1,y_1)=(-4,4)$, $m=1/4$

$y-4=1/4(x-(-4)) \quad \rightarrow \quad y-4=1/4(x+4)$

$y-4 = \frac{1}{4}x + \frac{4}{4} \quad \rightarrow \quad y-4 = \frac{1}{4}x + 1 \quad \rightarrow \quad y - \frac{1}{4}x = 5$

b) passing through the points (-3,3) and (-4,4):

$(x_1,y_1)=(-4,4)$ and $(x_2,y_2)=(-3,3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-4}{-3-(-4)} = \frac{-1}{-3+4} = \frac{-1}{1} = -1$$

Let $(x_1,y_1)=(-3,3)$ and $m=-1$

$y-y_1 =m(x-x_1) \quad \rightarrow \quad y-3=1-(x-(-3))$

$y-3 = -(x+3) \quad \rightarrow \quad y-3=-x-3 \quad \rightarrow \quad y=-x$

3-4 Equations for Horizontal and Vertical lines.

The equation for horizontal line , passing through (-3,4) is: y=4.

The equation for vertical line , passing through (-3,4) is : x=-3.

Given the linear equation 4y+6x=12. Find the slope and y-intercept of its graph.

$4y+6x=12 \quad \rightarrow \quad 4y=12-6x \quad \rightarrow \quad y=3-\frac{3}{2}x$

$m = -\frac{3}{2}$, y-intercept is b=3.

3-5 Parallel and Perpendicular lines.

Let L_1 and L_2 be two given lines, where m_1 is the slope of L_1 and m_2 is the slope of L_2. Then, L_1 and L_2 are said to be parallel, written as $L_1 \parallel L_2$, if and only if $m_1=m_2$. On the other hand, L_1 and L_2 are said to be perpendicular, written as $L_1 \perp L_2$, if and only if $m_1m_2=-1$ or $m_2=\frac{-1}{m_1}$.

Example:1

Given the equation line as x-2y=4. Find the equation of a line that passes through (3,-3) and is: a) Parallel to the given line. b) Perpendicular to the given line.

$x-2y=4 \quad \rightarrow \quad -2y=4-x \quad \rightarrow \quad y=-2+\frac{1}{2}x \quad \rightarrow \quad y=\frac{1}{2}x-2 \quad \rightarrow \quad m=1/2$

a) Parallel to the given line:

$m_1=m_2=1/2$

$y-y_1=m(x-x_1) \quad \rightarrow \quad y-(-3)=1/2(x-3) \quad \rightarrow \quad y+3=\frac{1}{2}x-3/2 \quad \rightarrow \quad y=\frac{1}{2}x-9/2$

b) Perpendicular to the given line:

$m_2=-1/m_1 \quad \rightarrow \quad m_2=\frac{-1}{\frac{1}{2}} = -2$

$y-y_1=m(x-x_1) \rightarrow y-(-3)=-2(x-3) \rightarrow y+3=-2x+6 \rightarrow y=-2x+3$

Example:2

Find the equations of the lines passing through (2,3) that are:

a) Parallel to the line $4x+3y=6$

b) Perpendicular to the line $x-3y+1=0$

Solution:

a) $4x+3y=6 \rightarrow 3y=6-4x \rightarrow y=2-\frac{4}{3}x \rightarrow m=-4/3$

$m_2=m_1=-4/3$

$y-y_1=m(x-x_1) \rightarrow y-3=-4/3(x-2) \rightarrow y-3=\frac{-4}{3}x-\frac{8}{3} \rightarrow y=\frac{-4}{3}x+1/3$

b) $x-3y+1=0 \rightarrow -3y=-x-1 \rightarrow y=\frac{1}{3}x+\frac{1}{3}$. $m_2=\frac{-1}{m_1}=\frac{-1}{\frac{1}{3}}=-3$

$y-y_1=m(x-x_1) \rightarrow y-3=-3(x-2) \rightarrow y-3=-3x+6 \rightarrow y=-3x+9$

Example:3

Determine whether the following pairs of lines are parallel, perpendicular, or neither:

a) $2x+3y=6$ and $3x-2y=6$

b) $2y+4x+1=0$ and $y-2+2x=0$

Solution a):

$2x+3y=6 \rightarrow 3y=6-2x \rightarrow y=2-\frac{2}{3}x \rightarrow m_1=-2/3$

$3x-2y=6 \rightarrow -2y=6-3x \rightarrow y=-3+\frac{3}{2}x \rightarrow m2=3/2$

The pair of lines are perpendicular.

Solution b):

$2y+4x-1=0 \rightarrow 2y=-4x+1 \rightarrow y=-2x+1/2 \rightarrow m_1=-2$

$y-2+2x=0 \rightarrow y=-2x+2 \rightarrow m_2=-2$

The pair of lines are parallel.

3-6 Systems of linear equations in two variables.

The general form for the Systems of linear equations in two variables are:

$A_1X+B_1Y=C_1$

$A_2X+B_2Y=C_2$

$A_1, A_2, B_1, B_2, C_1, C_2$ are real constants.

The solution method of the Systems of linear equations in two variables are:

1) Elimination by addition.

2) Substitution.

Solve the following system of two equations by:

1) $2x+y=9$

$x+3y=12$

a) Elimination by addition:

$2x+y=9$ …(1)

$\underline{-2x-6y=-24 \quad …(2)}$

$-5y=-15 \rightarrow y=3 \rightarrow 2x+3=9 \rightarrow 2x=6 \rightarrow x=3$

b) Substitution:

$y=9-2x \rightarrow x+3(9-2x)=12 \rightarrow x+27-6x=12 \rightarrow -5x=-15 \rightarrow x=3$

$y=9-2(3) \rightarrow y=3$

2) $x-2y=2 \quad ...(1)$

$\underline{x+y=5 \quad ...(2)}$

a) $-x+2y=-2$

$\underline{x+y=5}$

$3y=3 \rightarrow y=1 \rightarrow x+1=5 \rightarrow x=4$

b) $x=5-y \rightarrow (5-y)-2y=2 \rightarrow 5-y-2y=2 \rightarrow -3y=-3 \rightarrow y=1 \rightarrow x=4$

3) $x+2y=-4 \quad ...(1)$

$\underline{2x+4y=8} \quad ...(2)$

a) $-2x-4y=8$

$\underline{2x+4y=8}$

zero = zero

3-7 Systems of linear equations in three variables

$$a_1x+b_1y+c_1z=k_1$$
$$a_2x+b_2y+c_2z=k_2$$
$$a_3x+b_3y+c_3z=k_3$$

1) Solve the following equations:

$3x-2y+4z=6 \quad ...(1)$

$2x+3y-5z=-8 \quad ...(2)$

$5x-4y+3z=7 \quad ...(3)$

(1)and(2) $9x-6y+12z=18$

$\underline{4x+6y-10z=-16}$

$13x+2z=2 \quad ...(4)$

(1)and(3) $-6x+4y-8z=-12$

$\underline{5x-4y+3z=7}$

$-x-5z=-5 \quad ...(5)$

(4)and(5) $13x+2z=2$

$\underline{-13x-65z=-65}$

$-63z=-63 \rightarrow z=1$

by(5) $-x-5z=-5 \rightarrow -x-5(1)=-5 \rightarrow -x=0 \rightarrow x=0$

by(1) $3x-2y+4z=5 \rightarrow 3(0)-2y+4(1)=6 \rightarrow -2y=2 \rightarrow y=-1$

2) Solve the following system of equations:

