First Lecture

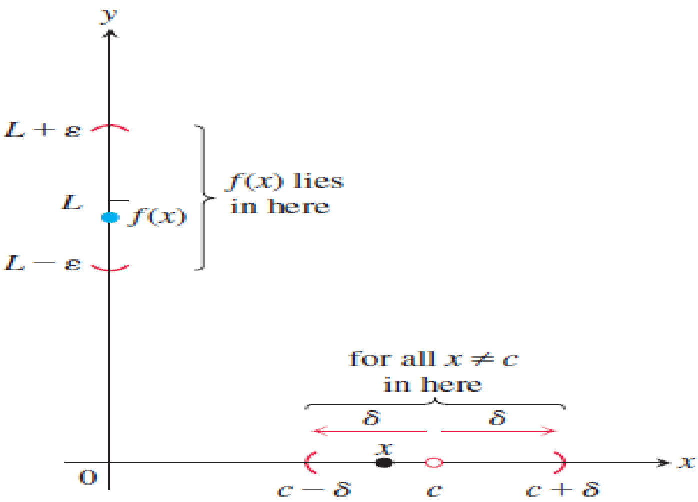
Limits and Theory of Derivative

The Limits

Definition: If f be a function defined at each point of some interval containing a, except possibly at itself. Then a number L is the limit of $f(x)$ as x approaches a (or is the limit of f at a) it for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that If

$$0 < |x - a| < \delta, then\ |f(x) - L| < \varepsilon$$

تعريف: اذا كانت الدالة معرفة عند كل نقطة في الفترة التي تضم النقطة a ، ربما ليست معرفة عند النقطة a . فالعدد L هو غاية الدالة $f(x)$ عندما x تقترب من a يوجد لكل عدد



If L is the limit of $f(x)$ as x approaches a, then we write

$$\lim_{x \to a} f(x) = L$$

If such an L can be found, then we say that the limit of f at a is exists, or that f has a limit at a, or that $\lim_{x \to a} f(x)$ exists. (The limit of the function f at a unique)

آذأ امكن ايجاد غاية قيمتها L ، سوف نقول ان غاية الدالة موجودة، او الدالة لها غاية عندما الغاية

تقترب من العدد a. (الغاية للدالة عند العدد a تكون وحيدة).

One sided limit

If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than **a**), then we write

$$\lim_{x\to a^+} f(x) = L$$

Similarly, If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x\to a^-} f(x) = L$$

Note: we say that the functions have limits

$$\lim_{x\to a} f(x) = L \text{, if and only if } \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$$

Theorem (Limit Law):

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

Examples: -

a) Evaluate the limitations for the following functions:

1) $\lim_{x\to 0}(x^2+3x-1)=(0^2+3(0)-1)=-1$

2) $\lim_{x\to -1}\left(\frac{x^3-2x^2-3x+4}{x^2-5x-5}\right)=\frac{(-1)^3-2(-1)^2-3(-1)+4}{(-1)^2-5(-1)-5}=\frac{-1-2+3+4}{1+5-5}=4$

3) $\lim_{x\to -2}\left(\sqrt{4x^2-3}\right)=\sqrt{4(-2)-3}=\sqrt{13}$

4) $\lim_{x\to -2}(|x|+3)=|-2|+3=2+3=5$

b) Evaluate the following limits:

1) $\lim_{x\to 1}\frac{1}{x-1}$

We cannot substitute $x = 1$, because it makes the denominator zero. So, we are to test the limits from the two sided of this function.

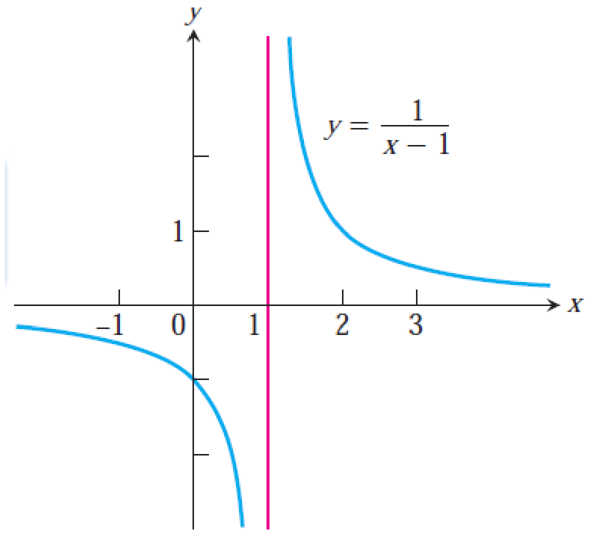
لاحظ اننا لانستطيع نعوض في الغاية مباشرة، وذلك لا المقام سوف تصبح قيمته صفر. ولهذا علينا ان نختبر الغاية من الطرفين.

$$\Longrightarrow \lim_{x\to 1^-}\frac{1}{x-1}=-\infty,\ \lim_{x\to 1^+}\frac{1}{x-1}=+\infty$$

This function is **one-sided limits**, but no limits of this function, because the value of $f(x)$ do not approach a single number as $x\to 1$.

هذه الدالة تملك غاية من جهة وذلك لان الغاية من الجهتين مختلفتين (الغاية ليست وحيدة).

$$i.e\ \lim_{x\to 1^-}\frac{1}{x-1}=-\infty\neq\lim_{x\to 1^+}\frac{1}{x-1}=+\infty$$



2) $\lim_{x\to1}(\frac{x^2+x-2}{x^2-x})$

We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$.

$$\Longrightarrow \lim_{x\to1}\left(\frac{x^2+x-2}{x^2-x}\right) = \lim_{x\to1}\left(\frac{(x-1)(x+2)}{x(x-1)}\right) = \lim_{x\to1}\left(\frac{(x+2)}{x}\right) = \frac{(1+2)}{1} = 3$$

3) $\lim_{x\to0}(\frac{|x|}{x})$

We cannot substitute $x = 0$ because it makes the denominator zero. So, we are to test the limits from the two sided of this function, by the definition of $|x|$, we have

$$\lim_{x\to0^-}\left(\frac{|x|}{x}\right) = \lim_{x\to0^-}\left(\frac{-x}{x}\right) = -1\,,\ \lim_{x\to0^+}\left(\frac{|x|}{x}\right) = \lim_{x\to0^+}\left(\frac{x}{x}\right) = 1$$

This function is one-sided limits, but no limits of this function, because the value of f(x) do not approach a single number as $x \to 0.\ i.e.$

$$\lim_{x\to0^-}\left(\frac{-x}{x}\right) = -1 \neq \lim_{x\to0^+}\left(\frac{x}{x}\right) = 1$$

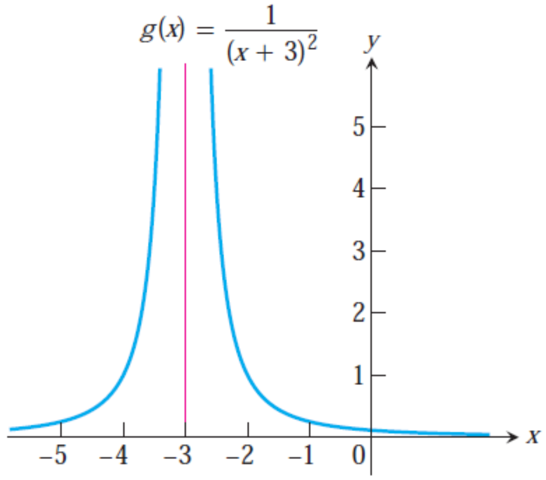
4) $\lim_{x\to-3}\left(\frac{1}{(x+3)^2}\right) = +\infty$

We can verify our solution is correct or not, we are to take the limits for both sided at $x = 3$. We are going to solve it as the above example, so we cannot substitute $x = 3$ because it makes the denominator zero. So, we are to test the limits from the two sided of this function.

$$\lim_{x\to-3^+}\left(\frac{1}{(x+3)^2}\right) = +\infty,\ \lim_{x\to-3^-}\left(\frac{1}{(x+3)^2}\right) = +\infty$$

Since the limits of both sided are equal, so the function has limit at $x = 4$.

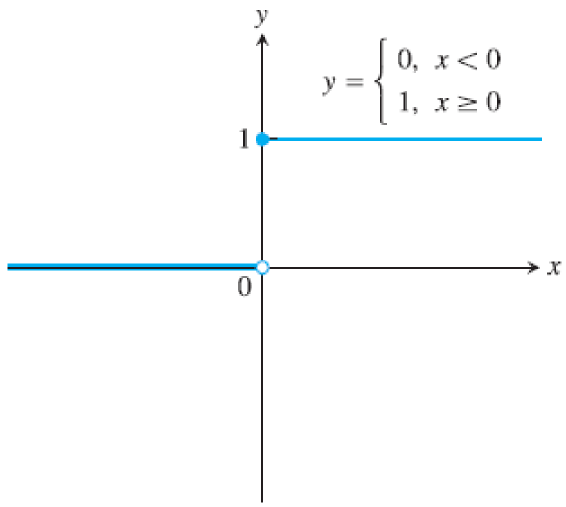
$$\lim_{x\to-3^+}\left(\frac{1}{(x+3)^2}\right) = +\infty = \lim_{x\to-3^-}\left(\frac{1}{(x+3)^2}\right)$$



5) Find the limit for the function $f(x) = \begin{cases} 0, if\ x < 0 \\ 1, if\ x \geq 0 \end{cases}$

We know that $\lim\limits_{x \to 0^-} f(x) = 0$, because $\lim\limits_{x \to 0^-} 0 = 0$. Similarly, $\lim\limits_{x \to 0^+} f(x) = 1$, because $\lim\limits_{x \to 0^+} 1 = 1$.

Since the limits of both sided are equal, so the function has limit at $x = 0$, thus both one sided limit exists, and they are unequal. Therefore, f has no two-sided limit at **0** (**has no limit**).



The Theory of Derivative

Definition: Suppose that x_0 is a number in the domain of the function $\boldsymbol{f}$. If

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

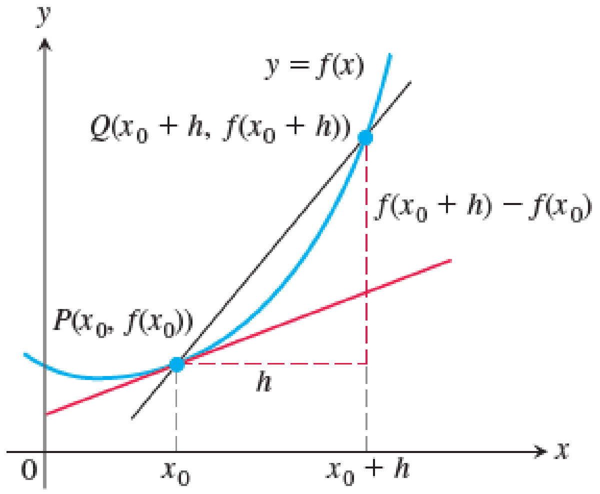
Then the value of this ***limit*** is called **the derivative of f at $x = x_0$**, and is denoted by $(\acute{f}(x), \acute{y}, \frac{dy}{dx}, D_x y)$.

i.e.,

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

This called ***(The Mathematical Definition for Derivative)***

Note: we can replace $\boldsymbol{h}$ by $\Delta \boldsymbol{x}$, $[\Delta x = (x_0 + h) - x_0]$



Definition: The ***slope (ميل)*** of the $y = f(x)$ at the point $p(x_0, f(x_0))$ is the number

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

The ***tangent line (خط المماس)*** to the curve at p is the line through p with this slope. The equation of the tangent line is: $y - f(x_0) = \acute{f}(x_0)(x - x_0)$.

Examples:

خطوات الحل:

نكتب القانون العام للاشتقاق

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

نجد مايلي:

a) تعطى $f(x)$ في السؤال.

b) نجد $f(x_0+h)$ وذلك بتحويل كل x موجودة في الدالة الاصلية الى (x_0+h) ثم نعوض بالقانون العام.

c) نبسط المسألة الى ابسط مايكون.

1) Find the derivative for the following function by using *the definition of derivative.*

a) $f(x) = 7 - 2x$

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f(x) = 7 - 2x$$

$$f(x+h) = 7 - 2(x+h)$$

$$\acute{f}(x) = \lim_{h\to 0}\frac{7-2(x+h)-(7-2x)}{h}$$

$$= \lim_{h\to 0}\frac{7-2x-2h-7+2x}{h}$$

$$= \lim_{h\to 0}\frac{-2h}{h} = -2$$

b) $f(x) = \frac{1}{x}$

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f(x) = \frac{1}{x}$$

$$f(x+h) = \frac{1}{x+h}$$

$$\acute{f}(x) = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{x-x-h}{x.h.(x+h)}$$

$$= \lim_{h\to 0} \frac{-1}{x.(x+h)} = \frac{-1}{x^2}$$

HOMEWORK

Q) By using the definition of the derivative find to $\frac{dy}{dx} = \acute{f}(x)$

1) $3x^3$, **2)** x^2+1, **3)** x^5,

4) $\frac{1}{x+1}$, **5)** $\frac{1}{x^2}$, **6)** $\sqrt{x+1}$.