$x+3y-z=4 \quad ...(1)$

$2x+y+2z=10 \quad ...(2)$

$3x-y+z=4 \quad ...(3)$

by(1)and(2) $-2x-6y+2z=-8$

$\underline{2x+y+2z=10}$

$-5y+4z=2 \quad ...(4)$

$-3x+9y+3z=-12$

$\underline{3x-y+z=4}$

$-10y+4z=-8 \quad ...(5)$

by(4)and(5) $10y-8z=-4$

$\underline{-10y+4z=-8}$

$-4z=-12 \quad \rightarrow \quad z=3$

$-5y+4z=2 \quad \rightarrow \quad -5y+5(3)=2 \quad \rightarrow \quad -5y=-10 \quad \rightarrow \quad y=2$

by(1) $x+3y-z=4 \quad \rightarrow \quad x+3(2)-3=6 \quad \rightarrow \quad x+3=4 \quad \rightarrow \quad x=1$

3) Solve the following system of equations:

$x_1+x_2+x_3=6 \quad ...(1)$

$2x_1-x_2+3x_3=9 \quad ...(2)$

$-x_1+2x_2+x_3=6 \quad ...(3)$

(1) and (3) $x_1+x_2+x_3=6$

$-x_1+2x_2+x_3=6$

$3x_2+2x_3=12 \quad .....(4)$

(2) and 2(3) $2x_1-x_2+3x_3=9$

$-2x_1+4x_2+2x_3=12$

$3x_2+5x_3=21 \quad ...(5)$

(4) – (5) $3x_2+2x_3=12$

$-3x_2-5x_3=-21$

$-3x_3=-9 \quad \rightarrow \quad x_3=3 \quad \rightarrow \quad 3x_2+6=12 \quad \rightarrow \quad 3x_2=6 \quad \rightarrow \quad x_2=2$

$x_1+2+3=6 \quad \rightarrow \quad x_1=6-5 \quad \rightarrow \quad x_1=1$

الفصل الرابع
Functions الدوال

A function expresses the idea of one quantity depending on or being determined by another.

4-1 Function:

Let A and B are two nonempty sets. Then, a function, say f, from A to B is a role that assigns to each element in A a unique element in B.

Let f denote a given function, The set A for which f assigns a unique value in B is called the Domain of the function f, The corresponding set of values in B is called the range of the function. If for each x their exists exactly one value of y, we say that y is a function of x, and we write y=f(x).

Domain:

If the function f and y=f(x), then the domain of f can be viewed as the set of allowable values for the independent variable x.

Range:

Is the set of all possible values of f(x) as x varies over the domain of f.

Find the domain and the range for the following functions:

a) f(x)=4x+1

$D_f=\{x|\ -\infty<x<\infty\}$, $R_f=\{y|y\in R\}$

b) $f(x)=x^2$

$D_f=\{x|\ x\in R\}$, $R_f=\{y|\ y\geq 0\}$

c) $f(x)=\sqrt{x}$

$D_f=\{x|\ x\geq 0\}$, $R_f=\{y|\ y\geq 0\}$

d) $f(x)=\frac{1}{(x-1)(x-3)}$

$D_f=\{x|\ x\neq 1,3\}$, $R_f=\{y|y\in R\}$

e) $f(x)=\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2$

$D_f=\{x|\ x\in R\}$, $R_f=\{y|\ y\in R\}$

f) $f(x)=\frac{x+1}{x-1}$

$D_f=\{x|\ x\neq 1\}$, $R_f=\{y|\ y\in R\}$

) $f(x)=\begin{cases} x+2, & x>2 \\ 4, & x\leq 2 \end{cases}$

h) $f(x)=\begin{cases} 2-2x, & x\leq -1 \\ 4, & -1\leq x\leq 3 \\ 2x-2, & x\geq 3 \end{cases}$

Let f(x)=2x+1 , $0\leq x\leq 1$. Find: f(0), f(1), f(1/2), f(a), f(a+h), $\frac{f(a+h)-f(a)}{h}$.

F(0)=1 , f(1)=3 , f(1/2)=2 , f(a)= 2a+1 , f(a+h)= 2(a+h)+1=2a+2h+1

$$\frac{f(\mathrm{a}+\mathrm{h})-\mathrm{f(a)}}{h}=\frac{2a+2h+1-(2a+1)}{h}=\frac{2a+2h=1-2a-1}{h}=\frac{2h}{h}$$
$$=2$$

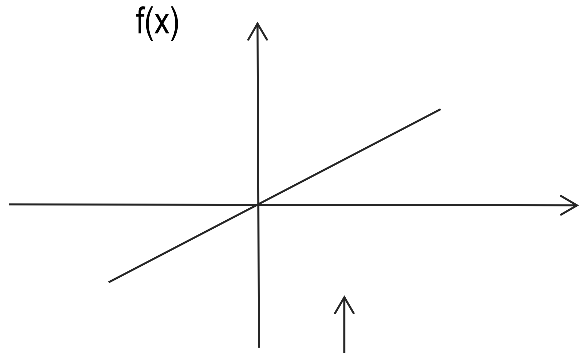
4-2 Graphs of Functions

Graph the following functions:

a) f(x)=x

x: …,-3, -2, -1, 0, 1, 2, 3,…

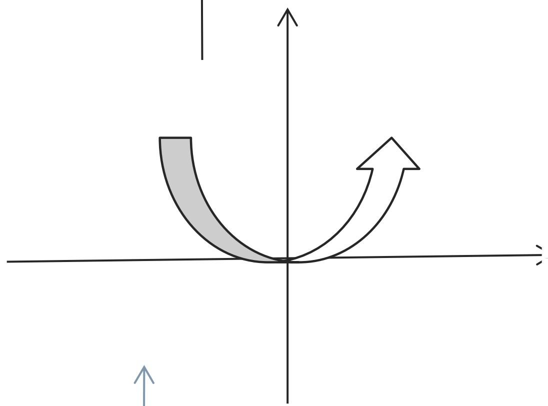
f(x): …,-3, -2, -1, 0, 1, 2, 3,…



b) f(x)=x^2

x: …, -3, -2, -1, 0, 1, 2, 3, …

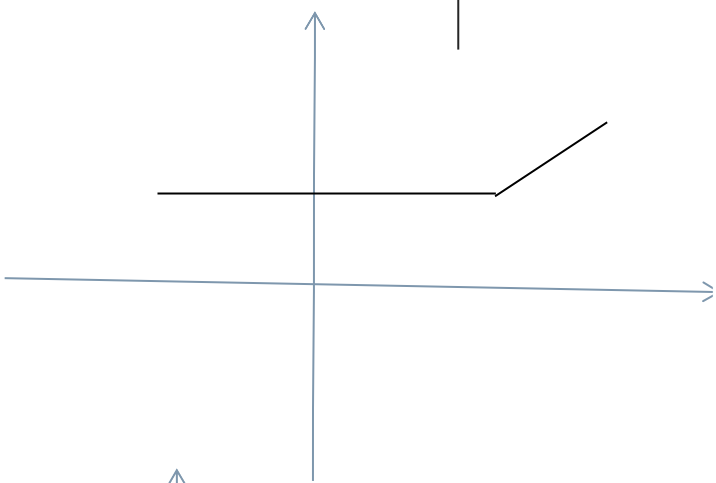
f(x): …, 9, 4, 1,0, 1, 4, 9, …



c) $\mathrm{f(x)} \;=\begin{cases}4 & x<2\\ x+2 & x\geq 2\end{cases}$

x: …-2,-1,o,1,2,3,4,…

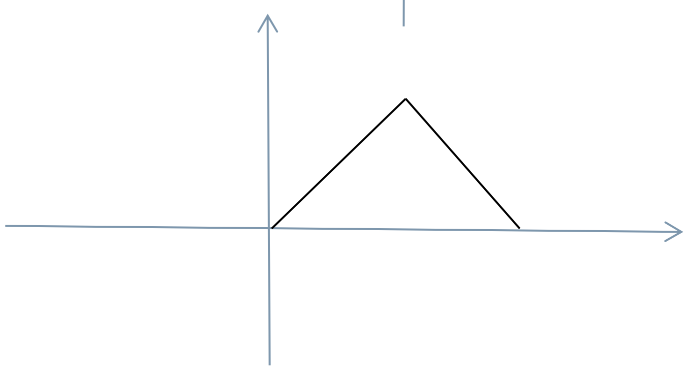
f(x)= …,4,4,4,4,4,5,6,…



d) f(x) = $\begin{cases}x & 0\leq x\leq 1\\ 2-x & 1\leq x\leq 2\\ 0 & other\ wise\end{cases}$

x: …,-1,0,1,2,3,…

f(x)= …,0,0,1,0,0,…



4-3 Kinds of functions

a) Constant functions: f(x)=a
b) Linear function: f(x)= ax +b
c) Quadratic function: $f(x)=ax^2+bx+c$
The **Vertex** which is the lowest point when a>0 and the highest point when a<0.
Is at the point: $x=\frac{-b}{2a}$ *and* $y=\frac{4ac-b^2}{4a}$ for the function is $f(x)=ax^2+bx+c$.
d) Absolute value function:
$$f(x)=|x|=\begin{cases} x\,, & x\geq 0 \\ -x\,, & x<0 \end{cases}$$
e) Exponential function: f(x)= e^x
f) Logarithmic function: f(x)=ln x
1) ln xy=lnx + lny
2) ln x/y=lnx – lny
3) ln x^n= n lnx
4) ln 1/x =-lnx

Find the vertex for the following function:
$F(x)=2x^2-8x+5$
a=2 , b=-8 , c=5
$x=\frac{-b}{2a}=\frac{-(-8)}{2(2)}=\frac{8}{4}=2 \quad \rightarrow$ y=-3

4-4 Combinations of functions
1) Arithmetic operations on functions:
If f(x) and g(x) are two functions on the same variable x. Then:
a) (f+g)(x)=f(x)+g(x)
b) (f-g)(x)=f(x)-g(x)
c) (f.g)(x)=f(x).g(x)
d) (f/g)(x)=f(x)/g(x) , g(x)≠0.

Let f(x)=x and $g(x)=x^2$. Find (f+g)(x), (f-g)(x), (f.g)(x), (f/g)(x) and state the domains.
$(f+g)(x)= x+x^2$
$(f-g)(x)= x-x^2$
$(f.g)(x)=x^3$
(f/g)(x)=1/x
2) Composition of functions
fog (x)=f(g(x)) , gof (x)=g(f(x))

Let f(x)=x-7, and g(x)=x^2 . Find fog(x) , gof(x):
fog(x)=f(g(x))=f(x^2)=x^2-7.   gof(x)=g(f(x))=g(x-7)=$(x-7)^2$

Let f(x)=x-1, and g(x)=1+$\sqrt{x-2}$, find fog(x) and gof(x):
f(x)=x-1, and g(x)=1+$\sqrt{x-2}$,then:
fog(x) =f(g(x))
=f(1+$\sqrt{x-2}$)
=1+$\sqrt{x-2}-1=\sqrt{x-2}$
gof(x)=g(f(x))
=g(x-1)= 1+$\sqrt{x-1-2}=1+x-3$

3) Inverse function:
To find the inverse function we have to:
a) solve the function y=f(x) for x in terms of y.
b) switch x and y. the resulting formula will be y=f^{-1}(x).
f^{-1}(x) is called the inverse function of f(x).

Example:
Let f(x)=$\frac{1}{2}\ x+1$ find the inverse function:
f(x)=y= $\frac{1}{2}\ x+1$
2y=x+2 ⟶ x=2y-2 ⟶ x=2(y-1)
f^{-1}(x)=2(x-1).

Example:
Let f(x)=e^x , find f^{-1}(x):
f(x)=y=e^x ⟶ lny=xlne ⟶ x=lny
f^{-1}(x)=lnx

f(x)=x^2 ⟶ x=±$\sqrt{y}$, there are no inverse.

الفصل الخامس
The derivatives and its المشتقات وتطبيقاتها Application

5-1 Derivative definition:

Let y=f(x) , then, the derivative of f are:

$f'(x)= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Example:

Find $f'(x)$ for the function: $f(x)=4x^2-3x+4$, and find $f'(3)$ and $f'(-3)$:

$f(x+\Delta x) = 4(x+\Delta x)^2-3(x+\Delta x)+4$

$f(x+\Delta x) - f(x) = [4(x+\Delta x)^2-3(x+\Delta x)+4] - (4x^2-3x+4)$

$= 8x\Delta x-3\Delta x+4(\Delta x)^2$

$=8x-3+4\Delta x \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$f'(x)= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}= 8x-3$

$f'(3)=21$, $f'(-3)=-27$

5-2 Derivatives Rules:

1) Let y=c, then $\frac{dy}{dx}$=zero

2) The power formula:

Let $y=x^n$, then $\frac{dy}{dx}=nx^{n-1}$

Example:

Find $\frac{dy}{dx}$ for the following:

a) $y=x^5 \longrightarrow \frac{dy}{dx}=5x^4$

b) $y=x \longrightarrow \frac{dy}{dx}=x^{1-1}=x^0=1$

c) $y=\sqrt{x} \longrightarrow y=x^{1/2} \longrightarrow \frac{dy}{dx}=\frac{1}{2}\,x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

d) $y=x^{1/3} \longrightarrow \frac{dy}{dx}=\frac{1}{3}\,x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} =\frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$

e) $y= \frac{1}{x^2} \longrightarrow y=x^2 \longrightarrow \frac{dy}{dx}=-2x^{-3}= \frac{-2}{x^3}$

f) $y= \frac{1}{\sqrt{x}} \longrightarrow y=x^{-\frac{1}{2}}$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}} = \frac{-1}{2x^{\frac{3}{2}}} = \frac{-1}{2\sqrt{x^3}}$$

3) Let y=cx then if cx is differentiable function of x, and c is a constant, then $\frac{\mathbf{dy}}{\mathbf{dx}} = c\frac{dy}{dx}$

Example:

Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ for the following:

a) $y=cx^n \longrightarrow \frac{\mathrm{dy}}{\mathrm{dx}}=cnx^{n-1}$

b) $y=10x^4 \longrightarrow \frac{\mathrm{dy}}{\mathrm{dx}}=40x^3$

c) $y=\frac{5}{x} \longrightarrow \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{-5}{x^2}$

d) $y=2\sqrt{x} \longrightarrow y=2x^{\frac{1}{2}}$

$\frac{\mathrm{dy}}{\mathrm{dx}} = 2\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}} = \frac{1}{\sqrt{x}}$