SECOND LECTURE

Example: ***(Slop of Curve & Tangent Line)***

1)

a) ***Find the slope of the curve*** $\left(y = \frac{1}{x}\right)$ ***at any point*** $x = a \neq 0$.

What is the slope at the point $(x = -1)$?

b) ***Where does the slope equal*** $\left(\frac{-1}{4}\right)$***?***

c) ***What happens to the tangent at the point*** $\left(a, \frac{1}{a}\right)$ ***as a change?***

Solution:

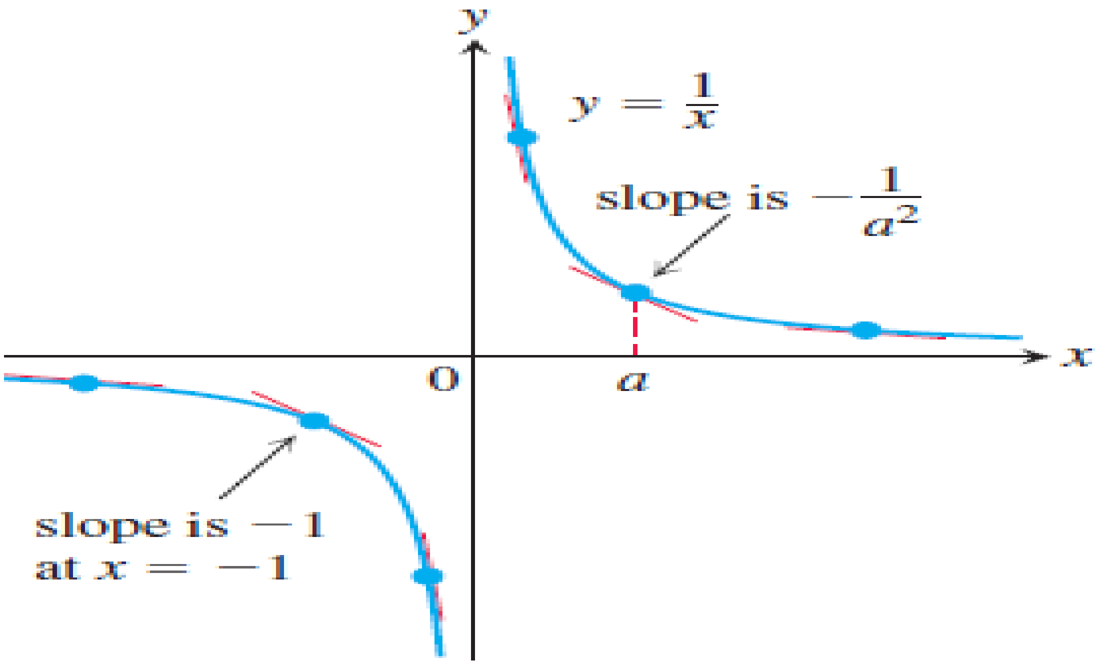
a) Here $f(x) = \frac{1}{x}$, the slope at $\left(a, \frac{1}{a}\right)$ is

$$\acute{f}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

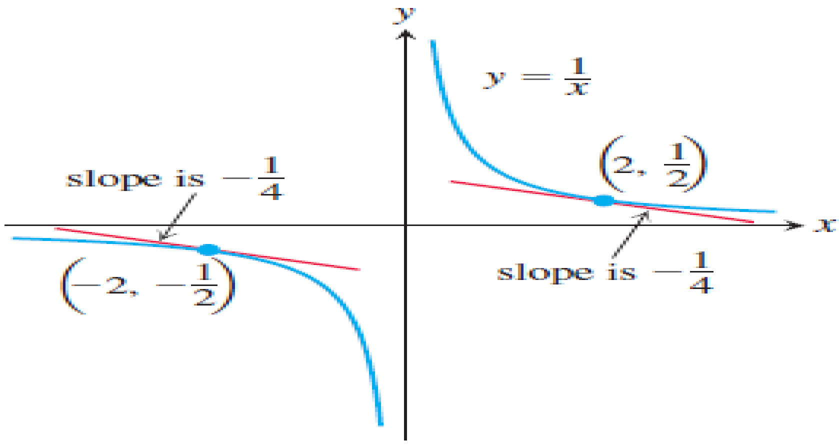
$$\Rightarrow \acute{f}(x) = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{\frac{a-(a+h)}{a(a+h)}}{h}$$

$$= \lim_{h \to 0} \frac{a-a-h}{a.h.(a+h)} = \lim_{h \to 0} \frac{-1}{a.(a+h)} = \frac{-1}{a^2}$$

The slope at the point $(x = -1)$ is $\acute{f}(-1) = \frac{-1}{(-1)^2} = -1$. See the graph below



b) The slope of $y = \frac{1}{x}$ at the point where $(x = a\ is\ \frac{-1}{a^2})$. It will be $(\frac{-1}{4})$ provided that $(\frac{-1}{a^2} = \frac{-1}{4})$. This equation is equivalent to $a^2 = 4$, so $a = \pm 2$. The curve has slope $(\frac{-1}{4})$ at the two points $(2, \frac{1}{2})$ and $(-2, \frac{-1}{2})$. See the graph below



c)*H.W*

2) *Find the slope of the graph of $y = x^2 + 1$, at the point $(2, 5)$ and find the equation of the tangent line at this point.*

Solution

$$\acute{f}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{((x+h)^2+1))-(x^2+1)}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h} = 2x$$

$$\therefore \acute{f}(x) = 4$$

The tangent line is the line through the point (2, 5) with slope 4.

$$\boldsymbol{y - 5 = 4(x-2) \Rightarrow y = 4x - 3.}$$

3) *The function $f(x) = |x|$ has a corner at $x = 0$, prove that f is not differentiable at $x = 0$.*

Solution

$$\acute{f}(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|h|}{h},$$

$$But\ \frac{|h|}{h} = \begin{cases} 1, & if \quad h > 0 \\ -1, & if \quad h < 0 \end{cases},$$

So that $\lim_{h\to 0^-}\left(\frac{|h|}{h}\right) = -1$, $\lim_{h\to 0^+}\left(\frac{|h|}{h}\right) = 1$.

Thus, $\acute{f}(0) = \lim_{h\to 0}\frac{|h|}{h}$ does not exists.

4)

a) *Find by definition of the derivative of*

$$\boldsymbol{f(x) = \sqrt{x}, for\ x > 0}$$

b) *Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$*

Solution:

a) $\acute{f}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$$\acute{f}(x) = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x})}$$

$$\therefore \acute{f}(x) = \lim_{h\to 0}\frac{1}{2\sqrt{x}}$$

The slop at $x = 4 \Longrightarrow \acute{f}(x) = m = \frac{1}{2\sqrt{4}}$

b) The tangent line is the line through $(x = 4, y = \sqrt{4} = 2)$,then the point is $(4,2)$ with slop $\frac{1}{4}$,

$$m = \frac{y - y_1}{x - x_1}$$

$$\Longrightarrow y = 2 + \frac{1}{4}(x - 4) = 2 + \frac{x}{4} - 1 = 1 + \frac{x}{4} = \frac{4 + x}{4}$$

Notation of Differentiation

$$\grave{f}(x) = \grave{y} = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

- $\frac{dy}{dx}$ = *the derivative of y with respect to x*
- $\grave{y}$ = *prime notation*
- *To indicate the value of a derivative at a specified number* $x = a = \cdots$

$$\grave{f}(a) = \frac{dy}{dx}\Big|_{x=a} = \frac{df}{dx}\Big|_{x=a} = \frac{d}{dx}f(x)\Big|_{x=a}$$

For example, $\grave{y} = \grave{f}(x) = \frac{1}{2\sqrt{x}}$.

When $x = 4$

$\grave{f}(x) = \frac{1}{2\sqrt{x}}\Big|_{x=a} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

Techniques of Differentiation

(Rules of Derivative)

Theorem: If f and g are two differentiations, and c is any constant, then

1) $\frac{d}{dx}c = 0$

2) $\frac{d}{dx}x = 1$

3) $\frac{d}{dx}x^n = n.x^{n-1}$

4) $\frac{d}{dx}c.f(x) = c.\frac{d}{dx}f(x)$

5) $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

6) $\frac{d}{dx}[f(x).g(x)] = f(x).\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f\grave{g} + g.\grave{f}$

7) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x).\frac{d}{dx}g(x)}{(g(x))^2} = \frac{g.\grave{f} - f\grave{g}}{g^2}$

8) $\frac{d}{dx}c.(g(x))^n = n.c\big(g(x)\big)^{n-1}.\frac{d}{dx}g(x)$

Examples:

Find $\grave{y}$ for the following function

a) $f(x) = \sqrt{x}$

$\Longrightarrow f(x) = x^{\frac{1}{2}} = \grave{f}(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$

b) $f(x) = (4x^2 - 1)(7x^3 + x)$

$\Rightarrow \grave{f}(x) = (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x)$

$= 140x^4 - 9x^2 - 1$

c) $f(x) = \frac{x^2-1}{x^4+1}$

$\Longrightarrow \grave{f}(x) = \frac{(x^4+1).2x - (x^2-1).4x^3}{(x^4+1)^2} = \frac{2x^5 - 4x^3 - 2x}{(x^4+1)^2}$

Higher Derivative

Definition: If $y = f(x)$ is a differentiable function, then its derivative $\grave{f}(x)$ is also a function. If $\grave{f}$ is also differentiable, then we can differentiate $\grave{f}$ to get a new function of x denoted by $\grave{\grave{f}}$. So, $\grave{\grave{f}} = (\grave{f})'$. The function $\grave{\grave{f}}$ is called the ***second derivative*** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

Examples:

1) If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, find $f^{(5)}$

Solution:

$\grave{f}(x) = 12x^3 - 6x^2 + 2x - 4,$

$f''(x) = 36x^2 - 12x + 2,$

$f^{(3)}(x) = 72x - 12,$

$f^{(4)}(x) = 72,$

$f^{(5)}(x) = 0.$

2) $f(x) = \frac{1}{x}$, find $f^{(3)}$

Solution:

$\grave{f}(x) = \frac{-1}{x^2},$

$f''(x) = \frac{2}{x^3}, \quad f^{(3)}(x) = \frac{-6}{x^4}$

THIRD LECTURE

Implicit Differentiation

Definition: We will say that a given equation in x and y defines the function $\boldsymbol{f}$ implicitly $\boldsymbol{y}$ if the graph of $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ coincides with a portion of the graph of the equation.

نستخدم الاشتقاق الضمني في حالة اعطاء الدالة بمتغيرين او اكثر حيث نشتق التغير المستقل (Independent) اشتقاقا صريحا, اما المتغير المعتمد (Dependent) نشتقه اشتقاقا ضمنيا, اذ ان المتغير المعتمد هو المتغير الذي يعتمد على متغيرات اخرى اما المتغير المستقل فهو المتغير الذي يُشتق بالنسبة له.

Examples:

Find the $\acute{f}(x)$ for the following functions.