4) If u(x)and v(x) are two differentiable functions of x, then f(x)=u±v,

And $f'(x)=\frac{du}{dx} \pm \frac{dv}{dx}$

Example:

a) $f(x)=x^3+\sqrt[3]{x} \longrightarrow f'(x)=3x^2+\frac{1}{3}x^{-\frac{2}{3}}$

b) $y=4x^3+\frac{4}{x^2} \longrightarrow \frac{\mathrm{dy}}{\mathrm{dx}} = 12x^2-8x^{-3} \longrightarrow \frac{\mathrm{dy}}{\mathrm{dx}}=12x^2-\frac{8}{x^3}$

c) $y=3x^3-5x^2+7x-10$

$\frac{\mathrm{dy}}{\mathrm{dx}} = 9x^2-10x+7$

d) $y=\frac{7x^4-5x^3+5}{3x^2} \longrightarrow y=\frac{7}{3}x - \frac{5}{3} + \frac{5}{3}x^{-3} \longrightarrow \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{7}{3} - 5x^{-4}$

5) Product Rule:

If u(x)and v(x) are two differentiable functions of x, then:

$$\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example:

Find $f'(x)$ if:

a) $f(x)=(4x^3-2x)(3x^2+4x+7)$

b) $f(x)=(2\sqrt{x}+1)(x^2+3)$

c) $f(x)=(3x^2+2x+1)(x^2-2)$

Solution:

a) Let $u=4x^3-2x$, $v=3x^2+4x+7$

$f(x)= u.v \longrightarrow u'=12x^2-2$, $v'=6x+4$, then:

$f'(x)= uv'+vu'$

$=(4x^3-2x)(6x+4)+(3x^2+4x+7)(12x^2-2) =60x^4+64x^3+66x^2-16x-14$

طريقة ثانية:
أيجاد نتيجة حاصل ضرب الدالتين ثم نشتق النتيجة.

b) $f(x)=(2\sqrt{x}+1)(x^3+3)$

$u=2x^{1/2}+1 \rightarrow u'=x^{-\frac{1}{2}}$

$v=x^3+3 \rightarrow v'=2x$

$f'(x)= uv'+vu'$

$=(2x^{1/2}+1)(2x)+(x^2+3)(x^{-1/2}) =4x^{3/2}+2x+x^{3/2}+3x^{-1/2} =5x^{3/2}+2x+ \frac{3}{\sqrt{x}}$

c) $f(x)=(3x^2+2x+1)(x^2-2)$

$f'(x)=(3x^2+2x+1)(2x)+(x^2-2)(6x+2) =12x^3+6x^2-10x-4$

6) Quotient Rule:

If u(x)and v(x) are two differentiable functions of x, then:

$' = \frac{vu'-uv'}{v^2}\left(\frac{u}{v}\right)$

Example:

Use the quotient rule to differentiate the following function:

a) $f(x) = \frac{2x+5}{2x-5} \rightarrow f'(x)= \frac{(2x-5)(2)-(2x+5)(2)}{(2x-5)^2} = \frac{-20}{(2x-5)^2}$

b) $f(x) = \frac{1-x^3}{1+x^3} \rightarrow f'(x)= \frac{(1+x^3)(-3x^2)-(1-x^3)(3x^2)}{(1+x^3)^2} = \frac{-6x^2}{(1+x^3)^2}$

c) $f(x) = \frac{x}{x-1} \rightarrow f'(x)= \frac{(x-1)^2-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$

d) $f(x) = \frac{(x+1)(x^2-2x)}{x-1} \rightarrow f'(x)= \frac{(x-1)(4x^3+3x^2-4x-2)-(x+1)(x^3-2x)}{(x-1)^2}$

$= \frac{3x^4-2x^3-5x^2+4x+2}{(x-1)^2}$

Example:

Find the slope of the tangent and the equation of the tangent line to the function $f(x)=\sqrt{x}$ for the points (9,4) , (1/9,1/6):

By the derivative definition: $f'(x)=\frac{1}{2\sqrt{x}}$

for x=9, then, $f'(9)=\frac{1}{2\sqrt{9}} = 1/6$

$y-y_1=m(x-x_1) \rightarrow y-4=1/6(x-9)$

$y-4=\frac{1}{6}x - \frac{9}{6}$ $y=\frac{1}{6}x + \frac{15}{6}$

for x=1/9

$f'(1/9)=\frac{1}{2\sqrt{\frac{1}{9}}}=\frac{1}{2\frac{1}{3}}$=3/2

y-1/6=3/2(x-1/9) $\rightarrow$ y-1/6= $\frac{3}{2}$x-1/6 $\rightarrow$ y= $\frac{3}{2}$ x

Example:

Find the slope of the tangent and the equation of the tangent line to the function

F(x)= $\frac{1}{x+1}$, at the point (3,o)?

$f'(x)=\frac{-1}{(x+1)^2}$ $\rightarrow$ $f'(3)=\frac{-1}{16}$=m

y-o= $\frac{-1}{16}$ (x-3)

7) The Chain Rule:

$\frac{\partial y}{\partial x}=\frac{dy}{du}.\frac{du}{dx}$

Example:

Find the derivative of the following functions, use the chain rule:

a) y=$(1-x^3)^4$

$\frac{dy}{du}=4u^3$, $\frac{dy}{dx}=-3x^2$ $\frac{\partial y}{\partial x}=(4u^3)(-3x^2)$

=$4(1-x^3)^3(-3x^2)$ $\rightarrow$ =$-12x^2(1-x^3)$

b) y= $\sqrt{4x+4}$ = $(4x+4)^{1/2}$

, $\frac{du}{dx}=4\frac{dy}{du}=\frac{1}{2}u^{-1/2}$

$\rightarrow$ $\frac{1}{2}(4x+4)^{-1/2}.4\frac{dy}{dx}=\frac{1}{2}u^{-1/2}$ .4

c) $\left(\frac{x-1}{x+1}\right)^4$

$\frac{dy}{du}=4u^3$, $\frac{du}{dx}=\frac{(x+1)-(x-1)}{(x+1)^2}$

$\rightarrow$ = 8 $\frac{(x-1)^3 dy}{(x+1)^5 dx}=4\left(\frac{x-1}{x+1}\right)^3.\frac{2}{(x+1)^2}$

8) Derivative of Exponential function:

Let y= e^x , then $\frac{dy}{dx}=e^x$

Example:

a) y= xe^x $\rightarrow$ $y'=\frac{dy}{dx}=xe^x+e^x=(x+1)e^x$

b) y= e^{x^4} $\rightarrow$ $\frac{dy}{dx}=4x^3e^{x^4}$

c) y= x^3e^x $\rightarrow$ $\frac{dy}{dx}$ = $x^3e^x+e^x.3x^2=(x^3+3x^2)e^x$

d) y= $\frac{e^x}{x+1}$ $\rightarrow$ $\frac{dy}{dx} = \frac{(x+1)e^x - e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$

e) y= e^{3x} $\rightarrow$ $\frac{\mathrm{dy}}{\mathrm{dx}} = 3e^{3x}$

f) y= $e^{x^3-3x^2}$ $\rightarrow$ $\frac{dy}{dx} = (3x^2 - 6x)e^{x^3-3x^2}$

g) y= $xe^{1/2}$ $\rightarrow$ $\frac{\mathrm{dy}}{\mathrm{dx}} = xe^{1/x}(-x^{-2}) + e^{1/x} = -x^{-1}e^{1/x} + e^{1/x} = (1-x^{-1})e^{1/x} = (1-\frac{1}{x})e^{1/x}$

9) Derivative of the logarithmic function:

Let y=lnx, then $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{x}$

a) y=ln(x^4+2x-10) $\rightarrow$ $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{(x^4+2x-10)}(4x^3+2) = \frac{4x^3+2}{(x^4+2x-10)}$

b) y= $\frac{lnx}{x^2}$ $\rightarrow$ $\frac{dy}{dx} = \frac{x^2.\frac{1}{x} - (lnx)(2x)}{(x^2)^2} = \frac{x - 2xlnx}{x^3}$

$= \frac{x(1-2lnx)}{x^4} = \frac{1-2lnx}{x^3}$

c) y= xln(x+1) $\rightarrow$ $\frac{\mathrm{dy}}{\mathrm{dx}} = x.\frac{1}{(x+1)} + \ln(x+1) = \frac{x}{(x+1)} + \ln(x+1)$

d) y= $\log_{10} x^2$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{lnx^2}{ln10} = \frac{1}{ln10}.\frac{1}{x^2}.2x = \frac{2x}{x^2 ln10}.$$

الفصل السادس

Integration التكامل

6-1 An indefinite integral:

The following are some useful for integration, where n is a Real number. a_1 and a_2 are real constants. And c is the constant of the integration method.

6-1-1 Rules for integration:

1) $\int x^x dx = \frac{1}{n+1} x^{n+1} + c \qquad , n \neq -1$

2) $\int \frac{1}{x} dx = \ln|x| + c$

3) $\int a_1 f(x)dx = a_1 \int f(x)dx$

4) $\int [a_1 f_1(x) + a_2 f_2(x)]dx = a_1 \int f_1(x)dx + a_2 \int f_2(x)dx$

5) $\int [f(x)]^n f'(x)dx = \frac{1}{n+1} [f(x)]^{n+1} + c$

6) $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

7) $\int e^x dx = e^x + c$

8) $\int e^{f(x)} f'(x)dx = e^{f(x)} + c$

مثال 1

Find the integral for the following:

a) $\int dx = x + c$

b) $\int x dx = \frac{1}{2}x^2 + c$

c) $\int x^2 dx = \frac{1}{3}x^3 + c$

d) $\int 3x^2 dx = 3\int x^2 dx = (3)(\frac{1}{3})x^3 + c = x^3 + c$

e) $\int \sqrt{x dx} = \int x^{\frac{1}{2}} dx \frac{x^{\frac{3}{2}}}{3/2} c = \frac{2}{3} x^{\frac{3}{2}} + c$

f) $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} + c = \frac{-1}{x} + c$

g) $\int \frac{1}{\sqrt{x}} dx = \int x^{\frac{-1}{2}} dx = \frac{x^{\frac{1}{2}}}{1/2} + c = 2x^{\frac{1}{2}} + c = 2\sqrt{x} + c$

h) $\int \frac{2}{x^3} dx = 2\int x^{-3} dx = (2)(\frac{x^{-2}}{-2}) + c = \frac{-1}{x^2} + c$

i) $\int (3x-1)dx = \int 3x dx - \int dx = 3\int x dx - \int dx = \frac{3}{2}x^2 - x + c$

$$j)\int\left[16x^7-\sqrt{x}+\frac{1}{x}\right]dx=16\int x^7dx-\int x^{\frac{1}{2}}dx+\int\frac{1}{x}dx$$

$$=(16)\left(\frac{x^8}{8}\right)-\frac{x^{\frac{3}{2}}}{3/2}+\ln|x|+c$$

$$=2x^8-\frac{2}{3}x^{\frac{3}{2}}+\ln|x|+c$$

k) $\int\left[e^x-3x^2\right]dx=\int e^xdx+3\int x^2dx$

$$=e^x+(3)\frac{x^3}{3}+c \quad\longrightarrow\quad =e^x+x^3+c$$

Find the integral for the following: **مثال 2**

a) $\int x(x+1)dx$

$$\int x(x+1)dx=\int(x^2+x)dx$$

$$=\int x^2dx+\int xdx$$

$$=\frac{x^3}{3}+\frac{x}{2}+c$$

b) $\int(x^3+1)(x^2-1)dx=\int(x^5-x^3+x^2-1)dx$

$$=\int x^5dx-\int x^3dx+\int x^2dx-\int dx$$

$$=\frac{1}{6}x^6-\frac{1}{4}x^4+\frac{1}{3}x^3-x+c$$

c) $\int(x-\frac{1}{x})(x+\frac{1}{x})dx$

$$\int\left[x^2-\left(\frac{1}{x}\right)^2\right]dx=\int x^2dx-\int\frac{1}{x^2}dx$$

$$=\int x^2dx-\int x^{-2}dx \qquad =\frac{1}{3}x^3-\left(\frac{-1}{3}\right)x^{-3}dx+c \qquad =\frac{1}{3}x^3+\frac{1}{3x^3}+c$$

d) $\int (x+5)^2dx$

$$\int (x+5)^2dx=\int (x^2+10x+25)dx$$

$$=\int x^2dx+10\int xdx+25\int dx$$

$$=\int \frac{1}{3}x^3+\frac{10}{2}x^2+25x+c$$

$$=\frac{1}{3}x^3+5x^2+25x+c$$

Find the integral for the following: **مثال 3**

a) $\int (x^2+x+1)(2x+1)dx$

$$=\int (2x^3+x^2+2x^2+x+2x+1)dx \qquad =\int (2x^3+3x^2+3x+1)dx$$

$$=2\int x^3dx+3\int x^2dx+3\int xdx+\int dx$$

$$=(2)\left(\frac{x^4}{4}\right)=(3)\left(\frac{x^3}{4}\right)+(3)\left(\frac{x^2}{4}\right)+x+c$$

$$=\frac{1}{2}x^4+x^3+\frac{3}{2}x^2+x+c$$

:طريقة ثانية

$$\int (x^2+x+1)(2x+1)dx = \frac{1}{2}(x^2+x+1)^2 + c$$

$$= \frac{1}{2}(x^2+x+1)(x^2+x+1) + c$$

$$= \frac{1}{2}\left[x^4 + x^3 + x^2 + x \ \ + x + x^2 + x + 1\right] + c$$

$$= \frac{1}{2}\left[x^4 + 2x^3 + 3x^2 + 2x + 1\right] + c \quad \longrightarrow \quad = \frac{1}{2}x^4 + x^3 + \frac{3}{2}x^2 + x + c$$

b) $\int (x^2+x+1)^{\frac{1}{3}}(2x+1)dx \quad \longrightarrow \quad = \frac{3}{4}(x^2+x+1)^{\frac{3}{4}} + c$

c) $\int (x^3+1)^4 x^2 dx \quad \longrightarrow \quad \int \frac{1}{3}(x^3+1)^4 3x^2 dx$

$$= \frac{1}{3}\cdot\frac{1}{5}(x^3+1)^5 + c \quad \longrightarrow \quad = \frac{1}{15}(x^3+1)^5 + c$$

d) $\int \frac{1}{x}(\ln x)^2 dx = \frac{1}{3}(\ln x)^3 + c$

مثال 4

Find the integral for the following:

a) $\int \frac{2x+1}{x^2+x+1}dx \quad \longrightarrow \quad \ln\left|x^2+x+1\right| + c$

b) $\int \frac{2x+1}{(x^2+x+1)^3}dx$

$$\frac{-1}{2}(x^2+x+1)^{-2} + c = \frac{-1}{2(x^2+x+1)^2} + c$$

c) $\int \frac{x}{\sqrt{1-3x^2}}$

$$\frac{-1}{6}\int \frac{-6x}{\sqrt{1-3x^2}}dx \qquad = \frac{-1}{6}\int (1-3x^2)^{\frac{-1}{2}}(-6x)dx$$

$$=\frac{-2}{6}(1-3x^2)^{\frac{1}{2}}+c \quad \longrightarrow \quad =\frac{-1}{3}\sqrt{1-3x^2}+c$$