1) $\boldsymbol{x^2 + y^2 = 1}$

$$2x + 2y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

2) $\boldsymbol{\frac{1}{y} + xy = x}$

$$-\frac{1}{y^2}\frac{dy}{dx} + \left(x\frac{dy}{dx} + y\right) = 1 \Longrightarrow \frac{dy}{dx}\left(x - \frac{1}{y^2}\right) = 1 - y$$

$$\Longrightarrow \frac{dy}{dx} = \frac{1-y}{\left(\frac{xy^2-1}{y^2}\right)} \Longrightarrow \frac{dy}{dx} = \frac{y^2(1-y)}{xy^2-1}$$

3) $\boldsymbol{(x^2 + y^2)^{\frac{1}{2}} = 1}$

$$\frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}.\left(2x + 2y\frac{dy}{dx}\right) = 0 \Longrightarrow \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dx} = 0$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Chain Rule

Definition: Let $y = f(t)$ and $x = g(t)$, ***the chain rule*** may be written as:

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Examples:

1) *If* $y = t^3 + 6$ *and* $x = 2t + 4$*, find* $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$$

$$\Longrightarrow \frac{dy}{dt} = 3t^2 \quad , \quad \frac{dx}{dt} = 2 \Longrightarrow \frac{dt}{dx} = \frac{1}{2}$$

$$\Longrightarrow \frac{dy}{dx} = 3t^2 . \frac{1}{2} = \frac{3}{2} t^2$$

2) *If* $y = \frac{t^2}{t^2+1}$ *and* $t = \sqrt{2x+1}$*, find* $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$$

$$\Longrightarrow \frac{dy}{dt} = \frac{(t^2+1).(2t) - (t^2).(2t)}{(t^2+1)^2} = \frac{2t^3 + 2t - 2t^3}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$\Longrightarrow \frac{dt}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}} . 2 = \frac{1}{\sqrt{2x+1}}$$

$$\therefore \frac{dy}{dx} = \frac{2t}{(t^2+1)^2} . \frac{1}{\sqrt{2x+1}}$$

$$\Longrightarrow \frac{2\sqrt{2x+1}}{((\sqrt{2x+1})^2+1)^2} . \frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2} . \frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2(x+1)^2}$$

HOMEWORK

a) *Find the first & second derivative for the following equations.*

1) $y = \frac{x^3+7}{x}$

2) $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3}$

3) $w = (\frac{1+3z}{3z})(3 - z)$

4) $w = 3z^2e^{2z}$

b) *Find* $\frac{dy}{dx}$ ***for the function:*** $y^3 + xy^2 + y + x - 8 = 0.$

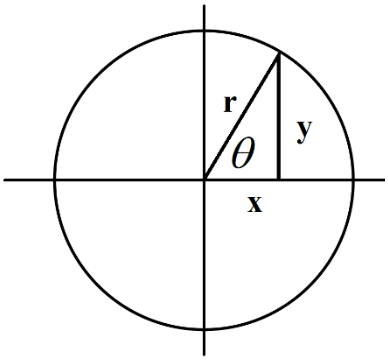
c) *Find* $\frac{dy}{dt}$ ***by using the chain rule for the given equation***

$y = 2u^4 + 4u^2$ **&** $u = 2t^3 + t^2 - t$, ***and evaluate*** $\frac{dy}{dt}$ ***at*** $t = 0.$

FOURTH LECTURE

Transcendental Functions

The Trigonometric Functions



$\boldsymbol{sin\theta = \frac{Oppsite}{Hypotenuse} = \frac{y}{r}}$

$\boldsymbol{cos\theta = \frac{Adjance}{Hypotenuse} = \frac{x}{r}}$

$\boldsymbol{tan\theta = \frac{Oppsite}{Adjance} = \frac{y}{x}}$

$\boldsymbol{sec\theta = \frac{1}{cos\theta} = \frac{r}{x}}$

$\boldsymbol{csc\theta = \frac{1}{sin\theta} = \frac{r}{y}}$

$\boldsymbol{cot\theta = \frac{1}{tan\theta} = \frac{x}{y}}$

By Pythagorean theorem we get:

$sin\theta = \frac{y}{r} \Longrightarrow sin^2\theta = \frac{y^2}{r^2}\ , cos\theta = \frac{x}{r} \Longrightarrow cos^2\theta = \frac{x^2}{r^2}$

$sin^2\theta + cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2+x^2}{r^2} = \frac{r^2}{r^2} = 1$

$$\therefore \boldsymbol{sin^2\theta + cos^2\theta = 1}$$

1) $\theta = 0$

$cos\theta = \frac{x}{r} = \frac{r}{r} = 1 \Longrightarrow cos\theta = 1$

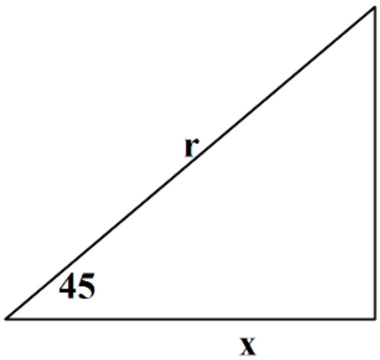
$sin\theta = \frac{y}{r} = \frac{0}{r} = 0 \Longrightarrow sin\theta = 0$

2) $\theta = 90^\circ$

$cos\theta = \frac{x}{r} = \frac{0}{r} = 0 \Longrightarrow cos\theta = 0$

$sin\theta = \frac{y}{r} = \frac{r}{r} = 1 \Longrightarrow sin\theta = 1$

3) $\theta = 45^\circ$



$x^2 + y^2 = r^2 \quad , y = x$

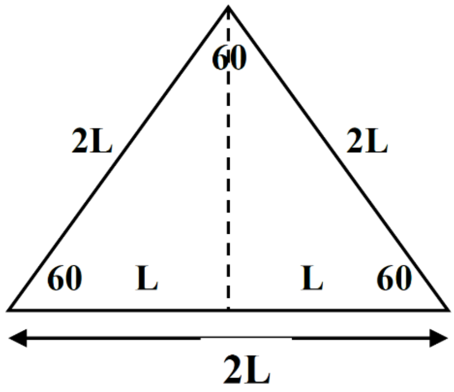
$x^2 + x^2 = r^2 \Longrightarrow 2x^2 = r^2 \Longrightarrow r = x\sqrt{2}$

$y^2 + y^2 = r^2 \Longrightarrow 2y^2 = r^2 \Longrightarrow r = y\sqrt{2}$

$cos\theta = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} \Longrightarrow cos\theta = \frac{1}{\sqrt{2}}$

$sin\theta = \frac{y}{y\sqrt{2}} = \frac{1}{\sqrt{2}} \Longrightarrow sin\theta = \frac{1}{\sqrt{2}}$

4) $\theta = 30^{\circ}$



$x^2 + y^2 = r^2$

$L^2 + y^2 = 4L^2 \Longrightarrow y^2 = 3L^2 \Longrightarrow y = \sqrt{3}L$

by same we get: $x = \sqrt{3}L$

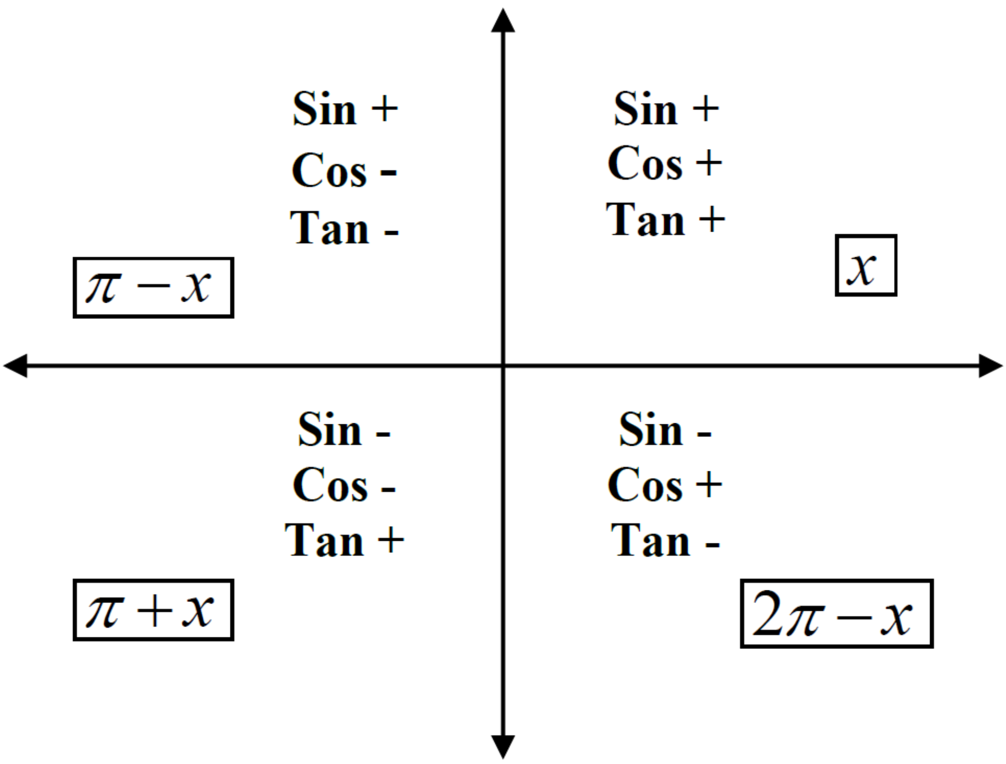
$cos\theta = \frac{x}{r} = \frac{\sqrt{3}L}{2L} \Longrightarrow cos\theta = \frac{\sqrt{3}}{2}$

$sin\theta = \frac{y}{r} = \frac{L}{2L} \Longrightarrow sin\theta = \frac{1}{2}$

5) $\theta = 60^{\circ}$

$cos\theta = \frac{x}{r} = \frac{L}{2L} \Longrightarrow cos\theta = \frac{1}{2}$

$sin\theta = \frac{y}{r} = \frac{\sqrt{3}L}{2L} \Longrightarrow sin\theta = \frac{\sqrt{3}}{2}$



	0	$45=\frac{\pi}{4}$	$30=\frac{\pi}{6}$	$60=\frac{\pi}{3}$	$90=\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin	0	$1/\sqrt{2}$	$1/2$	$\sqrt{3}/2$	1	0	-1	0
Cos	1	$1/\sqrt{2}$	$\sqrt{3}/2$	$1/2$	0	-1	0	1
Tan	0	1	$1/\sqrt{3}$	$\sqrt{3}$	∞	0	∞	0

Some Rule of Trigonometric Function

1) $sin^2\theta + cos^2\theta = 1$

2) $sec^2\theta = 1 + tan^2\theta$

3) $csc^2\theta = 1 + cot^2\theta$

4) $sin(A \pm b) = sinA\, cosB \pm cosA\, sinB$

5) $cos(A \pm b) = cosA\, cosB \mp sinA\, sinB$

6) $tan(A \pm b) = \frac{tan\, A \pm tan\, B}{1 \mp tan^2 A}$

7) $tan(2A) = \frac{2tan\, A}{1 - tan^2 A}$

8) $sin\, 2A = 2\, sin\, A\, cos\, A$

9) $cos\, 2A = cos^2 A - sin^2 A$

10) $cos\, 2A = 1 - 2sin^2 A$

11) $\cos 2A = 2cos^2 A - 1$

12) $sin^2 A = \frac{1}{2}(1 - \cos 2A)$

13) $cos^2 A = \frac{1}{2}(1 + \cos 2A)$

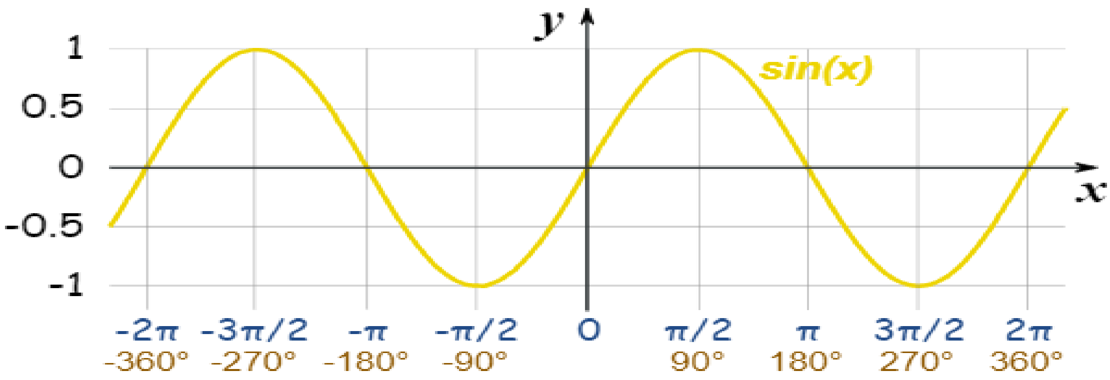
14) $\sin A - \sin B = 2sin\frac{A-B}{2} cos\frac{A+B}{2}$

15) $\sin(A + b) - \sin(A - B) = 2\cos A\, sinB$

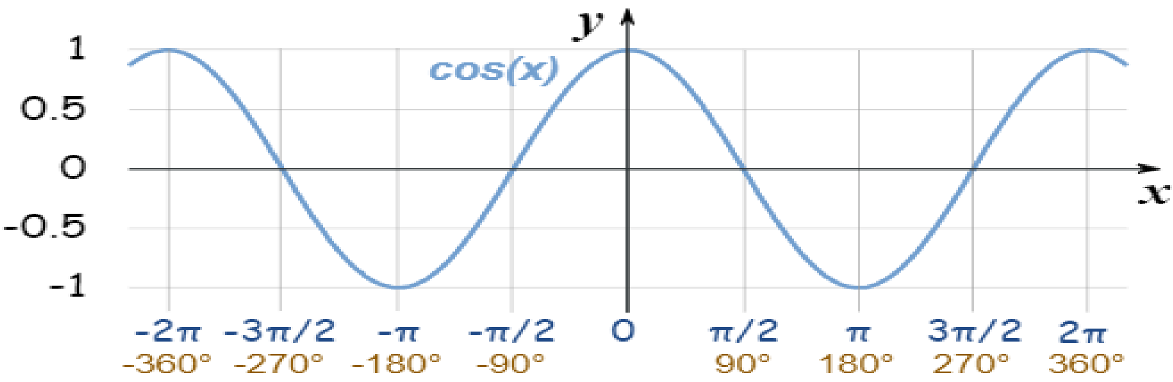
Derivative of Trigonometric Functions

The Rule Derivative

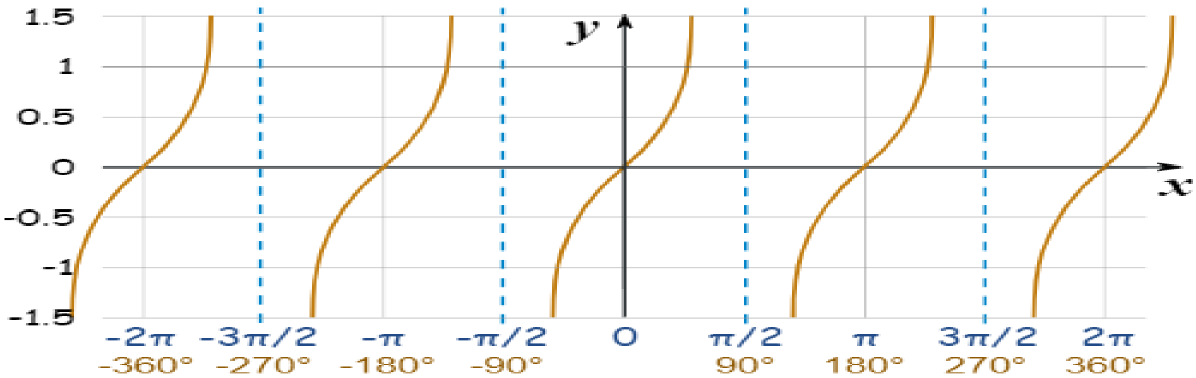
1) $\frac{d}{dx}\sin x = \cos x \Longrightarrow in\ Gen. \Rightarrow \frac{d}{dx}\sin u = \cos u \frac{du}{dx}$