6-1-2 Integration Methods

1) التكامل بطريقة التعويض

Transformation of variables

Find the integral for the following: مثال 5

a) $\int (2x+3)^4 dx \quad \longrightarrow \quad \int (2x=3)^4 dx = \int u^4 . \frac{1}{2} du$

$$=\frac{1}{2}\int u^4 du \quad \longrightarrow \quad =\frac{1}{2}.\frac{1}{5}u^5+c \quad \longrightarrow \quad =\frac{1}{10}u^4+c \quad \longrightarrow \quad \frac{1}{10}(2x+3)^5+c$$

b) $\int \frac{x}{\sqrt{3x^2-2}}$

$$\int (3x^2-2)^{\frac{-1}{2}} x dx = \int u^{\frac{-1}{2}} . \frac{1}{6} du$$

$$=\frac{1}{6}.2u^{\frac{1}{2}}+c \quad \longrightarrow \quad =\frac{1}{3}(3x^2-2)^{\frac{1}{2}}+c \quad \longrightarrow \quad =\frac{1}{3}\sqrt{2x^2-2}+c$$

Integration by parts 2) التكامل بطريقة التجزئة

Find the integral for the مثال 6

following:

a) $\int x\sqrt{x+5}dx$

$\int \sqrt{x+5dx} = \int h_2^1(x)dx \quad \longrightarrow \quad \int (x+5)^{\frac{1}{2}}dx = \int h_2^1(x)dx$

$\frac{3}{2}(x+)^{\frac{3}{2}} = h_2(x) \quad \longrightarrow \quad \int x\sqrt{x+5} = x.\frac{2}{3}(x+5)^{\frac{3}{2}} - \int \frac{3}{2}(x+5)^{\frac{3}{2}}dx \quad \longrightarrow \quad \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{4}{15}(x-5)^{\frac{5}{2}} + c$

b) $\int \ln x dx$

بافتراض أن $h_1(x) = \ln x$ فإن $h'(x) = \frac{1}{x}dx$ وبافتراض ان $\int dx = \int h_2^1(x)dx$ فإن $x = h_2(x)$ وبالتالي

فان ق قيمة التكامل:

$\int \ln x dx = x\ln x - \int x.\frac{1}{x}dx \quad \longrightarrow \quad = x\ln - \int dx \quad \longrightarrow \quad x\ln x - x + c$

c) $\int xe^2 dx$

وبافتراض أن $h_1(x) = x$ فإن $h_1'(x) = dx$ وبافتراض ان $\int e^x dx = \int h_2'(x)dx$ فإن $e^x = h_2(x)$ وبالتالي فإن

قيمة التكامل هي:

$\int xe^x dx = xe^x - \int e^x dx$

$= xe^x - e^x + c$

Integration by using fraction **3) التكامل بالتفريق الى كسور بسيطة**

Find the integral for the following: **مثال (7)**

a) $\int \frac{1}{x^2-2x-3}dx$

$x_2 - 2x - 3 = (x-3)(x+1)$

$$\int \frac{1}{x^2-2x-3}dx = \int \frac{1}{(x-3)(x+1)}dx$$

وبما ان الكسر $\frac{1}{(x-3)(x+1)}$ يمكن كتابته على الشكل:

$$\frac{A}{x-3}+\frac{B}{x+1}=\frac{A(x+1)+B(x-3)}{(x-3)(x+1)}=\frac{(A+B)x+(A-3B)}{(x-3)(x+1)}$$

$$\frac{1}{(x-3)(x+1)}=\frac{(A+B)x+(A-3b)}{(x-3)(x+1)} \quad \rightarrow \quad A=\frac{1}{4}\ ,\ B=\frac{-1}{4}$$

وكذلك يمكن الحصول على قيمة كل من A و B باستخدام العلاقة $A(x+1) + (x-3)= 1$ والتعويض عن قيمة x بأصفار المقام. أي عندما $x= 3$ فإن $4A=1$ ويعني $A=\frac{1}{4}$ ، أما عندما $X= -1$ فإن $-4B= 1$ ويعني $B=-\frac{1}{4}$. وعند التعويض عن تلك القيمتين نحصل على التكامل التالي:

$$\int \frac{1}{x^2-2x-3}dx = \int \frac{1}{(x-3)(x+1)}dx$$

$$= \int \frac{A}{x-3}dx + \int \frac{B}{x+1}dx \quad \rightarrow \quad = \int \frac{\frac{1}{4}}{x-3}dx + \int \frac{\frac{1}{4}}{x+1}dx$$

$$= \frac{1}{4}\int \frac{dx}{x-3} - \frac{1}{4}\int \frac{dx}{x+1} \quad \rightarrow \quad = \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1| + c$$

$$= \frac{1}{4}\ln\frac{|x-3|}{|x+1|} + c \quad \rightarrow \quad = \frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + c$$

b) $\int \frac{x^2+5x-1}{x+3}dx$

$$\int \frac{x^2+5x-1}{x+3}dx = \int (x+2)dx - \int \frac{7}{x+3}dx$$

$$= \int x dx + \int 2dx - \int \frac{7}{x+3}dx \quad \longrightarrow \quad = \frac{1}{2}x^2 + 2x - 7\ln|x+3| + c$$

c) $\int \frac{x^4-2x}{4x^2-9}dx$

$$\begin{array}{r|l}
 & \frac{1}{4}x^2 + \frac{9}{16} \\
\hline
4x^2-9 & x^4 - 2x \\
 & \pm x^4 \pm \frac{9}{4}x^2 \\
\hline
 & \frac{9}{4}x^2 - 2x \\
 & \pm \frac{9}{4}x^2 \pm \frac{81}{16} \\
\hline
 & -2x + \frac{81}{16}
\end{array}$$

$$\int \frac{x^4-2x}{4x^2-9}dx = \int \left(\frac{1}{4}x^2 + \frac{9}{16}\right)dx + \int \frac{-2x+\frac{81}{16}}{4x^2-9}$$

$$= \frac{1}{4}\int x^2 dx + \frac{9}{6}\int dx + \int \frac{-2x+\frac{81}{16}}{4x^2-9}dx$$

$$\int \frac{-2x+\frac{81}{16}}{4x^2-9}dx = \int \frac{-2x+\frac{81}{16}}{(2x-3)(2x+3)}dx$$

$$\frac{A}{2x-3} + \frac{B}{2x+3} = \frac{A(2x+3)+B(2x-3)}{(2x-3)(2x+3)}$$

$$A(2x+3)+B(2x-3)=-2x+\frac{81}{16}$$

$$2Ax+3A+2Bx-3B=-2x+\frac{81}{16}$$

$$(2A+3B)x+(3A-3B)=-2x+\frac{81}{16}$$

$$2A+2B=-2$$

$$3A-3B=\frac{81}{16}$$

وبحل المعادلتين الأخيرتين حلاً مشتركاً نجد أن $B=\frac{-43}{32}$ و $A=\frac{11}{32}$

$$\int\frac{-2x+\frac{81}{16}}{4x^2-9}dx=\int\frac{A}{2x-3}dx+\int\frac{B}{2x+3}dx$$

$$\frac{11}{(32)(2)}\int\frac{2}{2x-3}dx+\frac{43}{(32)(2)}\int\frac{2}{2x+3}dx$$

$$=\frac{11}{64}\ln|2x-3|-\frac{43}{64}\ln|2x+3|+c$$

$$\int\frac{-2x+\frac{81}{16}}{4x^2-9}dx=\int\frac{A}{2x-3}dx+\int\frac{B}{2x+3}dx$$

$$=\frac{11}{(32)(2)}\int\frac{2}{2x-3}dx+\frac{43}{(32)(2)}\int\frac{2}{2x+3}dx$$

$$=\frac{11}{64}\ln|2x-3|-\frac{43}{64}\ln|2x+3|+c$$

$$\int\frac{x^4-2x}{4x^2-9}dx=\frac{1}{12}x^3+\frac{19}{16}x+\frac{11}{64}\ln|2x-3|-\frac{43}{64}\ln|2x+3|+c$$

Let $\int f(x)dx = g(x)$

Then $\int_a^b f(x)dx = g(b) - g(a)$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

مثال 8

Find the integral for the following:

a) $\int_3^5 (x^2 - x + 1)dx = \int_3^5 x^2 dx - \int_3^5 x dx + \int_3^5 dx$

$$= \left.\frac{x^3}{3}\right]_3^5 - \left.\frac{x^2}{2}\right]_3^5 + x\Big]_3^5$$

$$= \frac{1}{3}(5^3 - 3^3) - \frac{1}{2}(5^2 - 3^2) + (5-3) = 26.67$$

b) $\int_1^\infty \frac{dx}{x^2}$

$$\int_1^\infty \frac{dx}{x^2} = \lim_{x\to\infty} \int_1^x \frac{dx}{x^2}$$

$$= \underset{x\to\infty}{lin} \frac{-1}{x}$$

$$= \underset{x\to\infty}{lin} \frac{-1}{x} - (-1)$$

$$= \frac{-1}{\infty} + 1$$

$$= 0 + 1 = 1$$

الفصل السابع
Matrices المصفوفات

7-1 Kinds of matrices

1) Square matrix**:**

المصفوفة المربعة: وهي المصفوفة التي يتساوى فيها عدد الصفوف مع عدد الاعمدة.

Example:

$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times2}$, $B=\begin{bmatrix}2 & -3 & 0\\ -4 & 0 & 6\\ 3 & 1 & -3\end{bmatrix}_{3\times3}$

2) Zero matrix**:**

المصفوفة الصفرية: وهي المصفوفة التي جميع عناصرها اصفار ويرمز لها عادة بالرمز **O**.

Example:

$O=\begin{bmatrix}0\\ 0\end{bmatrix}$ or $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

3) Diagonal matrix**:**

المصفوفة القطرية: هي مصفوفة مربعة square يكون جميع عناصرها اصفار عدا القطر الرئيسي $\neq 0$.

Example:

$B=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix} \rightarrow$ B=diag $[1 \quad 2 \quad -1]$

7-2 Equal matrices

المصفوفات المتساوية: هي المصفوفات التي لها نفس العدد من الصفوف والاعمدة.

Example:

$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times2}$, $B=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times2} \rightarrow$ A=B

Example:

Determine the values of the variables :

$$\begin{bmatrix} x+1 & 2 & 3 \\ 4 & y-1 & 5 \\ u & -1 & z+2 \end{bmatrix} = \begin{bmatrix} 2x-1 & t+1 & 3 \\ v+1 & -3 & 5 \\ -4 & w-1 & 2z-1 \end{bmatrix}$$

الحل: من تساوي المصفوفات ينتج تساوي العناصر المتناظرة (أي العناصر التي لها نفس الموقع في المصفوفتين).

$x+1=2x-1 \rightarrow 2x-x=1+1 \rightarrow x=2$
$t+1=2 \rightarrow t=1$
$v+1=4 \rightarrow v=3$
$y-1=-3 \rightarrow y=-2$
$u = -4$
$w-1=-1 \rightarrow w=0$
$z+2=2z-1 \rightarrow 2z-z=2+1 \rightarrow z=3$

7-3 Addition and subtraction of matrices

الجمع والطرح للمصفوفات: يمكن اجراء عملية الجمع والطرح للمصفوفات اذا كان لها نفس الحجم. وتتم عملية الجمع والطرح للمصفوفات وذلك بجمع أو طرح العناصر المتناظرة (أي العناصر التي لها نفس الموقع في المصفوفتين).

Example:

Let $A=\begin{bmatrix} 4 & 3 & 5 \\ 10 & 7 & 6 \end{bmatrix}_{2\times3}$, $B=\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix}_{2\times3}$, $C=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$

الحل: نلاحظ هنا انه يمكن جمع A و B لان لهما نفس الحجم, اما C فلا يمكن جمعها مع A و B لانها مختلفة بالحجم.

$A+B= \begin{bmatrix} 4 & 3 & 5 \\ 10 & 7 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 9 \\ 14 & 10 & 8 \end{bmatrix}_{2\times3}$

$A-B= \begin{bmatrix} 4 & 3 & 5 \\ 10 & 7 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 4 & 4 \end{bmatrix}_{2\times3}$

Determine the values of the variables :

$$\begin{bmatrix} 1 & -1 \\ u & 2 \end{bmatrix} + \begin{bmatrix} x & 4 \\ z & y \end{bmatrix} = \begin{bmatrix} 2 & v+1 \\ 0 & -1 \end{bmatrix}$$

الحل: من تساوي المصفوفات ينتج

$1+x=2 \rightarrow x=1$

$-1+4=v+1 \rightarrow v=2$
$u+z=0 \rightarrow u=z=0$ or $u=-z$
$2+y=-1 \rightarrow y=-3$

7-4 Multiplication of matrices

ضرب المصفوفات :هناك نوعين من الضرب,

1) ضرب المصفوفة في ثابت, وذلك بضرب جميع عناصر المصفوفة بذلك الثابت.

2) ضرب مصفوفتين , يشترط ان يكون اعمدة المصفوفة الاولى يساوي صفوف المصفوفة الثانية.

مثلا", $A_{2\times3}$ $B_{3\times3}$

يجب ان يكونا متساويين

أبعاد المصفوفة الناتجة

1) Scalar multiplication:

Let B= $\begin{bmatrix}4 & 3\\2 & 5\\1 & 10\end{bmatrix}$, c=5 , d=$\frac{1}{5}$

cB= 5 $\begin{bmatrix}4 & 3\\2 & 5\\1 & 10\end{bmatrix} = \begin{bmatrix}20 & 15\\10 & 25\\5 & 50\end{bmatrix}$

$\mathrm{dB}= \frac{1}{5}\begin{bmatrix}4 & 3\\2 & 5\\1 & 10\end{bmatrix} = \begin{bmatrix}4/5 & 3/5\\2/5 & 1\\1/5 & 2\end{bmatrix}$

2) Multiplication of two matrices:

Examples:

_Find the product AB and BA?