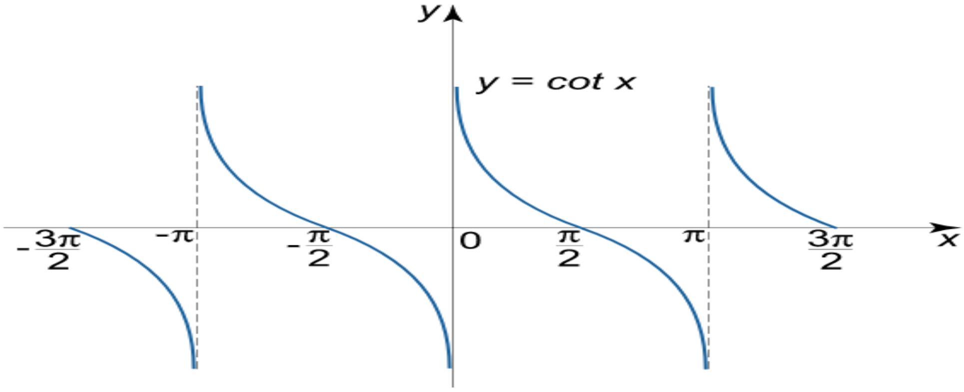
2) $\frac{d}{dx}\cos x = -\sin x \Longrightarrow in\ Gen. \Rightarrow \frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$



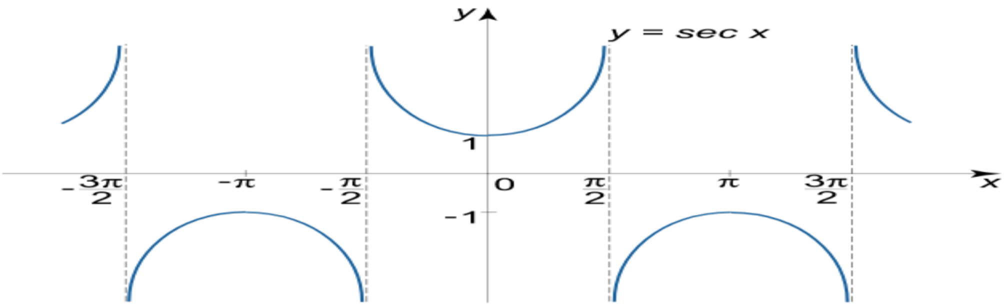
3) $\frac{d}{dx}\tan x = \sec^2 x \Longrightarrow in\ Gen. \Rightarrow \frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$



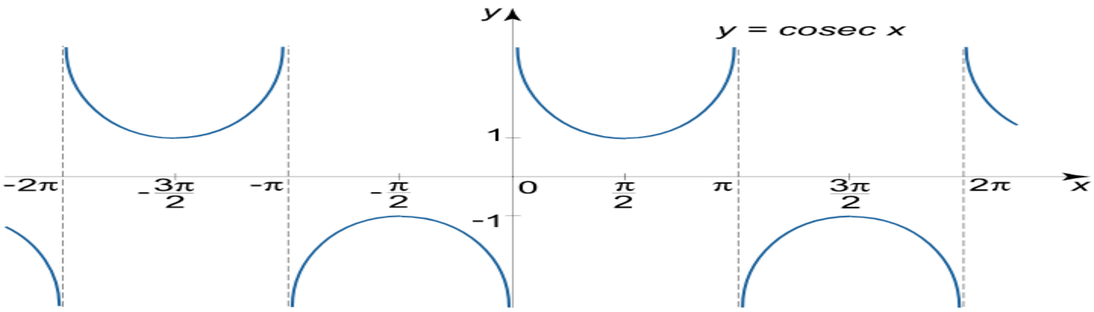
4) $\frac{d}{dx}\cot x = -\csc^2 x \Longrightarrow in\ Gen. \Rightarrow \frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$



5) $\frac{d}{dx}\sec x = \sec x \tan x \Longrightarrow in\ Gen. \Rightarrow \frac{d}{dx}\sec u = \sec u \tan\ u\frac{du}{dx}$



6) $\frac{d}{dx}\csc x = -\csc x \cot x \Longrightarrow in\ Gen. \Rightarrow \frac{d}{dx}\csc u = -\csc u \cot\ u\frac{du}{dx}$



Examples:

Find the derivative of the following functions:

1) $y = \tan 3x$

$\Longrightarrow \frac{dy}{dx} = \grave{y} = \sec^2 3x\,.3$

2) $y = \cos(3x^2 + 1)$

$\Longrightarrow \frac{dy}{dx} = \grave{y} = -\sin(3x^2 + 1).6x$

$3)\ y = x^2 . \tan x$

$\Longrightarrow \frac{dy}{dx} = \grave{y} = x^2 . \sec^2 x + 2x . \tan x$

$4)\ y = \frac{\sin x}{1+\cos x}$

$\Longrightarrow \frac{dy}{dx} = \grave{y} = \frac{(1+\cos x).\cos x - \sin x.(-\sin x)}{(1+\cos x)^2}$

$\Longrightarrow = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$

$5)\ y = \sec x . \tan x$

H.W

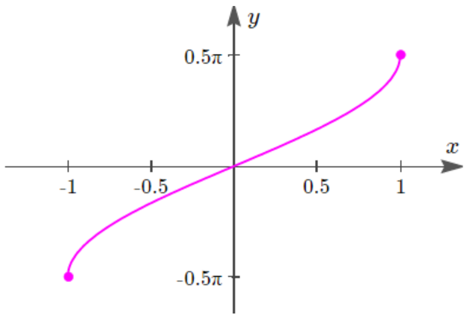
FIFTH LECTURE

The Inverse Trigonometric Functions

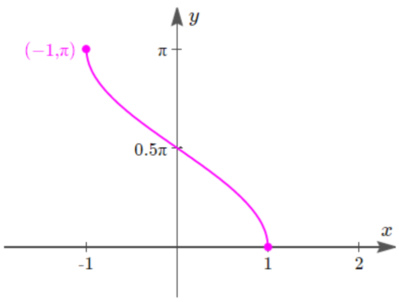
The Sin and Cosine functions can be written as:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad , \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

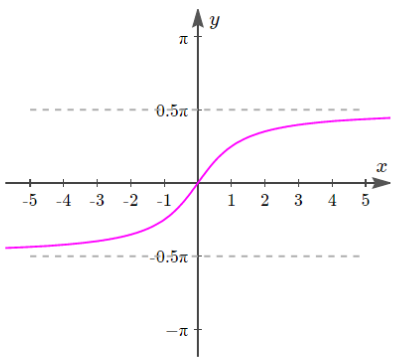
Definition: The inverse **sine fun**. Denoted by $\sin^{-1} x$ is defined to be the inverse of restricted sine fun. $\sin x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



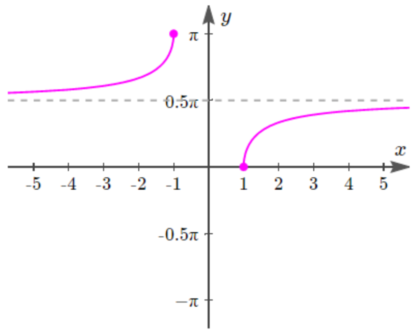
Definition: The inverse **cosine fun**. Denoted by $\cos^{-1} x$ is defined to be the inverse of restricted cosine fun. $\cos x \quad 0 \le x \le \pi$



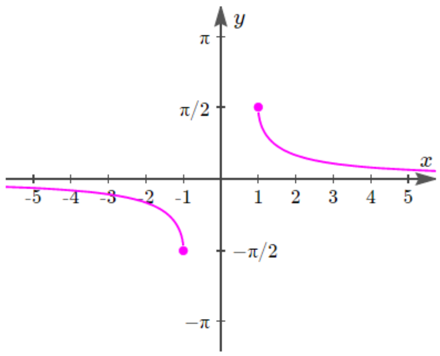
Definition: The inverse **tangent fun**. Denoted by $\tan^{-1} x$ is defined to be the inverse of restricted tangent fun. $\tan x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



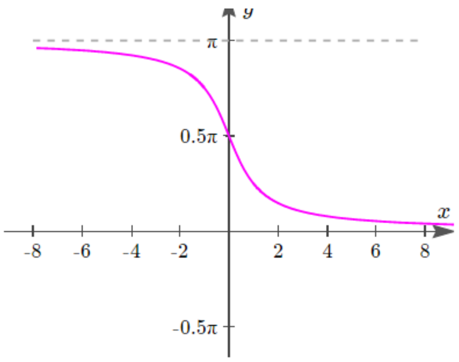
Definition: The inverse **secant fun**. Denoted by $sec^{-1}x$ is defined to be the inverse of restricted secant fun. $sec\, x \quad 0 \le x \le \pi \;, x \ne \frac{\pi}{2}$



Definition: The inverse **cosecant fun**. Denoted by $csc^{-1}x$ is defined to be the inverse of restricted cosecant fun. $csc\, x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



Definition: The inverse **cotangent fun**. Denoted by $cot^{-1}x$ is defined to be the inverse of restricted cosecant fun. $cot\, x \quad 0 \le x \le \pi$



Note: The (-1) in the expression for the inverse means **"inverse "**, it does not mean **reciprocal.** for example, the reciprocal of $\boldsymbol{sin\, x}$ is $(sin\, x) = \frac{1}{sin\, x} = csc\, x,\; sin^{-1}x \ne \frac{1}{sin\, x}$

To get rid any trigonometric function we will take the inverse function to the original function, and to get rid any inverse trigonometric function we will take the inverse function to the original function. The inverse of inverse function is the function the same.