Where A= $\begin{bmatrix}4 & 3 & 2\\5 & 6 & 0\end{bmatrix}$ and B= $\begin{bmatrix}1 & -1\\0 & 2\\4 & 3\end{bmatrix}$

$A_{2\times3}B_{3\times2} = \begin{bmatrix}4 & 3 & 2\\5 & 6 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 2\\4 & 3\end{bmatrix} \rightarrow$

$=\begin{bmatrix}(4)(1)+(3)(0)+(2)(4) & (4)(-1)+(3)(2)+(3)(2)\\(5)(1)+(6)(0)+(0)(4) & (5)(1)+(6)(2)+(0)(3)\end{bmatrix} \rightarrow$ $AB_{2\times2}= \begin{bmatrix}12 & 8\\5 & 7\end{bmatrix}$

BA= $\begin{bmatrix}1 & -1\\0 & 2\\4 & 3\end{bmatrix}\begin{bmatrix}4 & 3 & 2\\5 & 6 & 0\end{bmatrix} =$

$$\begin{bmatrix}(1)(4)+(-1)(5) & (1)(3)+(-1)(6) & (1)(2)+(-1)(0)\\(0)(4)+(2)(5) & (0)(3)+(2)(6) & (0)(2)+(2)(0)\\(4)(4)+(3)(5) & (4)(3)+(3)(6) & (4)(2)+(3)(0)\end{bmatrix}$$

$$BA_{3\times3}=\begin{bmatrix}-1 & -3 & -2\\ 10 & 12 & 0\\ 31 & 30 & 8\end{bmatrix}$$

_Find the product of the matrices C and D?

Let $C_{1\times3}=\begin{bmatrix}3 & 0 & -1\end{bmatrix}$, $D_{3\times1}=\begin{bmatrix}5\\3\\2\end{bmatrix}$

$CD_{1\times1}$= [(5)(3)+(0)(3)+(-1)(2)]=16

$$D_{3\times1}C_{1\times3}=\begin{bmatrix}5\\3\\2\end{bmatrix}\begin{bmatrix}3 & 0 & -1\end{bmatrix}$$

$$DC_{3\times3}=\begin{bmatrix}(5)(3) & (5)(0) & (5)(-1)\\ (3)(3) & (3)(0) & (3)(-1)\\ (2)(3) & (2)(0) & (2)(-1)\end{bmatrix} \rightarrow =\begin{bmatrix}19 & 0 & -5\\ 9 & 0 & -3\\ 6 & 0 & -2\end{bmatrix}$$

_ Find the product of the following matrices A, B, C, if possible?

$$A=\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix},\ B=\begin{bmatrix}4 & 3\\ 0 & 5\\ 0 & 5\end{bmatrix},\ C=\begin{bmatrix}3 & 1 & 2\\ 4 & 5 & 6\\ -1 & 4 & 0\end{bmatrix}$$

It is impossible to multiply the matrices:
$A_{2\times2}\times B_{3\times2}$, and $A_{2\times2}\times C_{3\times3}$, and $B_{3\times2}\times C_{3\times2}$

7-5 Identity Matrix:

المصفوفة الاحادية (المتماثلة): هي مصفوفة مربعة جميع عناصرها تساوي صفر عدا القطر الرئيسي diagonal عناصره تساوي واحد .

Examples:

$$,\quad \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\ \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Let $A=\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\rightarrow$ AI=IA=

$$\begin{bmatrix}(2)(1)+(4)(0) & (2)(0)+(4)(1)\\ (-1)(1)+(3)(0) & (-1)(0)+(3)(1)\end{bmatrix}$$

$= \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = A$

7-6 Determinants

المحددات: تعتبر من الخصائص المهمة في دراسة المصفوفات المربعة.

Examples:
Find the determinants for the matrices A and B, where:

$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ -1 & 2 \end{bmatrix}$

det(A)= |A|= 1×4 - 2×3=-2
det(B)= |B|= 5×2 – (-1×6)=16

Find the following determinants: $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \\ -2 & 0 & 6 \end{bmatrix}$

+ + +

det(A)= |A|= $\begin{vmatrix} 1 & 0 & 2 & 1 & 0 \\ 3 & 4 & 5 & 3 & 4 \\ 5 & 6 & 7 & 5 & 6 \end{vmatrix}$, |A|= (20+0+36)-(40+30+0)= -6

\- - -

+ + +

det(B)=|B|= $\begin{vmatrix} 2 & -1 & 0 & 2 & -1 \\ 3 & 4 & 5 & 3 & 4 \\ -2 & 0 & 6 & -2 & 0 \end{vmatrix}$, |B|= (48+10+0)-(0+0-18)=76

\- - -

Transpose of a matrix :المبدلة للمصفوفة

It is obtained by interchanging the rows and columns of A: A_{ij} $A^{T}_{ji} \longrightarrow$

For example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 5 & -1 & 0 \\ 2 & 3 & 4 \\ 6 & 7 & -10 \end{bmatrix}$, $B^T = \begin{bmatrix} 5 & 2 & 6 \\ -1 & 3 & 7 \\ 0 & 4 & -10 \end{bmatrix}$

Properties for transpose:

1) $(A^T)^T=A$

2) $(A+B)^T=A^T+B^T$

) $(AB)^T=B^TA^T$

Let A be a square matrix of order n×n . Then, if $A = A^T$, A is called a **symmetric matrix** .

For example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $A^T= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

Invers of a matrix: معكوس المصفوفة

If A is a square matrix of order n. And if we can find a square matrix of order n, say B, Such that $AB=BA=I_n$ then we say that A is invertable and B is the invers of a matrix A is Denoted by A^{-1}.

1) Inverse matrices are defined for square matrices only.

يعرف المعكوس فقط للمصفوفات المربعة.

2) If A is the inverse of B then B is the inverse of A.

أذا كان A معكوس للمصفوفة B فان B هي ايضا" معكوس للمصفوفة A .

3) If A has an inverse then say that A is invertable .

اذا كان للمصفوفة A معكوس عندئذ يقال بأن A قابلة للأنعكاس.

4) If A has an inverse then it is unique.

أذا كان للمصفوفة A معكوس فهناك معكوس واحد فقط.

5) Not all square matrices are invertable.

ليست جميع المصفوفات المربعة لها قابلية الانعكاس.

Example:

Prove that $B= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, is the inverse of $A= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$AB= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =I_2=BA$

Find the inverse matrix for:

$B= \begin{bmatrix} 4 & 3 \\ 10 & 8 \end{bmatrix}$, $B^{-1}= \frac{adj(B)}{\det(B)}$

$adj(B)= \begin{bmatrix} 8 & -3 \\ -10 & 4 \end{bmatrix}$, $\det(B)= (4)(8)-(3)(10)=32-30=2$

$B^{-1}= \frac{adj(B)}{\det(B)} =\frac{1}{2} \begin{bmatrix} 8 & -3 \\ -10 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -3/2 \\ -5 & 2 \end{bmatrix}$