Properties of Inverse Trigonometric Function

1) $sin^{-1} + cos^{-1} = \frac{\pi}{2}$

2) $tan^{-1} + cot^{-1} = \frac{\pi}{2}$

3) $sin^{-1}(-x) = -sin^{-1}(x)$

4) $tan^{-1}(-x) = -tan^{-1}(x)$

5) $cos^{-1}(x) = \frac{\pi}{2} - sin^{-1}(x)$

6) $sec^{-1}(x) = cos^{-1}(\frac{1}{x})$

7) $sec^{-1}(-x) = \pi - sec^{-1}(x)$

8) $cot^{-1}(x) = tan^{-1}(\frac{1}{x})$

9) $csc^{-1}(x) = sin^{-1}(\frac{1}{x})$

10) $cot^{-1}(x) = \frac{\pi}{2} - tan^{-1}(x)$

Note:

- $cos(sin^{-1}x) = \sqrt{1-x^2}$
- $sin(cos^{-1}x) = \sqrt{1-x^2}$
- $tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$
- $sin(tan^{-1}x) = \frac{x}{\sqrt{1+x^2}}$

Derivative of The Inverse Trigonometric Function

1) $\frac{d}{dx} sin^{-1}u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

2) $\frac{d}{dx} cos^{-1}u = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$

3) $\frac{d}{dx} tan^{-1}u = \frac{1}{1+u^2} \frac{du}{dx}$

4) $\frac{d}{dx} cot^{-1}u = \frac{-1}{1+u^2} \frac{du}{dx}$

5) $\frac{d}{dx} sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$,$(1 < |u|)$

6) $\frac{d}{dx} csc^{-1}u = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$,$(1 < |u|)$

Examples:

1) **Let** $y = sin^{-1}u$. **Prove** $\frac{d}{dx} sin^{-1}u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

Solution:

Let $u = sin\, y$, $\Longrightarrow \frac{du}{dx} = \cos y . \frac{dy}{dx}$

From $sin^2 y + cos^2 y = 1$

$\Longrightarrow cos^2 y = 1 - sin^2\, y$

$\Longrightarrow \cos y = \sqrt{1 - sin^2\, y} = \sqrt{1 - u^2}$

$\frac{dy}{dx} = \frac{1}{cos\, y} \frac{du}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-sin^2\, y}} \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$

2) $y = tan^{-1}(\frac{x+1}{x-1})$, **find** $\frac{dy}{dx}$

Solution:

$$\because \frac{d}{dx} tan^{-1}u = \frac{1}{1+u^2} \frac{du}{dx}$$

Where $u = \frac{x+1}{x-1} \Longrightarrow u^2 = (\frac{x+1}{x-1})^2$

$$\frac{du}{dx} = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$\therefore \frac{d}{dx} tan^{-1}u = \frac{1}{1+(\frac{x+1}{x-1})^2} \cdot \frac{-2}{(x-1)^2}$$

Note:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

3) Find $\frac{dy}{dx}$, if

a) $y = \sin^{-1}(x^3)$

Solution:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2 = \frac{3x^2}{\sqrt{1-x^6}}$$

b) $y = \sec^{-1}(e^x)$

Solution:

$$\because \frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}e^x$$

SIXTH LECTURE

The Logarithm and Exponential Functions

The Logarithm Function

Definition: The natural logarithm of a positive number x, written as ln x, is defined as an integral.

$$Ln(x) = \int_{1}^{x} \frac{1}{t} dt, \quad x > 0$$

Definition: The number e is the number in the domain of the natural logarithm that satisfies

$$Ln(e) = \int_{1}^{e} \frac{1}{t} dt = 1$$

Properties of Logarithms

Theorem: For any numbers $b > 0$ and $x > o$, the natural logarithm satisfies the following rules:

1) **Product Rule:** $ln\, ab = ln\, a + ln\, b$

2) **Quotient Rule:** $ln \frac{a}{b} = ln\, a - ln\, b$

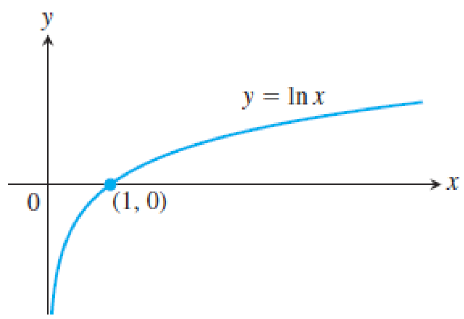
3) **Reciprocal Rule:** $ln \frac{1}{a} = -ln\, a$

4) **Power Rule:** $ln\, a^r = r\, ln\, a,\ for\ r\ rational\ number.$

5) $Ln\ 1 = 0$

6) $Ln\ e = 1, \quad e = 2.178 \ldots$

7) Graph of $y = \ln(x)$ is:



Example:

Apply the result in above theorem

a) $ln\ 4 + ln\ sin\ x = ln\ 4\ sin\ x$ *Product Rule*

b) $ln\ \frac{x+1}{2x-3} = ln\ (x+1) - ln\ (2x-3)$ *Quotient Rule*

c) $l\,n\frac{1}{8} = -l\,n\,8$ *Reciprocal Rule*

$= -l\,n\,2^3$ *Power Rule*

$= -3ln\ 2$

The Derivative of $y = ln\ x$

$$\frac{d}{dx} ln\ x = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}$$

In general form:

$$\frac{d}{dx} ln(u) = \frac{1}{x}\frac{du}{dx}$$

Example:

Find the $\grave{y}$ for the following functions:

1) $y = ln\ x \Longrightarrow \grave{y} = \frac{1}{x}$

2) $y = ln(x^3+1) \Longrightarrow \grave{y} = \frac{3x^2}{x^3+1}$

3) $y = 1 + ln(x-1) \Longrightarrow \grave{y} = \frac{1}{x-1}$

4) $y = ln\ \sqrt[3]{x^2-5} \Longrightarrow y = ln(x^2-5)^{\frac{1}{3}} \Longrightarrow \grave{y} = \frac{1}{3}ln(x^2-5)$

$\Longrightarrow \grave{y} = \frac{2x}{3(x^2-5)}$

5) $y = ln\ sin\ x \Longrightarrow \grave{y} = \frac{1}{sin\ x}cos\ x = cot\ x$

6) $y = ln^2\ 3x \Longrightarrow y = (ln3x)^2$

$\Longrightarrow \grave{y} = 2\ln 3x\,.\frac{1}{3x}3 = \frac{2\ln 3x}{x}$

The Logarithm Differentiation

الاشتقاق اللوغاريتمي

يستخدم الاشتقاق اللوغاريتمي في حالة عدم استطاعتنا اشتقاق الدالة المعطاة بالطرق السابقة وعادة ما تستخدم هذه الطريقة في الدوال المضروبة ببعضها وتحوي على حدود كثيرة وتتلخص الطريقة كالاتي:

1) اخذ لوغاريتم Ln الطرفين.
2) نبسط الدالة باستخدام خواص اللوغاريتمات.
3) نشتق الطرفين ضمنيا بالنسبة ل x.
4) ضرب الطرفين ب y.
5) تعويض قيمة y بما يعادلها بالمتغير x.

Examples:

Find $\frac{dy}{dx}$ the following functions by using Logarithmic Method

1) $y = x^x$

solution

$Ln\, y = ln\, x^x = x\, ln\, x$

$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + Ln\, x(1) = 1 + Ln\, x$

$\frac{dy}{dx} = y(1 + Ln\, x) = x^x(1 + Ln\, x)$

2) $y = \frac{\sqrt[3]{2x^2+1}.\sqrt{2x+1}}{(x^2+1)^4}$

solution

$lny = ln\frac{(2x^2+1)^{\frac{1}{3}}.(2x+1)^{\frac{1}{2}}}{(x^2+1)^4}$

$lny = \ln\left[(2x^2+1)^{\frac{1}{3}}.(2x+1)^{\frac{1}{2}}\right] - \ln(x^2+1)^4$

$lny = \frac{1}{3}\ln(2x^2+1) + \frac{1}{2}\ln(2x+1) - 4\ln(x^2+1)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\frac{1}{(2x^2+1)}.4x + \frac{1}{2}\frac{1}{(2x+1)}.2 - 4\frac{1}{(x^2+1)}.2x$

$y' = y\left[\frac{4x}{3(2x^2+1)} + \frac{1}{(2x+1)} - \frac{8x}{(x^2+1)}\right]$

$y' = \left(\frac{\sqrt[3]{2x^2+1}.\sqrt{2x+1}}{(x^2+1)^4}\right)\left(\frac{4x}{3(2x^2+1)} + \frac{1}{(2x+1)} - \frac{8x}{(x^2+1)}\right)$

3) $y = x^{\sin x}$

Solution

$$lny = lnx^{\sin x}$$

$$lny = \sin x \, lnx$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin(x) \cdot \frac{1}{x} + lnx \cos(x)$$

$$\frac{dy}{dx} = y \left[\frac{\sin(x)}{x} + lnx \cos(x) \right]$$

$$\frac{dy}{dx} = \left(x^{\sin x} \right) \left[\frac{\sin(x)}{x} + lnx \cos(x) \right]$$

4) $y = x^x + x^{lnx}$

Solution:

$$y = y_1 + y_2$$

$$y' = y_1' + y_2'$$

$$y_1 = x^x$$

$$\ln y_1 = \ln x^x \quad \Rightarrow \ln y_1 = x \ln x$$

$$\frac{1}{y_1} \frac{dy_1}{dx} = x \frac{1}{x} + \ln x \, .1$$

$$y_1' = y_1 [1 + \ln x] \quad \Longrightarrow \; y_1' = (x^x)[1 + \ln x]$$

$$\Longrightarrow \; y_1' = [x^x + x^x \ln x]$$

$$y_2 = x^{\ln(x)}$$

$$\ln y_2 = \ln x^{\ln x}$$

$$\ln y_2 = \ln x \ln x$$

$$\frac{1}{y_2} \frac{dy_2}{dx} = \left[\ln x \, \frac{1}{x} + \ln x \, \frac{1}{x} \right]$$

$$\frac{dy_2}{dx} = y_2 \left[\frac{\ln x}{x} + \frac{\ln x}{x} \right]$$

$$y_2' = \left(\ln x^{\ln x} \right) \left[\frac{2 \ln x}{x} \right]$$

$$y' = y_1' + y_2'$$

$$y' = [x^x + x^x \ln x] + \left(\ln x^{\ln x} \right) \left[\frac{2 \ln x}{x} \right]$$

The Logarithm with base a

اللوغاريتم الطبيعي هو اللوغاريتم الذي اساسه e ويمكن ان يكتب بصيغة اخر هي:

$$\log_e x = Ln\, x$$

اما اللوغاريتم الاعتيادي الذي اساسه ليس e مثل اللوغاريتم العشري واللوغاريتم الذي اساسه a, وهنالك علاقة مهمة تربط اللوغاريتم الاعتيادي باللوغاريتم الطبيعي وهي:

$$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a} \;,\; a \neq 1$$

For example: $\log_7 x = \frac{\ln x}{\ln 7}$

The Derivative of $\log_a x$

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{x \ln a}$$

$$\therefore \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

And in general form:

$$\frac{d}{dx}\log_a u = \frac{du}{dx}\frac{1}{u \ln a}$$

Examples:

1) Find $\frac{dy}{dx}$ to $y = \log_2(3x+1)$

Solution

a) $y = \frac{Ln\,(3x+1)}{Ln\,2} \Longrightarrow \frac{dy}{dx} = \frac{3}{(3x+1)Ln\,2}$

b) $\frac{dy}{dx} = \frac{3}{(3x+1)Ln\,2}$

2) Find $\frac{dy}{dx}$ to $y = Ln\,Ln\,x + \log_3(x^2+5)$

Solution

$$\frac{dy}{dx} = \frac{1}{x\,Ln\,x} + \frac{2x}{(x^2+5)Ln\,3}$$

SEVENTH LECTURE

The Exponential Functions

Definition: The function $y = e^x$, is the exponential function, where $e \approx$ 2.178…

Note: the exponential function is the inverse of Log. Fun.

➢ $y = \ln x = \log_e x \Rightarrow e^y = e^{\ln x} = x$

$\therefore y = x$

➢ $e^y = x$

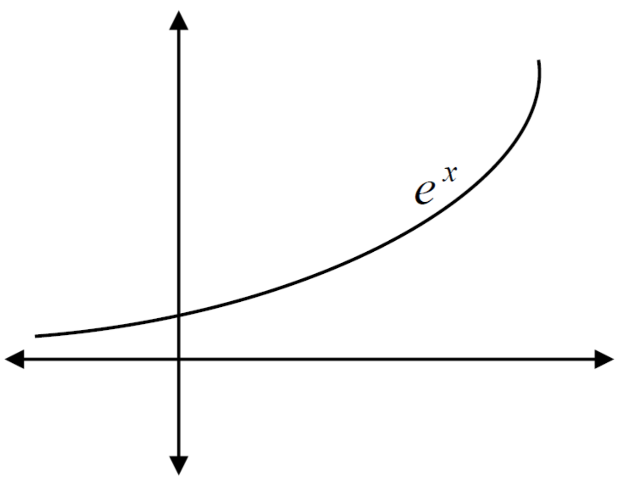
$\ln x = \ln e^y$

$= y \ln e \Rightarrow \ln x = y$

$\therefore y = \ln x$

Properties of Exponential Function

1) $\ln e = 1$
2) $e^0 = 1$
3) $e^{a+b} = e^a . e^b$
4) $e^{-a} = \frac{1}{e^a}$
5) $e^{ab} = (e^a)^b$
6) $e^{m \ln a} = a^m$
7) $e^{\ln x^2} = x^2$
8) ***The graph of*** $y = e^x$ ***is:***



The Derivative of $y = e^x$

In general form:

$$\frac{d}{dx}e^u = e^u.\frac{du}{dx}$$

Examples:

Find the $\grave{y}$ for the following functions

1) $y = e^{2x} \Longrightarrow \grave{y} = 2.e^{2x}$

2) $y = e^{x^3} \Longrightarrow \grave{y} = 3x^2e^{x^3}$

3) $y = e^{\sqrt{1+5x^3}} \Longrightarrow \grave{y} = (\frac{15x^2}{2.\sqrt{1+5x^3}})e^{\sqrt{1+5x^3}}$

4) $y = x^2.e^x \Longrightarrow \grave{y} = x^2.e^x + e^x(2x)$

5) $y = e^{sin^{-1}(x)} \Longrightarrow \grave{y} = \frac{e^{sin^{-1}(x)}}{\sqrt{1-x^2}}$, $\frac{d}{dx}\boldsymbol{sin^{-1}(x)} = \frac{1}{\sqrt{1-x^2}}$

6) $y = e^x.\ln x \Longrightarrow \grave{y} = e^x.\frac{1}{x} + \ln x\,.e^x.1$

$$= e^x\left(\frac{1}{x} + \ln x\right) = e^x\frac{(1+xlnx)}{x}$$

7) $e^{2x} = \sin(x+3y)$

$e^{2x}.2 = \cos(x+3y).(1+3\grave{y})$

$1 + 3\grave{y} = \frac{2e^{2x}}{\cos(x+3y)}$

$\grave{y} = \frac{2e^{2x}}{3\cos(x+3y)} - \frac{1}{3}$

The Exponential Function with Base Other Than a

Definition: If $\boldsymbol{a}$ is any number which is positive, and $\boldsymbol{x}$ is any real number, the function $\boldsymbol{f}$ defined by $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{a}^{\boldsymbol{x}} = \boldsymbol{e}^{\boldsymbol{x}\,\boldsymbol{ln}\,\boldsymbol{a}}, a \text{ is any constant}$, is called ***the exponential function to base a.***

For example: $\mathbf{3}^{x^4}, \mathbf{7}^{x^2}, \ \mathbf{10}^{ln\,x}$

The Derivative of $y = a^x$

$$\frac{d}{dx} a^u = a^u \,.\, ln\, a \,.\, \frac{du}{dx}$$

Examples: ***Find $\frac{dy}{dx}$ to:***

1) $\quad y = 7^{x^2}$

$y' = 7^{x^2}(ln7)(2x)$

$\quad = (ln7)(2x)\, 7^{x^2}$

2) $\quad y = 3^{tan\,(x)}$

$\quad y' = 3^{\tan\,(x)}(ln3)(\sec^2(x))$

3) $\quad y = 2^{sec\,(x)}$

$y' = 2^{\sec(x)}(ln2)\,(\sec(x)\tan\,(x))$

$\Longrightarrow (ln2)\sec(x)\tan(x)\,.\,2^{\sec(x)}$

EIGHTH LECTURE

Application of Derivative

Drawing of curves

لرسم الشكل العام لأي منحني بطريقة استخدام التفاضلات نتبع الخطوات التالية:

1) اوسع مجال للدالة اذا كانت :

a) دالة اعتيادية (متعددة حدود) اوسع مجال لها هو $\boldsymbol{R}$.

b) دالة كسرية (في مقامها متغير) اوسع مجال لها هو $\boldsymbol{R}$ ماعدا القيم الي تجل المقام صفر.

2) ايجاد نقاط التقاطع المحورين وذلك

a) مع محور السينات x وذلك بوضع $\boldsymbol{y=0}$ في الدالة الاصلية ونجد $\boldsymbol{x}$.

b) مع محور الصادات y وذلك بوضع $\boldsymbol{x=0}$ في الدالة الاصلية ونجد $\boldsymbol{y}$.

3) التناظر (Symmetry) ويكون حول:

a) حول المحور الصادات y : عندما نعوض $-x$ بدلا من x في الدالة الاصلية تبقى كما هي

بعبارة اخرى (اذا كانت جميع اسس x زوجية فقط فالدالة متناظرة حول محور الصادات)

$$f(-x)=f(x)$$

b) حول نقطة الاصل: عندما نعوض $-x$ بدلا من x في الدالة الاصلية تتغير اشارتها فقط

بعبارة اخرى (اذا كانت جميع اسس x فردية فقط فالدالة متناظرة حول نقطة الاصل)

$$f(-x)=-f(x)$$

ملاحظة: اذا كانت اسس $\boldsymbol{x}$ زوجية وفردية فانه لايوجد تناظر.

4) المحاذيات (يخص الدوال الكسرية فقط) وتكون:

a) المحاذي العمودي معادلته x=c حيث c كمية ثابتة.

b) المحاذي الافقي معادلته y=c حيث c كمية ثابتة.

لأيجاد معادلة اي محاذي نساوي المقام تلك الدالة للصفر ثم نجد قيم المجهول بحل تلك المعادلة الناتجة فتمثل معادلة المحاذي.

ملاحظة: يمكن ايجاد معادلة المحاذي الافقي

$$y=\frac{\text{معامل } x \text{ في البسط}}{\text{معامل } x \text{ في المقام}} \text{ او } \frac{\text{معامل } x^2 \text{ في البسط}}{\text{معامل } x^2 \text{ في المقام}}=\cdots$$

(اس x في البسط = اس x في المقام والقياس لاعلى اس)

5) النقطة الحرجة وانواعها ومناطق التزايد والتناقص وذلك:

بوضع المشتقة الاولى مساوية لصفر حيث $\acute{f}(x) = 0$

وتكون :

a) نقطة نهاية عظمة محلية وتتغير اشارتها من $+ + +$ الى $- - -$(الاختبار من اليسار الى اليمين).

b) نقطة نهاية صغرى محلية وتتغير اشارتها من $- - -$ الى $+ + +$ (الاختبار من اليسار الى اليمين).

c) مجرد نقطة حرجة اي لاتتغير اشارتها.

ملاحظة:

- اذا كانت : $\acute{f}(x) > 0$ فان y متزايدة
- اذاكانت $\acute{f}(x) < 0$ فان y متناقصة.
- الاتجاه من اليمين اكبر من {$x: x >$ العدد}
- الاتجاه من اليسار اصغر من {$x: x <$ العدد}
- والمنطقة المحددة بين عددين تقرا بشكل فترة فتوحة (العدد الكبير, العدد الصغير)

طريقة ايجاد النهايات ومناطق التزايد والتناقص للدالة:

1) نجد $\acute{f}(x) = 0$

2) نجد المعادلة الناتجة ونجد قيم x.

3) نضع قيم x على الاعداد (يسمى خط اشارة $\acute{y}$ وهو محور السينات)

4) نختار قيم اكبر من x واصغر من x (جوارات x) ونعوضها في $(\acute{y})$ لمعرفة اشارة $\acute{y}$.

5) نحدد مناطق التزايد والتناقص.

6) نحدد العظمى او الصغرى حسب نوع تغير اشارة $\acute{y}$ لاننسى الاختبار من اليسار لليمين

7) نعوض قيم x بالدالة الاصلية $y = f(x)$ فنحصل على قيم y المناظرة لقيم x

فتكون (x, y) هي النهاية العظمة او الصغرى او مجرد حرجة.

6) نقطة الانقلاب

هي نقطة تنتمي للدالة وعندها $\grave{y} = 0$ وتتغير اشارة $\grave{y}$ من + الى – او بالعكس. في الدوال المستمرة تكون نقطة الانقلاب بين كل عظمى وصغرى اوبالعكس, او كل نقطة مجرد حرجة هي نقطة انقلاب.

ملاحظة: اذا كانت $\grave{y} > 0$ (موجبة) فان y مقعرة.

اذا كانت $\grave{y} < 0$ (سالبة) فان yمحدبة.

طريقة ايجاد نقاط الانقلاب ومناطق التحدب والتقعر.

1) نجد $\acute{y}$ ثم $\grave{y} = 0$
2) نحل المعادلة الناتجة ونجد قيم x ونضعها على خط الاعداد (اشارة خط $\grave{y}$)
3) نختار قيم اكبر من x واصغر من x (جوارات x) ونعوضها في ($\grave{y}$) لمعرفة اشارة $\grave{y}$.
4) نحدد مناطق الحدب (----) ومناطق التقعر (++++).
5) تُعرف نقطة الانقلاب من تغير الاشارة حولها, ولاتوجد انقلاب في حالة عدم تغير الاشارة.
6) نعوض قيم x بالدالة الاصلية $y = f(x)$ فنحصل على قيم y المناظرة لقيم x

فتكون (x, y) هي نقطة الانقلاب

Examples:

Draw the following function by using differentiation

1) $f(x) = x^5$

Solution:

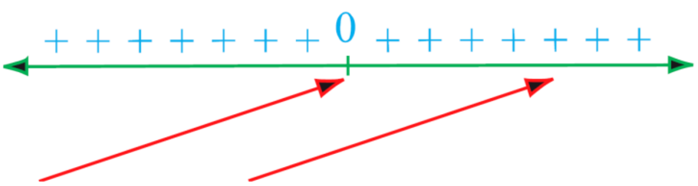
1. Since $f(x)$ is polynomial function the domain= R
2. **The intersection points are:**

   Set $x = 0 \Longrightarrow y = 0^5 = 0$ ... (1)

   $y = 0 \Longrightarrow x^5 = 0 \Longrightarrow x = 0$ ... (2)

   Then the intersection point is $(0,0)$
3. **Symmetry**: There is symmetry about the origin (0, 0) because the exponents of x **are odd**.
4. **Asymptotes**: $f(x)$ has no asymptotes. Because $f(x)$ is not rational function
5. **Maximum and Minimum**:

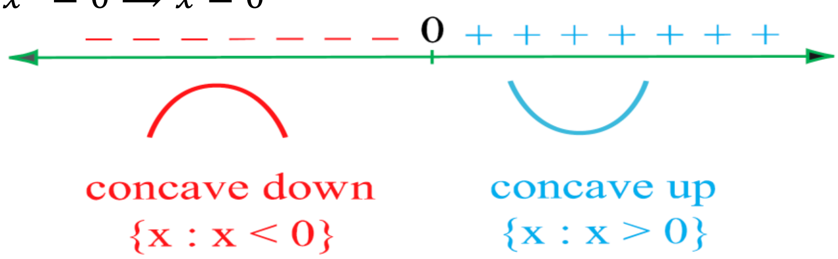
   $\grave{f}(x) = 5x^4$ $\grave{f}(x) = 0 \Longrightarrow 5x^4 = 0 \Longrightarrow x = 0$



$\{x: x < 0\}, \{x: x > 0\}$ increasing $f(x)$

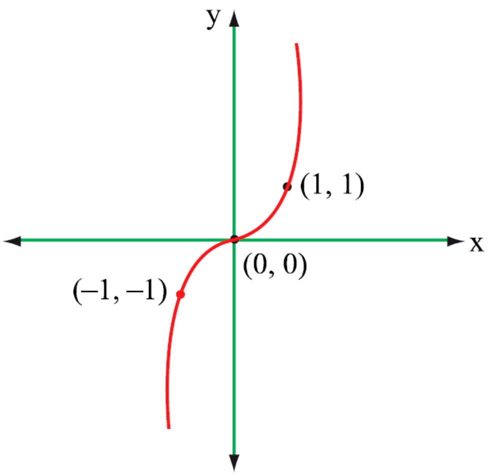
The point $(0,0)$ is critical point

6. $\grave{\grave{f}}(x) = 20x^3 = 0 \Longrightarrow x = 0$



The point $(0,0)$ is inflection point.

x	-2	-1	0	1	2
y	-32	-1	0	1	32



2) $\boldsymbol{y = f(x) = x^3 - 3x^2 + 4}$

Solution:

1. Since $f(x)$ is polynomials function the domain= R
2. **The intersection points are:**

   Set $x = 0 \Longrightarrow y = 0^3 - 3(0)^2 + 4 = 4$

   $y = 0 \Longrightarrow x^3 - 3x^2 + 4 = 0 \Longrightarrow x^2(x-3) = -4$

   $x^2 = -4 \Longrightarrow x = 2 \quad or\ x - 3 = -4 \Longrightarrow x = -1$

   Then the intersection points are $(0,4), (2,0), (-1,0)$
3. **Symmetry**: $f(x)$ is not symmetry. Because the exponents of x **neither even nor odd**.
4. **Asymptotes**: $f(x)$ has no asymptotes. Because $f(x)$ is not rational function.
5. **Maximum and Minimum**:

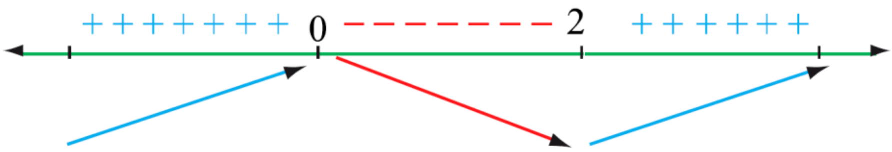
   $\grave{f}(x) = 3x^2 - 6x = 0 \Longrightarrow x(3x - 6) = 0$

   $\Longrightarrow x = 0 \quad or\ 3x - 6 = 0 \Longrightarrow x = 2$

   If $x = 0$ then $f(0) = 4$

   If $x = 2$ then $f(2) = 0$

نعوض قيم x في الدالة الاصلية



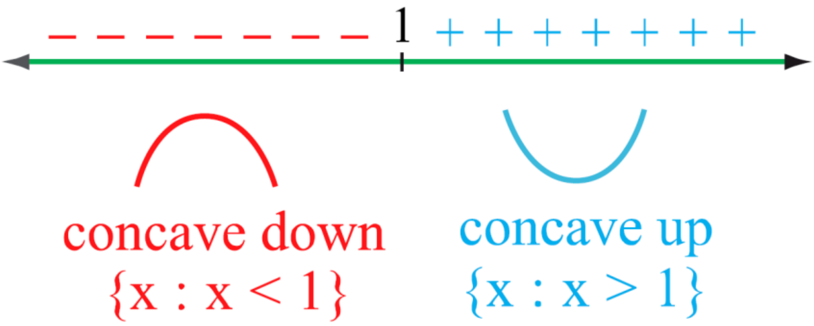
$f(x)$ is increasing on $\{x: x < 0\}, \{x: x > 2\}$

$f(x)$ is decreasing on $(0,2)$

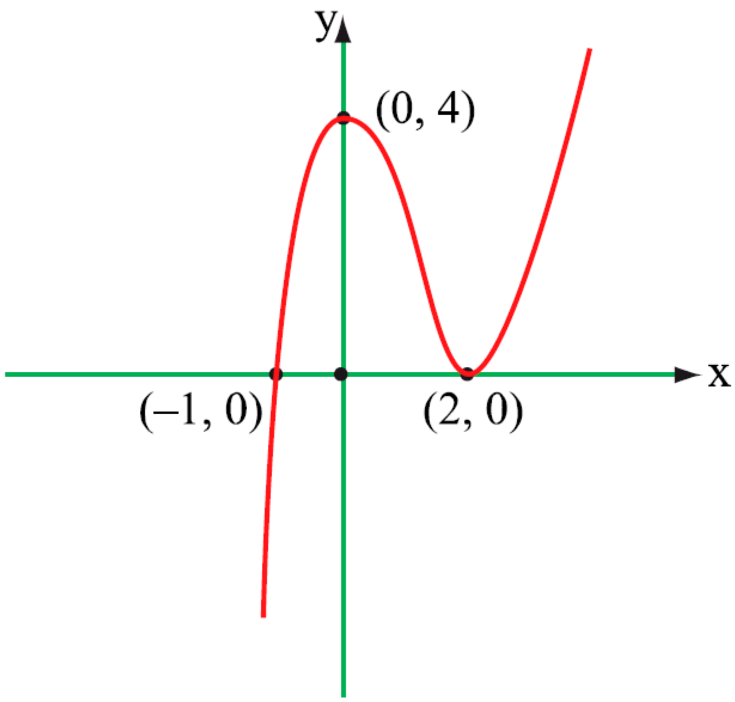
6. $\grave{\grave{f}}(x) = 6x - 6 = 0 \Longrightarrow x = 1$

   $f(1) = 2$

   (1,2) is inflection point.



x	***-1***	***0***	***1***	***2***	***3***
y	***0***	***4***	***2***	***0***	***4***



3) $y = \frac{3x-1}{x+1}$

Solution:

1. Denominator cannot be $0.\ x + 1 = 0 \Longrightarrow x = -1$

Domain $= R/\{-1\}$

2. Since 1 belongs to domain of f but (-1) doesn't belong to domain of f. then the curve is not symmetric to $y\ axis$ and not symmetric to origin point.

3. $x = 0 \Longrightarrow y = -1 \Longrightarrow (0,1) \rightarrow y$ –intercept.

   $y = 0 \Longrightarrow x = \frac{1}{3} \Longrightarrow (\frac{1}{3}, 0) \rightarrow x$ –intercept.

4. $x + 1 = 0 \Longrightarrow x = -1$ (Vertical asymptote)

   $\frac{3x-1}{1x+1} = \frac{3}{1} = 3$ (Horizontal asymptote)

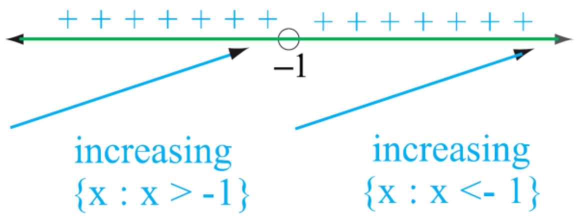
   نجد المحاذي الافقي من خلال

   $y = \frac{\text{معامل } x \text{ في البسط}}{\text{معامل } x \text{ في المقام}}$

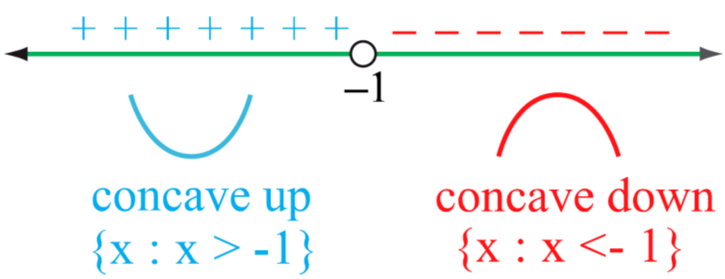
5. $y' = \frac{(x+1).(3)-(3x-1).(1)}{(x+1)^2} = \frac{3x+3-3x+1}{(x+1)^2} = \frac{4}{(x+1)^2}$

   $\forall x \in R/\{-1\}, f'(x) > 0$

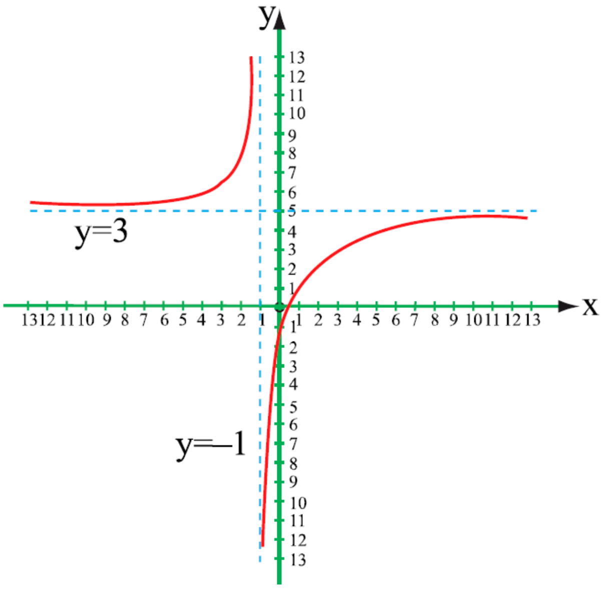
   So f is increasing on $\{x: x < -1\}, \{x: x > -1\}$



6. $y'' = \frac{-8}{(x+1)^3} \neq 0$



Function has no inflection point because (-1) doesn't belong to domain of f.



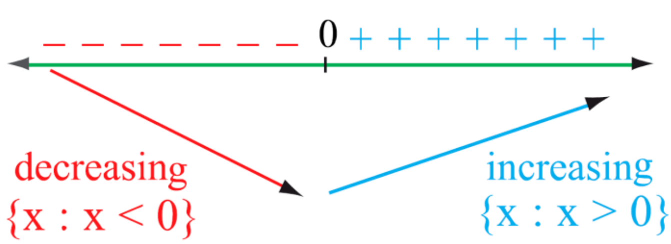
4) $\boldsymbol{y = \frac{x^2}{x^2+1}}$

Solution:

1. Domain of $f = R$.
2. $x = 0 \Rightarrow y = 0$ $\therefore$ (0,0) is x-y- intercepts.
3. There is symmetry about the $\boldsymbol{y-axis}$ because the exponents of x **are even**.
4. $x^2 + 1$ is always different from 0. Then there is no vertical asymptote.

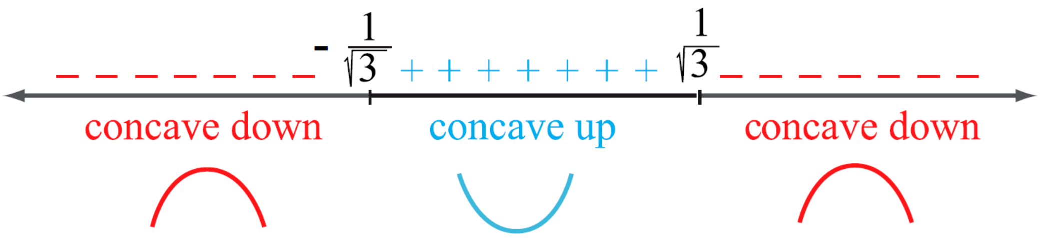
   $\boldsymbol{y} = \frac{\text{معامل } x^2 \text{ بالبسط}}{\text{معامل } x^2 \text{ بالمقام}} = \frac{1}{1} = 1$

5. $\grave{y} = \frac{(x^2+1).(2x)-x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2} \Rightarrow x = 0 \Rightarrow f(0) = 0 \Rightarrow (0,0)$



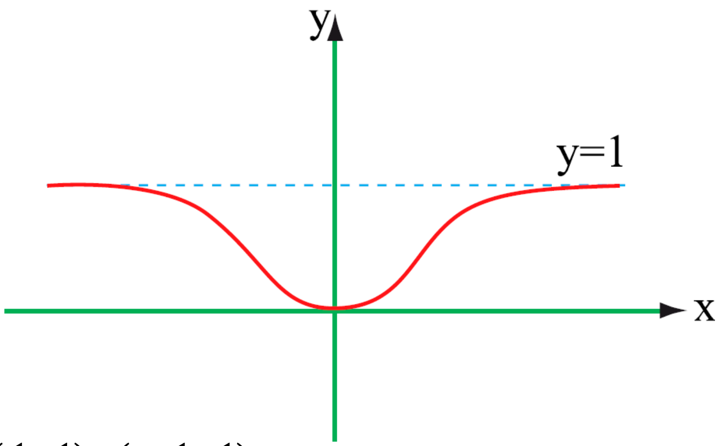
(0,0) is article point

6. $\grave{\grave{y}} = \frac{(x^2+1)^2.2-2x(x^2+1)2x}{(x^2+1)^4} = \frac{2-6x^2}{(x^2+1)^3} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$



$f(x)$ is concave down $\left\{x: x < -\frac{1}{\sqrt{3}}\right\}, \{x: x > \frac{1}{\sqrt{3}}\}$

$f(x)$ is concave up $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$



$f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{1}{4} \Longrightarrow \left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$. Are inflection points.

NINTH LECTURE

Optimization (Maximum, Minimum) Problems

By using differential calculus, we can solve many problems that call for minimizing or maximizing a function to get ***maximum area, minimum velocity, … etc.***

لحل مثل هذه المسائل نتبع الخطوات الاتية:

1) نرسم شكلا توضيحيا للمسالة نعين عليه معاليم المسالة ونفرض عليه مجاهيل المسالة.
2) نكون الدالة التي تقع عليها كلمة اكبر مايمكن او اصغر مايمكن ونشتق عذع الدالة ان كانت بمتغير واحد.
3) اذا كانت الدالة بمتغيرين او اكثر نجد علاقة من بين المتغيرات ونوحد المتغيرات ونجعلها متغير واحد.
4) نساوي المشتقة للصفر ونجد قيم المجاهيل.

Examples:

1) Find two numbers whose product is as large as possible and their sum=20.

solution

First number $= x$

second number $= y$

The product $A = x.y$

The sum $x + y = 20$

$\Rightarrow y = 20 - x$

$\therefore A = x.(20 - x)$

$A = 20x - x^2$

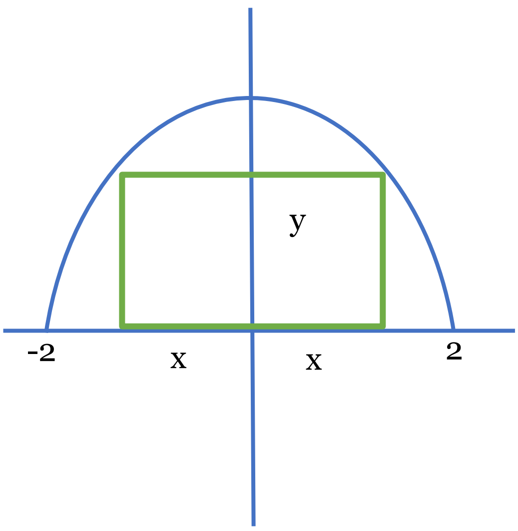
$\frac{dA}{dx} = 20 - 2x = 0$

$\Rightarrow x = 10$

$\Rightarrow y = 20 - 10 = 10$

2) Find maximum rectangle that can be inscribed between the curve $y = 4 - x^2$ and the x-axis and it's vertices lie on the curve and it's base on the x-axis.

solution



$y = 4 - x^2$

When

$y = 0 \Rightarrow 4 - x^2 = 0$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$Area = 2xy$

$A = 2x.(4 - x^2) \Rightarrow A = 8x - 2x^3$

$\frac{dy}{dx} = 8 - 6x^2 = 0$

$\Rightarrow 8 = 6x^2 \Rightarrow x^2 = \frac{8}{6} = \frac{4}{3}$

$x = \pm\frac{2}{\sqrt{3}} \quad \Rightarrow y = 4 - \frac{4}{3} \Rightarrow y = \frac{12-4}{3} = \frac{8}{3}$

3) What is the volume of the largest cone that can be inscribed in a sphere with radius r.

Solution

$v = \frac{\pi}{3}x^2.h$

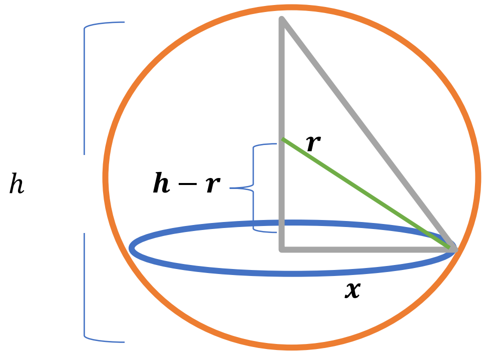
$v = \frac{\pi}{3}[r^2 - (h-r)^2].h$

$v = \frac{\pi}{3}r^2h - \frac{\pi}{3}(h-r)^2.h$

$v = \frac{\pi}{3}r^2h - \frac{\pi}{3}h.(h^2 - 2hr + r^2)$

$v = \frac{\pi}{3}r^2h - \frac{\pi}{3}h^3 + \frac{2\pi}{3}h^2r - \frac{\pi}{3}r^2h$

$\therefore v = \frac{2\pi}{3}h^2r - \frac{\pi}{3}h^3$



$r^2 = x^2 + (h-r)^2$
$x^2 = r^2 - (h-r)^2$

4) A square sheet 50 x 50 cm is used to make an open top box by cutting small square area from its corner and bending up. Find maximum volume that can be got from this sheet

Solution

$v = (50-2x)^2.x$

$= (2500 - 200x + 4x^2).x$

$= 2500x - 200x^2 + 4x^3$

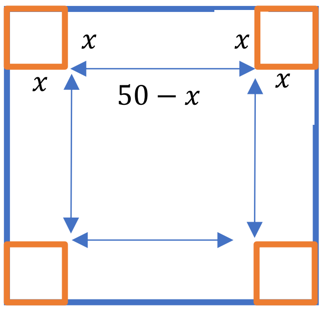
$\frac{dv}{dx} = 2500 - 400x + 12x^2 \qquad \} \div 4$

$= 3x^2 - 100x + 625$

$(x-25)(3x-25) = 0$

either $x - 25 = 0 \rightarrow \quad x = 25$

or $\quad 3x - 25 = 0 \rightarrow \quad x = \frac{25}{3}$



TENTH LECTURE

Related Rates

(Rate of Change)

المعدلات المرتبطة:

مجموعة من المتغيرات تعتمد على نتغير واحد (يسمى بارامیتر) حيث العلاقة هي ارتباط. فأن المعدلات الزمنية تسمى بالمعدلات المرتبطة بالزمن.

اذا كان المتغيران x, y, s, v تابعين كل منهما يرتبط بالمتغير t (مستقل) فأن:

$\frac{dy}{dt}$ = Rate of change of y w.r.t. time

$\frac{dx}{dt}$ = Rate of change of x w.r.t. time

$\frac{ds}{dt}$ = Rate of change of distance with time = velocity = V

$\frac{dv}{dt}$ = Rate of change of velocity with time = acceleration $= \frac{d^2s}{dx^2}$

لحل المسائل بالمعدلات الزمنية نتبع ما يلي:

1) نرسم مخطط للمسألة ان امكن ونحدد عليه الثوابت ونفرض عليه المتغيرات بالرموز.
2) نحدد العلاقة الرئيسية في حل السؤال مثل الحجم او المساحة او اي علاقة ...
3) اذا كان هنالك اكثر من متغيرين فنجد علاقة من معلومات السؤال للتخلص من كثرة المتغيرات.
4) نعوض معطيات السؤال من المتغيرات ولكن بعد الاشتقاق

ملاحظة: عند اشتقاق اي متغير نضربه في $\frac{d\,\text{المتغير}}{dt}$ =المعدل, ويمكن معرفة هذه الاسئلة من ذكر جد المعدل او سرعة او من معطيات المسالة باسم اذا علمت ان المعدل او السرعة ...

Examples:

1) ***How fast does the radius of spherical soap bubble change when a cylindrical tank is drained at a rate of (3 Liter/sec).***

Solution

$v = \pi r^2 h$

$\Longrightarrow v = \pi x^2 h \quad s.t. \quad \boldsymbol{x\ is\ constant}$

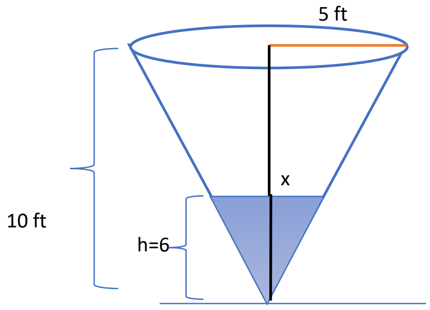
$\frac{dv}{dt} = \pi x^2 \frac{dh}{dt}$ $\qquad\qquad$ $\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt}$

$3 = \pi x^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{3}{\pi x^2}$

2) ***Water runs into the conical tank at the rate*** $(9\ ft^3/min)$ ***, How fast the water level rising when the water is (6 ft) deep all dimensions of the tank as shown in figure.***

Solution



$v = \frac{\pi}{3} x^2 h \ldots\ldots (1)$

$\frac{5}{10} = \frac{x}{h} \ldots\ldots (2)$

$\Rightarrow \quad x = \frac{h}{2}$

$v = \frac{\pi}{3} (\frac{h}{2})^2 h = \frac{\pi}{12}\ h^3 \quad \Rightarrow \frac{dv}{dt} = \frac{\pi 3h^2}{12} \cdot \frac{dh}{dt}$

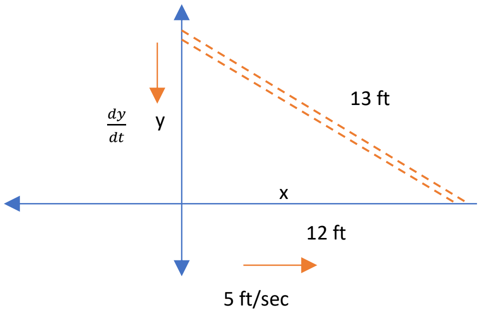
$\therefore \frac{dh}{dt} = \frac{4\frac{dv}{dt}}{\pi h^2} = \frac{4\ (9)}{\pi\ (6)^2} = \frac{1}{\pi}$ $\qquad\qquad$ $\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt}$

3) A (13 ft) ladder is standing against a wall when it's base starts to slide away. By the time, the base is 12-ft from the wall, the base is moving at the rate of (5 ft/sec):

a) How fast the top of the ladder sliding down?

b) How fast the area of the triangle formed by the ladder, wall and the ground are change?

Solution



a) $13^2 = x^2 + y^2$

$x^2 = 13^2 - y^2$

$x = \sqrt{169 - y^2}$

$12 = \sqrt{169 - y^2}$

$144 = 169 - y^2 \quad \Rightarrow \quad y^2 = 169 - 144 = 25 \quad \Rightarrow y = 5\ ft$

$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$0 = 2\,(12)(5) + 2\,(5)\frac{dy}{dt} \quad \Rightarrow \frac{dy}{dt} = \frac{-120}{10} = -12$

b) $A = \frac{x.y}{2}$

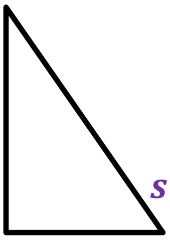
$\frac{dA}{dt} = \frac{1}{2}\left(x.\frac{dy}{dt} + y\frac{dx}{dt}\right)$

$\frac{dA}{dt} = \frac{1}{2}\left(\,12 * (-12) + 5 * (5)\right) \quad \Rightarrow \frac{dA}{dt} = -59.5\ ft^2/sec$

ملحق بالقوانين العامة

قوانين المساحات والحجوم للاشكال الهندسية الثنائية

المثلث القائم الزاوية



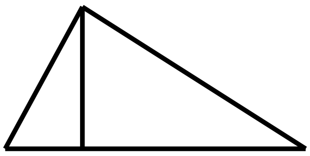
مبرهن فيثاغورس $z^2 = x^2 + y^2$

$\tan\theta = \frac{\text{المقابل}}{\text{المجاور}} = \frac{y}{x}$, $\cos\theta = \frac{\text{المجاور}}{\text{الوتر}} = \frac{x}{z}$, $\sin\theta = \frac{\text{المقابل}}{\text{الوتر}} = \frac{y}{z}$

المساحة = حاصل ضرب ضلعيه القائمين $A = \frac{1}{2}xy$

المحيط = مجموع اطوال اضلاعه الثلاثة $P = x + y + z$

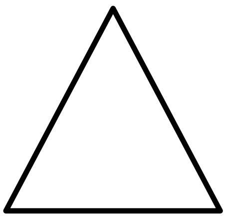
المثلث الاعتيادي



المساحة = نصف القاعدة × الارتفاع ← $A = \frac{1}{2}xh$

المحيط = مجموع اطوال اضلاعه الثلاثة ← $p = x + y + z$

المثلث المتساوي الاضلاع



المساحة = $(\text{طول الضلع})^2 * \frac{\sqrt{3}}{4}$ ← $A = \frac{\sqrt{3}}{4}x^2$

المحيط = (مجموع اطوال اضلاعه الثلاثة) ← $p = 3x$

المربع

المساحة = $(\text{طول الضلع})^2$ ← $A = x^2$

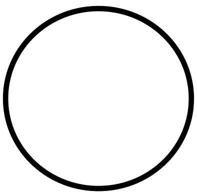
المحيط = طول الضلع × 4 ← $P = 4x$

المستطيل

المساحة = الطول × العرض ← $A = x \cdot y$

المحيط = 2 (الطول + العرض) ← $P = 2(x + y)$

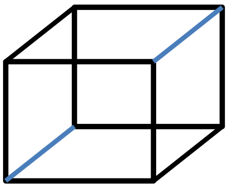
الدائرة



المساحة = $(\text{نصف القطر})^2 \times \pi$ ← $A = \pi r^2$

المحيط = القطر × π ← $P = 2\pi r$

قوانين المساحات والحجوم للاشكال الهندسية ثلاثية الابعاد

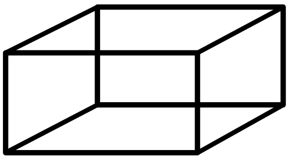


المكعب

الحجم = (طول الضلع)^3 ← $V = x^3$

المساحة الجانبية = 4 × (طول الضلع)^2 ← $L.A = 4x^2$

المساحة السطحية = 6 × (طول الضلع)^2 ← $T.A = 6X^2$

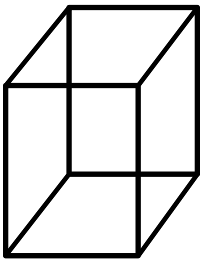


متوازي المستطيلات قاعدته مستطيلة (متوازي السطوح المستطيلة)

الحجم = مساحة القاعدة × الارتفاع ← $V = x\,y\,z$

المساحة الجانبية = محيط القاعدة × الارتفاع ← $L.A = 2(x+y)z$

المساحة السطحية = المساحة الجانبية + ضعف مساحة القاعدة ← $T.A = 2(x+y)\,z + 2xy$

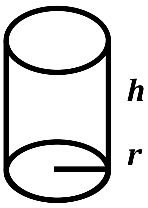


متوازي مستطيلات قاعدته مربعة

الحجم = مساحة القاعدة × الارتفاع ← $V = x^2\,z$

المساحة الجانبية = محيط القاعدة × الارتفاع ← $L.A = 4x\,z$

المساحة السطحية = المساحة الجانبية + ضعف مساحة القاعدة ← $T.A = 4x\;z + 2x^2$

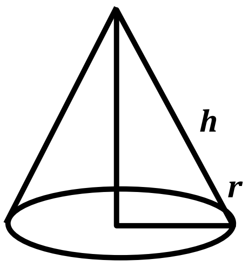


الاسطوانة الدائرية القائمة

الحجم = مساحة القاعدة × الارتفاع ← $V = \pi\,r^2\,h$

المساحة الجانبية = محيط القاعدة × الارتفاع ← $L.A = 2\pi\,r\,h$

المساحة السطحية (الكلية) = المساحة الجانبية + ضعف مساحة القاعدة $T.A = 2\pi rh + 2\pi r^2$

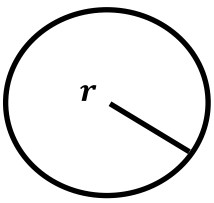


المخروط الدائري القائم

الحجم = ثلث مساحة القاعدة × الارتفاع ← $V = \frac{1}{3}\pi r^2 h$

المساحة الجانبية = نصف محيط القاعدة × طول المولد ← $L.A = \pi r L$

المساحة السطحية = المساحة الجانبية + مساحة القاعدة ← $T.A = \pi r L + \pi r^2$



الكرة

المساحة السطحية ← $A = 4\pi\,r^2$

الحجم ← $V = \frac{4\pi}{3} r^3$